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Some properties of the free stable distributions

Takahiro Hasebe[a], Thomas Simon[b] and Min Wang[b]

[a]*Department of Mathematics, Hokkaido University, North 10 West 8, Kita-Ku, Sapporo 060-0810, Japan. E-mail: thasebe@math.sci.hokudai.ac.jp*
[b]*Laboratoire Paul Painlevé, Université de Lille, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France.*
E-mail: thomas.simon@univ-lille.fr; min.wang@math.univ-lille1.fr

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Abstract. We investigate certain analytical properties of the free α-stable densities on the line. We prove that they are all classically infinitely divisible when $\alpha \leq 1$ and that they belong to the extended Thorin class when $\alpha \leq 3/4$. The Lévy measure is explicitly computed for $\alpha = 1$, showing that free 1-stable distributions are not in the Thorin class except in the drifted Cauchy case. In the symmetric case we show that the free stable densities are not infinitely divisible when $\alpha > 1$. In the one-sided case we prove, refining unimodality, that the densities are whale-shaped, that is their successive derivatives vanish exactly once on their support. We also derive several fine properties of spectrally one-sided free stable densities, including a detailed analysis of the Kanter random variable, complete asymptotic expansions at zero, and several intrinsic features of whale-shaped functions.

Résumé. Nous étudions certaines propriétés analytiques des densités α-stables libres sur la droite. Nous montrons qu'elles sont classiquement infiniment divisibles pour $\alpha \leq 1$ et qu'elles appartiennent à la classe de Thorin étendue pour $\alpha \leq 3/4$. La mesure de Lévy est calculée explicitement pour $\alpha = 1$ et ce calcul entraîne que les lois 1-stables libres n'appartiennent pas à la classe de Thorin, sauf dans le cas de la loi de Cauchy avec dérive. Dans le cas symétrique, nous montrons que les densités α-stables libres ne sont pas infiniment divisibles quand $\alpha > 1$. Dans le cas de signe constant nous montrons que les densités stables libres ont une courbe en baleine, autrement dit que leurs dérivées successives ne s'annulent qu'une seule fois sur leurs supports, ce qui constitue un raffinement de l'unimodalité. Nous établissons enfin plusieurs propriétés précises des densités stables libres spectralement de signe constant, parmi lesquelles une analyse détaillée de la variable aléatoire de Kanter, des expansions asymptotiques complètes en zéro, ainsi que plusieurs propriétés intrinsèques des courbes en baleine.

MSC: 33E20; 46L54; 60E07; 62E15

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1. Introduction

In this paper, we investigate certain properties of real free stable random variables. We say that a real random variable $\mathbf{X}$ is free stable, if for any $a, b > 0$ there exists $c > 0, d \in \mathbb{R}$ such that

$$a\mathbf{X}_1 + b\mathbf{X}_2 \stackrel{d}{=} c\mathbf{X} + d, \tag{1}$$

where $\mathbf{X}_1, \mathbf{X}_2$ are free independent copies of $\mathbf{X}$. As in the classical framework, when $\mathbf{X}$ is not constant it turns out that there exist solutions to (1) only if $c = (a^\alpha + b^\alpha)^{1/\alpha}$ for some fixed $\alpha \in (0, 2]$ which is called the stability parameter.

We will be mostly concerned with free strictly stable densities, which correspond to the case $d = 0$. Every free strictly stable distribution turns out to be equivalent to a distribution whose Voiculescu transform is of the form

$$\phi_{\alpha,\rho}(z) = -e^{\mathrm{i}\pi\alpha\rho} z^{-\alpha+1}, \qquad \Im(z) > 0, \tag{2}$$

where (α, ρ) belongs to the following set $\mathcal{A}$ of admissible parameters:

$$\mathcal{A} = \big\{\alpha \in (0, 1], \rho \in [0, 1]\big\} \cup \big\{\alpha \in (1, 2], \rho \in [1 - 1/\alpha, 1/\alpha]\big\}.$$

Above, we have used the standard terminology that two measures μ, ν on the line are equivalent if there exist real numbers $a > 0, b \in \mathbb{R}$ such that $\mu(S) = \nu(aS + b)$ for every Borel set S. We refer e.g. to [34] for some background on the free additive convolution, to [11] for the original solution to the equation (1), and to the introduction of [24] for the above parametrization (α, ρ), which mimics that of the strict classical framework. Let us also recall that free stable laws appear as limit distributions of spectra of large random matrices with possibly unbounded variance – see [8,16], and that their domains of attraction have been fully characterized in [9,10]. In the following, we will denote by $\mathbf{X}_{\alpha,\rho}$ the random variable whose Voiculescu transform is given by (2), and set $f_{\alpha,\rho}$ for its density. The analogy with the classical case extends to the fact, observed in Corollary 1.3 of [24], that with our parametrization one has

$$\mathbb{P}[\mathbf{X}_{\alpha,\rho} \geq 0] = \rho.$$

For this reason, we will call ρ the positivity parameter of the free strictly random variable $\mathbf{X}_{\alpha,\rho}$. Clearly one has $\mathbb{P}[\mathbf{X}_{\alpha,\rho} \leq 0] = 1 - \rho$ and the Voiculescu transform also shows that $\mathbf{X}_{\alpha,\rho} \stackrel{d}{=} -\mathbf{X}_{\alpha,1-\rho}$. In this paper, some focus will be put on the one-sided case and we will use the shorter notations $\mathbf{X}_{\alpha,1} = \mathbf{X}_\alpha$ and $f_{\alpha,1} = f_\alpha$. Throughout, the random variable $\mathbf{X}_{\alpha,\rho}$ will be mostly handled as a classical random variable via its usual Fourier, Laplace and Mellin transforms, except for a few situations where the free independence is discussed.

Several analytical properties of free stable densities have been derived in the Appendix to [10], where it was shown in particular that they can be expressed in closed form via the inverse of certain trigonometric functions. It is also indicated in [10] that every free stable distribution of stability index $\alpha \neq 1$ is equivalent to a free strictly stable distribution of the same index. The density $f_{\alpha,\rho}$ turns out to be a truly explicit function in three specific situations only, which is again reminiscent of the classical case:

- $f_{2,1/2}(x) = \frac{\sqrt{4-x^2}}{2\pi}$ for $x \in [-2, 2]$, (semi-circular density),
- $f_{1/2}(x) = \frac{\sqrt{4x-1}}{2\pi x^2}$ for $x \geq 1/4$, (inverse Beta density),
- $f_{1,\rho}(x) = \frac{\sin(\pi\rho)}{\pi(x^2+2\cos(\pi\rho)x+1)}$ for $x \in \mathbb{R}$, (standard Cauchy density with drift).

The study of $f_{\alpha,\rho}$ was carried on further in [22,24] where, among other results, several factorizations and series representations were obtained. Our purpose in this paper is to deduce from these results several new and non-trivial properties. Our first findings deal with the infinite divisibility of $\mathbf{X}_{\alpha,\rho}$. Since this random variable is freely infinitely divisible (FID), it is a natural question whether it is also classically infinitely divisible (ID).

Theorem 1. *One has*

(a) *For every* $\alpha \in (0, 1]$ *and* $\rho \in [0, 1]$, *the random variable* $\mathbf{X}_{\alpha,\rho}$ *is* ID.
(b) *For every* $\alpha \in (1, 2]$, *the random variable* $\mathbf{X}_{\alpha,1/2}$ *is not* ID.

Above, the non ID character of $\mathbf{X}_{2,1/2}$ is plain from the compactness of its support. Observe also that by continuity of the law of $\mathbf{X}_{\alpha,\rho}$ in (α, ρ) and closedness in law of the ID property – see e.g. Lemma 7.8 in [38], for every $\alpha \in (1, 2)$ there exists some $\epsilon(\alpha) > 0$ such that $\mathbf{X}_{\alpha,\rho}$ is not ID for all $\rho \in [1/2 - \epsilon(\alpha), 1/2 + \varepsilon(\alpha)]$. We believe that one can take $\varepsilon(\alpha) = 1/\alpha - 1/2$, that is our above result is optimal with respect to the ID property. Unfortunately, we found no evidence for this fact as yet – see Remark 3 for possible approaches.

As it will turn out in the proof, for $\alpha \leq 1$ the ID random variables $\mathbf{X}_{\alpha,\rho}$ have no Gaussian component. A natural question is then the structure of their Lévy measure. We will say that the law of a positive ID random variable is a generalized Gamma convolution (GGC) if its Lévy measure has a density φ such that $x\varphi(x)$ is a completely monotonic (CM) function on $(0, +\infty)$. There exists an extensive literature on such positive distributions, starting from the seventies with the works of O. Thorin. The denomination comes from the fact that up to translation, these laws are those of the random integrals

$$\int_0^\infty a(t)\, d\mathbf{\Gamma}_t,$$

where $a(t)$ is a suitable deterministic function and $\{\mathbf{\Gamma}_t, t \geq 0\}$ is the standard Gamma subordinator. We refer to [12] for a comprehensive monograph with an accent on the Pick functions representation and to the more recent survey [25] for the above Wiener–Gamma integral representation, among other topics. See also Chapters 8 and 9 in [39] for their relationship with Stieltjes functions. In Chapter 7 of [12], this notion is extended to distributions on the real line. Following (7.1.5) therein, we will say that the law of a real ID random variable is an extended GGC if its Lévy measure has a density φ such that $x\varphi(x)$ and $x\varphi(-x)$ are CM as a function of x on $(0, +\infty)$. In order to simplify our presentation, we will also use the notation GGC for extended GGC.

Theorem 2. *For every $\alpha \in (0, 3/4]$ and $\rho \in [0, 1]$, the law of $\mathbf{X}_{\alpha,\rho}$ is a* GGC.

Contrary to the above, we think that this result is not optimal and that the random variable $\mathbf{X}_{\alpha,\rho}$ has a GGC law at least for every $\alpha \in (0, 4/5]$ and $\rho \in [0, 1]$ – see Conjecture 1. During our proof, we will see that for every $\alpha, \rho < 1$ the GGC character of $\mathbf{X}_{\alpha,\rho}$ is a consequence of that of $\mathbf{X}_\alpha$. Unfortunately this simpler question, which is connected to the hyperbolically completely monotonic (HCM) character of negative powers of the classical positive stable distribution, is rather involved. Moreover, we will see in Corollary 1 that the law of $\mathbf{X}_\alpha$ is not a GGC for α close enough to 1.

Our next result deals with the case $\alpha = 1$. According to the Appendix of [10], every free 1-stable distribution is equivalent to a unique distribution whose Voiculescu transform writes

$$\phi_\rho(z) = -2\rho\mathrm{i} - \frac{2(1-2\rho)}{\pi}\log z, \qquad \Im(z) > 0,$$

for some $\rho \in [0, 1]$. By (2), this means that a free 1-stable distribution is equivalent to the law of the free independent sum

$$\mathbf{C}_{a,b} \overset{d}{=} a\mathbf{X}_{1,1/2} + b\mathbf{T},$$

for $a \geq 0, b \in \mathbb{R}$, where $\mathbf{T}$ has Voiculescu transform $-\log z$ and will be called henceforth the exceptional free 1-stable random variable. More precisely, for any $a > 0$ and $b \neq 0$, the random variable $C_{a,b}$ is equivalent to the 1-free non strictly stable random variable whose Voiculescu transform is ϕ_ρ, where $\rho \neq 1/2$ is determined by

$$\frac{a}{\pi b} = \frac{\min(\rho, 1-\rho)}{1-2\rho}.$$

The case $b = 0$ corresponds to $\rho = 1/2$ and to the 1-free symmetric strictly stable random variable, which is the standard Cauchy random variable: one sees from (2) that $\phi_{1/2}$ is the Voiculescu transform of $\mathbf{X}_{1,1/2}$. Observe also that ϕ_0 is the Voiculescu transform of $\frac{2}{\pi}(\mathbf{T} + \log(\pi/2))$ whereas ϕ_1 is that of $-\frac{2}{\pi}(\mathbf{T} + \log(\pi/2))$. Notice finally that in the above parametrization of free 1-stable distributions, the parameter ρ is not a positivity parameter. Actually, as in the classical framework there does not seem to exist a closed formula for $\mathbb{P}[\mathbf{C}_{a,b} \geq 0]$ when $b \neq 0$.

The density of $\mathbf{C}_{a,b}$ can be retrieved from Proposition A.1.3 of [10], in an implicit way. In this paper, taking advantage of a factorization due to Zolotarev for the exceptional classical 1-stable random variable, we obtain the following explicit result.

Theorem 3. *The random variable $\mathbf{C}_{a,b}$ is* ID *without Gaussian component and with Lévy measure*

$$\frac{1}{x^2}\left(\frac{a}{\pi}\mathbf{1}_{\{x\neq 0\}} + |b|\left(1 - \frac{|b^{-1}x|e^{-2|b^{-1}x|}}{1-e^{-|b^{-1}x|}}\right)\mathbf{1}_{\{bx<0\}}\right)dx,$$

where the second term is assumed to be zero if $b = 0$.

This computation implies that the random variable $\mathbf{C}_{a,b}$ is self-decomposable (SD) and that the associated Lévy process has CM jumps, but that its law is not a GGC except for $b = 0$ – see Remark 7. A key-tool for the proof is an identity connecting $\mathbf{T}$ and the free Gumbel random variable – see Proposition 2, providing an analogue of Zolotarev's factorization in the free setting, and which is interesting in its own right.

Our last main result concerns the shape of the densities $f_{\alpha,\rho}$. It was shown in the Appendix to [10] that the latter are analytic on the interior of their support, and strictly unimodal i.e. they have a unique local maximum. These basic properties mimic those of the classical stable densities displayed in the monograph [44]. A refinement of strict unimodality was recently investigated in [29], where it is shown that all classical stable densities are bell-shaped (BS), that is their nth derivative vanishes exactly n times on the interior of their support, as is the case for the standard Gaussian density. The freely strictly 1-stable density $f_{1,\rho}$ is BS, but it is visually clear that this property is not fulfilled neither by $f_{2,1/2}$ nor by $f_{1/2}$. Let us introduce the following alternative refinement of strict unimodality.

Definition. A non-negative function f on $\mathbb{R}$ is said to be *whale-shaped* if its support is a closed half-line, if it is smooth in the interior of its support and vanishes at both ends of its support, and if

$$\sharp\{x \in \mathrm{Supp}\, f,\ f^{(n)}(x) = 0\} = 1$$

for every $n \geq 1$.

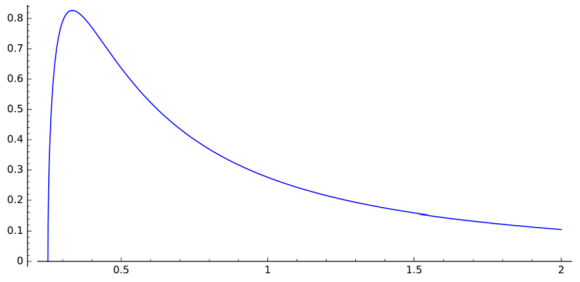

Fig. 1. The free positive 1/2-stable density (WS).

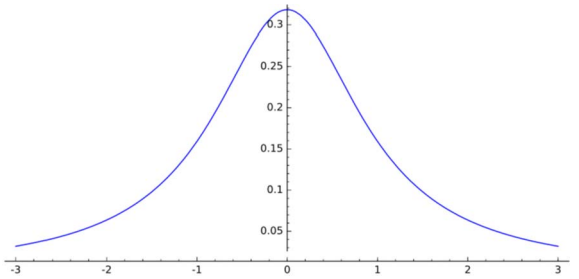

Fig. 2. The free symmetric 1-stable density (BS).

The denomination comes from the visual aspect of such functions – see Figure 1 and compare with the visual aspect of a bell-shaped density given in Figure 2. We will denote by WS the whale-shaped property and set WS_+ (resp. WS_-) for those whale-shaped functions whose support is a positive half-line $[x_0, +\infty)$ for some $x_0 \in \mathbb{R}$, resp. a negative half-line $(-\infty, x_0]$. Observe that if $f \in \mathrm{WS}_+$, then $x \mapsto f(-x)$ belongs to WS_-. It is easy to see that if $f \in \mathrm{WS}_+$ has support $[x_0, +\infty)$, then f is positive on $(x_0, +\infty)$, $f^{(n)}(+\infty) = 0$ and $(-1)^{n-1} f^{(n)}(x_0+) > 0$ for every $n \geq 1$. In particular, the class WBS_0 introduced in the main definition of [40] corresponds to those WS_+ functions whose support is $(0, +\infty)$. Observe finally that the sequence of vanishing places of the successive derivatives of a function in WS_+ increases, by Rolle's theorem. Other less immediate properties of WS functions will be established in Section 3.3.

Theorem 4. *One has*

(a) *For every* $\alpha \in (0, 1)$, *the density* f_α *is* WS_+.
(b) *For every* $\alpha \in (0, 3/4]$ *and* $\rho \in (0, 1)$, *the density* $f_{\alpha,\rho}$ *is* BS.
(c) *The density of* $\mathbf{T}$ *is* WS_-.
(d) *For* $a \neq 0$ *and for* $b = 0$ *or* $ab^{-1} \in \pi\mathbb{Z}$, *the density of* $\mathbf{C}_{a,b}$ *is* BS.

This result leaves open the question of the exact shape of the density for all $\alpha > 1$. Observe that the limiting case $\alpha = 2$ is rather peculiar since it can be elementally shown that its even derivatives never vanish, whereas its odd derivatives vanish only once and at zero. But since the BS property is not closed under pointwise limits, it might be true that $f_{\alpha,\rho}$ is BS whenever its support is $\mathbb{R}$. On the other hand, in spite of Theorem 4(c) we think that for $\alpha \in (1, 2)$ the visually whale-shaped density $f_{\alpha,1/\alpha}$, whose support is a negative half-line, is not WS_-. Indeed, we will see in Proposition 10 that otherwise it would be ID, and we know that this is not true at least for α close enough to 2.

Our four theorems are proved in Section 2. In the last section, we derive further results related to the analysis of the one-sided free stable densities. First, we analyze in more detail the Kanter random variable $\mathbf{K}_\alpha$, which plays an important role in the proof of all four theorems. The range $\alpha < 1/5$ is particularly investigated, and two open questions made in [26] and [14] are answered in the negative. We also derive the full asymptotic expansion of the densities of $\mathbf{X}_\alpha$, $\mathbf{X}_{\alpha,1-1/\alpha}$ and $1 - \mathbf{T}$ at the left end of their support, completing the series representation at infinity which were given in (1.16) of [24]. Finally, we display some properties of whale-shaped functions and densities.

Notation. Throughout, unless otherwise explicitly stated, in any factorization of the type $X \stackrel{d}{=} Y + Z$ or $X \stackrel{d}{=} Y \times Z$, the random variables Y, Z on the right-hand side will be assumed to be classically independent.

2. Proofs of the main results

2.1. *Preliminaries*

The proofs of all four theorems rely on the following result by Haagerup and Möller [22] who, using a general property of the S-transform, have computed the fractional moments of $\mathbf{X}_\alpha$. They obtain

$$\mathbb{E}\big[\mathbf{X}_\alpha^s\big] = \frac{\Gamma(1-s/\alpha)}{\Gamma(2-(1/\alpha-1)s)\Gamma(1-s)} = \left(\frac{1}{1+(1-1/\alpha)s)}\right) \times \left(\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)\Gamma(1-(1/\alpha-1)s)}\right)$$

for $s<\alpha$. Identifying the two factors, we get the following multiplicative identity in law

$$\mathbf{X}_\alpha \stackrel{d}{=} \mathbf{U}^{1-1/\alpha} \times \mathbf{K}_\alpha, \tag{3}$$

where $\mathbf{U}$ is uniform on $(0,1)$ and $\mathbf{K}_\alpha$ is the so-called Kanter random variable. The latter appears in the following factorization due to Kanter – see Corollary 4.1 in [27]:

$$\mathbf{Z}_\alpha \stackrel{d}{=} \mathbf{L}^{1-1/\alpha} \times \mathbf{K}_\alpha, \tag{4}$$

where $\mathbf{L}$ has unit exponential distribution and $\mathbf{Z}_\alpha$ is a classical positive α-stable random variable with Laplace transform $\mathbb{E}[e^{-\lambda \mathbf{Z}_\alpha}] = e^{-\lambda^\alpha}$ and fractional moments

$$\mathbb{E}\big[\mathbf{Z}_\alpha^s\big] = \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}$$

for $s<\alpha$. Observe that the random variable $\mathbf{K}_\alpha$ has fractional moments

$$\mathbb{E}\big[\mathbf{K}_\alpha^s\big] = \frac{\Gamma(1-s/\alpha)}{\Gamma(1-(1/\alpha-1)s)\Gamma(1-s)} \tag{5}$$

for $s<\alpha$, and in particular a support $[b_\alpha,+\infty)$ which is bounded away from zero, with

$$b_\alpha^{-1} = \alpha^{-1}(1-\alpha)^{1-\frac{1}{\alpha}} = \lim_{n\to+\infty} \mathbb{E}\big[\mathbf{K}_\alpha^{-n}\big]^{1/n},$$

by Stirling's formula. The density of $\mathbf{K}_\alpha$ is explicit for $\alpha = 1/2$, with

$$\mathbf{K}_{\frac{1}{2}} \stackrel{d}{=} \frac{1}{4\cos^2(\pi \mathbf{U}/2)} \stackrel{d}{=} \frac{1}{4\mathbf{B}_{\frac{1}{2},\frac{1}{2}}}$$

and where, here and throughout, $\mathbf{B}_{a,b}$ stands for a standard $\beta(a,b)$ random variable with density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$$

on $(0,1)$. Plugging this in (3) yields easily

$$\mathbf{X}_{\frac{1}{2}} \stackrel{d}{=} \frac{1}{4\mathbf{B}_{\frac{1}{2},\frac{3}{2}}}$$

and we retrieve the aforementioned closed expression of $f_{1/2}$. Several analytical properties of the density of $\mathbf{K}_\alpha$ have been obtained in [26]. In particular, Corollary 3.2 in [26] shows that the density of $\mathbf{K}_\alpha - b_\alpha$ is CM, a fact which we will use repeatedly in the sequel.

Remark 1.
(a) Specifying Haagerup and Möller's result to the negative integers yields

$$\mathbb{E}\big[\mathbf{X}_\alpha^{-n}\big] = \frac{1}{n\alpha^{-1}+1}\binom{n\alpha^{-1}+1}{n}, \quad n\geq 0.$$

The latter is a so-called Fuss–Catalan sequence, and it falls within the scope of more general positive-definite sequences studied in [30,32]. With the notations of these papers, one has $\mathbf{X}_\alpha \overset{d}{=} W^{-1}_{1/\alpha,1}$. This implies that f_α can be written explicitly, albeit in complicated form, for $\alpha = 1/3$ and $\alpha = 2/3$ – see (40) and (41) in [32]. It is also interesting to mention that $\mathbf{X}^{-1}_{\frac{1}{2}}$ has Marchenko–Pastur (or free Poisson) distribution, with density

$$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$$

on $(0,4)$. More generally, Proposition A.4.3 in [10] – see also (8) in [32] – shows that $\mathbf{X}^{-1}_{\frac{1}{n}}$ is distributed for each $n \geq 2$ as the $(n-1)$th free multiplicative convolution power of the Marchenko–Pastur distribution.

(b) The negative integer moments of $\mathbf{K}_\alpha$ are given by the simple binomial formula

$$\mathbb{E}\big[\mathbf{K}_\alpha^{-n}\big] = \binom{n\alpha^{-1}}{n}, \quad n \geq 0.$$

This shows that the law of $\mathbf{K}_\alpha^{-1}$ is of the type studied in [31], more precisely it is $\nu(1/\alpha, 0)$ with the notations therein. By Gauss's multiplication formula – see e.g. Theorem 1.5.2 in [1] – and Mellin inversion, this also implies the identity

$$\mathbf{K}^{-1}_{\frac{1}{3}} \overset{d}{=} \mathbf{K}^{-2}_{\frac{2}{3}} \overset{d}{=} 27\mathbf{B}_{\frac{1}{3},\frac{2}{3}}(1-\mathbf{B}_{\frac{1}{3},\frac{2}{3}})$$

in terms of a single random variable $\mathbf{B}_{\frac{1}{3},\frac{2}{3}}$. In particular, the density of $\mathbf{K}_\alpha$ can be written in closed form for $\alpha = 1/3$ and $\alpha = 2/3$ as a two-to-one transform of the density of $\mathbf{B}_{\frac{1}{3},\frac{2}{3}}$ – see also Theorems 5.1 and 5.2 in [31]. As seen above, $\mathbf{K}^{-1}_{\frac{1}{2}} \overset{d}{=} 4\mathbf{B}_{\frac{1}{2},\frac{1}{2}}$ is arc-sine distributed, with density

$$\frac{1}{\pi\sqrt{x(4-x)}}$$

on $(0,4)$. It is well-known that this is the distribution of the rescaled free independent sum of two Bernoulli random variables with parameter $1/2$. It turns out that in general, $\mathbf{K}^{-1}_{\frac{1}{n}}$ is distributed for each $n \geq 2$ as the $(n-1)$th free multiplicative convolution power of a free Bernoulli process at time $n/(n-1)$ – see (6.9) in [31].

(c) The random variable $\mathbf{K}_\alpha$ can be expressed as the following explicit deterministic transformation of a single uniform variable $\mathbf{U}$ on $(0,1)$:

$$\mathbf{K}_\alpha \overset{d}{=} \frac{\sin(\pi\alpha\mathbf{U})\sin^{\frac{1-\alpha}{\alpha}}(\pi(1-\alpha)\mathbf{U})}{\sin^{\frac{1}{\alpha}}(\pi\mathbf{U})}. \tag{6}$$

This is Kanter's original observation – see Section 4 in [27], and it will play an important role in the proof of Theorem 3. Notice that the deterministic transformation involved in (6) appears in the implicit expression of the densities f_α, which is given in the second part of Proposition A.I.4 in [10] – see also (11) in [32] for the case when α is the reciprocal of an integer. There does not seem to exist any computational explanation of this fact. We refer to equation (1) in [18], and also to Proposition 1 and Proposition 2 therein for further results on this transformation.

2.2. *Proof of Theorem 1*

2.2.1. *The case $\alpha \leq 1$*

We begin with the one-sided situation $\rho = 1$. We deduce from (3) and the multiplicative convolution formula that, for any $x > 0$,

$$\begin{aligned} f_\alpha(x+b_\alpha) &= \frac{\alpha}{1-\alpha}\int_1^{1+\frac{x}{b_\alpha}} y^{-\frac{2-\alpha}{1-\alpha}} f_{\mathbf{K}_\alpha}\big(y^{-1}(x+b_\alpha)\big)\,dy \\ &= \frac{\alpha b_\alpha^{\frac{1}{1-\alpha}}}{1-\alpha}\int_0^1 \frac{x}{(b_\alpha+tx)^{\frac{2-\alpha}{1-\alpha}}} f_{\mathbf{K}_\alpha}\left(b_\alpha + \frac{b_\alpha(1-t)x}{b_\alpha+tx}\right)dt. \end{aligned}$$

On the one hand, for every $t \in (0,1)$, the function

$$x \mapsto \frac{(1-t)x}{b_\alpha + tx}$$

is a Bernstein function, using the terminology of [39]. On the other hand, by the aforementioned Corollary 3.2 in [26], the function $z \mapsto f_{\mathbf{K}_\alpha}(b_\alpha + z)$ is CM. It follows from Theorem 3.7 in [39] that $g \circ f \in \mathrm{CM}$ for every $g \in \mathrm{CM}$ and $f \in$ Bernstein. This implies that the function

$$x \mapsto f_{\mathbf{K}_\alpha}\left(b_\alpha + \frac{b_\alpha(1-t)x}{b_\alpha + tx}\right)$$

is CM, and so is

$$x \mapsto (b_\alpha + tx)^{-\frac{2-\alpha}{1-\alpha}} f_{\mathbf{K}_\alpha}\left(b_\alpha + \frac{b_\alpha(1-t)x}{b_\alpha + tx}\right)$$

as the product of two CM functions. Integrating in t shows that $x \mapsto x^{-1} f_\alpha(x + b_\alpha)$ is CM on $(0,\infty)$ and it is a standard and easy fact following from Theorem 1.4 in [39] that this implies the factorization

$$\mathbf{X}_\alpha \overset{d}{=} b_\alpha + \mathbf{\Gamma}_2 \times \tilde{\mathbf{X}}_\alpha$$

for some positive random variable $\tilde{\mathbf{X}}_\alpha$ where, here and throughout, $\mathbf{\Gamma}_t$ stands for a standard $\Gamma(t)$ random variable with density

$$\frac{x^{t-1}e^{-x}}{\Gamma(t)}$$

on $(0,+\infty)$. We can finally deduce from Kristiansen's well-known $\mathbf{\Gamma}_2$ theorem – see [28] – that $\mathbf{X}_\alpha$ is ID.

To handle the two-sided situation $\rho \in (0,1)$, we appeal to the following identity in law which was observed in [24] – see (2.8) therein:

$$\mathbf{X}_{\alpha,\rho} \overset{d}{=} \mathbf{X}_{1,\rho} \times \mathbf{X}_\alpha. \tag{7}$$

Since $\mathbf{X}_{1,\rho}$ has a drifted Cauchy law and since the underlying Cauchy process $\{\mathbf{X}_t^{(1,\rho)}, t \geq 0\}$ is 1-self-similar, the latter identity transforms into

$$\mathbf{X}_{\alpha,\rho} \overset{d}{=} \mathbf{X}_{\mathbf{X}_\alpha}^{(1,\rho)}, \tag{8}$$

which is a Bochner's subordination identity. By e.g. Theorem 30.1 in [38], this finally shows that $\mathbf{X}_{\alpha,\rho}$ is ID for every $\alpha \in (0,1]$ and $\rho \in [0,1]$.

Remark 2.

(a) The above proof shows that

$$s \mapsto \frac{\mathbb{E}[(\mathbf{X}_\alpha - b_\alpha)^s]}{\Gamma(2+s)}$$

is the Mellin transform of some positive random variable. On the other hand, it seems difficult to find a closed formula for the Mellin transform $\mathbb{E}[(\mathbf{X}_\alpha - b_\alpha)^s]$, except in the case $\alpha = 1/2$ where

$$\mathbb{E}\big[(\mathbf{X}_{\frac{1}{2}} - b_{\frac{1}{2}})^s\big] = \frac{2^{1-2s}}{\pi}\Gamma(3/2+s)\Gamma(1/2-s), \quad s \in (-3/2, 1/2).$$

(b) We believe that $\mathbf{X}_\alpha - b_\alpha$ is a $\Gamma_{3/2}$-mixture for every $\alpha \in (0,1)$, that is

$$s \mapsto \frac{\mathbb{E}[(\mathbf{X}_\alpha - b_\alpha)^s]}{\Gamma(3/2+s)}$$

is the Mellin transform of some positive random variable. This more stringent property is actually true for $\alpha \leq 3/4$, as a consequence of the above proof and Theorem 2 – see Remark 10(b).

2.2.2. *The case $\alpha > 1$ and $\rho = 1/2$*
We first derive a closed expression for the Fourier transform of $\mathbf{X}_{\alpha,\rho}$, which has independent interest. It was already obtained as Theorem 1.8 in [24] in a slightly different manner. Our proof is much simpler and so we include it here. Introduce the so-called Wright function

$$\phi(a, b, z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(b + an)n!}$$

with $a > -1$, $b \in \mathbb{R}$, and $z \in \mathbb{C}$. This function was thoroughly studied in the [42,43] for various purposes, and is referenced in Formula 18.1(27) in the encyclopedia [21]. It will play a role in other parts of the present paper.

Lemma 1. *For all $(\alpha, \rho) \in \mathcal{A}$, one has*

$$\mathbb{E}\big[e^{\mathrm{i}t\mathbf{X}_{\alpha,\rho}}\big] = \phi\big(\alpha - 1, 2, -(\mathrm{i}t)^\alpha e^{-\mathrm{i}\pi\alpha\rho\,\mathrm{sgn}(t)}\big), \quad t \in \mathbb{R}.$$

Proof. The case $\alpha = 1$ is an easy and classic computation, since $\mathbf{X}_{1,\rho}$ has a drifted Cauchy distribution and $\phi(0, 2, z) = e^z$. When $\alpha \neq 1$, we first observe that since $\mathbf{X}_{\alpha,\rho} \stackrel{d}{=} -\mathbf{X}_{\alpha,1-\rho}$, it is enough to consider the case $t > 0$.

Let $\mathbf{Z}_{\alpha,\rho}$ be a classical strictly stable random variable having characteristic function

$$\mathbb{E}\big[e^{\mathrm{i}x\mathbf{Z}_{\alpha,\rho}}\big] = e^{-(\mathrm{i}x)^\alpha e^{-\mathrm{i}\pi\alpha\rho}}, \quad x > 0.$$

Using the multiplicative factorization $\mathbf{Z}_{\alpha,\rho} \stackrel{d}{=} \mathbf{X}_{\alpha,\rho} \times \mathbf{\Gamma}_2^{1-1/\alpha}$ of Corollary 1.5 in [24], we obtain

$$e^{-(\mathrm{i}x)^\alpha e^{-\mathrm{i}\pi\alpha\rho}} = \int_0^\infty te^{-t}\mathbb{E}\big[e^{\mathrm{i}xt^{1-1/\alpha}\mathbf{X}_{\alpha,\rho}}\big]\,dt = x^{\frac{2\alpha}{1-\alpha}} \int_0^\infty te^{-tx^{\frac{\alpha}{1-\alpha}}}\mathbb{E}\big[e^{\mathrm{i}t^{1-1/\alpha}\mathbf{X}_{\alpha,\rho}}\big]\,dt$$

for all $x > 0$. On the other hand, a straightforward term-by-term integration implies

$$x^{\frac{2\alpha}{1-\alpha}} \int_0^\infty te^{-tx^{\frac{\alpha}{1-\alpha}}}\phi\big(\alpha - 1, 2, -\big(\mathrm{i}t^{1-1/\alpha}\big)^\alpha e^{-\mathrm{i}\pi\alpha\rho}\big)\,dt = e^{-(\mathrm{i}x)^\alpha e^{-\mathrm{i}\pi\alpha\rho}}, \quad x > 0.$$

The result follows then by uniqueness of the Laplace transform. □

We can now finish the proof of the case $\alpha > 1$, $\rho = 1/2$, where the above lemma reads

$$\mathbb{E}\big[e^{\mathrm{i}t\mathbf{X}_{\alpha,1/2}}\big] = \phi\big(\alpha - 1, 2, -|t|^\alpha\big), \quad t \in \mathbb{R}.$$

Applying Theorem 1 in [42] and some trigonometry, we obtain the asymptotic behaviour

$$\phi\big(\alpha - 1, 2, -t^\alpha\big) \sim \kappa_\alpha t^{-3/2} e^{\cos(\pi/\alpha)\alpha(\alpha-1)^{1/\alpha-1}t}\cos\big(3\pi/2\alpha + \sin(\pi/\alpha)\alpha(\alpha - 1)^{1/\alpha-1}t\big)$$

as $t \to +\infty$, for some $\kappa_\alpha > 0$. This implies that $t \mapsto \mathbb{E}[e^{\mathrm{i}t\mathbf{X}_{\alpha,1/2}}]$ vanishes on $\mathbb{R}$, and hence cannot be the characteristic function of an ID distribution – see e.g. Lemma 7.5 in [38].

□

Remark 3.

(a) It was recently shown in Theorem 1 of [5] that for any $a, \beta > 0$, the function $\phi(a, \beta, -z)$ has only positive zeroes on $\mathbb{C}$. Combined with Lemma 1, this entails that the function $t \mapsto \mathbb{E}[e^{\mathrm{i}t\mathbf{X}_{\alpha,\rho}}]$ never vanishes on $\mathbb{R}$ for $\alpha > 1$ and $\rho \neq 1/2$, so that the above simple argument cannot be applied. Nevertheless, we conjecture that $\mathbf{X}_{\alpha,\rho}$ is not ID for all $\alpha > 1$ and $\rho \in [1 - 1/\alpha, 1/\alpha]$.

(b) When $\rho = 1/\alpha$, Lemma 1 also gives the moment generating function

$$\mathbb{E}\big[e^{\lambda\mathbf{X}_{\alpha,1/\alpha}}\big] = \phi\big(\alpha - 1, 2, \lambda^\alpha\big) = \prod_{n\geq 1}\left(1 + \frac{\lambda^\alpha}{\lambda_{\alpha,n}}\right), \quad \lambda \geq 0,$$

where $0 < \lambda_{\alpha,1} < \lambda_{\alpha,2} \ldots$ are the positive zeroes of $\phi(\alpha - 1, 2, -z)$. Above, the product representation is a consequence of the Hadamard factorization for the entire function $\phi(\alpha - 1, 2, z)$ which is of order < 1 – see again Theorem 1 in [42],

whereas the simplicity of the zeroes follows from the Laguerre theorem on the separation of zeroes for $\phi(\alpha-1,2,z)$, which has genus 0. Consider now the random variable

$$\mathbf{Y}_\alpha = b_{1/\alpha}^{-1/\alpha} - \mathbf{X}_{\alpha,1/\alpha} = \alpha(\alpha-1)^{1/\alpha-1} - \mathbf{X}_{\alpha,1/\alpha},$$

whose support is $(0,\infty)$ by Proposition A.1.2 in [10], and whose infinite divisibility amounts to that of $\mathbf{X}_{\alpha,1/\alpha}$. Its log-Laplace transform reads

$$\begin{aligned} -\log \mathbb{E}\big[e^{-\lambda \mathbf{Y}_\alpha}\big] &= \alpha(\alpha-1)^{1/\alpha-1}\lambda - \sum_{n\geq 1}\log\left(1+\frac{\lambda^\alpha}{\lambda_{\alpha,n}}\right) \\ &= \int_0^\infty \big(1-e^{-\lambda^\alpha x}\big)\left(\frac{(\alpha-1)^{1/\alpha-1}x^{-1/\alpha}}{\Gamma(1-1/\alpha)} - \sum_{n\geq 1} e^{-\lambda_{\alpha,n}x}\right)\frac{dx}{x}, \end{aligned} \tag{9}$$

where in the second equality we have used Frullani's identity repeatedly and the well-known change of variable formula (1) p.viii in [39]. Putting everything together shows that $\mathbf{X}_{\alpha,1/\alpha}$ is ID if and only if the function on the right-hand side is Bernstein. Unfortunately, this property seems difficult to check at first sight.

2.3. *Proof of Theorem 2*

2.3.1. *The case $\rho = 1$ and $\alpha \in (0,3/4]$*
Here, we need to show that the law of $\mathbf{X}_\alpha$ is a true GGC. To do so, we first observe that by (3) and the main result of [13] which states that the independent product of two random variables having a GGC distribution has a GGC distribution, it is enough to show that the law of $\mathbf{K}_\alpha$ itself is a GGC. Alternatively, this fact is also a consequence of the formula

$$\mathbb{E}\big[e^{-\lambda \mathbf{X}_\alpha}\big] = \frac{\alpha\lambda^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\int_\lambda^\infty \mathbb{E}\big[e^{-x\mathbf{K}_\alpha}\big]x^{\frac{1}{\alpha-1}}\,dx, \quad \lambda \geq 0, \tag{10}$$

which follows from (3) and of Theorem 6.1.1 and Properties (iv) and (xi) p.68 in [12]. To analyze the law of $\mathbf{K}_\alpha$, we will use the identity in law

$$\mathbf{K}_\alpha \overset{d}{=} \mathbf{K}_{1-\alpha}^{\frac{1}{\alpha}-1}, \tag{11}$$

an easy consequence of (5) which shows that both random variables have the same fractional moments. Plugging (11) again into (4) implies that the Laplace transform of $\mathbf{K}_{1-\alpha}$ is the survival function of the power transformation $\mathbf{Z}_\alpha^{\frac{-\alpha}{1-\alpha}}$. In other words, one has

$$F_\alpha(x) := \mathbb{E}\big[e^{-x\mathbf{K}_{1-\alpha}}\big] = \mathbb{P}[\mathbf{L} \geq x\mathbf{K}_{1-\alpha}] = \mathbb{P}\big[\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}} \geq x\big], \quad x \geq 0. \tag{12}$$

Setting $F_\alpha(x)$ for the function defined in (12), we next observe that since $\mathbf{K}_\alpha$ has a CM density and support $[b_\alpha,+\infty)$, this function F_α has by Theorem 9.5 in [39] an analytic extension on $\mathbb{C}\setminus(-\infty,0]$ which is given by

$$F_\alpha(z) = \exp -\left[b_{1-\alpha}z + \int_0^\infty \frac{z}{t+z}\frac{\theta_\alpha(t)}{t}\,dt\right] \tag{13}$$

for some measurable function $\theta_\alpha : (0,\infty) \to [0,1]$ such that $\int_0^1 \theta_\alpha(t)t^{-1}\,dt < \infty$. See also Theorem 51.12 in [38]. Applying now Theorem 9.10 (i) in [39], we see that the GGC property of $\mathbf{K}_{1-\alpha}$ is equivalent to the non-decreasing character of θ_α on $(0,\infty)$, and the following proposition allows us to conclude the proof of the case $\rho = 1$.

Proposition 1. *The function θ_α has a continuous version on $(0,\infty)$, which is non-decreasing for every $\alpha \in [1/4,1)$.*

Proof. The analysis of θ_α depends, classically, on the behaviour of F_α near $(-\infty,0]$. Assume for a moment that θ_α is continuous. For every $r>0$ and $\delta \in (0,1)$, we have, after some simple rearrangements,

$$\begin{aligned} \frac{F_\alpha(re^{\mathrm{i}\pi(1-\delta)})}{F_\alpha(re^{\mathrm{i}\pi(\delta-1)})} &= \exp -2\mathrm{i}\left[\sin(\pi\delta)b_{1-\alpha}r + \int_0^\infty \frac{\sin(\pi(1-\delta))\theta_\alpha(rt)}{1+2\cos(\pi(1-\delta))t+t^2}\,dt\right] \\ &= \exp -\big[2\mathrm{i}\sin(\pi\delta)b_{1-\alpha}r + 2\mathrm{i}\pi\,\mathbb{E}\big[\theta_\alpha\big(r(\mathbf{X}_{1,1-\delta})^+\big)\big]\big] \\ &\to \exp -2\mathrm{i}\pi\theta_\alpha(r) \end{aligned}$$

as $\delta \to 0$ since $\mathbf{X}_{1,1-\delta} \to \mathbf{1}$ in law as $\delta \to 0$ and θ_α is bounded continuous. On the other hand, it follows from the third expression of F_α in (12) and the first formula of Corollary 1 p.71 in [44], after a change of variable, that

$$F_\alpha(x) = 1 + \frac{1}{2\mathrm{i}\pi} \int_0^\infty e^{-t} \big(e^{-e^{\mathrm{i}\pi\alpha} t^\alpha x^{1-\alpha}} - e^{-e^{-\mathrm{i}\pi\alpha} t^\alpha x^{1-\alpha}} \big) \frac{dt}{t}, \quad x > 0.$$

The analytic continuations of F_α near $(-\infty, 0]$ are then expressed, changing the variable backwards, as

$$F_\alpha\big(re^{\mathrm{i}\pi}\big) = 1 + \frac{1}{2\mathrm{i}\pi} \int_0^\infty e^{-ru} \big(e^{ru^\alpha} - e^{ru^\alpha e^{-2\mathrm{i}\pi\alpha}} \big) \frac{du}{u}$$

and

$$F_\alpha\big(re^{-\mathrm{i}\pi}\big) = 1 + \frac{1}{2\mathrm{i}\pi} \int_0^\infty e^{-ru} \big(e^{ru^\alpha e^{2\mathrm{i}\pi\alpha}} - e^{ru^\alpha} \big) \frac{du}{u} = \overline{F_\alpha\big(re^{\mathrm{i}\pi}\big)}.$$

Therefore, we obtain

$$\frac{F_\alpha(re^{\mathrm{i}\pi})}{F_\alpha(re^{-\mathrm{i}\pi})} = e^{-2\mathrm{i}\pi\eta_\alpha(r)}$$

for every $r > 0$, with the notation

$$\eta_\alpha(r) = \frac{1}{\pi} \arg\big[F_\alpha\big(re^{-\mathrm{i}\pi}\big) \big].$$

Since

$$\Im\big(F_\alpha\big(re^{-\mathrm{i}\pi}\big)\big) = \frac{1}{2\pi} \int_0^\infty e^{-ru} \Re\big(e^{ru^\alpha} - e^{ru^\alpha e^{2\mathrm{i}\pi\alpha}} \big) \frac{du}{u} > 0$$

for every $r > 0$, the function η_α takes its values in $[0, 1]$ and is clearly continuous. By construction, the functions $t^{-1}\eta_\alpha(t)$ and $t^{-1}\theta_\alpha(t)$ have the same Stieltjes transform, and it follows by uniqueness that θ_α has a continuous version, which is η_α.

It remains to study the monotonous character of η_α on $(0, \infty)$. A first observation is that, expanding the exponentials inside the brackets and using Euler's reflection formula – see e.g. Theorem 1.2.1 in [1], the following absolutely convergent series representation holds:

$$F_\alpha\big(re^{-\mathrm{i}\pi}\big) = \sum_{n \geq 0} \frac{z^n e^{\mathrm{i}\pi n\alpha}}{n!\Gamma(1 - n\alpha)} = \phi\big(-\alpha, 1, ze^{\mathrm{i}\pi\alpha}\big) \tag{14}$$

with $z = r^{1-\alpha}$. In particular, the function

$$r \mapsto r^{\alpha-1} \Im\big(F_\alpha\big(re^{-\mathrm{i}\pi}\big)\big) = \frac{1}{\pi} \sum_{n \geq 1} \frac{\Gamma(n\alpha)}{n!} r^{(n-1)(1-\alpha)}$$

is absolutely monotonous on $(0, \infty)$, and the non-decreasing character of θ_α will hence be established as soon as $r \mapsto r^{\alpha-1}\Re(F_\alpha(re^{-\mathrm{i}\pi}))$ is non-increasing on $(0, \infty)$. We use the representation

$$\Re\big(F_\alpha\big(re^{-\mathrm{i}\pi}\big)\big) = 1 + \frac{1}{2\pi} \int_0^\infty e^{-r^{1-1/\alpha}u} \Im\big(e^{u^\alpha e^{2\mathrm{i}\pi\alpha}} \big) \frac{du}{u}$$

and divide this last part of the proof into two parts.

- The case $\alpha \in [1/2, 1)$. If $\alpha = 1/2$, we simply have $\Re(F_{1/2}(re^{-\mathrm{i}\pi})) \equiv 1$. If $\alpha > 1/2$ we rewrite, using again the first part of Corollary 1 p.71 in [44],

$$\begin{aligned} \Re\big(F_\alpha\big(re^{-\mathrm{i}\pi}\big)\big) &= \frac{1}{2} + \frac{1}{2}\bigg(1 - \frac{1}{\pi} \int_0^\infty e^{-r^{1-1/\alpha}u} \Im\big(e^{-u^\alpha e^{-\mathrm{i}\pi\rho\alpha}} \big) \frac{du}{u} \bigg) \\ &= \frac{1}{2}\big(1 + \mathbb{P}\big[\mathbf{Z}_{\alpha,\rho} \leq r^{1-1/\alpha} \big] \big), \end{aligned}$$

where $\rho = 2 - 1/\alpha \in (0,1)$ and $\mathbf{Z}_{\alpha,\rho}$ is a real α-stable random variable with positivity parameter ρ as in Lemma 1. Thus, $\Re(F_\alpha(re^{-\mathrm{i}\pi}))$ decreases on $(0,\infty)$ and $r^{\alpha-1}\Re(F_\alpha(re^{-\mathrm{i}\pi}))$ also decreases on $(0,\infty)$, as required.

- The case $\alpha \in [1/4, 1/2)$. Contrary to the above, the argument is here entirely analytic. We consider

$$\begin{aligned} G_\alpha(r) = \Re\big(F_\alpha\big(r^{\frac{\alpha}{\alpha-1}}e^{-\mathrm{i}\pi}\big)\big) &= 1 + \frac{1}{2\pi}\int_0^\infty e^{-ru}\Im\big(e^{u^\alpha e^{2\mathrm{i}\pi\alpha}}\big)\frac{du}{u} \\ &= 1 + \frac{1}{2\pi}\int_0^\infty e^{\cos(2\pi\alpha)u^\alpha - ru}\sin\big(\sin(2\pi\alpha)u^\alpha\big)\frac{du}{u} \\ &= 1 + \frac{1}{2\pi\alpha}\int_0^\infty g_{\alpha,r}(t)\sin(t)\,dt \end{aligned}$$

with $g_{\alpha,r}(t) = t^{-1}e^{\cot(2\pi\alpha)t - r(\sin(2\pi\alpha))^{-1/\alpha}t^{1/\alpha}}$. Observe that the positive function $t \mapsto g_{\alpha,r}(t)$ decreases on $(0,+\infty)$, because $\alpha \in [1/4, 1/2)$ and

$$\frac{g'_{\alpha,r}(t)}{g_{\alpha,r}(t)} = \cot(2\pi\alpha) - \frac{1}{t} - \frac{r}{\alpha}\big(\sin(2\pi\alpha)\big)^{-1/\alpha}t^{1/\alpha-1} < 0.$$

For every $k \geq 0$ we have

$$\int_{2k\pi}^{2(k+1)\pi} g_{\alpha,r}(t)\sin(t)\,dt = \int_0^\pi \big(g(t+2k\pi) - g\big(t+(2k+1)\pi\big)\big)\sin(t)\,dt > 0,$$

so that $G_\alpha(r) > 1$ for every $r > 0$. We next compute

$$\begin{aligned} \big(r^\alpha G_\alpha(r)\big)' &= \alpha r^{\alpha-1}\left(G_\alpha(r) + \frac{r}{\alpha}G'_\alpha(r)\right) \\ &> \alpha r^{\alpha-1}\left(1 - \frac{r}{2\pi\alpha}\int_0^\infty e^{\cos(2\pi\alpha)u^\alpha - ru}\sin\big(\sin(2\pi\alpha)u^\alpha\big)\,du\right) \\ &= \frac{r^\alpha}{2\pi}\int_0^\infty e^{-ru}\big(2\pi\alpha - e^{\cos(2\pi\alpha)u^\alpha}\sin\big(\sin(2\pi\alpha)u^\alpha\big)\big)\,du > 0, \end{aligned}$$

since $2\pi\alpha > 1 \geq e^{\cos(2\pi\alpha)u^\alpha}\sin(\sin(2\pi\alpha)u^\alpha)$ for every $u > 0$. Changing the variable backwards, this finally shows that $r \mapsto r^{\alpha-1}\Re(F_\alpha(re^{-\mathrm{i}\pi}))$ decreases on $(0,\infty)$.

□

Remark 4.

(a) The above argument shows that the survival function $x \mapsto \mathbb{P}[\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}} \geq x]$ is HCM for every $\alpha \geq 1/4$, with the terminology of [12]. A consequence of Corollary 2 is that this is not true anymore for $\alpha < 1/5$, and we believe – see Conjecture 1 – that the right domain of validity of this property is $\alpha \in [1/5, 1)$. The more stringent property that $\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}}$ is a HCM random variable for $\alpha \leq 1/2$ was conjectured in [14] and some partial results were obtained in [14,15]. In Remark 9 below, we will see that this property is not true for $\alpha < 1/5$.

(b) We do not know if the representation (13) holds for the Laplace transform of $\mathbf{X}_\alpha$. Since the latter is a Γ_2-mixture we obtain, similarly as above,

$$\mathbb{E}\big[e^{-x\mathbf{X}_\alpha}\big] = e^{-b_{1-\alpha}x}\int_0^\infty \frac{\nu_\alpha(dt)}{(x+t)^2}$$

for some positive measure ν_α on $[0,+\infty)$. This representation would suffice if we could show that the generalized Stieltjes functions on the right-hand side is the product of two standard Stieltjes functions, applying Theorem 6.17 in [39] as in the proof of Theorem 9.5 therein. However, this is not true in general, for example when ν_α is the sum of two Dirac masses. Alternatively, because of (10) one would like to prove that if f has representation (13), then so has $x \mapsto \int_x^\infty f(y)\,dy$. This is true in the GGC case by Property (xi) p.68 in [12], but we were not able to prove this in general.

2.3.2. *The case $\rho < 1$ and $\alpha \in (0, 3/4]$*

The case $\rho = 0$ follows from $\mathbf{X}_{\alpha,0} \stackrel{d}{=} -\mathbf{X}_\alpha$. For $\rho \in (0, 1)$ we appeal to (8), the previous case, and the Huff–Zolotarev subordination formula which is given e.g. in Theorem 30.1 of [38]. Since the law of $\mathbf{X}_\alpha$ is a GGC for $\alpha \leq 3/4$, its Laplace transform reads

$$\mathbb{E}\big[e^{-\lambda \mathbf{X}_\alpha}\big] = \exp -\left[b_\alpha \lambda + \int_0^\infty \big(1 - e^{-\lambda x}\big) k_\alpha(x) \frac{dx}{x} \right]$$

for some CM function k_α. Formula (30.8) in [38] and the closed expression of the density of $\mathbf{X}_{1,\rho}$ imply that the Lévy measure $\nu_{\alpha,\rho}$ of $\mathbf{X}_{\alpha,\rho}$ has density

$$\begin{aligned}\psi_{\alpha,\rho}(x) &= b_\alpha \psi_{1,\rho}(x) + \frac{\sin(\pi\rho)}{\pi} \int_0^\infty \frac{k_\alpha(u)}{x^2 + 2\cos(\pi\rho)xu + u^2}\, du \\ &= \frac{\sin(\pi\rho)}{\pi |x|} \left(\frac{b_\alpha}{|x|} + \int_0^\infty \frac{k_\alpha(|x|u)}{1 + 2\cos(\pi\rho)\,\mathrm{sgn}(x)u + u^2}\, du \right)\end{aligned}$$

over $\mathbb{R}^*$, where the closed expression for $\psi_{1,\rho}$ can be deduced e.g. from Theorem 14.10 and Lemma 14.11 in [38]. Both functions $x\psi_{\alpha,\rho}(x)$ and $x\psi_{\alpha,\rho}(-x)$ are hence CM on $(0, \infty)$.

Remark 5. For $\alpha \in (0, 1)$, since $b_\alpha > 0$ and the ID random variable $\mathbf{X}_{1,\rho}$ has no Gaussian component, the Huff–Zolotarev subordination formula shows that $\mathbf{X}_{\alpha,\rho}$ does not have a Gaussian component either, and that for $\rho \in (0, 1)$ its Lévy measure is such that

$$\int_{|x| \leq 1} |x| \nu_{\alpha,\rho}(dx) = +\infty.$$

With the terminology of [38] – see Definition 11.9 therein, this means that the Lévy process associated with $\mathbf{X}_{\alpha,\rho}$ is of type C. This contrasts with the classical α-stable Lévy process which is of type B for $\alpha < 1$. When $\rho = 1$ and $\alpha \leq 3/4$, the GGC property shows that the Lévy process corresponding to $\mathbf{X}_\alpha$ is of type B. We believe that this is true for all $\alpha \in (0, 1)$.

2.4. *Proof of Theorem 3*

It is well-known and easy to see from the Voiculescu transform

$$\phi_{1,1/2}(z) = -1$$

that the free independent sum of $\mathbf{X}_{1,1/2}$ with any random variable is also a classical independent sum. Hence, the ID character of $\mathbf{C}_{a,b}$ follows from that of $\mathbf{T}$, which is a consequence of Theorem 1 and the convergence in law

$$\frac{(1-\alpha)^{1-\alpha} - \mathbf{X}_\alpha}{1-\alpha} \xrightarrow{d} \mathbf{T} \quad \text{as } \alpha \uparrow 1, \tag{15}$$

the latter being easily obtained in comparing the two Voiculescu transforms. More precisely, since for any random variable X having Voiculescu transform $\phi_X(z)$ and any $p, q \in \mathbb{R}$ with $p \neq 0$ the Voiculescu transform of $pX + q$ is given by $q + p\phi_X(z/p)$, we have

$$\begin{aligned}\phi_{\frac{(1-\alpha)^{1-\alpha} - \mathbf{X}_\alpha}{1-\alpha}}(z) &= (1-\alpha)^{-\alpha} - \frac{1}{1-\alpha} \phi_{\mathbf{X}_\alpha}\big(-(1-\alpha)z\big) \\ &= (1-\alpha)^{1-\alpha} \frac{1 - z^{1-\alpha}}{1-\alpha} \to \log z = \phi_{\mathbf{T}}(z).\end{aligned}$$

This concludes the first part of the theorem. Moreover, it is clear that neither $\mathbf{X}_{1,1/2}$ nor $\mathbf{T}$, whose support is a half-line by Proposition A.1.3 in [10], have a Gaussian component, and this property conveys hence to $\mathbf{C}_{a,b}$. Finally, since the Lévy measure of $\mathbf{X}_{1,1/2}$ is

$$\frac{1}{\pi x^2} \mathbf{1}_{\{x \neq 0\}}$$

as seen in the above proof, we are reduced to show by independence and scaling that the Lévy measure of $\mathbf{T}$ has density

$$\frac{1}{x^2}\left(1-\frac{|x|e^{-2|x|}}{1-e^{-|x|}}\right)\mathbf{1}_{\{x<0\}}.$$

This last computation will be done in two steps. Consider the random variable

$$\mathbf{W}=\frac{\sin(\pi\mathbf{U})}{\pi\mathbf{U}}e^{\pi\mathbf{U}\cot(\pi\mathbf{U})}$$

and the exceptional 1-stable random variable $\mathbf{S}$ characterized by

$$\mathbb{E}\big[e^{s\mathbf{S}}\big]=s^s,\quad s>0.$$

Proposition 2. *One has the identities*

$$\mathbf{S}\overset{d}{=}\log\mathbf{L}+\log\mathbf{W}\quad\textit{and}\quad\mathbf{T}\overset{d}{=}\log\mathbf{U}+\log\mathbf{W}.$$

Proof. We begin with the first identity. Using (4), we decompose

$$\frac{(1-\alpha)^{1-\alpha}-\mathbf{Z}_\alpha}{1-\alpha}\overset{d}{=}\mathbf{K}_\alpha\times\left(\frac{1-\mathbf{L}^{1-\frac{1}{\alpha}}}{1-\alpha}\right)+\left(\frac{(1-\alpha)^{1-\alpha}-\mathbf{K}_\alpha}{1-\alpha}\right).\tag{16}$$

On the one hand, a comparison of the two moment generating functions yields

$$\frac{(1-\alpha)^{1-\alpha}-\mathbf{Z}_\alpha}{1-\alpha}\xrightarrow{d}\mathbf{S}\quad\text{as }\alpha\uparrow 1.$$

On the other hand, the right-hand side of (16) is a deterministic transformation, depending on α, of $(\mathbf{L},\mathbf{U})$ independent. It is easy to see from (6) that

$$\mathbf{K}_\alpha\times\left(\frac{1-\mathbf{L}^{1-\frac{1}{\alpha}}}{1-\alpha}\right)\xrightarrow{a.s.}\log\mathbf{L}\quad\text{as }\alpha\uparrow 1.$$

To study the second term, we use the elementary expansions

$$\begin{aligned}
&\sin(\pi\alpha\mathbf{U})=\sin(\pi\mathbf{U})+(\alpha-1)\pi\mathbf{U}\cos(\pi\mathbf{U})+O\big((1-\alpha)^3\big)\\
&\sin^{\frac{1-\alpha}{\alpha}}\big(\pi(1-\alpha)\mathbf{U}\big)=1+(1-\alpha)\log\sin\big(\pi\mathbf{U}(1-\alpha)\big)+O\big((1-\alpha)^2\log^2(1-\alpha)\big)\\
&\sin^{\frac{1}{\alpha}}(\pi\mathbf{U})=\sin(\pi\mathbf{U})\big(1+(1-\alpha)\log\sin(\pi\mathbf{U})\big)+O\big((1-\alpha)^2\big)\\
&(1-\alpha)^{1-\alpha}=1+(1-\alpha)\log(1-\alpha)+O\big((1-\alpha)^2\log^2(1-\alpha)\big),
\end{aligned}$$

which, combined with (6), yield the almost sure asymptotics

$$\frac{(1-\alpha)^{1-\alpha}-\mathbf{K}_\alpha}{1-\alpha}=\log\left(\frac{\sin(\pi\mathbf{U})}{\pi\mathbf{U}}e^{\pi\mathbf{U}\cot(\pi\mathbf{U})}\right)+O\big((1-\alpha)\log^2(1-\alpha)\big).$$

Putting everything together completes the proof of the first identity. The second one is derived exactly in the same way, using (3) and (15). □

Remark 6.

(a) The first identity in Proposition 2 is actually the consequence of an integral transformation due to Zolotarev – see (2.2.19) with $\beta=1$ in [44]. We have offered a separate proof which is perhaps clearer, and which enhances the similarities between the free and the classical case. Observe in particular the identity

$$\mathbf{S}\overset{d}{=}\mathbf{T}+\log\mathbf{\Gamma}_2,\tag{17}$$

which is reminiscent of Corollary 1.5 in [24]. The latter identity is also a consequence of Proposition 2 and the standard identities

$$\mathbf{L}^{\beta} \overset{d}{=} \mathbf{U}^{\beta} \times \mathbf{\Gamma}_2^{\beta} \tag{18}$$

valid for every $\beta \in \mathbb{R}^*$ and their limit as $\beta \to 0$, which is

$$\log \mathbf{L} \overset{d}{=} \log \mathbf{U} + \log \mathbf{\Gamma}_2. \tag{19}$$

(b) It is interesting to look at these standard identities (18) and (19) in the context of extreme value distributions. Indeed, the three classical extreme value distributions are Fréchet $\mathbf{L}^{\beta}$ for $\beta < 0$, Weibull $-\mathbf{L}^{\beta}$ for $\beta > 0$ and Gumbel $-\log \mathbf{L}$ for $\beta \to 0$, whereas the free counterparts are $\mathbf{U}^{\beta}$ for $\beta < 0$, $-\mathbf{U}^{\beta}$ for $\beta > 0$ and $-\log \mathbf{U}$ for $\beta \to 0$ according to the classification of [7]. Besides, recall that a $\mathbf{\Gamma}_2$ factor also appears in the formula $\mathbf{Z}_{\alpha,\rho} \overset{d}{=} \mathbf{X}_{\alpha,\rho} \times \mathbf{\Gamma}_2^{1-1/\alpha}$ which connects classical and free stable random variables.

(c) Recently Vargas and Voiculescu have introduced Boolean extreme value distributions [41]. The result is the Dagum distribution, which is indexed by $\beta > 0$ and has density function

$$\frac{x^{1/\beta - 1}}{\beta(1 + x^{1/\beta})^2}$$

on $(0, \infty)$. Hence, the Dagum distribution is the law of

$$\left(\mathbf{U}^{-1} - 1\right)^{\beta} \overset{d}{=} \left(\frac{\mathbf{L}}{\mathbf{L}}\right)^{\beta},$$

which is the independent quotient of two Fréchet distributions, and an example of the generalized Beta distribution of the second kind (GB2). On the other hand, by Proposition 4.12 (b) in [2], the Boolean α-stable distribution has for $\alpha \leq 1$ the law of the independent quotient

$$\frac{\mathbf{Z}_{\alpha,\rho}}{\mathbf{Z}_{\alpha}}$$

and it is interesting to notice that by Zolotarev's duality – see (3.3.16) in [44] – and scaling, the positive part of this random variable is distributed as

$$\left(\frac{\mathbf{Z}_{\alpha\rho}}{\mathbf{Z}_{\alpha\rho}}\right)^{\rho} \overset{d}{\longrightarrow} \left(\frac{\mathbf{L}}{\mathbf{L}}\right)^{\frac{1}{\alpha}} \quad \text{as } \rho \to 0.$$

Finding an interpretation about why such quotients appear in those two Boolean cases is left to future work.

(d) The second identity in Proposition 2 can be rewritten as

$$e^{\mathbf{T}} \overset{d}{=} \mathbf{U} \times \mathbf{W}.$$

In [3], it is pointed out that the law of $e^{\mathbf{T}}$ is the Dykema–Haagerup distribution, which appears as the eigenvalue distribution of $A_N^* A_N$ as $N \to \infty$, where A_N is an $N \times N$ upper-triangular random matrix with independent complex Gaussian entries – see [20].

Our second step is to compute the Mellin transform of $\mathbf{W}$.

Proposition 3. *For all $s > 0$, one has*

$$\mathbb{E}\left[\mathbf{W}^s\right] = \frac{s^s}{\Gamma(1+s)} = \exp\left[s - \int_0^{\infty} \left(1 - e^{-sx}\right)\left(1 - \frac{x}{e^x - 1}\right)\frac{dx}{x^2}\right].$$

Proof. The first equality follows from

$$s^s = \mathbb{E}\left[e^{s\mathbf{S}}\right] = \mathbb{E}\left[\mathbf{L}^s\right]\mathbb{E}\left[\mathbf{W}^s\right] = \Gamma(1+s)\mathbb{E}\left[\mathbf{W}^s\right], \quad s > 0,$$

a consequence of the first identity in Proposition 2. To get the second one, we proceed as in the proof of Lemma 14.11 of [38] and start from the standard Frullani identity

$$\log s = \int_0^\infty \frac{e^{-x} - e^{-sx}}{x}\, dx,$$

which transforms, dividing the integral at 1 and making an integration by parts, into

$$s \log s = \int_0^\infty \big(e^{-sx} - 1 + sx\mathbf{1}_{\{x\le 1\}}\big)\frac{dx}{x^2} - s\left(\int_0^\infty \big(e^{-x} - 1 + x\mathbf{1}_{\{x\le 1\}}\big)\frac{dx}{x^2}\right).$$

On the other hand, it is well-known – see e.g. Formulae 1.7.2(21) and 1.9(1) in [21] – that

$$\log \Gamma(1+s) = -\gamma s + \int_0^\infty \big(e^{-sx} - 1 + sx\big)\frac{dx}{x(e^x - 1)},$$

where $\gamma = -\Gamma'(1)$ is Euler's constant. Putting everything together yields

$$\begin{aligned}\log \mathbb{E}\big[\mathbf{W}^s\big] &= cs + \int_0^\infty \big(e^{-sx} - 1 + sx\mathbf{1}_{\{x\le 1\}}\big)\left(1 - \frac{x}{e^x - 1}\right)\frac{dx}{x^2}\\ &= \tilde{c}s - \int_0^\infty \big(1 - e^{-sx}\big)\left(1 - \frac{x}{e^x - 1}\right)\frac{dx}{x^2},\end{aligned}$$

where $c, \tilde{c}$ are two constants to be determined. But it is clear that $\tilde{c}$ is the right end of the support of $\log \mathbf{W}$ which we know, by Remark 6(c), to be one. Alternatively, one can use Binet's formula

$$\gamma = \int_0^\infty \left(\frac{e^{-x}}{1 - e^{-x}} - \frac{e^{-x}}{x}\right) dx,$$

which is 1.7.2(22) in [21] for $z = 1$, and rearrange the different integrals, to retrieve $\tilde{c} = 1$. This completes the proof. □

We can now finish the proof of Theorem 3. Putting together Propositions 2 and 3, we get

$$\begin{aligned}\log \mathbb{E}\big[e^{s\mathbf{T}}\big] &= \log \mathbb{E}\big[\mathbf{U}^s\big] + \log \mathbb{E}\big[\mathbf{W}^s\big] = -\log(1+s) + \log \mathbb{E}\big[\mathbf{W}^s\big]\\ &= s - \int_0^\infty \big(1 - e^{-sx}\big)\left(1 - \frac{xe^{-2x}}{1 - e^{-x}}\right)\frac{dx}{x^2},\end{aligned}$$

where the third equality follows from rearranging Frullani's identity and the second equality in Proposition 3. All of this shows that the ID random variable $\mathbf{T}$ has support $(-\infty, 1]$ – in accordance with Proposition A.1.3 in [10], and that its Lévy measure has density

$$\frac{1}{x^2}\left(1 - \frac{|x|e^{-2|x|}}{1 - e^{-|x|}}\right)\mathbf{1}_{\{x<0\}}$$

as required.

Remark 7.

(a) The first equality in Proposition 3 shows that $\mathbf{W}$ has the distribution ν_0 studied in Theorem 6.1 of [30]. This distribution also appears in Sakuma and Yoshida's limit theorem – see [37]. Finally, combining this equality and the second identity in Proposition 2 implies

$$\mathbb{E}\big[e^{s\mathbf{T}}\big] = \frac{s^s}{\Gamma(2+s)}$$

for all $s > 0$, which was previously obtained in [3] by other methods.

(b) It is easy to see that the function

$$x \mapsto \frac{1}{x} - \frac{1}{e^x - 1}$$

decreases from $1/2$ to zero on $(0,\infty)$. By Corollary 15.11 in [38], this shows that $\log \mathbf{W}$ is SD. A further computation yields

$$\frac{1}{x^2}\left(1-\frac{x}{e^x-1}\right)=\int_0^\infty e^{-ux}\big(u-[u]\big)\,du,\quad x>0. \tag{20}$$

This implies that the Lévy process associated to $\log \mathbf{W}$ has CM jumps. By Theorem 3, so does the Lévy process associated to $\mathbf{T}$, whose Lévy measure has density

$$\frac{e^{-|x|}}{|x|}+\frac{1}{x^2}\left(1-\frac{|x|}{e^{|x|}-1}\right)=\int_0^\infty e^{-u|x|}\big(u-[u-1]_+\big)\,du,\quad x<0.$$

By Theorem 51.12 in [38], the latter computation also implies that the law of the positive random variable $1-\log \mathbf{W}$ has a CM density.

(c) Making an integration by parts in (20) yields

$$\frac{1}{x}\left(1-\frac{x}{e^x-1}\right)=\int_0^\infty e^{-ux}\left(du-\sum_{n\geq 1}\delta_n(du)\right),$$

where δ stands for the Dirac mass. By (7.1.5) in [12], this implies that the law of $\log \mathbf{W}$ is not a GGC, and the same is true for $\mathbf{T}$ because

$$\frac{1}{x}\left(1-\frac{xe^{-2x}}{1-e^{-x}}\right)=\int_0^\infty e^{-ux}\left(du-\sum_{n\geq 2}\delta_n(du)\right).$$

By (15) and Theorem 7.1.1 in [12], this yields the following negative counterpart to Theorem 2.

Corollary 1. *There exists $\alpha_0<1$ such that for every $\alpha\in(\alpha_0,1)$, the law of $\mathbf{X}_\alpha$ is not a* GGC.

This also implies that there is a function $\delta\colon(\alpha_0,1)\to[0,1)$ such that $\mathbf{X}_{\alpha,\rho}$ is not a GGC for $\alpha\in(\alpha_0,1)$ and $\rho\in[\delta(\alpha),1]$. Observe on the other hand that it does not seem possible to apply our methods to $\mathbf{X}_{\alpha,\rho}$ with a fixed $\rho\in(0,1)$. Indeed, as in the classical case, the possible limit laws of affine transformations of $\mathbf{X}_{\alpha,\rho}$ with $\rho\in(0,1)$ fixed and $\alpha\to 1$ are given only in terms of $\mathbf{X}_{1,\rho}$, whose law is a GGC.

2.5. *Proof of Theorem 4*

2.5.1. *The one-sided case*

By (3) and Corollary 3.2 in [26], we have the independent factorisation

$$\mathbf{X}_\alpha \overset{d}{=} b_\alpha \mathbf{U}^{-1/\beta}(1+\mathbf{X}),$$

where $1/\beta=1/\alpha-1$ and $\mathbf{X}$ has a CM density on $(0,\infty)$. We will now show the WS property for all positive random variables of the type

$$\mathbf{Y}=\mathbf{U}^{-1/\beta}(1+\mathbf{X})-1$$

with $\beta>0$ and $\mathbf{X}$ having a CM density on $(0,\infty)$. Setting f,g for the respective densities of $\mathbf{Y},\mathbf{X}$, the multiplicative convolution formula shows that

$$f(x)=\frac{\beta}{(x+1)^{\beta+1}}\int_0^x (y+1)^\beta g(y)\,dy=\frac{\beta}{(1+1/x)^{\beta+1}}\int_0^1 (y+1/x)^\beta g(xy)\,dy$$

for every $x>0$. In particular, one has $f(0+)=f(+\infty)=0$. Moreover, the first equality and an induction on n imply that f is smooth with

$$(x+1)f^{(n+1)}(x)=\beta g^{(n)}(x)-(\beta+n+1)f^{(n)}(x) \tag{21}$$

for every $n\geq 0$. Hence, we also have $f^{(n)}(+\infty)=0$ for all $n\geq 0$ and a successive application of Rolle's theorem yields

$$\#\big\{x\in(0,\infty)\,|\,f^{(n)}(x)=0\big\}\geq 1$$

for every $n \geq 1$. Fix now $n \geq 1$ and suppose that there exist $0 < x_n^{(1)} < x_n^{(2)} < \infty$ such that

$$f^{(n)}\big(x_n^{(1)}\big) = f^{(n)}\big(x_n^{(2)}\big) = 0.$$

By (21) and the complete monotonicity of g, we have

$$(-1)^n f^{(n+1)}\big(x_n^{(i)}\big) > 0$$

for $i = 1, 2$. An immediate analysis based on the intermediate value theorem shows then that there must exist $x_n^{(3)} \in (x_n^{(1)}, x_n^{(2)})$ with

$$f^{(n)}\big(x_n^{(3)}\big) = 0 \quad \text{and} \quad (-1)^n f^{(n+1)}\big(x_n^{(3)}\big) \leq 0,$$

which is impossible again by (21) and the complete monotonicity of g. All in all, we have proved that

$$\#\big\{x \in (0,\infty) \,|\, f^{(n)}(x) = 0\big\} = 1$$

for all $n \geq 1$, which is the WS property.

2.5.2. *The two-sided case*
We know by Proposition A.1.4 in [10] that $f_{\alpha,\rho}$ is an analytic integrable function on $\mathbb{R}$, and by Theorem 1.7 in [24] that it converges to zero at $\pm\infty$, decreases near $+\infty$ and increases near $-\infty$. Moreover, we have shown in Theorem 2 that if $\alpha \leq 3/4$, it is the density of an ID distribution on $\mathbb{R}$ with Lévy measure $\varphi_{\alpha,\rho}(x)\,dx$ such that $x\varphi_{\alpha,\rho}(x)$ and $x\varphi_{\alpha,\rho}(-x)$ are CM on $(0,\infty)$. We are hence in position to apply Corollary 1.2 in [29], which shows that $f_{\alpha,\rho}$ is BS.

2.5.3. *The exceptional* 1*-stable case*
We use the second identity in Proposition 2, which rewrites

$$1 - \mathbf{T} \overset{d}{=} (1 - \log \mathbf{W}) + \mathbf{L}.$$

We have seen in Remark 7(b) that the random variable $1 - \log \mathbf{W}$ has a CM density on $(0, +\infty)$, in other words that it belongs to the class ME* with the notations of [40]. Applying the Proposition in [40] with $n = 1$ shows that $1 - \mathbf{T}$ has a WBS_0 density, with the notation of the main definition in [40]. As mentioned in the Introduction, this means that the density of $\mathbf{T}$ is WS_-.

2.5.4. *The two-sided* 1*-stable case with* $b = 0$ *or* $ab^{-1} \in \pi\mathbb{Z}$
We may suppose $a > 0$ by symmetry. If $b = 0$ the statement is clear since it is elementally shown that the Cauchy density

$$\frac{1}{\pi(1+x^2)}$$

is BS – see also Corollary 1.3 in [29]. If $b \neq 0$, we may suppose $b < 0$ by symmetry. By independence, we have

$$\log \mathbb{E}\big[e^{-\mathrm{i}\xi \mathbf{C}_{a,b}}\big] = -a|\xi| + \log \mathbb{E}\big[e^{\mathrm{i}|b|\xi \mathbf{T}}\big], \quad \xi \in \mathbb{R}.$$

A further computation using Lemma 14.11 in [38] and Remark 7(b) yields

$$\log \mathbb{E}\big[e^{-\mathrm{i}\xi \mathbf{C}_{a,b}}\big] = c_1 + c_2 \mathrm{i}\xi + \int_{\mathbb{R}} \left(\frac{1}{\mathrm{i}\xi + s} - \left(\frac{1}{s} - \frac{\mathrm{i}\xi}{s^2} \right) \mathbf{1}_{\mathbb{R}\setminus(-1,1)}(s) \right) \varphi_{a,b}(s)\, ds$$

for some $c_1, c_2 \in \mathbb{R}$ and

$$\varphi_{a,b}(s) = \frac{a}{\pi} s + \big(|b|s - \big[|b|s - 1\big]_+\big) \mathbf{1}_{\{s \geq 0\}}.$$

This function satisfies (1.1) and (1.2) in [29] and is such that $s\varphi_{a,b}(s) \geq 0$. Moreover, for $ab^{-1} \in \pi\mathbb{Z}$ the function $\varphi_{a,b}(s) - k$ changes its sign only once for every $k \in \mathbb{Z}$. Finally, we know from Propositions A.1.3 and A.2.1 in [10] that the density of $\mathbf{C}_{a,b}$ is smooth, converges to zero at $\pm\infty$, decreases near $+\infty$ and increases near $-\infty$. We can hence apply Theorem 1.1 in [29] and conclude the proof.

Remark 8.

(a) If the random variable $1 - \log \mathbf{W}$ had a PF_∞ density as $\mathbf{L}$ does, then the BS character of $f_{1,1/2}$ and the additive total positivity arguments used in [29] would show that $\mathbf{C}_{a,b}$ has a BS density on $\mathbb{R}$ for $a \neq 0$. But $1 - \log \mathbf{W}$ cannot have a PF_∞ density, since its law is not a GGC – see e.g. Example 3.2.2 in [12].

(b) If $ab^{-1} \notin \pi\mathbb{Z}$, the function $\varphi_{a,b}(s) - k$ changes its sign at least three times for every negative integer k, so that we cannot use Theorem 1.1 in [29]. It is not clear to the authors whether the density of $\mathbf{C}_{a,b}$ is always BS for $a \neq 0$, and the case $ab^{-1} \in \pi\mathbb{Z}$ might be more the exception than the rule.

3. Further results

3.1. *Some properties of the function θ_α*

In this paragraph we consider further aspects of the function

$$\theta_\alpha(r) = \frac{1}{\pi} \arg\bigl[F_\alpha\bigl(re^{-\mathrm{i}\pi}\bigr)\bigr], \tag{22}$$

whose non-decreasing character amounts to the GGC property for the law of $\mathbf{K}_{1-\alpha}$. We first prove the following asymptotic result.

Proposition 4. *For every $\alpha \in [1/5, 1)$, one has*

$$\lim_{r\to+\infty} \Re\bigl(F_\alpha\bigl(re^{-\mathrm{i}\pi}\bigr)\bigr) = \frac{1}{2\alpha}.$$

For every $\alpha \in (0, 1/5)$, one has

$$\liminf_{r\to+\infty} \Re\bigl(F_\alpha\bigl(re^{-\mathrm{i}\pi}\bigr)\bigr) = -\infty \quad \textit{and} \quad \limsup_{r\to+\infty} \Re\bigl(F_\alpha\bigl(re^{-\mathrm{i}\pi}\bigr)\bigr) = +\infty.$$

The second part of this proposition has an immediate corollary, which answers in the negative an open problem stated in [26] – see Conjecture 3.1 therein.

Corollary 2. *The function θ_α is not monotonous on $(0, \infty)$ for $\alpha < 1/5$. In particular, the law of $\mathbf{K}_\alpha$ is not a* GGC *for $\alpha > 4/5$.*

Proof of Proposition 4. We have seen during the proof of Proposition 1 that $\Re(F_{1/2}(re^{-\mathrm{i}\pi})) \equiv 1$ and that

$$\Re\bigl(F_\alpha\bigl(re^{-\mathrm{i}\pi}\bigr)\bigr) \to \frac{1}{2\alpha}$$

as $r \to +\infty$ for all $\alpha \in (1/2, 1)$. We next consider the case $\alpha \in [1/5, 1/2)$ introducing, as above, the function

$$G_\alpha(r) = \Re\bigl(F_\alpha\bigl(r^{\frac{\alpha}{\alpha-1}} e^{-\mathrm{i}\pi}\bigr)\bigr) = 1 + \frac{1}{2\pi} \Im\left(\int_0^\infty e^{-ru+u^\alpha e^{2\mathrm{i}\pi\alpha}} \frac{du}{u}\right).$$

Note that the integral converges only in its imaginary part, but we keep this notation. Setting $\theta = \frac{5}{6}(1 - 2\alpha) \in (0, 1/2]$, we have $2\alpha + \alpha\theta \in [1/2, 1)$ and by Cauchy's theorem, we can rewrite

$$G_\alpha(r) = 1 + \frac{\theta}{2} + \frac{1}{2\pi} \Im\left(\int_0^\infty e^{-rue^{i\pi\theta}+u^\alpha e^{i\pi(2\alpha+\alpha\theta)}} \frac{du}{u}\right).$$

The latter converges to

$$1 + \frac{\theta}{2} + \frac{1}{2\pi} \Im\left(\int_0^\infty e^{u^\alpha e^{i\pi(2\alpha+\alpha\theta)}} \frac{du}{u}\right) = 1 + \frac{\theta}{2} + \frac{1}{2\pi\alpha} \Im\left(\int_0^\infty e^{-ue^{-i\pi(1-2\alpha-\alpha\theta)}} \frac{du}{u}\right)$$

as $r \to 0$. The evaluation of the oscillating integral on the right-hand side is given e.g. in Formula 1.6(36) p.13 in [21], and we finally obtain

$$\lim_{r\to 0} G_\alpha(r) = 1 + \frac{\theta}{2} + \frac{1}{2\alpha}(1 - 2\alpha - \alpha\theta) = \frac{1}{2\alpha}.$$

We finally consider the case $\alpha \in (0, 1/5)$, which is much more technical and requires several steps. Setting $\theta = 2\alpha/(1-\alpha) \in (0, 1/2)$, we have $2\alpha + \alpha\theta = \theta$ and the same argument as above implies

$$G_\alpha(r) = 1 + \frac{\theta}{2} + \frac{1}{2\pi}\Im\left(\int_0^\infty e^{(-rt+t^\alpha)e^{\mathrm{i}\pi\theta}}\frac{dt}{t}\right).$$

Hence, we are reduced to show that

$$\liminf_{r\to 0} H_\alpha(r) = -\infty \quad \text{and} \quad \limsup_{x\to 0} H_\alpha(r) = +\infty$$

with the notations $f_r(t) = \sin(\pi\theta)(-rt + t^\alpha)$ and

$$H_\alpha(r) = \Im\left(\int_0^\infty e^{(-rt+t^\alpha)e^{\mathrm{i}\pi\theta}}\frac{dt}{t}\right) = \int_0^\infty e^{\cot(\pi\theta) f_r(t)}\sin\big(f_r(t)\big)\frac{dt}{t}.$$

Let us begin with the liminf. Setting

$$r_k = \alpha\left(\frac{(1-\alpha)\sin(\pi\theta)}{2k\pi}\right)^{1/\alpha - 1} \quad \text{and} \quad m_k = \left(\frac{2k\pi}{(1-\alpha)\sin(\pi\theta)}\right)^{1/\alpha},$$

it is clear that the function $f_{r_k}(t)$ increases on $(0, m_k)$ and decreases on $(m_k, +\infty)$, and that its global maximum equals $f_{r_k}(m_k) = 2k\pi$. This yields

$$\int_{m_k}^\infty e^{\cot(\pi\theta) f_{r_k}(t)}\sin\big(f_{r_k}(t)\big)\frac{dt}{t} < 0 \quad \text{for every } k \geq 1.$$

Considering now the unique $a_k \in (0, m_k)$ such that $f_{r_k}(a_k) = \pi$, we have $\lim_{k\to\infty} a_k = (\pi/\sin(\pi\theta))^{1/\alpha}$, so that

$$\int_0^{a_k} e^{\cot(\pi\theta) f_{r_k}(t)}\sin\big(f_{r_k}(t)\big)\frac{dt}{t} \to \int_0^{(\pi/\sin(\pi\theta))^{1/\alpha}} e^{\cos(\pi\theta)t^\alpha}\sin\big(t^\alpha\sin(\pi\theta)\big)\frac{dt}{t} < \infty$$

as $k \to +\infty$. Hence it suffices to show that $A_k \to -\infty$ as $k \to +\infty$, with

$$A_k = \int_{a_k}^{m_k} e^{\cot(\pi\theta) f_{r_k}(t)}\sin\big(f_{r_k}(t)\big)\frac{dt}{t} = \frac{1}{\sin(\pi\theta)}\int_\pi^{2k\pi} \frac{e^{\cot(\pi\theta)u}}{-r_k\varphi_k(u) + \alpha(\varphi_k(u))^\alpha}\sin(u)\,du,$$

where the second equality comes from a change of variable, having set $\varphi_k(u)$ for the inverse function of f_{r_k} on $[\pi, 2k\pi]$ and written

$$\varphi_k'(u) = \frac{1}{f_{r_k}'(\varphi_k(u))} = \frac{1}{\sin(\pi\theta)(-r_k + \alpha(\varphi_k(u))^{\alpha-1})} > 0.$$

We next define $p_k(u) := e^{-\cot(\pi\theta)u}(-r_k\varphi_k(u) + \alpha(\varphi_k(u))^\alpha)$ and prove its strict unimodality on $[\pi, 2k\pi]$, computing

$$p_k'(u) = e^{-\cot(\pi\theta)u}\frac{\varphi_k'(u)}{\varphi_k(u)}\big(-r_k t + \alpha^2 t^\alpha - \cos(\pi\theta)\big(-r_k t + \alpha t^\alpha\big)^2\big)$$

with $t = \varphi_k(u)$. The strict unimodality of $p_k(u)$ on $(\pi, 2k\pi)$ amounts to the fact that

$$q_k(t) = -r_k t + \alpha^2 t^\alpha - \cos(\pi\theta)\big(-r_k t + \alpha t^\alpha\big)^2$$

has at most one zero point on $[a_k, m_k]$. It is clear by construction that there exists $c_k \in (0, m_k)$ such that $g_k(t) = -r_k t + \alpha t^\alpha$ increases on $(0, c_k)$ and decreases on (c_k, m_k), and for all $t \in (c_k, m_k)$ we have $q_k(t) = t g_k'(t) - \cos(\pi\theta)(-r_k t + \alpha t^\alpha)^2 < 0$. On the other hand, the function $g_k(t)$ is increasing and concave on $[0, c_k)$, so that its inverse function $\psi_k(v)$ is increasing and convex on $[0, g_k(c_k))$. Now since

$$q_k(t) = 0 \quad \Leftrightarrow \quad \cos(\pi\theta)v^2 - \alpha v + (1-\alpha)r_k\psi_k(v) = 0,$$

we see that there are at most two solutions of $q_k(t) = 0$ on $[0, c_k)$, one of them being zero, and hence at most one solution on $[a_k, m_k)$, as required. We now denote by z_k the unique mode of $p_k(u)$ on $[a_k, m_k]$ and, setting $l_k = \inf\{l \geq 1, z_k \leq 2l\pi\}$, decompose

$$A_k = \frac{1}{\sin(\pi\theta)}\left(\int_{\pi}^{2l_k\pi} p_k^{-1}(u)\sin(u)\,du + \int_{2l_k\pi}^{2k\pi} p_k^{-1}(u)\sin(u)\,du\right).$$

Since $z_k \to \tan(\pi\theta)$ viz. $l_k \to l_\infty < +\infty$ as $k \to \infty$, it is easy to see that the first term in the decomposition is bounded, and we are finally reduced to show that

$$B_k = \int_{2l_k\pi}^{2k\pi} p_k^{-1}(u)\sin(u)\,du \to -\infty \quad \text{as } k \to +\infty.$$

Since $p_k^{-1}(u)$ increases on $[2l_k\pi, 2k\pi]$, we have

$$B_k = \sum_{j=l_k}^{k-1}\left(\int_{2j\pi}^{(2j+1)\pi} \left(p_k^{-1}(u) - p_k^{-1}(u+\pi)\right)\sin(u)\,du\right)$$

for every $k \geq 1$ and since $p_k(u) \to \frac{\alpha}{\sin(\pi\theta)} u e^{-\cot(\pi\theta)u}$ pointwise as $k \to +\infty$, Fatou's lemma implies

$$\limsup_{k\to+\infty} B_k \leq \frac{\sin(\pi\theta)}{\alpha}\sum_{j=l_\infty}^{\infty}\int_{2j\pi}^{(2j+1)\pi}\left(\frac{e^{\cot(\pi\theta)u}}{u} - \frac{e^{\cot(\pi\theta)(u+\pi)}}{u+\pi}\right)\sin(u)\,du.$$

Using the inequality

$$\frac{1+e^{\pi\cot\pi\theta}}{2u} \leq \frac{e^{\pi\cot\pi\theta}}{u+\pi},$$

which holds for $u \geq \frac{\pi(e^{\pi\cot\pi\theta}+1)}{e^{\pi\cot\pi\theta}-1}$, we deduce that for j_∞ large enough, one has

$$\begin{aligned}\limsup_{k\to+\infty} B_k &\leq \frac{\pi\sin(\pi\theta)}{\alpha}\sum_{j=j_\infty}^{\infty}\frac{1-e^{\pi\cot\pi\theta}}{2}\int_{2j\pi}^{(2j+1)\pi}\frac{e^{\cot(\pi\theta)u}}{u}\sin(u)\,du\\ &\leq -\frac{(e^{\pi\cot\pi\theta}-1)\pi\sin(\pi\theta)}{4\alpha}\sum_{j=j_\infty}^{\infty}\int_{2j\pi+\pi/6}^{2j\pi+5\pi/6}\frac{1}{u}\,du = -\infty.\end{aligned}$$

All of this shows that

$$\liminf_{r\to 0} H_\alpha(r) = -\infty.$$

The argument for the limsup follows exactly along the same lines, considering the subsequence

$$\tilde{r}_k = \alpha\left(\frac{(1-\alpha)\sin(\pi\theta)}{(2k+1)\pi}\right)^{1/\alpha-1}.$$

□

Remark 9. As mentioned in Remark 4(a), the above proof shows that $x \mapsto \mathbb{P}[\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}} \geq x]$ is not HCM for every $\alpha < 1/5$. By Theorem 6.3.5 in [12], this implies that $\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}}$ is not HCM for $\alpha < 1/5$ either. This shows that Conjecture 1.2 in [14] is not true in general.

We believe that θ_a is non-decreasing for $\alpha \in [1/5, 1)$, which is equivalent to the following

Conjecture 1. *The law of* $\mathbf{K}_\alpha$ *is a* GGC *if and only if* $\alpha \leq 4/5$.

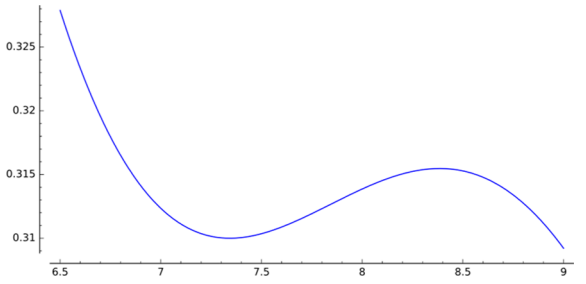

Fig. 3. $x^{-1}G_{1/5}(x^{-5})$.

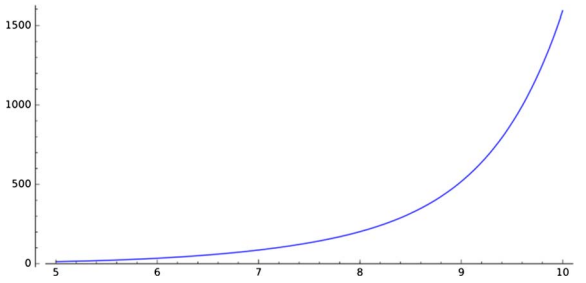

Fig. 4. $\tilde{G}_{1/5}(x)$.

The above Corollary 2 shows the only if part, and in the proof of Theorem 2 we have shown the if part for $\alpha \leq 3/4$. However, it seems that our methods fail to handle the remaining case $\alpha \in (3/4, 4/5]$, because some simulations show that $r^{\alpha-1}G_\alpha(r^{\frac{\alpha-1}{\alpha}}) = r^{\alpha-1}\Re(F_\alpha(re^{-\mathrm{i}\pi}))$ is not monotonous anymore, at least for α close enough to $1/5$ – see Figure 3. Observe from (14) that the problem can be reformulated in terms of the monotonicity of the ratio of two power series, the non-decreasing character of θ_α being equivalent to that of

$$\tilde{G}_\alpha : x \longmapsto \frac{\Im(F_\alpha(xe^{-\mathrm{i}\pi}))}{\Re(F_\alpha(xe^{-\mathrm{i}\pi}))} = \frac{\sum_{n\geq 0} \frac{\sin(n\pi\alpha)}{n!\Gamma(1-n\alpha)}x^n}{\sum_{n\geq 0} \frac{\cos(n\pi\alpha)}{n!\Gamma(1-n\alpha)}x^n}$$

on $(0, \infty)$. A necessary condition for $\tilde{G}_\alpha$ to be non-decreasing is that its denominator does not vanish on $(0, \infty)$, which is false for $\alpha < 1/5$ by Proposition 4 and true for $\alpha \geq 1/4$ by the proof of Theorem 2. But the case $\alpha \in [1/5, 1/4)$ still eludes us. Let us mention that monotonicity properties of ratios of power series are studied in the literature on special functions – see e.g. Chapter 3.1 in [4]. However, we could not find any clue in this literature for our problem, and it is not easy to understand why the value $\alpha = 1/5$ should be critical for the monotonicity of the above ratio. See Figure 4 for a simulation.

We finally turn to the behaviour of $F_\alpha(re^{-\mathrm{i}\pi})$ at infinity, which implies that of $\theta_\alpha(r)$.

Proposition 5. *For all $\alpha \in (0, 1)$, one has*

$$F_\alpha\big(re^{-\mathrm{i}\pi}\big) \sim \frac{\mathrm{i}c_\alpha e^{b_{1-\alpha}r}}{\sqrt{r}} \quad \text{as } r \to +\infty,$$

with $c_\alpha = \frac{\alpha^{\frac{1}{2(\alpha-1)}}}{\sqrt{2\pi(1-\alpha)}}$. In particular, one has $\theta_\alpha(r) \to 1/2$ as $r \to +\infty$.

Proof. From (14), we can write

$$F_\alpha\big(re^{-\mathrm{i}\pi}\big) = \phi\big(-\alpha, 1, r^{1-\alpha}e^{\mathrm{i}\pi\alpha}\big), \quad r > 0.$$

We now use the asymptotic expansion for large $z \in \mathbb{C}$ and $a \in (-1, 0)$ of the Wright function $\phi(a, b, z)$, which has been obtained in [43]. Applying therein Theorem 1 for $\alpha \leq 1/3$ resp. Theorem 5 for $\alpha > 1/3$ and taking the first term in (1.3)

implies the required asymptotic for $F_\alpha(re^{-i\pi})$, since we have here

$$A_0 = \frac{1}{\sqrt{2\pi\alpha}} \quad \text{and} \quad Y = b_{1-\alpha} r e^{-i\pi}$$

in the notation of [43], the first equality being a consequence of Stirling's formula. From (22), we then readily deduce that $\theta_\alpha(r) \to 1/2$ as $r \to +\infty$. □

Remark 10.

(a) Taking the first two terms in the series representation (14) yields at once the asymptotic behaviour of $\theta_\alpha(r)$ at zero, which is

$$\theta_\alpha(r) \sim \frac{r^{1-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)^2}.$$

On the other hand, the complete asymptotic expansion (1.3) in [43] has only purely imaginary terms in our framework, so that we cannot deduce from it the asymptotics of $\theta_\alpha(r) - 1/2$ at infinity. It follows from Proposition 1 that $\theta_\alpha(r) \in [0, 1/2)$ for $\alpha \geq 1/4$, and from Proposition 4 that $\theta_\alpha(r) - 1/2$ crosses zero an infinite number of times for $\alpha < 1/5$, as $r \mapsto +\infty$. For $\alpha \in [1/5, 1/4)$, we are currently unable to prove that $\theta_\alpha(r) \in [0, 1/2)$ for every $r > 0$, which would be a first step to show that it increases from 0 to 1/2. Recall that the latter is equivalent to the fact that the denominator of the above $\tilde{G}_\alpha$ does not vanish on $(0, \infty)$.

(b) If $\alpha \leq 3/4$, it follows from (13), Theorem 8.2 and Remark 8.3 in [39], and the above proposition, that the Thorin mass of the GGC random variable $\mathbf{K}_\alpha$ equals 1/2. Hence, $\mathbf{K}_\alpha - b_\alpha$ is a $\Gamma_{1/2}$-mixture by Theorem 4.1.1. in [12], which is a refinement of Corollary 3.2 in [26]. Since this property amounts to the CM character of $x \mapsto \sqrt{x} f_{\mathbf{K}_\alpha}(b_\alpha + x)$, a perusal of the proof of Theorem 1 shows that $\mathbf{X}_\alpha - b_\alpha$ is a $\Gamma_{3/2}$-mixture as soon as $\alpha \leq 3/4$. We believe that this is true for every $\alpha \in (0, 1)$.

3.2. *Asymptotic expansions for the free extreme stable densities*

In this paragraph we derive the full asymptotic expansion at zero of the density $f_{\mathbf{Y}_\alpha}$ of the random variable

$$\mathbf{Y}_\alpha = \begin{cases} \mathbf{X}_\alpha - b_\alpha & \text{if } \alpha \in (0, 1), \\ \mathbf{X}_{\alpha, 1-1/\alpha} + b_{1/\alpha}^{-1/\alpha} \overset{d}{=} b_{1/\alpha}^{-1/\alpha} - \mathbf{X}_{\alpha, 1/\alpha} & \text{if } \alpha \in (1, 2], \end{cases}$$

and $\mathbf{Y}_1 = 1 - \mathbf{T}$. We will use the standard notation of Definition C.1.1 in [1] for asymptotic expansions. Our expansions complete the estimates of Proposition A.1.2 in [10] and the series representations of Theorem 1.7 in [24], from which one can only infer that the random variable $\mathbf{Y}_\alpha$ is positive. They can also be viewed as free analogues of Linnik's expansions (14.35) in [38] – see also Theorem 2.5.3 in [44] – for the classical extreme stable distributions. Observe that in the classical case, the expansion for $\alpha > 1$ is deduced from that of the case $\alpha \in [1/2, 1)$ by the Zolotarev's duality which is discussed in Section 2.3 of [44]. Even though the very same duality relationship holds in the free case – see Proposition A.3.1 in [10] and Corollary 1.4 in [24], for $\mathbf{Y}_\alpha$ this duality only yields

$$f_{\mathbf{Y}_{1/\alpha}}(x) = \frac{1}{\alpha}\big(b_\alpha^{-\alpha} - x\big)^{-1/\alpha - 1} f_{\mathbf{Y}_\alpha}\big(\big(b_\alpha^{-\alpha} - x\big)^{1/\alpha} - b_\alpha\big)$$

for every $\alpha \in [1/2, 1)$, and does not seem particularly helpful to connect explicitly the two expansions at zero. When $\alpha \neq 1$, our method hinges on Wright's original papers [42] for the case $\alpha > 1$ and [43] for the case $\alpha < 1$. It is remarkable that the two expansions turn out to have the same parametrization.

Proposition 6. *For every $\alpha \in (0, 1) \cup (1, 2]$, one has*

$$f_{\mathbf{Y}_\alpha}(x) \sim \sum_{n=0}^{\infty} a_n(\alpha) x^{n+1/2} \quad \text{as } x \to 0,$$

with

$$a_n(\alpha) = \left(\frac{2}{\alpha}\right)^{n+1/2} \frac{(-1)^n}{\pi |\alpha - 1|^{(n+3/2)/\alpha} (2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}} \left((1-v)^{-2} {}_2F_1 \left[\begin{matrix} \alpha + 1 & 1 \\ 3 \end{matrix} ; v \right]^{-n-1/2} \right)_{v=0}.$$

Proof. We begin with the case $\alpha > 1$, writing down first $f_{\mathbf{Y}_\alpha}$ with the help of Bromwich's integral formula

$$f_{\mathbf{Y}_\alpha}(x) = \frac{1}{2\pi \mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^{zx} \mathcal{L}_\alpha(z)\, dz,$$

where

$$\mathcal{L}_\alpha(z) = \mathbb{E}\big[e^{-z\mathbf{Y}_\alpha}\big] = e^{-\alpha(\alpha-1)^{\frac{1}{\alpha}-1} z} \times \mathbb{E}\big[e^{z\mathbf{X}_{\alpha,1/\alpha}}\big]$$

is well-defined and analytic on the open right half-plane. Combining next Theorem 1.8 in [24] and Theorem 2 in [42], we obtain

$$\mathcal{L}_\alpha(z) = e^{-\alpha(\alpha-1)^{\frac{1}{\alpha}-1} z} \times \phi\big(\alpha - 1, 2, z^\alpha\big) = \mathrm{O}\big(|z|^{-3/2}\big)$$

uniformly on the right half-plane. Making a change of variable and applying Cauchy's theorem, we deduce

$$f_{\mathbf{Y}_\alpha}(x) = \frac{1}{2\pi \mathrm{i} x} \int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty} e^{z} \mathcal{L}_\alpha\big(zx^{-1}\big)\, dz = \frac{1}{2\pi \mathrm{i} x} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^{z} \mathcal{L}_\alpha\big(zx^{-1}\big)\, dz.$$

Using now the full asymptotic expansion of Theorem 2 in [42], we get

$$f_{\mathbf{Y}_\alpha}(x) \sim \sum_{n=0}^{\infty} a_n(\alpha) x^{n+1/2} \quad \text{as } x \to 0,$$

where

$$a_n(\alpha) = \frac{(-1)^n a_n}{(\alpha-1)^{(n+3/2)/\alpha}} \left(\frac{1}{2\pi \mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^{z} z^{-3/2-n}\, dz \right) = \frac{(-1)^n a_n}{(\alpha-1)^{(n+3/2)/\alpha} \Gamma(n+3/2)}$$

and a_n is defined at the beginning of p.258 in [42] for $\rho = \alpha - 1$ and $\beta = 2$. Above, the interchanging of the contour integral and the expansion is easily justified – alternatively one can use the generalized Watson's lemma which is mentioned at the top of p.615 in [1], whereas the second equality follows from Hankel's formula – see e.g. Exercise 1.22 in [1]. To conclude the proof of the case $\alpha > 1$, it remains to evaluate the coefficients $a_n(\alpha)$, which is done in observing that the function in (1.21) of [42] is here

$$\sqrt{{}_2F_1\left[\begin{matrix}\alpha+1 & 1 \\ 3 & \end{matrix}; v\right]},$$

and making some simplifications.

We now consider the case $\alpha < 1$. The argument is analogous but it depends on the expansions of [43] which, the author says, cannot be simply deduced from those of [42]. We again write

$$f_{\mathbf{Y}_\alpha}(x) = \frac{1}{2\pi \mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^{zx} \mathcal{L}_\alpha(z)\, dz,$$

where

$$\mathcal{L}_\alpha(z) = e^{\alpha(1-\alpha)^{\frac{1}{\alpha}-1} z} \times \mathbb{E}\big[e^{-z\mathbf{X}_\alpha}\big] = e^{\alpha(1-\alpha)^{\frac{1}{\alpha}-1} z} \times \phi\big(\alpha - 1, 2, -z^\alpha\big) = \mathrm{O}\big(|z|^{-3/2}\big)$$

uniformly in the open right half-plane, the second equality following from Theorem 1.8 in [24] and the estimate from the Lemma p.39 in [43]. Reasoning as above, we get

$$f_{\mathbf{Y}_\alpha}(x) = \frac{1}{2\pi \mathrm{i} x} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^{z} \mathcal{L}_\alpha\big(zx^{-1}\big)\, dz \sim \sum_{n=0}^{\infty} a_n(\alpha) x^{n+1/2} \quad \text{as } x \to 0,$$

where

$$a_n(\alpha) = \frac{a_n}{(\alpha(1-\alpha)^{\frac{1}{\alpha}-1})^{n+3/2} \Gamma(n+3/2)}$$

and a_n is defined at the bottom of p.38 in [43] for $\sigma = 1-\alpha$ and $\beta = 2$. After some simplifications, we also obtain the required expression for $a_n(\alpha)$. □

Remark 11.

(a) It does not seem that a simple closed formula can be obtained for the coefficients $a_n(\alpha)$ in general. We can compute

$$a_0(\alpha) = \sqrt{\frac{2}{\alpha}} \times \frac{1}{\pi|\alpha-1|^{3/(2\alpha)}} \quad \text{and} \quad a_1(\alpha) = -\sqrt{\frac{2}{\alpha}} \times \left(\frac{2\alpha^2 - 23\alpha + 47}{36\pi\alpha|\alpha-1|^{5/(2\alpha)}}\right).$$

Observe that $a_1(\alpha)$ is always negative. We believe that in general, one has

$$a_n(\alpha) = \sqrt{\frac{2}{\alpha}} \times \frac{Q_{2n}(\alpha)}{\pi\alpha^n|\alpha-1|^{(2n+3)/(2\alpha)}}$$

for some $Q_{2n} \in \mathbb{Q}_{2n}[X]$. This would again mimic the classical situation, save for the fact that here the polynomial Q_{2n} does not seem to have symmetric coefficients – see Remark 2 p.101 in [44].

(b) For $\alpha = 2$, the involved hypergeometric function becomes the standard geometric series and we simply get

$$a_n(2) = \frac{(-1)^n}{\pi(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\big((1-v)^{n-3/2}\big)_{v=0} = \frac{-1}{\pi(2n-1)16^n}\binom{2n}{n},$$

which is always negative except for $n = 0$. Of course, this can be retrieved via the binomial theorem for the explicit density

$$f_{\mathbf{Y}_2}(x) = \frac{\sqrt{x}}{\pi}\sqrt{1-\frac{x}{4}}.$$

(c) For $\alpha = 1/2$, the involved hypergeometric function simplifies with the help of Exercise 3.39 in [1], and we get

$$a_n(1/2) = \frac{(-1)^n 4^{n+2}}{2\pi(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\big((1+\sqrt{1-v})^{2n+1}(1-v)^{-2}\big)_{v=0} = \frac{(-1)^n(n+1)4^{n+2}}{\pi},$$

whose signs alternate. This again can be retrieved via the binomial theorem for the explicit density

$$f_{\mathbf{Y}_{1/2}}(x) = \frac{16\sqrt{x}}{\pi(1+4x)^2}.$$

(d) As already observed in Remark 1(a), the densities of $\mathbf{Y}_{1/3}$ and $\mathbf{Y}_{2/3}$ can be written in closed form with the help of Formulae (40) and (41) in [32]. In principle, a full asymptotic expansion can also be derived from these expressions, but the task seems too painful. Notice that here, the involved hypergeometric functions do not seem to simplify.

(e) The above proof shows that the following functions

$$\lambda \mapsto e^{-\alpha(\alpha-1)^{\frac{1}{\alpha}-1}\lambda}\phi\big(\alpha-1, 2, \lambda^\alpha\big) \quad \text{resp. } \lambda \mapsto e^{\alpha(1-\alpha)^{\frac{1}{\alpha}-1}\lambda}\phi\big(\alpha-1, 2, -\lambda^\alpha\big)$$

on $(0,\infty)$, which are obtained in removing Wright's exponential term at infinity, are CM functions for $\alpha \in (1,2]$ resp. for $\alpha \in (0,1)$.

We now complete the picture and derive the asymptotic expansion of $\mathbf{Y}_1 = 1 - \mathbf{T}$. To state our result, we need to introduce the Stirling series $\{c_n, n \geq 0\}$ appearing in the expansion

$$\left(\frac{\mathrm{e}}{x}\right)^x \sqrt{\frac{x}{2\pi}}\Gamma(x) \sim \sum_{n\geq 0} c_n x^{-n} \quad \text{as } x \to +\infty,$$

which is given e.g. in Exercise 23 p.267 of [17]. One has $c_0 = 1, c_1 = 1/12, c_2 = 1/288$ and $c_3 = -139/51{,}840$. In general, c_n is a rational number and the corresponding sequences of numerators and denominators are referenced under A00163 and A00164 in the online version of [35].

Proposition 7. *One has*

$$f_{\mathbf{Y}_1}(x) \sim \sum_{n=0}^{\infty} a_n(1) x^{n+1/2} \quad \text{as } x \to 0,$$

with

$$a_n(1) = \frac{(-1)^n 2^{2n+1/2} n!(c_0 + \cdots + c_n)}{\pi(2n+1)!}.$$

Proof. Applying Remark 7(a), we first compute the Laplace transform

$$\mathbb{E}\big[e^{-z\mathbf{Y}_1}\big] = \frac{1}{z(1+z)} \left(\frac{z}{\mathrm{e}}\right)^z \frac{1}{\Gamma(z)}$$

for every z in the open right half-plane. Comparing next (2.15) and (2.21) in [33], we get the expansion

$$\begin{aligned}
\mathbb{E}\big[e^{-z\mathbf{Y}_1}\big] &\sim \frac{1}{\sqrt{2\pi z}(1+z)} \sum_{n\geq 0} (-1)^n c_n z^{-n} \\
&= \frac{1}{\sqrt{2\pi z^3}} \left(\sum_{n\geq 0} (-1)^n z^{-n}\right)\left(\sum_{n\geq 0} (-1)^n c_n z^{-n}\right) \\
&\sim \frac{1}{\sqrt{2\pi z^3}} \sum_{n\geq 0} (-1)^n (c_0 + \cdots + c_n) z^{-n}
\end{aligned}$$

as $|z| \to \infty$, uniformly in the open right half-plane $|\arg(z)| < \pi/2$. Reasoning as in Proposition 6, we finally obtain

$$f_{\mathbf{Y}_1}(x) \sim \sum_{n=0}^{\infty} a_n(1) x^{n+1/2} \text{as } x \to 0,$$

with

$$a_n(1) = \frac{(-1)^n (c_0 + \cdots + c_n)}{\sqrt{2\pi}\Gamma(n+3/2)} = \frac{(-1)^n 2^{2n+1/2} n!(c_0 + \cdots + c_n)}{\pi(2n+1)!}.$$

□

Remark 12. It is easy to see from (15) that

$$(1-\alpha)^{-1}\mathbf{Y}_\alpha \xrightarrow{d} \mathbf{Y}_1 \quad \text{as } \alpha \uparrow 1,$$

and it is natural to infer from this and Proposition 6 that

$$\begin{aligned}
a_n(1) &= \frac{(-1)^n 2^{n+1/2}}{\pi(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}} \left((1-v)^{-2}{}_2F_1\left[\begin{matrix}2 & 1\\ 3\end{matrix}; v\right]^{-n-1/2}\right)_{v=0} \\
&= \frac{(-1)^n}{\pi(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}} \left(\frac{v^{2n+1}}{(1-v)^2(-v-\log(1-v))^{n+1/2}}\right)_{v=0},
\end{aligned}$$

except that we cannot interchange a priori the asymptotic expansion at zero and the convergence in law. We have checked the correspondence for $n = 0$ and $n = 1$, with

$$a_0(1) = \frac{\sqrt{2}}{\pi} \quad \text{and} \quad a_1(1) = -\frac{13\sqrt{2}}{18\pi}$$

to be compared with Remark 11(a). We believe that this formula is true for every $n \geq 1$. Observe that this is equivalent to the following expression of the Stirling series:

$$c_n = b_n - b_{n-1}, \quad n \geq 1,$$

with

$$b_n = \frac{1}{2^{2n+1/2}n!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}} \left(\frac{v^{2n+1}}{(1-v)^2(-v-\log(1-v))^{n+1/2}} \right)_{v=0},$$

which is different from the combinatorial expression given in Exercise 23 p.267 of [17], and which we could not locate in the literature.

3.3. *Further properties of whale-shaped functions*

In this paragraph we prove five analytical properties of WS functions and densities. Those five easy pieces apply all to the densities f_α, and have an independent interest. We restrict the study to the class WS_+, the corresponding properties for WS_- being deduced at once.

Proposition 8. *Let f be a WS_+ density with unique mode M. Then f is perfectly skew to the right, that is*

$$f(M+x) > f(M-x) \quad \textit{for every } x > 0.$$

Proof. Let x_0 be the left-extremity of $\mathrm{Supp}\, f$ and $M = x_1 < x_2 < x_3$ be the vanishing places of the three first derivatives of f. Suppose first $M - x_0 > x_2 - M$. Taylor's formula with integral remainder implies

$$f(M+x) - f(M-x) = \int_0^x (x-t)\big(f''(M+t) - f''(M-t)\big)\, dt.$$

On the one-hand, we have $f''(M+t) - f''(M-t) > 0$ for all $t > x_2 - M$ since $f''(M-t) \le 0$ for all $t \ge 0$ and $f''(M+t) > 0$ for all $t > x_2 - M$. On the other hand, writing

$$f''(M+t) - f''(M-t) = \int_0^t \big(f^{(3)}(M+s) + f^{(3)}(M-s)\big)\, ds,$$

which is valid for all $t < M - x_0$, we also have $f''(M+t) - f''(M-t) > 0$ for all $t \le x_2 - M$ since $f^{(3)}(u) > 0$ for all $x_0 < u < x_3$, by the WS_+ property. Putting everything together shows $f(M+x) > f(M-x)$ for all $x > 0$. Supposing next $M - x_0 \le x_2 - M$, the proof is analogous and easier; we just need to delete the corresponding arguments for $t > x_2 - M$. □

Remark 13. If we denote by $M_{\alpha,\rho}$ the unique mode of $f_{\alpha,\rho}$, the function

$$x \mapsto f_{\alpha,\rho}(M_{\alpha,\rho} + x) - f_{\alpha,\rho}(M_{\alpha,\rho} - x)$$

has constant and possibly zero sign on $(0,\infty)$ for $\rho = 0, 1/2, 1$ and for $\alpha = 1$, as seen from the above proposition, the explicit drifted Cauchy case and the symmetric case. One might wonder if this property of perfect skewness remains true in general. The perfect skewness of classical stable densities is a challenging open problem, which had been stated in the introduction to [23].

Proposition 9. *Let f be a WS_+ density and M, m, μ be its respective mode, median and mean. Then f satisfies the strict mean-median-mode inequality*

$$M < m < \mu.$$

Proof. We use the same notation of the proof of the previous proposition. First, the latter clearly implies $M < m$. To obtain the two strict inequalities together, let us now consider the function

$$g(x) = f(m+x) - f(m-x)$$

on $[0, m - x_0]$. If $m < x_2$, then the WS_+ property implies $f^{(2)}(m+x) > f^{(2)}(m) > f^{(2)}(m-x)$ for every $x \in (0, m-x_0]$, so that g is strictly convex on $(0, m-x_0]$. Since $g(0) = 0$, $g'(0) < 0$ and $g(m-x_0) > 0$, this shows that g vanishes only once on $(0, m-x_0]$ and from below, and hence also on the whole $(0,\infty)$. If $m \ge x_2$, then g is negative on $(0, m-M]$ and strictly convex on $[m-M, m-x_0]$ and we arrive at the same conclusion. We are hence in position to apply Lemma 1.9 (a) and (a strict, easily proved version of) Theorem 1.14 in [19], which implies the strict mean-median-mode inequality for f. □

Remark 14.

(a) It is well-known that $\mathbf{X}_\alpha$ has infinite mean, which can be seen e.g. from the power series formula for the density of Theorem 1.7 in [24]. Hence, in this framework the above result only reads $M < m < \infty$, and it is readily obtained from the previous proposition.

(b) In the relevant case $\alpha \in (1,2)$ it is natural to conjecture that the strict mean-median-mode inequality holds, in one or the other direction, for both free and classical stable densities. Observe that the three parameters clearly coincide for $\rho = 1/2$, whereas for $\rho = 1/\alpha$, easy computations show that the mean is zero and the mode and median are positive, so that it is enough to prove $m < M$. In general, this problem is believed to be challenging and beyond the scope of the present paper. We refer to [6] for a series of results on this interesting question, which however do not apply to non-explicit densities.

Proposition 10. *Let f be a* WS *density on $(0,+\infty)$ and X be the corresponding random variable. Then X is a Γ_2-mixture. In particular, it is* ID.

Proof. As in Theorem 1, we need to show that $g(x) = x^{-1} f(x)$ is a CM function, in other words that $(-1)^n g^{(n)}(x) > 0$ on $(0,\infty)$. By Leibniz's formula, we first compute

$$g^{(n)}(x) = n! \sum_{p=0}^{n} \frac{(-1)^p f^{(n-p)}(x)}{(n-p)! x^{p+1}}.$$

This implies, after some simple rearrangements,

$$h_n'(x) = (-1)^n x^n f^{(n+1)}(x), \tag{23}$$

where $h_n(x) = (-1)^n x^{n+1} g^{(n)}(x)$ has the same sign as $(-1)^n g^{(n)}(x)$. By the WS property, setting x_i for the vanishing places of the ith derivative of f for all $i \geq 1$, we see that

$$h_n(x) = n! \sum_{p=0}^{n} \frac{(-1)^{n-p} f^{(n-p)}(x)}{(n-p)!} x^{n-p}$$

is positive on $[x_n,\infty)$ since $(-1)^i f^{(i)}(x) > 0$ when $x \in (x_i,\infty)$ for all $i \geq 0$. Moreover, it follows from (23) and the whale-shape that $h_n'(x) > 0$ for $x \in (0, x_{n+1}]$. It is hence enough to show that $h_n(0+) = 0$ in order to conclude the proof, because $(0,\infty) = (0, x_{n+1}] \cup [x_n,\infty)$. But the whale-shape shows again that

$$0 \leq (-1)^{i-1} x^i f^{(i)}(x) \leq 2(-1)^{i-1} x^{i-1} \big(f^{(i-1)}(x) - f^{(i-1)}(x/2) \big)$$

for all $x \in (0, x_1]$ and an induction on i, starting from $f(0+) = 0$, implies $(x^i f^{(i)})(0+) = 0$ for all $i \geq 0$, so that $h_n(0+) = 0$ as well. □

Remark 15.

(a) The WS property is not satisfied by all densities of Γ_2-mixtures vanishing at zero. A simulation shows for example that the derivative of the density

$$f(x) = x\big(t a^2 e^{-ax} + (1-t) e^{-x} \big)$$

vanishes three times for $a = 20$ and $t = 4/5$. This contrasts with the densities of Γ_1-mixtures, which are characterized by their complete monotonicity – see e.g. Proposition 51.8 in [38].

(b) For a given smooth density f on $(0,\infty)$ and $n \geq 0$, let us introduce the following property: one has $f \in \mathrm{BS}_n$ if

$$\begin{cases} \sharp\{x > 0,\, f^{(i)}(x) = 0\} = i & \text{for } i \leq n, \\ \sharp\{x > 0,\, f^{(i)}(x) = 0\} = n & \text{for } i > n. \end{cases}$$

For $n \geq 1$, this property was introduced in [40] under the less natural denomination WBS_{n-1} – see the definition therein. Clearly, one has $\mathrm{BS}_0 = \mathrm{CM}$ and $\mathrm{BS}_1 = \mathrm{WS}$ for densities on $(0,\infty)$. Since the density of $\mathbf{\Gamma}_t$ has mth derivative

$$(-1)^m \left(\sum_{p=0}^{m} \binom{m}{p} (1-t)_p x^{-p} \right) \frac{x^{t-1} e^{-x}}{\Gamma(t)}$$

on $(0,\infty)$, it is an easy exercise using Rolle's theorem and Descartes' rule of signs to show that $\mathbf{\Gamma}_t \in \mathrm{BS}_n$ for $t \in (n, n+1]$. In this respect, the class BS_n can be thought of as an extension of the densities of $\mathbf{\Gamma}_t$ for $t \in (n, n+1]$. Moreover, we have just seen that the set of densities of Γ_{n+1}-mixtures contains the class BS_n for $n = 0, 1$. We actually believe that this is true for all $n \geq 0$. In view of the Proposition in [40] and of the general results of [29], it is also natural to conjecture that the densities in BS_n are infinitely divisible for all $n \geq 2$, contrary to the densities of Γ_{n+1}-mixtures. Taking the limit $n \to \infty$, this would imply that all bell-shaped densities on a half-line are ID. On the other hand, the class BS_n does not seem to share any interesting property related to perfect skewness, mean-median-mode inequality for $n \geq 2$.

(c) The above proposition entails that the case $\{\alpha \in (0,1], \rho = 1\}$ in Theorem 1(a) is a consequence of Theorem 4(a). On the other hand, as we saw above, the proof of Theorem 1 also shows that $\mathbf{X}_\alpha$ is a $\Gamma_{3/2}$-mixture for $\alpha \leq 3/4$, which is not a consequence of the whale-shape.

We next study the stability of the WS property under exponential tilting. Within ID densities on $\mathbb{R}$, this transformation amounts to the multiplication of the Lévy measure by $e^{-c|x|}$, allowing for models with finite positive moments and analogous small jumps. This is a particular instance of the general tempering transformation, where the exponential perturbation is replaced by a CM function, and we refer to [36] for a thorough study on tempered stable densities. If we restrict to ID densities on a positive half-line, it is seen from the Lévy–Khintchine formula that exponential tilting amounts to multiplying the density by the same e^{-cx} and renormalizing. In particular, the set of densities of Γ_t-mixtures with $t \in (0,2]$ is also stable under exponential tilting.

Proposition 11. *If $f \in \mathrm{WS}_+$, then $e^{-x} f \in \mathrm{WS}_+$.*

Proof. It is enough to consider the case Supp $f = (0,\infty)$. Set $g(x) = e^{-x} f(x)$. Considering $h_n(x) = (-1)^n e^x g^{(n)}(x)$ for each $n \geq 0$, we have $h_{n+1} = h_n - h_n'$ and an easy induction starting from $h_0 = f$ implies

$$(-1)^{p-1} h_{n+1}^{(p)}(0+) > 0 \quad \text{and} \quad h_{n+1}^{(p)}(+\infty) = 0$$

for all $n, p \geq 0$. We will now show that $h_{n+1}^{(p)}$ vanishes once on $(0,\infty)$ for all $n, p \geq 0$, and that the sequence $\{x_{p,n+1}, p \geq 0\}$ defined by $h_{n+1}^{(p)}(x_{p,n+1}) = 0$ is increasing. This is sufficient for our purpose, in taking $p = 0$.

Consider first the case $n = 0$, with $h_1^{(p)} = f^{(p)} - f^{(p+1)}$. It is clear that $(-1)^p h_1^{(p)}(x) > 0$ for $x \in [x_{p+1}, \infty)$ and that $(-1)^p h_1^{(p+1)}(x) > 0$ for $x \in (0, x_{p+1}]$. Since $(-1)^p h_1^{(p)}(0+) < 0$, this implies that $h_1^{(p)}$ vanishes once on $(0,\infty)$ for all $p \geq 0$, and Rolle's theorem entails that the sequence $\{x_{p,1}, p \geq 0\}$ defined by $h_1^{(p)}(x_{p,1}) = 0$ is increasing.

The induction step is obtained analogously from $h_{n+2}^{(p)} = h_{n+1}^{(p)} - h_{n+1}^{(p+1)}$, since $(-1)^p h_{n+2}^{(p)}(0+) < 0$ and, by the induction hypothesis, $(-1)^p h_{n+2}^{(p)}(x) > 0$ for $x \in [x_{p+1,n+1}, \infty)$ and $(-1)^p h_{n+2}^{(p+1)}(x) > 0$ for $x \in (0, x_{p+1,n+1}]$. □

Remark 16.

(a) The above proposition implies that $e^{-x} f_\alpha$, the "tilted free positive stable density", is WS_+ and ID. It would be interesting to know if it is also FID.

(b) The class WS_+ is not stable under the general tempering transformation introduced in [36]. For example, the random variable obtained from $\mathbf{\Gamma}_2$ in multiplying its Lévy measure by te^{-x} is easily seen to be $(1/2)\mathbf{\Gamma}_{2t}$, whose density belongs to WS_+ only for $t \in (1/2, 1]$.

Proposition 12. *Let $f \in \mathrm{WS}_+$ and $\{x_n, n \geq 0\}$ be the vanishing places of $\{f^{(n)}, n \geq 0\}$. Then f is analytic on (x_0,∞) and $x_n \to \infty$.*

Proof. Again we may suppose $x_0 = 0$. If f is a density, then Proposition 10 implies that $f = xg$ where g is CM and hence analytic on $(0,\infty)$, so that f is analytic on $(0,\infty)$ as well. If f is not a density, then Proposition 11 shows that $g = e^{-cx} f$ is a WS_+ density on $(0,\infty)$ for some normalizing $c > 0$, and f inherits the analyticity of g on $(0,\infty)$.

The second property is an easy consequence of the first one. Let x_∞ be the increasing limit of $\{x_n, n \geq 0\}$ and suppose $x_\infty < \infty$. By the whale-shape, we would then have $(-1)^n f^{(n)}(x) > 0$ for $x > x_\infty$, so that f would be CM on (x_∞,∞), and hence also on $(0,\infty)$ by Bernstein's theorem and analytic continuation, a contradiction since $f(0+) = 0$. □

Overview of random variables used in the paper

$\mathbf{X}_{\alpha,\rho}$	free strictly α-stable	Section 1
$\mathbf{X}_{\alpha}$	free positive α-stable	
$\mathbf{C}_{a,b}$	free 1-stable	
$\mathbf{T}$	exceptional free 1-stable	
$\mathbf{K}_{\alpha}$	Kanter	Section 2.1
$\mathbf{U}$	uniform on $(0,1)$	
$\mathbf{L}$	unit exponential	
$\mathbf{B}_{a,b}$	standard $\beta(a,b)$	
$\mathbf{Z}_{\alpha}$	classical positive α-stable	
$\mathbf{Z}_{\alpha,\rho}$	classical strictly α-stable	Section 2.2
$\mathbf{\Gamma}_t$	standard $\gamma(t)$	
$\mathbf{S}$	exceptional classical 1-stable	Section 2.4
$\mathbf{W}$	$\frac{\sin(\pi\mathbf{U})}{\pi\mathbf{U}} e^{\pi\mathbf{U}\cot(\pi\mathbf{U})}$	
	$\mathbf{X}_{\alpha} - b_{\alpha}$ if $\alpha \in (0,1)$	Section 3.2
$\mathbf{Y}_{\alpha}$	$1-\mathbf{T}$ if $\alpha = 1$	(Section 2.2)
	$b_{1/\alpha}^{-1/\alpha} - \mathbf{X}_{\alpha,1/\alpha}$ if $\alpha \in (1,2]$	

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