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On the thin-shell conjecture for the Schatten classes

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Abstract. We study the thin-shell conjecture for the Schatten classes. In particular, we establish the conjecture for the operator norm, and we also improve on the best known bound for the Schatten classes, due to Barthe and Cordero-Erausquin (*Proc. Lond. Math. Soc.* **106** (2013) 33–64) or Lee and Vempala (2017), for a few more cases. We also show that a necessary condition for the conjecture to be true for any of the Schatten classes is a rather strong negative correlation property: as a consequence of this we obtain the validity of this negative correlation property for all the cases for which we already know the conjecture is true (as for example for the operator norm), but moreover also for all the cases for which we can get a better estimate than the one in (*Proc. Lond. Math. Soc.* **106** (2013) 33–64) or (Lee and Vempala (2017)). For the proofs, our starting point is techniques that were employed for the Schatten classes in (*Math. Ann.* **312** (1998) 773–783) and (*Ann. Inst. Henri Poincaré Probab. Stat.* **43** (2007) 87–99) with regard to other problems.

Résumé. Nous étudions la conjecture de la variance (ou autrement dit, de la concentration du volume d'un convexe dans une petite couronne euclidienne) pour les classes de Schatten. En particulier, nous établissons la conjecture pour la norme d'opérateur, et nous améliorons également le meilleur majorant connu, grâce à Barthe et Cordero-Erausquin (*Proc. Lond. Math. Soc.* **106** (2013) 33–64) ou Lee et Vempala (2017), dans quelques cas de plus.

Nous montrons aussi qu'une condition nécessaire pour que la conjecture soit vraie dans une des classes de Schatten est une propriété de corrélation négative qui doit être suffisamment forte: ceci implique que nous obtenons la validité de cette propriété dans tous les cas pour lesquels on peut démontrer la conjecture (comme par exemple pour la norme d'opérateur), mais aussi dans tous les cas pour lesquels on peut obtenir une meilleure estimation que celle dans (*Proc. Lond. Math. Soc.* **106** (2013) 33–64) ou (Lee and Vempala (2017)).

En ce qui concerne les démonstrations, notre point de départ consiste en des techniques qui ont été utilisées dans (*Math. Ann.* **312** (1998) 773–783) et (*Ann. Inst. Henri Poincaré Probab. Stat.* **43** (2007) 87–99) pour les classes de Schatten dans le contexte d'autres problèmes connexes.

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1. Introduction

We work in real, finite-dimensional vector spaces (that can be identified with $\mathbb{R}^m$ for some $m \geq 1$) which are equipped with a fixed Euclidean structure (or inner product $\langle \cdot, \cdot \rangle$). A compact, convex subset K with non-empty interior is called a convex body. Furthermore, it is called *isotropic* if (i) its Lebesgue volume $\mathrm{vol}(K) = 1$, (ii) it is *centred*, that is, has barycentre at the origin, and (iii) its *covariance matrix* is a multiple of the identity, namely

$$\int_K x_i x_j \, dx = L_K^2 \mathbf{1}_{i=j} \quad \text{for all } 1 \leq i, j \leq m;$$

L_K here is called the *isotropic constant* of K.

Anttila, Ball and Perissinaki [4] conjectured the following type of concentration of volume result for isotropic convex bodies: there exists a sequence of positive numbers $\varepsilon_m \in (0, \frac{1}{2})$ decreasing to 0 such that

$$\mathrm{vol}\big(x \in K : \big| \|x\|_2 - \sqrt{m} L_K \big| \geq \varepsilon_m \sqrt{m} L_K \big) \leq \varepsilon_m. \tag{1}$$

In other words, a random point $\mathcal{X}$ distributed uniformly in K should typically have distance from the origin close to the average (in the L_2 sense) distance $\sqrt{m}L_K$, and be found inside a small annulus of average radius $\sqrt{m}L_K$, a *"thin shell"*. A quantitatively stronger form of the conjecture, which follows as a consequence of another problem, the KLS conjecture (see the next paragraphs for details and references), and also justifies the name *variance conjecture*, was suggested by Bobkov and Koldobsky in [8]: they considered the quantity $\sigma_K^2 := \mathrm{Var}_K(\|\mathcal{X}\|_2^2)/(mL_K^4)$, and asked whether it should be uniformly bounded from above for all isotropic convex bodies K in $\mathbb{R}^m$. Note also the following equivalent definition they gave:

$$\sigma_K^2 := m\frac{\mathrm{Var}_K(\|\mathcal{X}\|_2^2)}{[\mathbb{E}_K(\|\mathcal{X}\|_2^2)]^2}, \tag{2}$$

which allows one to also consider σ_K for bodies K which are not necessarily in isotropic position (in this paper we work with the latter definition).

Both forms of the conjecture are closely related and were initially stated with regard to central limit properties of marginals of the random vector $\mathcal{X}$: it was shown in [4] that a concentration condition in the form of (1) implies most of the 1-dimensional marginals of $\mathcal{X}$ are almost gaussian, and more precisely they are ε_m-close in distribution to a zero-mean gaussian variable with variance L_K^2 (in different settings such an idea can also be found in papers of Diaconis and Freedman [10] and of von Weizsäcker [32], which built on a very general method by Sudakov [31]). In fact, as can be seen in both [4] and [8], if one estimates accurately the order of magnitude of σ_K for a specific isotropic convex body K (as we do here for some cases of the Schatten classes), one immediately has the optimal distance rate of the "good" marginals of $\mathcal{X} \sim \mathrm{Unif}(K)$ to the Gaussian distribution.

The validity of such a concentration condition in the general case was first established by Klartag [19] via logarithmic in the dimension improvements of the trivial bounds (and shortly thereafter with similar estimates by Fleury, Guédon and Paouris [14]). Power-law improvements were then found by Klartag [20], Fleury [13], and Guédon and Milman [15]. More recently Lee and Vempala [24] gave the best known estimate: $\sigma_K \le C\sqrt[4]{m}$, with C an absolute constant, for every isotropic convex body K in $\mathbb{R}^m$. Their result gives in fact improved bounds for the KLS conjecture, which is more general and asks to similarly bound quantities defined analogously to (2) for all smooth (or locally Lipschitz) functions (this conjecture was put forth in an equivalent form, and with algorithmic applications in mind, by Kannan, Lovász and Simonovits [18]).

In a breakthrough result Eldan [12] has shown that, although the thin-shell conjecture is formally a special case of the KLS conjecture, whatever general estimates in $\mathbb{R}^m$ are known for the former immediately follow for the latter as well, with only perhaps logarithmic in m corrections. For more details on the above and on several other important connections of the thin-shell conjecture to problems in the asymptotic theory of convex bodies, the reader is referred to the monographs [1] and [9].

There are also special restricted classes of convex bodies for which the conjecture has been resolved optimally. These include the family of ℓ_p balls treated by Ball and Perissinaki [6] (see also [4], as well as a direct extension of this result to generalised Orlicz balls by Pilipczuk and Wojtaszczyk [29]), isotropic unconditional convex bodies treated by Klartag [21], and moreover isotropic convex bodies with enough, but fewer than those of unconditional bodies, symmetries, which were treated by Barthe and Cordero-Erausquin [7] through a refinement of Klartag's method from [21] (one prominent example here is the regular simplex). Another example of (*almost isotropic*) bodies satisfying the conjecture are hyperplane projections of the ℓ_p balls, studied by Alonso-Gutiérrez and Bastero [2]. Finally, one should obviously include all cases of bodies for which the KLS conjecture has been resolved optimally (see Kolesnikov and Milman's paper [22, Section 1.2] for a detailed list and also its main result for a class of such bodies).

In this paper we study one more special family of convex bodies with regard to the thin-shell conjecture. Let $\mathcal{M}_n(\mathbb{C})$ denote the space of all $n \times n$ matrices with complex entries (viewed as a real vector space, that is, so that $\dim(\mathcal{M}_n(\mathbb{C})) = 2n^2$). For $T \in \mathcal{M}_n(\mathbb{C})$ and $p \ge 1$, one defines the Schatten p-norm of T by

$$\|T\|_{S_p^n} := \|s(T)\|_p = \left(\sum_{i=1}^n s_i(T)^p\right)^{1/p},$$

where $s(T) = (s_1(T), \dots, s_n(T))$ is the non-increasing rearrangement of the singular values of T, that is, of the eigenvalues of $(T^*T)^{1/2}$. As usual, by $\|s(T)\|_\infty$ we mean just the maximum of these singular values, namely $s_1(T)$, and we set $\|T\|_{S_\infty^n} := \|s(T)\|_\infty$ to be the operator or spectral norm of T. Recall that $\|T\|_{S_2^n}$ coincides with the Hilbert–Schmidt norm of T, or in other words the Euclidean norm on $\mathcal{M}_n(\mathbb{C})$.

The convex bodies we study are unit balls $K_{p,E}$ of the Schatten p-norms in subspaces E of $\mathcal{M}_n(\mathbb{C})$ which include the whole space $\mathcal{M}_n(\mathbb{C})$, the subspace $\mathcal{M}_n(\mathbb{R})$ of all $n \times n$ matrices with real entries, or one of the following classical subspaces: of real self-adjoint (or more simply symmetric) matrices, of complex self-adjoint (or Hermitian) matrices, of anti-symmetric Hermitian matrices, or of complex symmetric matrices. (Recall that by self-adjoint matrices we mean matrices T which satisfy $T = T^*$ where T^* stands for the conjugate transpose of T in $\mathcal{M}_n(\mathbb{F})$; recall also that such matrices have n real eigenvalues $e_1(T), e_2(T), \dots, e_n(T)$, and that their singular values are just the moduli of their eigenvalues.) We can also consider the more general space $\mathcal{M}_n(\mathbb{H})$ of all $n \times n$ matrices with quaternion entries (viewed again as a real vector space, that is, so that $\dim(\mathcal{M}_n(\mathbb{H})) = 4n^2$), and its subspace of quaternionic self-adjoint matrices; there is a way of defining eigenvalues and singular values, and hence considering the Schatten p-norms on these last two spaces completely analogously to above (see e.g. [3, Appendix E]).

Our first main result is the following theorem.

Theorem 1. *Let $E = \mathcal{M}_n(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and let $K_{p,E}$ stand for the unit ball of the Schatten p-norm in E. Suppose that $p \geq n^t \log n$ for some $t > 0$. We have*

$$\sigma^2_{K_{p,E}} \leq C \max\{n^{2-t}, 1\}$$

for some absolute constant C, where $\sigma_{K_{p,E}}$ is defined as in (2). In particular $\sigma_{K_{p,E}} = O(1)$ for all $p \gtrsim n^2 \log n$ (this range includes the case of the operator norm S^n_∞). In the same range of p we have $\sigma_{K_{p,E}} \gtrsim 1$, and hence $\sigma_{K_{p,E}} \simeq 1$.

Moreover, the same conclusions hold true when E is the subspace of Hermitian matrices, or of anti-symmetric Hermitian matrices, or of complex symmetric matrices.

Note that, as follows from the various symmetries of the balls $K_{p,\mathcal{M}_n(\mathbb{F})}$ (see e.g. Lemma 21 below), their homothetic copies of volume 1 are in isotropic position. This may very well not be true in all the other subspaces E (see e.g. [5] where it is indicated that we do not have isotropicity in the case of the subspace of real self-adjoint matrices). Nevertheless, the last statement of Theorem 1 will follow without us needing to know if $K_{p,E}$ is isotropic.

For $p \gg n \log n$ Theorem 1 improves on the best known estimate: $\sigma_{K_{p,\mathcal{M}_n(\mathbb{F})}} = O(\sqrt{n}) = O(\sqrt[4]{\dim(\mathcal{M}_n(\mathbb{F}))})$ for all $p \geq 1$, for the isotropic $K_{p,\mathcal{M}_n(\mathbb{F})}$, which Barthe and Cordero-Erausquin [7] established using their method that applies to bodies with enough symmetries, or also follows from the result of Lee and Vempala [24].

The methods we use are more specific to the Schatten classes: the starting point is the fact that the uniform distribution on $K_{p,E}$ defines an *invariant* ensemble of 'random' matrices from E (see for example [25] or [3]). This means that the distribution depends only on the non-increasing rearrangement of the singular values $s_i(T)$ or of the eigenvalues $e_i(T)$ of $T \in E$ (the former case corresponds to $E = \mathcal{M}_n(\mathbb{F})$ or the subspaces of complex symmetric or anti-symmetric Hermitian matrices, while the latter to the subspaces of $\mathbb{F}$-self-adjoint matrices). If furthermore we need to estimate moments of a function that also only depends on the singular values of T (as for example here, where we are interested in moments of the Euclidean norm $\|T\|_2 \equiv \|T\|_{S^n_2}$), we can, by essentially "quotienting" out everything else, reduce the estimation of said moments to computing integrals over $\mathbb{R}^n$ of highly symmetric functions. Let us recall the relevant fact from Random Matrix Theory when $E = \mathcal{M}_n(\mathbb{F})$ or when E is the subspace of $\mathbb{F}$-self-adjoint matrices (see for example [3, Propositions 4.1.3 and 4.1.1]).

Fact A. Let $E = \mathcal{M}_n(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and let $\beta = 1$ or 2 or 4 respectively. There exists a constant $c_{n,E}$, depending only on n and the subspace E, such that, given any function $F : \mathbb{R}^n \to \mathbb{R}^+$ that is symmetric (namely, invariant under permutations of the coordinates of the input) and measurable, we have that

$$\int_E F\big(s(T)\big)\, dT = c_{n,E} \int_{\mathbb{R}^n} F\big(|x_1|, \dots, |x_n|\big) \cdot f_{2,\beta,\beta-1}(x)\, dx,$$

where $s(T)$ is the non-increasing rearrangement of the singular values of the matrix $T \in E$, and where, for non-negative integers a, b, c, we set

$$f_{a,b,c}(x) := \prod_{1 \leq i < j \leq n} \big|x_i^a - x_j^a\big|^b \cdot \prod_{1 \leq i \leq n} |x_i|^c. \tag{3}$$

Similarly, if E is the subspace of $\mathbb{F}$-self-adjoint matrices,

$$\int_E F\big(e(T)\big)\, dT = c_{n,E} \int_{\mathbb{R}^n} F(x_1, \dots, x_n) \cdot f_{1,\beta,0}(x)\, dx,$$

where $e(T)$ is the non-increasing rearrangement of the eigenvalues of the matrix $T \in E$.

In particular, if we set $d_n = \dim(E)$ (which equals $abn(n-1)/2 + (c+1)n$ with a, b, c corresponding to E as above), then for every $s > -d_n$ we have

$$\int_{K_{p,E}} \|T\|_2^s \, dT = c_{n,E} \int_{B_p^n} \|x\|_2^s f_{a,b,c}(x)\, dx = \frac{c_{n,E}}{\Gamma(1+\frac{d_n+s}{p})} \int_{\mathbb{R}^n} \|x\|_2^s e^{-\|x\|_p^p} f_{a,b,c}(x)\, dx. \tag{4}$$

Analogous results hold true for the subspaces of complex symmetric or of Hermitian anti-symmetric matrices (see Section 2).

The idea to use this key fact from Random Matrix Theory comes from previous arguments about the Schatten classes with respect to other, related questions from Convex Geometry, which also used Fact A as a starting point:

- In [23] König, Meyer and Pajor studied whether the unit balls of S_p^n satisfied the hyperplane conjecture (the reader is referred to [26] and [9] for details and literature regarding this conjecture), and showed they do.
- In [16] Guédon and Paouris studied the behaviour of the Schatten classes with respect to concentration of volume, and showed that all but an exponentially small (in the dimension) fraction of the unit balls $K_{p,E}$ is found in a Euclidean ball of radius twice the average distance of an element in $K_{p,E}$ from the origin. Note that in many ways this question is complementary to the thin-shell conjecture.

Not long after [16], Paouris [28] resolved the latter question in the affirmative for all convex bodies in isotropic position; however, as should probably be expected, the approach in [28] differs from the methods of [16] and [23], which are very specific to the Schatten classes.

We use a refinement of the latter methods. To begin with, let us recall that from the abovementioned papers one already has that

$$\mathbb{E}_{\overline{K}_{p,E}}\big(\|\mathcal{T}\|_2^2\big) := \int_{\overline{K}_{p,E}} \|T\|_2^2 \, dT \simeq d_n \simeq n^2$$

for all $p \geq 1$ and for all the subspaces E we consider, where $\overline{K}_{p,E}$ denotes the homothetic image of $K_{p,E}$ with volume 1. From this and definition (2) it follows that

$$\sigma_{K_{p,E}}^2 \simeq \frac{1}{d_n} \operatorname{Var}_{\overline{K}_{p,E}}\big(\|\mathcal{T}\|_2^2\big),$$

thus it suffices to find good bounds for the quantities $\operatorname{Var}_{\overline{K}_{p,E}}(\|\mathcal{T}\|_2^2)$. Furthermore, by Fact A, and (4) in particular, the study of these quantities can be reduced to estimating variances of the Euclidean norm with respect to the densities $f_{a,b,c,p}(x) := e^{-\|x\|_p^p} f_{a,b,c}(x)$ on $\mathbb{R}^n$. Let us set for brevity $M_{a,b,c,p}(f) = \int_{\mathbb{R}^n} f(x) f_{a,b,c,p}(x)\, dx$. We have the following

Proposition 2. *Let* $\operatorname{Var}_{M_{a,b,c,p}}(\|x\|_2^2)$ *(or more briefly* $\operatorname{Var}_{M_p}(\|x\|_2^2)$*) denote the quantity*

$$\frac{M_{a,b,c,p}(\|x\|_2^4)}{M_{a,b,c,p}(1)} - \left(\frac{M_{a,b,c,p}(\|x\|_2^2)}{M_{a,b,c,p}(1)}\right)^2,$$

where a, b, c depend on the subspace E we consider in the way we saw in Fact A or as we will see in Section 2 as well. The following relation is true:

$$\operatorname{Var}_{M_p}\big(\|x\|_2^2\big) \simeq \max\left\{\sigma_{K_{p,E}}^2, \frac{1}{p}\right\} \cdot n^{4/p}.$$

This shows that Theorem 1 will follow by proving $\operatorname{Var}_{M_p}(\|x\|_2^2) = O(n^{2-t} \cdot n^{4/p})$ for $p \geq n^t \log n$ (it also shows that the results of Barthe and Cordero-Erausquin [7], and of Lee and Vempala [24], lead to $\operatorname{Var}_{M_{2,\beta,\beta-1,p}}(\|x\|_2^2) = O(n \cdot n^{4/p})$ for all $p \geq 1$).

It is worth remarking that the common feature of the subspaces E for which Theorem 1 works is that the joint distribution of the singular values or eigenvalues of a matrix $T \in K_{p,E}$ can be expressed in terms of a density $f_{a,b,c,p}$ with $a = 2$; although this might seem like a mere technicality, it appears to be crucial for our arguments. Note in particular that this is valid for the subspace of Hermitian matrices as well, due to a trick from [11], even though it is not immediate from Fact A. To the best of our knowledge such a "trick"/reduction is not known, and may even be impossible to have, for the subspaces of real and quaternionic self-adjoint matrices, however this is not to say that Theorem 1 could not be true for these as well.

The second main result of the paper is a necessary condition for the thin-shell conjecture to hold true for any of the balls K_p in the subspaces E of Theorem 1. This necessary condition comes in the form of a rather strong negative correlation property that the densities $f_{2,b,c,p}$ should satisfy.

Theorem 3 (Negative correlation property for the densities $f_{2,b,c,p}$). *We have*

$$\begin{aligned}&\frac{M_p(x_1^4)}{M_p(1)} \geq \left(\frac{3}{2}+o(1)\right)\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2,\\ &\textit{and hence}\quad n\cdot \mathrm{Var}_{M_p}\big(x_1^2\big) \geq \left(\frac{n}{2}+o(n)\right)\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2 \simeq n\cdot n^{4/p}.\end{aligned} \tag{5}$$

Therefore, for

$$\mathrm{Var}_{M_p}\big(\|x\|_2^2\big) = n\cdot \mathrm{Var}_{M_p}\big(x_1^2\big) + n(n-1)\left[\frac{M_p(x_1^2x_2^2)}{M_p(1)} - \left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2\right], \tag{6}$$

to be bounded by $n^{4/p}$, or even by $o(n\cdot n^{4/p})$, we need the cross terms in (6) to be negative.

Corollary 4. *Combining Theorems 1 and 3, we conclude that the densities $f_{2,b,c,p}$ on $\mathbb{R}^n$ satisfy a negative correlation property for all $p\in[c_0 n\log n, +\infty)$, where c_0 is an absolute constant that can be computed explicitly. The same is true for the densities $\mathbf{1}_{B_p^n}(x)\cdot f_{2,b,c}(x)$ for all $p\in[c_0 n\log n, +\infty]$.*

The rest of the paper is organised as follows. In Section 2 we introduce further notation and preliminary results and outline how we will use them in our arguments. In Section 3 we establish Proposition 2 and a couple more technical lemmas, which we then employ in Section 4 to prove Theorems 1 and 3 for the balls K_p in the spaces $\mathcal{M}_n(\mathbb{F})$ and in the subspace of complex symmetric matrices. In Section 5 we explain how to also obtain the theorems in the cases of Hermitian and of Hermitian anti-symmetric matrices. Finally, in Section 6 we further explore the "desirable" negative correlation property mentioned above, with particular emphasis on what it entails for the original uniform densities over the isotropic balls $K_{p,\mathcal{M}_n(\mathbb{F})}$.

2. Further notation and preliminaries

As mentioned in the Introduction, a function $F:\mathbb{R}^m\to\mathbb{R}$ is called *symmetric* if it is invariant under permutations of the coordinates of the input. It is called *positively homogeneous of degree* k, where $k\in\mathbb{R}$, or more simply k*-homogeneous*, if $F(rx_1,\dots,rx_n) = r^k\cdot F(x_1,\dots,x_n)$ for all $r>0$. We write $\overline{A}$ for the homothetic copy of volume 1 of a set $A\subset\mathbb{R}^m$ (as long as this exists). Lebesgue volume will be denoted by $|A|$, and hence $\overline{A} := \frac{1}{|A|^{1/m}}A$ whenever $|A|\neq 0$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $A\simeq B$ (or $A\lesssim B$) for two quantities A, B related to objects in $\mathbb{R}^m$, we mean that there exist absolute constants $c_1, c_2>0$, independent of the dimension m, such that $c_1A\leq B\leq c_2A$ (or $A\leq c_1B$). We will also use the Landau notation: $A=O(B)$ means the same thing as $A\lesssim B$, whereas $A=o(B)$ means that the ratio A/B tends to 0 as the dimension grows to infinity.

The counterpart of Fact A for the subspace of complex symmetric matrices is the following (see [17, Chapter 3]).

Fact B. Let E be the subspace of $\mathcal{M}_n(\mathbb{C})$ of complex symmetric matrices, namely matrices T with complex entries and the property that $a_{j,i}(T)=a_{i,j}(T)$. There exists a constant $c_{n,E}$ such that, given any function $F:\mathbb{R}^n\to\mathbb{R}^+$ that is symmetric and measurable, we have that

$$\begin{aligned}\int_E F\big(s_1(T),\dots,s_n(T)\big)\,dT &= c_{n,E}\cdot\int_{\mathbb{R}^n} F\big(|x_1|,\dots,|x_n|\big)\cdot f_{2,1,1}(x)\,dx\\ &= c_{n,E}\cdot\int_{\mathbb{R}^n} F\big(|x_1|,\dots,|x_n|\big)\cdot\prod_{1\leq i<j\leq n}\big|x_i^2-x_j^2\big|\cdot\prod_{1\leq i\leq n}|x_i|\,dx,\end{aligned}$$

where $(s_1(T),\dots,s_n(T))$ is the non-increasing rearrangement of the singular values of T.

Therefore, if $d_n = n^2 + n$ is the dimension of E, for every $s > -d_n$ we have

$$\int_{K_{p,E}} \|T\|_2^s \, dT = c_{n,E} \int_{B_p^n} \|x\|_2^s f_{a,b,c}(x)\, dx = \frac{c_{n,E}}{\Gamma(1+\frac{d_n+s}{p})} \int_{\mathbb{R}^n} \|x\|_2^s e^{-\|x\|_p^p} f_{2,1,1}(x)\, dx.$$

On the other hand, for the subspace of $\mathcal{M}_n(\mathbb{C})$ of anti-symmetric Hermitian matrices, where antisymmetric means that $T^* = -T$, we have the following result (see [25, Chapter 13] or [11, Section 2] for an alternative proof). Recall that the eigenvalues of such a matrix come in pairs, and are of the form $\pm i\theta_1, \dots, \pm i\theta_s$ if $n = 2s$, where $\theta_1 \geq \cdots \geq \theta_s \geq 0$ are $s = \lfloor \frac{n}{2} \rfloor$ non-negative real numbers, while, if $n = 2s+1$, they are of the form $\pm i\theta_1, \dots, \pm i\theta_s, 0$ (that is, the matrix T has one additional eigenvalue which is equal to 0). Then the singular values of T are the numbers $\theta_1, \dots, \theta_s$ with multiplicity two, as well as the number 0 if $n = 2s+1$.

Fact C. Let E be the subspace of $\mathcal{M}_n(\mathbb{C})$ of anti-symmetric Hermitian matrices equipped with the standard Gaussian measure. Then the induced joint probability density of the singular values $\theta_1, \dots, \theta_s$ of the random matrix $T \in E$ is given by

$$\mathbb{P}_n\big((\theta_1, \dots, \theta_s) \in A\big) = c_{n,E} \cdot \int_A \prod_{1 \leq i < j \leq s} \big|x_i^2 - x_j^2\big|^2 \exp\big(-\|x\|_2^2\big)\, dx$$

if $n = 2s$, and by

$$\mathbb{P}_n\big((\theta_1, \dots, \theta_s) \in A\big) = c_{n,E} \cdot \int_A \prod_{1 \leq i < j \leq s} \big|x_i^2 - x_j^2\big|^2 \cdot \prod_{1 \leq i \leq s} |x_i|^2 \exp\big(-\|x\|_2^2\big)\, dx$$

if $n = 2s+1$, where A is a 1-symmetric measurable subset of $\mathbb{R}^s$, and $c_{n,E}$ is a constant depending only on n.

Fact C will allow us in Section 5 to show that the subspaces of anti-symmetric Hermitian matrices, as well as of Hermitian matrices, satisfy Theorems 1 and 3.

We will also need in the sequel the following result that gives us the order of magnitude of the volume radius of the balls $K_{p,E}$ (far more accurate estimates for the volume of the unit balls of the Schatten classes of real and complex matrices have been found by Saint-Raymond [30], but we won't need these here).

Fact D (See [16, Proposition 3]). Let $\mathbb{F} = \mathbb{R}$, or $\mathbb{C}$ or $\mathbb{H}$, and let E be any subspace of $\mathcal{M}_n(\mathbb{F})$ with dimension $d_n \simeq n^2$ (this includes all the classical subspaces we mentioned in the Introduction). Then for every $p \geq 1$ we have

$$|K_{p,E}|^{1/d_n} = \big|B\big(S_p^n\big) \cap E\big|^{1/d_n} \simeq d_n^{-\frac{1}{4}-\frac{1}{2p}} \simeq n^{-\frac{1}{2}-\frac{1}{p}}.$$

2.1. *Brief outline of the main argument*

As was explained in the Introduction, our starting point is Proposition 2 which we can prove using Facts A–D. Our attention then turns to studying the quantities $\mathrm{Var}_{M_p}(\|x\|_2^2)$, which, due to the symmetries of the densities $f_{a,b,c,p}$ and also of the Euclidean norm, we can expand as

$$\begin{aligned}\mathrm{Var}_{M_p}\big(\|x\|_2^2\big) &= n \cdot \frac{M_p(x_1^4)}{M_p(1)} + n(n-1) \cdot \frac{M_p(x_1^2x_2^2)}{M_p(1)} - n^2 \cdot \left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2 \\ &= n \cdot \mathrm{Var}_{M_p}\big(x_1^2\big) + n(n-1)\left[\frac{M_p(x_1^2x_2^2)}{M_p(1)} - \left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2\right]. \qquad (7)\end{aligned}$$

An advantage we gain by focusing on the above integrals, which was central to the methods of [23] and [16], is that we can now use analytic techniques more easily: via integration by parts with respect to each of the coordinates x_i we can obtain a series of recursive equivalences that seem to facilitate the estimation of terms such as the ones appearing in the right hand side of (7). The relevant lemma, used in both [23] and [16], is the following

Lemma 5. *For every $l = (\epsilon_l, \rho_l) \in \{+1, -1\}^n \times \{\rho$ is a permutation of $[n]\}$ we consider the following subsets of $\mathbb{R}^n$ that can be written as intersections of $2n-1$ halfspaces*:

$$\mathcal{P}_l := \big\{x : \epsilon_l(i)x_i \geq 0 \text{ for all } i, \text{ and } |x_{\rho_l(1)}| \leq |x_{\rho_l(2)}| \leq \cdots \leq |x_{\rho_l(n)}|\big\}.$$

Let $\xi \geq 0$ and $s > -d_n - \xi$, and let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be an s-homogeneous function with the property that the product

$$f(x) \cdot f_{a,b,c}(x) = f(x) \cdot \prod_{1 \leq i < j \leq n} \left|x_i^a - x_j^a\right|^b \cdot \prod_{1 \leq i \leq n} |x_i|^c$$

is C^1 in the interior of each of the subsets $\mathcal{P}_l$, and its partial derivatives can be continuously extended to the border of $\mathcal{P}_l$ (except perhaps at the origin). Then

$$(\xi + c + 1) M_p\left(f(x) \sum_{i=1}^{n} |x_i|^{\xi}\right)$$
$$= p M_p\left(\|x\|_{\xi+p}^{\xi+p} f(x)\right) - M_p\left(\sum_{i=1}^{n} |x_i|^{\xi} x_i \frac{\partial f}{\partial x_i}(x)\right) - ab M_p\left(f(x) \sum_{i=1}^{n} \sum_{j \neq i} \frac{|x_i|^{\xi} x_i^a}{x_i^a - x_j^a}\right). \tag{8}$$

Trying to optimise on the way this lemma can be used for our problem, we manage in Section 4 to obtain precise identities (in the place of inequalities or equivalences deduced in [23] and [16]) which involve the terms from (7) and which allow us to establish Theorems 1 and 3.

We now turn to the details.

3. Reduction to integrals over $\mathbb{R}^n$

In this section the main purpose is to prove Proposition 2. We start with detailed estimates about the Gamma function; note that in the sequel the only two different ranges of p and q for which the first part of the following lemma is applied are (i) when q is an absolute constant, and (ii) when p is at least as large as the dimension (we wish to thank the anonymous referee for suggesting we present the estimates in these two cases separately, and also for proposing a simpler proof for part (b) which allows to slightly strengthen its conclusion too).

Lemma 6. *For every $p \geq 1$, for every dimension $d_n \simeq n^2$ as above, and for every $q \in [2, d_n]$, the following estimates are true*:

(a)

$$\frac{\Gamma(1 + \frac{d_n}{p})}{\Gamma(1 + \frac{d_n + q}{p})} = \left(\frac{d_n + p + q}{p}\right)^{-q/p} \left(1 + O\left(\frac{q}{n^2}\right)\right)^q;$$

at the same time, when p is at least comparable to the dimension, say $p \geq d_n/2$, then

$$\frac{\Gamma(1 + \frac{d_n}{p})}{\Gamma(1 + \frac{d_n + q}{p})} = \left(1 + O\left(\frac{q}{p}\right)\right);$$

(b)

$$\frac{C_2}{p(p + d_n)} \left(\frac{\Gamma(1 + \frac{d_n}{p})}{\Gamma(1 + \frac{d_n + 2}{p})}\right)^2 \leq \left(\frac{\Gamma(1 + \frac{d_n}{p})}{\Gamma(1 + \frac{d_n + 2}{p})}\right)^2 - \frac{\Gamma(1 + \frac{d_n}{p})}{\Gamma(1 + \frac{d_n + 4}{p})} \leq \frac{C_3}{p(p + d_n)} \left(\frac{\Gamma(1 + \frac{d_n}{p})}{\Gamma(1 + \frac{d_n + 2}{p})}\right)^2,$$

where the O-notation in (a) *implies constants which may depend on p, d_n and q, but which are in absolute value less than some absolute constant $c_1 > 0$, and where C_2, C_3 are positive absolute constants.*

Proof. For part (a) we will use one of Binet's formulas for $\log \Gamma(x)$:

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log(2\pi) + 2 \int_0^{\infty} \frac{\tan(t/x)}{e^{2\pi t} - 1}\, dt$$

for every positive x. Hence

$$\begin{aligned}
&\log\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+q}{p})}\\
&\quad=\left(\frac{p+d_n}{p}-\frac{1}{2}\right)\log\left(\frac{p+d_n}{p}\right)-\frac{p+d_n}{p}+2\int_0^\infty\frac{\arctan(pt/(p+d_n))}{e^{2\pi t}-1}\,dt\\
&\qquad-\left(\frac{p+d_n+q}{p}-\frac{1}{2}\right)\log\left(\frac{p+d_n+q}{p}\right)+\frac{p+d_n+q}{p}-2\int_0^\infty\frac{\arctan(pt/(p+d_n+q))}{e^{2\pi t}-1}\,dt\\
&\quad=\frac{1}{2}\log\left(\frac{p+d_n+q}{p+d_n}\right)+\left[\frac{q}{p}-\frac{p+d_n}{p}\log\left(\frac{p+d_n+q}{p+d_n}\right)\right]-\frac{q}{p}\log\left(\frac{p+d_n+q}{p}\right)\\
&\qquad+2\int_0^\infty\frac{\arctan(pt/(p+d_n))-\arctan(pt/(p+d_n+q))}{e^{2\pi t}-1}\,dt.
\end{aligned}$$

By a second-order Taylor approximation of the logarithmic funtion, we obtain

$$\log\left(\frac{p+d_n+q}{p+d_n}\right)=\log\left(1+\frac{q}{p+d_n}\right)=\frac{q}{p+d_n}+O\left(\left(\frac{q}{p+d_n}\right)^2\right),$$

and so

$$\frac{1}{2}\log\left(\frac{p+d_n+q}{p+d_n}\right)=O\left(\frac{q}{p+d_n}\right)=O\left(\frac{q}{n^2}\right),$$

and

$$\frac{q}{p}-\frac{p+d_n}{p}\log\left(\frac{p+d_n+q}{p+d_n}\right)=O\left(\frac{q^2}{p(p+d_n)}\right).$$

On the other hand, by the mean value theorem we get, for every $t>0$,

$$\arctan\bigl(pt/(p+d_n)\bigr)-\arctan\bigl(pt/(p+d_n+q)\bigr)=\frac{1}{1+(t/x_t)^2}\left|\frac{pt}{p+d_n}-\frac{pt}{p+d_n+q}\right|$$

for some $x_t\in[(p+d_n)/p,(p+d_n+q)/p]$, which makes the difference above

$$\le\frac{(p+d_n+q)^2}{(p+d_n+q)^2+(pt)^2}\frac{pqt}{(p+d_n)(p+d_n+q)}.$$

Hence

$$\begin{aligned}
&\int_0^\infty\frac{\arctan(pt/(p+d_n))-\arctan(pt/(p+d_n+q))}{e^{2\pi t}-1}\,dt\\
&\quad\le\int_0^\infty\frac{\frac{pqt}{(p+d_n)(p+d_n+q)}}{2\pi t[1+(2\pi t)/2!+(2\pi t)^2/3!+\cdots]}\frac{(p+d_n+q)^2}{(p+d_n+q)^2+(pt)^2}\\
&\quad\le\int_0^\infty\frac{q}{2\pi(p+d_n)}\frac{1}{1+t^2}\,dt=O\left(\frac{q}{p+d_n}\right)
\end{aligned}$$

The first claim of part (a) follows (given that $q/n^2=O(1)$ for every $q\in[2,d_n]$, so $\exp(O(q/n^2))=1+O(q/n^2)$).

On the other hand, when $p\ge d_n/2$ say, then we can more simply apply the mean value theorem for the function $\log\Gamma(x)$:

$$\begin{aligned}
\log\frac{\Gamma(1+\frac{d_n+q}{p})}{\Gamma(1+\frac{d_n}{p})}&=\log\Gamma\left(1+\frac{d_n+q}{p}\right)-\log\Gamma\left(1+\frac{d_n}{p}\right)\\
&=\frac{q}{p}\cdot(\log\Gamma)'\left(1+\frac{d_n}{p}+w\right)
\end{aligned}$$

for some $w \in (0, q/p)$. Given that $1 + \frac{d_n}{p} + w \in (1,5)$ under the assumptions $p \geq d_n/2$, $q \leq d_n$, and that $(\log\Gamma)'$ is continuous on $[1,5]$, we conclude that

$$\log\Gamma\left(1+\frac{d_n+q}{p}\right) - \log\Gamma\left(1+\frac{d_n}{p}\right) = O\left(\frac{q}{p}\right),$$

which gives the second claim of part (a).

To prove part (b) now, we recall Weierstrass' definition of the Gamma function:

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\prod_{l=1}^{\infty}\left(1+\frac{x}{l}\right)^{-1} e^{x/l},$$

valid for all $x > 0$, where γ stands for the Euler–Mascheroni constant. Setting $F_1(x) = \log\Gamma(x)$, we can derive that

$$(F_1)'(x) = -\gamma - \frac{1}{x} + \sum_{l=1}^{\infty}\left(\frac{1}{l} - \frac{1}{x+l}\right),$$

$$(F_1)''(x) = \frac{1}{x^2} + \sum_{l=1}^{\infty}\frac{1}{(x+l)^2}.$$

We thus see that, for all $x \geq 1$,

$$\frac{1}{x+1} < \frac{1}{x^2} + \frac{1}{x+1} < (F_1)''(x) < \frac{1}{x^2} + \frac{1}{x} \leq \frac{2}{x}. \tag{9}$$

We now write

$$\left(\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+2}{p})}\right)^2 - \frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+4}{p})} = \left(\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+2}{p})}\right)^2\left(1 - \frac{(\Gamma(1+\frac{d_n+2}{p}))^2}{\Gamma(1+\frac{d_n}{p})\Gamma(1+\frac{d_n+4}{p})}\right), \tag{10}$$

which prompts us to set $u = 1 + \frac{d_n+2}{p}$ and

$$F_2(y) = \log\left(\frac{\Gamma(u+y)\Gamma(u-y)}{(\Gamma(u))^2}\right) = F_1(u+y) + F_1(u-y) - 2F_1(u).$$

Observe that $F_2(0) = 0 = (F_2)'(0)$. Thus, for any $y \in (0, d_n/p)$ say, a second order Taylor approximation gives

$$F_2(y) = \frac{y^2}{2}(F_2)''(h_y) = \frac{y^2}{2}\big((F_1)''(u+h_y) + (F_1)''(u-h_y)\big)$$

for some $h_y \in (0, y)$. Setting $y = \frac{2}{p}$ and making use of (9), we obtain

$$\begin{aligned}\frac{1}{p(p+d_n+2)} &< \frac{y^2}{2}\left(\frac{1}{u+y+1} + \frac{1}{u+1}\right) \\ &< F_2(y) = \log\left(\frac{\Gamma(1+\frac{d_n}{p})\Gamma(1+\frac{d_n+4}{p})}{(\Gamma(1+\frac{d_n+2}{p}))^2}\right) \\ &< \frac{y^2}{2}\left(\frac{2}{u} + \frac{1}{u-y}\right) < \frac{4}{p(p+d_n)}.\end{aligned}$$

It remains to take exponentials, and to note that $\exp(-C/(p(p+d_n))) = 1 - \frac{C'}{(p(p+d_n))}$ whenever $C > 0$ is an absolute constant: this leads to

$$\frac{C_2}{p(p+d_n)} < 1 - \frac{(\Gamma(1+\frac{d_n+2}{p}))^2}{\Gamma(1+\frac{d_n}{p})\Gamma(1+\frac{d_n+4}{p})} < \frac{C_3}{p(p+d_n)},$$

which put together with (10) completes the proof. □

We are now ready to prove Proposition 2. We make two remarks first.

Remark 7. Recall that immediate consequences of this proposition are the following:

- $K_{p,E}$ satisfies the thin-shell conjecture, in other words, $\sigma^2_{K_{p,E}} \le C$ for some absolute constant C, if and only if $\mathrm{Var}_{M_{a,b,c,p}}(\|x\|_2^2) \le C' n^{4/p}$ (for some other absolute constant depending linearly on C).
- Using the best bounds for $\sigma_{K_{p,E}}$ that we currently have, which are due to Barthe and Cordero-Erausquin [7] or also follow from the more general result of Lee and Vempala [24], we already deduce the following estimate: for all p and for all subspaces E for which we know that $K_{p,E}$ is in isotropic or almost isotropic position (these always include the spaces $\mathcal{M}_n(\mathbb{F})$),

$$\mathrm{Var}_{M_p}\big(\|x\|_2^2\big) \lesssim n \cdot n^{4/p} = o\big(n^2\big) \cdot n^{4/p}. \tag{11}$$

We are going to take advantage of this estimate in the sequel.

Remark 8. In the conclusion of the proposition the estimate

$$\mathbb{E}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big) \simeq d_n, \tag{12}$$

which follows from the arguments of [23] and [16], and is valid for all of the classical subspaces E we are considering, is already incorporated. If we do not use it yet, then, as will be clear from the ensuing proof, we will get

$$\begin{aligned}\mathrm{Var}_{M_p}\big(\|x\|_2^2\big) &\simeq \max\left\{\sigma^2_{K_{p,E}}, \frac{1}{p}\right\} \cdot d_n^{4/p-1} |K_{p,E}|^{4/d_n} \mathbb{E}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big)\\ &\simeq \max\left\{\sigma^2_{K_{p,E}}, \frac{1}{p}\right\} \cdot \frac{1}{d_n^2} \mathbb{E}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big) \cdot n^{4/p},\end{aligned}$$

where the last equivalence follows from the volume estimates for the unit balls of the Schatten classes provided by [16, Proposition 3] (see end of Section 2).

Proof. Note that by (12) we have

$$\mathrm{Var}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big) = \frac{1}{d_n}\big[\mathbb{E}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big)\big]^2 \cdot \sigma^2_{K_{p,E}} \simeq \sigma^2_{K_{p,E}} \cdot d_n.$$

But

$$\begin{aligned}\mathrm{Var}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big) &= \int_{\overline{K}_{p,E}} \|s(T)\|_2^4\, dT - \left(\int_{\overline{K}_{p,E}} \|s(T)\|_2^2\, dT\right)^2\\ &= \frac{1}{|K_{p,E}|^{1+\frac{4}{d_n}}} \int_{K_{p,E}} \|s(T)\|_2^4\, dT - \left(\frac{1}{|K_{p,E}|^{1+\frac{2}{d_n}}} \int_{K_{p,E}} \|s(T)\|_2^2\, dT\right)^2\\ &= \frac{1}{|K_{p,E}|^{4/d_n}}\left[\frac{1}{|K_{p,E}|} \int_{K_{p,E}} \|s(T)\|_2^4\, dT - \left(\frac{1}{|K_{p,E}|} \int_{K_{p,E}} \|s(T)\|_2^2\, dT\right)^2\right].\end{aligned}$$

Given that $|K_{p,E}|^{4/d_n} \simeq d_n^{-1-2/p}$, we therefore obtain that

$$\begin{aligned}\sigma^2_{K_{p,E}} \cdot d_n^{-2/p} &\simeq |K_{p,E}|^{4/d_n} \cdot \mathrm{Var}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big)\\ &= \frac{1}{|K_{p,E}|} \int_{K_{p,E}} \|s(T)\|_2^4\, dT - \left(\frac{1}{|K_{p,E}|} \int_{K_{p,E}} \|s(T)\|_2^2\, dT\right)^2,\end{aligned}$$

which, by use of Facts A and B, becomes

$$= \frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+4}{p})}\frac{M_p(\|x\|_2^4)}{M_p(1)} - \left(\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+2}{p})}\right)^2\left(\frac{M_p(\|x\|_2^2)}{M_p(1)}\right)^2$$

$$= \frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+4}{p})}\left[\frac{M_p(\|x\|_2^4)}{M_p(1)} - \left(\frac{M_p(\|x\|_2^2)}{M_p(1)}\right)^2\right] \tag{13}$$

$$-\left[\left(\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+2}{p})}\right)^2 - \frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+4}{p})}\right]\left(\frac{M_p(\|x\|_2^2)}{M_p(1)}\right)^2. \tag{14}$$

Now we use the estimates of Lemma 6 (without needing yet the more accurate form in which we stated them): the term in (13) can be rewritten as

$$\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+4}{p})}\operatorname{Var}_{M_p}\big(\|x\|_2^2\big) \simeq d_n^{-4/p}\cdot \operatorname{Var}_{M_p}\big(\|x\|_2^2\big),$$

and since the term in (14) is negative, we get

$$\operatorname{Var}_{M_p}\big(\|x\|_2^2\big) \gtrsim \sigma_{K_{p,E}}^2 \cdot d_n^{2/p} \simeq \sigma_{K_{p,E}}^2 \cdot n^{4/p}. \tag{15}$$

In addition, since the sum of the terms in (13) and (14) is equal to a positive quantity, we obtain

$$\begin{aligned} d_n^{-4/p}\cdot \operatorname{Var}_{M_p}\big(\|x\|_2^2\big) &\gtrsim \left[\left(\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+2}{p})}\right)^2 - \frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+4}{p})}\right]\left(\frac{M_p(\|x\|_2^2)}{M_p(1)}\right)^2 \\ &\geq \frac{C_2}{p(p+d_n)}\left(\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+2}{p})}\right)^2\left(\frac{M_p(\|x\|_2^2)}{M_p(1)}\right)^2 \\ &= \frac{C_2}{p(p+d_n)}|K_{p,E}|^{4/d_n}\big[\mathbb{E}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big)\big]^2 \\ &\simeq \frac{C_2}{p(p+d_n)}d_n^{1-2/p}. \end{aligned}$$

This shows that $\operatorname{Var}_{M_p}(\|x\|_2^2) \gtrsim \frac{1}{p}\cdot d_n^{2/p} \simeq \frac{1}{p}\cdot n^{4/p}$ if $1 \leq p \lesssim d_n$, whereas if $p \gtrsim d_n$ then $\max\{\sigma_{K_{p,E}}^2, 1/p\} = \sigma_{K_{p,E}}^2$ given that for every centred convex body in a d_n-dimensional space we have

$$\sigma_{K_{p,E}}^2 \geq \sigma_{B_2^{d_n}}^2 = \frac{4}{d_n+4}$$

(see [8, Theorem 2]). Combining with (15), we conclude that

$$\operatorname{Var}_{M_p}\big(\|x\|_2^2\big) \gtrsim \max\left\{\sigma_{K_{p,E}}^2, \frac{1}{p}\right\}\cdot n^{4/p}. \tag{16}$$

In the opposite direction, we have

$$\begin{aligned} d_n^{-4/p}\cdot \operatorname{Var}_{M_p}\big(\|x\|_2^2\big) &\lesssim \sigma_{K_{p,E}}^2\cdot d_n^{-2/p} + \left[\left(\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+2}{p})}\right)^2 - \frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+4}{p})}\right]\left(\frac{M_p(\|x\|_2^2)}{M_p(1)}\right)^2 \\ &\leq \sigma_{K_{p,E}}^2\cdot d_n^{-2/p} + \frac{C_3}{p(p+d_n)}|K_{p,E}|^{4/d_n}\big[\mathbb{E}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big)\big]^2 \\ &\simeq \sigma_{K_{p,E}}^2\cdot d_n^{-2/p} + \frac{C_3}{p(p+d_n)}d_n^{1-2/p}, \end{aligned}$$

whence we obtain the reverse inequality to (16). This completes the proof of the proposition. □

As mentioned in the Introduction, our task now becomes to find good estimates for the quantity $\mathrm{Var}_{M_p}(\|x\|_2^2)$, and ideally to show that it is $O(n^{4/p})$. One of our tools towards this goal is Lemma 5 that was stated in the Introduction; it will become apparent that one other thing we need is to be able to relate integrals of the form $M_p(\|x\|_p^l f(x))$, where l is some real number, to each other.

Lemma 9. *Let $l, s \in \mathbb{R}$ be such that $s > -d_n$ and $l + s > -d_n$. Suppose also that $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a continuous, s-homogeneous function. Then*

$$M_p\big(\|x\|_p^l f(x)\big) = \frac{\Gamma(\frac{d_n+l+s}{p})}{\Gamma(\frac{d_n+s}{p})} M_p(f).$$

Proof. We use a polar integration formula. Since both f and $\|x\|_p^l f(x)$ are positively homogeneous functions of order s and $l + s$ respectively, we have

$$\begin{aligned} M_p(f) &= \int_{\mathbb{R}^n} f(x) \cdot f_{a,b,c}(x) \exp\big(-\|x\|_p^p\big)\, dx \\ &= n \operatorname{vol}\big(B_p^n\big) \int_0^\infty r^{d_n+s-1} e^{-r^p} \int_{\partial B_p^n} f(u) f_{2,b,c}(u)\, d\mu_{B_p^n}(u)\, dr \\ &= n \operatorname{vol}\big(B_p^n\big) \frac{\Gamma(\frac{d_n+s}{p})}{p} \int_{\partial B_p^n} f(u) f_{2,b,c}(u)\, d\mu_{B_p^n}(u), \end{aligned}$$

and similarly

$$\begin{aligned} M_p\big(\|x\|_p^l f(x)\big) &= \int_{\mathbb{R}^n} \|x\|_p^l f(x) \cdot f_{a,b,c}(x) \exp\big(-\|x\|_p^p\big)\, dx \\ &= n \operatorname{vol}\big(B_p^n\big) \int_0^\infty r^{d_n+l+s-1} e^{-r^p} \int_{\partial B_p^n} \|u\|_p^l f(u) f_{2,b,c}(u)\, d\mu_{B_p^n}(u)\, dr \\ &= n \operatorname{vol}\big(B_p^n\big) \frac{\Gamma(\frac{d_n+l+s}{p})}{p} \int_{\partial B_p^n} f(u) f_{2,b,c}(u)\, d\mu_{B_p^n}(u), \end{aligned}$$

where $\mu_{B_p^n}$ is a type of cone-volume measure (see e.g. [27]), normalised so that it has total mass 1, on the boundary ∂B_p^n of B_p^n, which is defined by

$$\mu_{B_p^n}(A) := \frac{|\{tu : u \in A, 0 \le t \le 1\}|}{|B_p^n|}.$$

This completes the proof. □

Note that the case $l = p$ also follows from Lemma 5 applied with $\xi = 0$ (and at first with functions f that satisfy the hypotheses of that lemma; see also [16, Corollary 7(a)] for a different proof of the case $l = p$ that works directly for arbitrary continuous functions). However, in what follows, we will need to make use of other cases of Lemma 9 as well.

3.1. *Orders of magnitude of relevant quantities*

Given that, by symmetry,

$$\begin{aligned} \mathrm{Var}_{M_p}\big(\|x\|_2^2\big) &= \frac{M_p(\|x\|_2^4)}{M_p(1)} - \left(\frac{M_p(\|x\|_2^2)}{M_p(1)}\right)^2 \\ &= n \cdot \frac{M_p(x_1^4)}{M_p(1)} + n(n-1) \cdot \frac{M_p(x_1^2 x_2^2)}{M_p(1)} - n^2 \cdot \left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2 \\ &= n \cdot \mathrm{Var}_{M_p}\big(x_1^2\big) + n(n-1) \cdot \left[\frac{M_p(x_1^2 x_2^2)}{M_p(1)} - \left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2\right], \end{aligned} \tag{17}$$

our first objective thus becomes to study the order of magnitude of the quantities $M_p(x_1^2)/M_p(1)$, $M_p(x_1^4)/M_p(1)$ and $M_p(x_1^2x_2^2)/M_p(1)$.

To this end we recall Lemma 5 that was stated in the introduction, and has been used for the exact same purpose both in [23] and in [16].

Remark 10. In [23] the authors apply Lemma 5 with $\xi = 0$ or $\xi = p$ and with $f(x) = 1$. On the other hand, the authors of [16] have to apply the lemma in more general cases as well: they obtain recursive equivalences using the lemma with $\xi = 2$ or $\xi = p$ and with f being different powers of the Euclidean norm.

In both situations, it turns out that the most bothersome to deal with term in (5) is the last one: the way they estimate it in both of the abovementioned papers is by observing that

$$\zeta_1(a,\xi)\cdot\big(|x_i|^\xi + |x_j|^\xi\big) \le \frac{|x_i|^\xi x_i^a - |x_j|^\xi x_j^a}{x_i^a - x_j^a} \le \zeta_2(a,\xi)\cdot\big(|x_i|^\xi + |x_j|^\xi\big) \tag{18}$$

for all $x_i \neq x_j$, where $\zeta_1(a,\xi) = \min\{\frac{1}{2}, \frac{a+\xi}{2a}\}$ (or, if a is even, $\zeta_1(a,\xi) = \min\{1, \frac{a+\xi}{2a}\}$), and $\zeta_2(a,\xi) = \max\{1, \frac{a+\xi}{2a}\}$, and then by writing

$$\begin{aligned} M_p\left(f(x)\sum_{i=1}^n\sum_{j\neq i}\frac{|x_i|^\xi x_i^a}{x_i^a - x_j^a}\right) &= M_p\left(f(x)\sum_{i<j}\frac{|x_i|^\xi x_i^a - |x_j|^\xi x_j^a}{x_i^a - x_j^a}\right) \\ &\simeq \zeta_i(a,\xi)\cdot M_p\left(f(x)\sum_{i<j}\big(|x_i|^\xi + |x_j|^\xi\big)\right) \\ &= \zeta_i(a,\xi)\cdot(n-1)M_p\big(f(x)\|x\|_\xi^\xi\big). \end{aligned}$$

In this way Lemma 5 leads to

$$\begin{aligned} \zeta_1(a,\xi)\frac{d_n}{n}M_p\big(f(x)\|x\|_\xi^\xi\big) &\lesssim pM_p\big(\|x\|_{\xi+p}^{\xi+p}f(x)\big) - M_p\left(\sum_{i=1}^n |x_i|^\xi x_i\frac{\partial f}{\partial x_i}(x)\right) \\ &\lesssim \zeta_2(a,\xi)\frac{d_n}{n}M_p\big(f(x)\|x\|_\xi^\xi\big) \end{aligned} \tag{19}$$

for any positive function f satisfying the assumptions of the lemma.

Proposition 11. *Let M_p denote integration over $\mathbb{R}^n$ with respect to one of the densities $f_{a,b,c,p}$ of the form*

$$f_{a,b,c,p}(x) = \exp\big(-\|x\|_p^p)\big)\cdot\prod_{1\le i<j\le n}\big|x_i^a - x_j^a\big|^b\cdot\prod_{1\le i\le n}|x_i|^c$$

that we have considered, where $p\in[1,+\infty)$ and a, b are positive integers, c is a non-negative integer. We have

$$\frac{M_p(x_1^2)}{M_p(1)} \simeq n^{2/p} \quad \textit{and} \quad \frac{M_p(x_1^4)}{M_p(1)} \simeq \frac{M_p(x_1^2x_2^2)}{M_p(1)} \simeq n^{4/p}. \tag{20}$$

Proof. The first part of (20) is essentially the core result of [23]. For the reader's convenience, let us recall how one can easily deduce it from (19) with the help of Hölder's inequality and Lemma 9: we first apply (19) with $\xi = 2$ and $f(x) = \mathbf{1}$ to obtain that

$$\frac{d_n}{n}M_p\big(\|x\|_2^2\big) \gtrsim pM_p\big(\|x\|_{p+2}^{p+2}\big) \ge \frac{p}{n^{2/p}}M_p\big(\|x\|_p^{p+2}\big) \simeq \frac{d_n^{2/p}}{n^{2/p}}d_nM_p(1),$$

or in other words, that $M_p(x_1^2)/M_p(1) \gtrsim n^{2/p}$. To also get the reverse inequality, we apply (19) with $\xi = p$ and $f(x) = \mathbf{1}$: this gives

$$\frac{a+p}{2a}\frac{d_n}{n}M_p\big(\|x\|_p^p\big) \gtrsim pM_p\big(\|x\|_{2p}^{2p}\big)$$

and then, by a simple application of Hölder's inequality, we can conclude that

$$\frac{M_p(x_1^2)}{M_p(1)} \leq \left(\frac{M_p(|x_1|^{2p})}{M_p(1)}\right)^{1/p} \lesssim \left(n\frac{M_p(|x_1|^p)}{M_p(1)}\right)^{1/p} = \left(\frac{d_n}{p}\right)^{1/p}.$$

The second part of (20) can follow by very similar reasoning: in this case we have to apply (19) with $\xi = 2p$ or $\xi = 3p$ as well, to be able to compare $M_p(x_1^4)/M_p(1)$ to $M_p(|x_1|^{4p})/M_p(1)$.

Finally note that

$$\frac{M_p(x_1^2x_2^2)}{M_p(1)} \leq \sqrt{\frac{M_p(x_1^4)}{M_p(1)}\frac{M_p(x_2^4)}{M_p(1)}},$$

while

$$\begin{aligned} n(n-1)\frac{M_p(x_1^2x_2^2)}{M_p(1)} &= \frac{M_p(\|x\|_2^4)}{M_p(1)} - n\frac{M_p(x_1^4)}{M_p(1)} \\ &\geq \left(\frac{M_p(\|x\|_2^2)}{M_p(1)}\right)^2 - n\frac{M_p(x_1^4)}{M_p(1)} \\ &\geq c_1n^2\cdot n^{4/p} - c_2n\cdot n^{4/p} \simeq n^2\cdot n^{4/p}, \end{aligned}$$

which completes the proof. □

4. Proof of the main results

It is not difficult to convince ourselves that estimating the variance of the Euclidean norm with respect to the densities $f_{a,b,c,p}$ is a more delicate problem than merely finding the orders of magnitude of the terms $(M_p(x_1^2)/M_p(1))^2$, $M_p(x_1^4)/M_p(1)$ and $M_p(x_1^2x_2^2)/M_p(1)$ appearing when we write out the variance: we have just seen that they are all $\simeq n^{4/p}$, however it is obvious that we cannot extract any non-trivial information about the variance from these equivalences if we do not also find a way to estimate the constants appearing in them (or in other words, the coefficient of $n^{4/p}$ in each case). To this end, we will now attempt to estimate the contribution of the last term in (5) in a more precise manner: one way this can be done is through the following proposition.

Proposition 12. *Let $\xi \geq 0$ and $s > -d_n - \xi$, and let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ be an s-homogeneous function that satisfies the hypotheses of Lemma 5. Suppose moreover that f is a symmetric function. Then we have*

$$\begin{aligned} &\left(\frac{2d_n + (\xi - c - 1)n}{n}\right)M_p\big(\|x\|_\xi^\xi f(x)\big) \\ &\quad = pM_p\big(\|x\|_{\xi+p}^{\xi+p} f(x)\big) - M_p\left(\sum_{i=1}^n |x_i|^\xi x_i \frac{\partial f}{\partial x_i}(x)\right) + abn(n-1)M_p\left(\frac{|x_2|^\xi x_1^a}{x_1^a - x_2^a} f(x)\right), \end{aligned} \tag{21}$$

where by symmetry we can also write

$$M_p\left(\frac{|x_2|^\xi x_1^a}{x_1^a - x_2^a} f(x)\right) = \frac{1}{2}M_p\left(\frac{x_1^a|x_2|^\xi - x_2^a|x_1|^\xi}{x_1^a - x_2^a} f(x)\right).$$

If in addition a is an even integer, then

$$M_p\left(\frac{|x_2|^\xi x_1^a}{x_1^a - x_2^a} f(x)\right) = \begin{cases} \frac{1}{2}M_p\big(\frac{|x_1|^{a-\xi} - |x_2|^{a-\xi}}{|x_1|^a - |x_2|^a}|x_1x_2|^\xi f(x)\big) > 0 & \text{if } \xi < a, \\ 0 & \text{if } \xi = a, \\ \frac{1}{2}M_p\big(\frac{|x_2|^{\xi-a} - |x_1|^{\xi-a}}{|x_1|^a - |x_2|^a}|x_1x_2|^a f(x)\big) < 0 & \text{if } \xi > a. \end{cases} \tag{22}$$

Proof. We first prove (21). By Lemma 5 we can write

$$(\xi+c+1)M_p\left(f(x)\sum_{i=1}^n |x_i|^\xi\right)$$
$$= pM_p\big(\|x\|_{\xi+p}^{\xi+p} f(x)\big) - M_p\left(\sum_{i=1}^n |x_i|^\xi x_i \frac{\partial f}{\partial x_i}(x)\right) - abM_p\left(f(x)\sum_{i=1}^n\sum_{j\neq i}\frac{|x_i|^\xi x_i^a}{x_i^a - x_j^a}\right). \quad (23)$$

Note that by symmetry the last summand is equal to

$$abn(n-1)M_p\left(\frac{|x_1|^\xi x_1^a}{x_1^a - x_2^a} f(x)\right),$$

which we can rewrite as

$$M_p\left(\frac{(|x_1|^\xi + |x_2|^\xi)x_1^a}{x_1^a - x_2^a} f(x)\right) - M_p\left(\frac{|x_2|^\xi x_1^a}{x_1^a - x_2^a} f(x)\right).$$

Since the function $(|x_1|^\xi + |x_2|^\xi)f(x)$ is invariant under permuting the first two coordinates, it follows that

$$M_p\left(\frac{x_1^a}{x_1^a - x_2^a}\big(|x_1|^\xi + |x_2|^\xi\big)f(x)\right) = \frac{1}{2}M_p\left(\frac{x_1^a - x_2^a}{x_1^a - x_2^a}\big(|x_1|^\xi + |x_2|^\xi\big)f(x)\right)$$
$$= \frac{1}{2}M_p\big(\big(|x_1|^\xi + |x_2|^\xi\big)f(x)\big) = \frac{1}{n}M_p\big(\|x\|_\xi^\xi f(x)\big).$$

We thus conclude that the last summand in (23) is equal to

$$ab(n-1)M_p\big(\|x\|_\xi^\xi f(x)\big) - abn(n-1)M_p\left(\frac{|x_2|^\xi x_1^a}{x_1^a - x_2^a} f(x)\right)$$
$$= \left(\frac{2(d_n - (c+1)n)}{n}\right)M_p\big(\|x\|_\xi^\xi f(x)\big) - abn(n-1)M_p\left(\frac{|x_2|^\xi x_1^a}{x_1^a - x_2^a} f(x)\right).$$

This gives (21). The other two equations follow by symmetry and, in the case of (22), by the fact that $x_i^a = |x_i|^a$ when a is an even integer. This completes the proof. □

The following corollary summarises the three main identities that Proposition 12 gives us for densities of the form $f_{2,b,c,p}$ and which we need to use in the sequel.

Corollary 13. *Let M_p denote integration with respect to a density of the form $f_{2,b,c,p} = \prod_{i<j}|x_i^2 - x_j^2|^b \cdot \prod_i |x_i|^c e^{-\|x\|_p^p}$ (namely, let $a=2$). By applying Proposition 12 with $\xi = 2$ and $f(x) = \mathbf{1}$ we obtain*

$$\left(\frac{2d_n + (1-c)n}{n}\right)\frac{M_p(\|x\|_2^2)}{M_p(1)} = p\frac{M_p(\|x\|_{p+2}^{p+2})}{M_p(1)}. \quad (24)$$

By applying Proposition 12 with $\xi = 2$ and $f(x) = \|x\|_2^2$ we obtain

$$\left(\frac{2d_n + (1-c)n}{n}\right)\frac{M_p(\|x\|_2^4)}{M_p(1)} = p\frac{M_p(\|x\|_2^2\cdot\|x\|_{p+2}^{p+2})}{M_p(1)} - 2\frac{M_p(\|x\|_4^4)}{M_p(1)}. \quad (25)$$

Finally, by applying Proposition 12 with $\xi = 4$ and $f(x) = \mathbf{1}$ we obtain

$$\left(\frac{2d_n + (3-c)n}{n}\right)\frac{M_p(\|x\|_4^4)}{M_p(1)} = p\frac{M_p(\|x\|_{p+4}^{p+4})}{M_p(1)} - \big(d_n - (c+1)n\big)\frac{M_p(x_1^2x_2^2)}{M_p(1)}. \quad (26)$$

4.1. *On the cases of Schatten classes corresponding to $a = 2$ and to large p*

Here we prove the main claim of Theorem 1. We will combine identities (24) and (25) with a simple application of Hölder's inequality, by which we have

$$\|x\|_p^{p+2} \geq \|x\|_{p+2}^{p+2} \geq \|x\|_p^{p+2} \cdot n^{-2/p} = \left(1 - O\left(\frac{\log n}{p}\right)\right)\|x\|_p^{p+2} \tag{27}$$

for every $p \gg \log n$. Indeed, by (24), (27) and Lemma 9 we obtain that

$$\begin{aligned}\left(\frac{2d_n + (1-c)n}{n}\right)\frac{M_p(\|x\|_2^2)}{M_p(\mathbf{1})} &= p\frac{M_p(\|x\|_{p+2}^{p+2})}{M_p(\mathbf{1})} \\ &\geq \left(1 - O\left(\frac{\log n}{p}\right)\right)p\frac{M_p(\|x\|_p^{p+2})}{M_p(\mathbf{1})} \\ &= \left(1 - O\left(\frac{\log n}{p}\right)\right)d_n\frac{\Gamma(1+\frac{d_n+2}{p})}{\Gamma(1+\frac{d_n}{p})}.\end{aligned} \tag{28}$$

We obviously also have

$$\frac{M_p(\|x\|_4^4)}{M_p(\mathbf{1})} = n\frac{M_p(x_1^4)}{M_p(\mathbf{1})} \geq n\left(\frac{M_p(x_1^2)}{M_p(\mathbf{1})}\right)^2 = \frac{1}{n}\left(\frac{M_p(\|x\|_2^2)}{M_p(\mathbf{1})}\right)^2.$$

In view of the above estimates, as well as of part (a) of Lemma 6, (25) and Lemma 9 now give

$$\begin{aligned}\left(\frac{2d_n + (1-c)n}{n}\right)\frac{M_p(\|x\|_2^4)}{M_p(\mathbf{1})} &= p\frac{M_p(\|x\|_2^2 \cdot \|x\|_{p+2}^{p+2})}{M_p(\mathbf{1})} - 2\frac{M_p(\|x\|_4^4)}{M_p(\mathbf{1})} \\ &\leq p\frac{M_p(\|x\|_2^2 \cdot \|x\|_p^{p+2})}{M_p(\mathbf{1})} - \frac{2}{n}\left(\frac{M_p(\|x\|_2^2)}{M_p(\mathbf{1})}\right)^2 \\ &= (d_n+2)\frac{\Gamma(1+\frac{d_n+4}{p})}{\Gamma(1+\frac{d_n+2}{p})}\frac{M_p(\|x\|_2^2)}{M_p(\mathbf{1})} - \frac{2}{n}\left(\frac{M_p(\|x\|_2^2)}{M_p(\mathbf{1})}\right)^2 \\ &= \left(1 + O\left(\frac{\log n}{p}\right) + O\left(\frac{1}{n^2}\right)\right)\left(\frac{2d_n + (1-c)n}{n}\right)\left(\frac{M_p(\|x\|_2^2)}{M_p(\mathbf{1})}\right)^2.\end{aligned}$$

Thus, we conclude that

$$\mathrm{Var}_{M_p}\left(\|x\|_2^2\right) = \left(O\left(\frac{\log n}{p}\right) + O\left(\frac{1}{n^2}\right)\right)\left(\frac{M_p(\|x\|_2^2)}{M_p(\mathbf{1})}\right)^2 \leq C\max\left\{\frac{n^2\log n}{p}, 1\right\} \cdot n^{4/p},$$

where C is an absolute constant.

This combined with Proposition 2 establishes the main claim of Theorem 1. For the remaining statement that $\sigma_{K_p}^2 \gtrsim 1$ when $p \gtrsim n^2 \log n$, see Section 4.3.

Remark 14. Note that (24), (25) and Lemma 9 readily imply the thin-shell conjecture when $p = 2$ as well:

$$\begin{aligned}\left(\frac{2d_n + (1-c)n}{n}\right)\frac{M_2(\|x\|_2^4)}{M_2(\mathbf{1})} &= 2\frac{M_2(\|x\|_2^2 \cdot \|x\|_4^4)}{M_2(\mathbf{1})} - 2\frac{M_2(\|x\|_4^4)}{M_2(\mathbf{1})} \\ &= (d_n+4)\frac{M_2(\|x\|_4^4)}{M_2(\mathbf{1})} - \left(\frac{2d_n + (1-c)n}{n}\right)\frac{M_2(\|x\|_2^2)}{M_2(\mathbf{1})}\end{aligned}$$

$$= \frac{d_n+2}{2}\left(\frac{2d_n+(1-c)n}{n}\right)\frac{M_2(\|x\|_2^2)}{M_2(\mathbf{1})}$$
$$= \frac{d_n+2}{d_n}\left(\frac{2d_n+(1-c)n}{n}\right)\left(\frac{M_2(\|x\|_2^2)}{M_2(\mathbf{1})}\right)^2.$$

Although this is not interesting when $E=\mathcal{M}_n(\mathbb{F})$, given that in those cases we know that $\overline{K}_{2,E}$ is in isotropic position and therefore, since $p=2$ corresponds to the Euclidean norm, that it is exactly the Euclidean ball of volume 1 in E, it is perhaps worth noting in the case of Hermitian matrices, of anti-symmetric Hermitian, or of complex symmetric matrices (especially so if $\overline{K}_{2,E}$ turns out to not be isotropic for one or more of these three subspaces E).

Remark 15. Note that, since $\overline{K}_{p,E}$ is in isotropic position when $E=\mathcal{M}_n(\mathbb{R})$ or $\mathcal{M}_n(\mathbb{C})$, we have that

$$d_n L_{K_{p,E}}^2 = \frac{1}{|K_{p,E}|^{1+\frac{2}{d_n}}}\int_{K_{p,E}}\left\|s(T)\right\|_2^2\,dT$$
$$= \frac{1}{|K_{p,E}|^{2/d_n}}\left(\frac{1}{|K_{p,E}|}\int_{K_{p,E}}\left\|s(T)\right\|_2^2\,dT\right)$$
$$= \frac{1}{|K_{p,E}|^{2/d_n}}\frac{\Gamma(1+\frac{d_n}{p})}{\Gamma(1+\frac{d_n+2}{p})}\frac{M_p(\|x\|_2^2)}{M_p(\mathbf{1})}$$

which for large p can be rewritten, using (28), as

$$= \left(1+O\left(\frac{\log n}{p}\right)\right)\frac{d_n}{|K_{p,E}|^{2/d_n}}\left(\frac{2d_n+(1-c)n}{n}\right)^{-1}.$$

This shows that, for $p\gg \log n$,

$$L_{K_{p,E}} = \left(1+O\left(\frac{\log n}{p}\right)+O\left(\frac{1}{n}\right)\right)\sqrt{\frac{n}{2d_n}}\frac{1}{|K_{p,E}|^{2/d_n}}$$
$$= \left(1+O\left(\frac{\log n}{p}\right)+O\left(\frac{1}{n}\right)\right)\frac{1}{\sqrt{2\beta n}}\frac{1}{|K_{p,E}|^{1/d_n}},$$

where $\beta=1$ if $E=\mathcal{M}_n(\mathbb{R})$, and $\beta=2$ if $E=\mathcal{M}_n(\mathbb{C})$.

Recall now that Saint-Raymond [30] has found very precise estimates for the volume of $K_{p,\mathcal{M}_n(\mathbb{R})}$ and of $K_{p\mathcal{M}_n(\mathbb{C})}$, and in particular he has shown that

$$|K_{\infty,\mathcal{M}_n(\mathbb{R})}|^{1/n^2} = \big(1+o(1)\big)\frac{1}{2}\sqrt{\frac{2\pi e^{3/2}}{n}}, \qquad |K_{\infty,\mathcal{M}_n(\mathbb{C})}|^{1/(2n^2)} = \big(1+o(1)\big)\frac{1}{2}\sqrt{\frac{\pi e^{3/2}}{n}}.$$

We can thus conclude that

$$L_{K_{\infty,\mathcal{M}_n(\mathbb{R})}} = \big(1+o(1)\big)\frac{1}{\sqrt{\pi e^{3/2}}} = L_{K_{\infty,\mathcal{M}_n(\mathbb{C})}}.$$

4.2. *Necessity of a negative correlation property*

Here we prove Theorem 3. Let us set

$$\frac{M_p(x_1^2)}{M_p(1)} = c_2\frac{M_p(|x_1|^p)}{M_p(1)}$$

where c_2 depends both on p and on n. Then by (24) we can also write

$$\frac{M_p(|x_1|^{p+2})}{M_p(1)} = \frac{2d_n+(1-c)n}{pn}\cdot c_2\frac{M_p(|x_1|^p)}{M_p(1)}.$$

Using this and the fact that

$$M_p\left[|x_1|^p\left(x_1^2-\frac{2d_n+(1-c)n}{pn}c_2\right)^2\right]\geq 0,$$

we can deduce that

$$p\frac{M_p(|x_1|^{p+4})}{M_p(1)}\geq\frac{(2d_n+(1-c)n)^2}{d_n\cdot n}\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2. \tag{29}$$

Furthermore, as we mentioned in Remark 7, we have

$$\mathrm{Var}_{M_p}\big(\|x\|_2^2\big)\lesssim n\cdot n^{4/p}=o\big(n^2\big)\cdot n^{4/p},$$

which implies that

$$\frac{M_p(x_1^2x_2^2)}{M_p(1)}=\big(1+o(1)\big)\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2. \tag{30}$$

Combining (29)–(30) with identity (26) we obtain that

$$\begin{aligned}\left(\frac{2d_n+(3-c)n}{n}\right)\frac{M_p(x_1^4)}{M_p(1)}&=p\frac{M_p(|x_1|^{p+4})}{M_p(1)}-\left(\frac{d_n-(c+1)n}{n}\right)\frac{M_p(x_1^2x_2^2)}{M_p(1)}\\&\geq\frac{(2d_n+(1-c)n)^2}{d_n\cdot n}\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2-\frac{d_n}{n}\big(1+o(1)\big)\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2.\end{aligned}$$

This gives inequality (5) of Theorem 3, and the rest of the theorem readily follows, thus completing the proof.

4.3. *Precise estimates for σ_{K_p}*

It remains to estimate σ_{K_p} for large p more accurately and establish the final part of Theorem 1 with regard to the cases of $E=\mathcal{M}_n(\mathbb{F})$ or of the subspace of complex symmetric matrices. Combining identities (25) and (26), we get

$$\begin{aligned}\left(\frac{2d_n+(1-b-c)n}{n}\right)\frac{M_p(x_1^2x_2^2)}{M_p(1)}&=p\frac{M_p(x_1^2|x_2|^{p+2})}{M_p(1)}\\&=\frac{p}{n-1}\left(\frac{M_p(x_1^2\|x\|_{p+2}^{p+2})}{M_p(1)}-\frac{M_p(|x_1|^{p+4})}{M_p(1)}\right).\end{aligned}$$

Provided that p is large enough, and making use of Lemma 9 as well, we can compute the latter terms with great accuracy:

$$\begin{aligned}\frac{M_p(|x_1|^{p+4})}{M_p(1)}=\frac{1}{n}\frac{M_p(\|x\|_{p+4}^{p+4})}{M_p(1)}&=\left(1+O\left(\frac{\log n}{p}\right)\right)\cdot\frac{1}{n}\frac{M_p(\|x\|_p^{p+4})}{M_p(1)}\\&=\left(1+O\left(\frac{\log n}{p}\right)\right)\cdot\frac{1}{n}\frac{\Gamma(\frac{d_n+p+4}{p})}{\Gamma(\frac{d_n}{p})},\end{aligned}$$

while similarly

$$\begin{aligned}\frac{M_p(x_1^2\|x\|_{p+2}^{p+2})}{M_p(1)}&=\left(1+O\left(\frac{\log n}{p}\right)\right)\frac{M_p(x_1^2\|x\|_p^{p+2})}{M_p(1)}\\&=\left(1+O\left(\frac{\log n}{p}\right)\right)\frac{\Gamma(\frac{d_n+p+4}{p})}{\Gamma(\frac{d_n+2}{p})}\frac{M_p(x_1^2)}{M_p(1)}.\end{aligned}$$

Using also identity (24) now, we can continue by writing

$$\left(\frac{2d_n+(1-c)n}{n}\right)\frac{M_p(x_1^2)}{M_p(1)} = p\frac{M_p(|x_1|^{p+2})}{M_p(1)}$$
$$= \left(1+O\left(\frac{\log n}{p}\right)\right)\cdot\frac{d_n+2}{n}\frac{\Gamma(\frac{d_n+2}{p})}{\Gamma(\frac{d_n}{p})}.$$

In the end, gathering all error terms as well, we have

$$\left(\frac{2d_n+(1-b-c)n}{n}\right)\frac{M_p(x_1^2x_2^2)}{M_p(1)} = \frac{p}{n-1}\frac{\Gamma(\frac{d_n+p+4}{p})}{\Gamma(\frac{d_n}{p})}\left(\frac{d_n+2}{2d_n+(1-c)n}-\frac{1}{n}\right)+O\left(\frac{\log n}{n}\right)$$
$$= \frac{p}{n-1}\left(1+O\left(\frac{\log n}{p}\right)\right)\frac{\Gamma(\frac{d_n+p+4}{p})}{\Gamma(\frac{d_n}{p})}\left(\frac{d_n+2}{2d_n+(1-c)n}-\frac{1}{n}\right).$$

In other words

$$\frac{M_p(x_1^2x_2^2)}{M_p(1)} = \left(1+O\left(\frac{\log n}{p}\right)\right)\frac{\Gamma(1+\frac{d_n+4}{p})}{\Gamma(1+\frac{d_n}{p})}\frac{d_n(nd_n-2d_n+(1+c)n)}{(n-1)(2d_n+(1-c)n)(2d_n+(1-b-c)n)}, \tag{31}$$

while

$$\frac{M_p(x_1^2)}{M_p(1)} = \left(1+O\left(\frac{\log n}{p}\right)\right)\frac{\Gamma(1+\frac{d_n+2}{p})}{\Gamma(1+\frac{d_n}{p})}\frac{d_n}{2d_n+(1-c)n}. \tag{32}$$

Returning to (26), and observing that $\frac{d_n}{n} = bn + O(1)$ when $a = 2$ (as we consider here), we also see that

$$\left(\frac{2d_n+(3-c)n}{n}\right)\frac{M_p(x_1^4)}{M_p(1)}$$
$$= p\frac{M_p(|x_1|^{p+4})}{M_p(1)} - b(n-1)\frac{M_p(x_1^2x_2^2)}{M_p(1)}$$
$$= \frac{\Gamma(1+\frac{d_n+4}{p})}{\Gamma(1+\frac{d_n}{p})}\left[\frac{d_n}{n}-\frac{bd_n(nd_n-2d_n+(1+c)n)}{(2d_n+(1-c)n)(2d_n+(1-b-c)n)}\right]+O\left(\frac{n\log n}{p}\right)$$
$$= \left(1+O\left(\frac{\log n}{p}\right)\right)\frac{\Gamma(1+\frac{d_n+4}{p})}{\Gamma(1+\frac{d_n}{p})}\left[\frac{d_n}{n}-\frac{bd_n(nd_n-2d_n+(1+c)n)}{(2d_n+(1-c)n)(2d_n+(1-b-c)n)}\right].$$

In other words

$$\frac{M_p(x_1^4)}{M_p(1)} = \left(1+O\left(\frac{\log n}{p}\right)\right)\frac{\Gamma(1+\frac{d_n+4}{p})}{\Gamma(1+\frac{d_n}{p})}\left[\frac{d_n}{2d_n+(3-c)n}\right.$$
$$\left.-\frac{bnd_n(nd_n-2d_n+(1+c)n)}{(2d_n+(1-c)n)(2d_n+(1-b-c)n)(2d_n+(3-c)n)}\right]. \tag{33}$$

Combining all these with Lemma 6(a) (when applied with $p > d_n$), we conclude that

$$n\cdot\frac{M_p(x_1^4)}{M_p(1)} + n(n-1)\cdot\frac{M_p(x_1^2x_2^2)}{M_p(1)} - n^2\cdot\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2$$
$$= \left(1+O\left(\frac{\log n}{p}\right)\right)n\cdot\left(\frac{d_n}{2d_n+(3-c)n}-\frac{bnd_n(nd_n-2d_n+(1+c)n)}{(2d_n+(1-c)n)(2d_n+(1-b-c)n)(2d_n+(3-c)n)}\right)$$

$$
\begin{aligned}
&+\left(1+O\left(\frac{\log n}{p}\right)\right)n(n-1)\cdot\frac{d_n(nd_n-2d_n+(1+c)n)}{(n-1)(2d_n+(1-c)n)(2d_n+(1-b-c)n)}\\
&-\left(1+O\left(\frac{\log n}{p}\right)\right)n^2\cdot\left(\frac{d_n}{2d_n+(1-c)n}\right)^2\\
&=\Bigg[n\cdot\left(\frac{d_n}{2d_n+(3-c)n}-\frac{bnd_n(nd_n-2d_n+(1+c)n)}{(2d_n+(1-c)n)(2d_n+(1-b-c)n)(2d_n+(3-c)n)}\right)\\
&\quad+n\cdot\frac{d_n(nd_n-2d_n+(1+c)n)}{(2d_n+(1-c)n)(2d_n+(1-b-c)n)}-n^2\cdot\left(\frac{d_n}{2d_n+(1-c)n}\right)^2\Bigg]+O\left(\frac{n^2\log n}{p}\right)\\
&\simeq\frac{1}{16b}+o(1),
\end{aligned}
$$

as long as $O(n^2\log n/p)\in[-\frac{1}{16b},\frac{1}{16b}]$ say. Given in addition that

$$\mathrm{Var}_{M_p}\big(\|x\|_2^2\big)\simeq\max\left\{\sigma_{K_p}^2,\frac{1}{p}\right\}\cdot n^{4/p},$$

and that for all d_n-dimensional bodies K we have $\sigma_K^2\gtrsim 1/d_n$, we obtain that

$$\sigma_{K_p}^2\simeq\mathrm{Var}_{M_p}\big(\|x\|_2^2\big)\simeq\frac{1}{b}+o(1)$$

for all $p\gtrsim n^2\log n$.

5. The cases of complex anti-symmetric and of Hermitian matrices

We turn to showing why we can have analogues of Proposition 2, Theorem 1 and Theorem 3 for the Schatten classes in the subspaces of anti-symmetric Hermitian matrices and of Hermitian matrices. We stated in Section 2, Fact C, that, if the subspace of anti-symmetric Hermitian matrices is equipped with the standard Gaussian measure, then the induced joint probability density of the singular values $\theta_1,\dots,\theta_s$ of a random matrix $T\in E$ is given by

$$\mathbb{P}_n\big((\theta_1,\dots,\theta_s)\in A\big)=c_{n,E}\cdot\int_A\prod_{1\le i<j\le s}\big|x_i^2-x_j^2\big|^2\exp\big(-\|x\|_2^2\big)\,dx$$

if $n=2s$, and by

$$\mathbb{P}_n\big((\theta_1,\dots,\theta_s)\in A\big)=c_{n,E}\cdot\int_A\prod_{1\le i<j\le s}\big|x_i^2-x_j^2\big|^2\cdot\prod_{1\le i\le s}|x_i|^2\exp\big(-\|x\|_2^2\big)\,d\theta$$

if $n=2s+1$, where A is any 1-symmetric measurable subset of $\mathbb{R}^s$. This of course implies that for every symmetric measurable function $F:\mathbb{R}^s\to\mathbb{R}^+$ we must have

$$
\begin{aligned}
&\int_E F(\theta_1,\dots,\theta_s)\exp\big(-\|T\|_{S_2^n}^2/2\big)\,dT\\
&\quad=c_{n,E}\cdot\int_{\mathbb{R}^s}F\big(|x_1|,\dots,|x_s|\big)\cdot\prod_{1\le i<j\le s}\big|x_i^2-x_j^2\big|^2\cdot\prod_{1\le i\le s}|x_i|^{2r}\exp\big(-\|x\|_2^2\big)\,dx,
\end{aligned}
\tag{34}
$$

where $n=2s+r$, $r\in\{0,1\}$. This allows us to prove the following

Lemma 16. *Let $F:\mathbb{R}^s\to\mathbb{R}^+$ be a measurable, symmetric and k-homogeneous function. Then*

$$
\begin{aligned}
&\int_{K_{p,E}}F(\theta_1,\dots,\theta_s)\,dT\\
&\quad=\frac{c_{n,E}2^{-(n^2+k)/p}}{\Gamma(1+\frac{n^2+k}{p})}\int_{\mathbb{R}^s}F\big(|x_1|,\dots,|x_s|\big)\cdot\prod_{1\le i<j\le s}\big|x_i^2-x_j^2\big|^2\cdot\prod_{1\le i\le s}|x_i|^{2r}\exp\big(-\|x\|_p^p\big)\,dx,
\end{aligned}
$$

where $n^2=\dim(E)$.

Proof. We first note that, for every anti-symmetric Hermitian matrix T and for each $p \geq 1$,

$$\|T\|_{S_p^n}^p = 2\sum_{i=1}^{s} |\theta_i|^p = 2\big\|(\theta_1, \dots, \theta_s)\big\|_p^p,$$

therefore, applying (34) with the function

$$F(x_1, \dots, x_s) \cdot \frac{\exp(-\|x\|_p^p)}{\exp(-\|x\|_2^2)},$$

we see that

$$\begin{aligned}&\int_E F(\theta_1, \dots, \theta_s) \exp\big(-\|T\|_{S_p^n}^p/2\big)\, dT \\ &\quad = c_{n,E} \cdot \int_{\mathbb{R}^s} F\big(|x_1|, \dots, |x_s|\big) \cdot \prod_{1\leq i<j\leq s} \big|x_i^2 - x_j^2\big|^2 \cdot \prod_{1\leq i\leq s} |x_i|^{2r} \exp\big(-\|x\|_p^p\big)\, dx.\end{aligned}$$

Furthermore, given that $K_{p,E} = \{T \in E : \|T\|_{S_p^n} \leq 1\}$, we can write

$$\begin{aligned}\int_E F(\theta_1, \dots, \theta_s) \exp\big(-\|T\|_{S_p^n}^p/2\big)\, dT &= \int_0^{+\infty} e^{-t} \int_{(2t)^{1/p} K_{p,E}} F(\theta_1, \dots, \theta_s)\, dT\, dt \\ &= 2^{(n^2+k)/p}\Gamma\left(1 + \frac{n^2+k}{p}\right) \cdot \int_{K_{p,E}} F(\theta_1, \dots, \theta_s)\, dT.\end{aligned}$$

This concludes the proof. □

Note now that, if $M_{2,2,2r,p}$ denotes integration on $\mathbb{R}^s$ with respect to the density

$$\exp\big(-\|x\|_p^p\big)) \cdot f_{2,2,2r}(x) = \exp\big(-\|x\|_p^p\big)) \cdot \prod_{1\leq i<j\leq s} \big|x_i^2 - x_j^2\big|^2 \cdot \prod_{1\leq i\leq s} |x_i|^{2r},$$

where $r \in \{0, 1\}$, then

$$\begin{aligned}\mathbb{E}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big) &= \frac{1}{|K_{p,E}|^{2/d_n}} \frac{\int_{K_{p,E}} 2\|(\theta_1, \dots, \theta_s\|_2^2\, dT}{|K_{p,E}|} \\ &= \frac{1}{|K_{p,E}|^{2/d_n}} \cdot \frac{\Gamma(1+\frac{n^2}{p})}{2^{2/p}\Gamma(1+\frac{n^2+2}{p})} \frac{M_{2,2,2r,p}(\|x\|_2^2)}{M_{2,2,2r,p}(1)},\end{aligned}$$

and similarly

$$\begin{aligned}&|K_{p,E}|^{4/d_n} \cdot \mathrm{Var}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big) \\ &\quad = \frac{1}{|K_{p,E}|}\int_{K_{p,E}} \|s(T)\|_2^4\, dT - \left(\frac{1}{|K_{p,E}|}\int_{K_{p,E}} \|s(T)\|_2^2\, dT\right)^2 \\ &\quad = \frac{1}{2^{4/p}}\left[\frac{\Gamma(1+\frac{n^2}{p})}{\Gamma(1+\frac{n^2+4}{p})} \frac{M_{2,2,2r,p}(\|x\|_2^4)}{M_{2,2,2r,p}(1)} - \left(\frac{\Gamma(1+\frac{n^2}{p})}{\Gamma(1+\frac{n^2+2}{p})}\right)^2 \left(\frac{M_{2,2,2r,p}(\|x\|_2^2)}{M_{2,2,2r,p}(1)}\right)^2\right].\end{aligned}$$

From this point on, we can proceed as in Sections 3 and 4 to prove that

$$\mathbb{E}_{\overline{K}_{p,E}}\big(\|s(T)\|_2^2\big) \simeq n^{1-2/p} \cdot s \cdot \frac{M_{2,2,2r,p}(x_1^2)}{M_{2,2,2r,p}(1)} \simeq n^2 = \dim(E),$$

as well as Theorems 1 and 3 for the subspace of anti-symmetric Hermitian matrices (note that this time, when we apply Lemma 5 and Propositions 11 and 12, d_n is replaced by $d_s = 2s(s-1) + (2r+1)s$, which is not equal to $\dim(E)$; still the conclusions we obtain are of the same form given that $s = \lfloor \frac{n}{2} \rfloor \simeq n$).

Let us finally see why Theorems 1 and 3 hold true when E is the subspace of Hermitian matrices too, even though by Fact A we know that the density we have to work with when we reduce integrals over the balls $K_{p,E}$ to integrals over $\mathbb{R}^n$ is the density $f_{1,2,0}(x)$.

Proposition 17. *Let $f : \mathbb{R}^n \to \mathbb{R}^+$ be a symmetric function. Then there exists a constant c_n depending only on n such that*

$$\int_{\mathbb{R}^n} f\big(|x_1|, \dots, |x_n|\big) \cdot f_{1,2,0}(x) e^{-\|x\|_p^p}\, dx$$
$$= \sum_{\substack{A \subset [n] \\ |A| = \lceil n/2 \rceil}} c_n \cdot \int_{\mathbb{R}^n} f\big(|x_1|, \dots, |x_n|\big) e^{-\|x\|_p^p} \cdot \prod_{i,j \in A; i<j} \big|x_i^2 - x_j^2\big|^2 \cdot \prod_{l,k \notin A; l<k} \big|x_l^2 - x_k^2\big|^2 \cdot \prod_{l \notin A} |x_l|^2\, dx.$$

Proof. Since the integrand $f(|x_1|, \dots, |x_n|) e^{-\|x\|_p^p}$ is invariant under permutations of the coordinates or flipping of their signs, we can make use of the exact argument of Edelman and La Croix from [11, Section 4] to obtain the result. □

We now remark that, with $f(x) = \|x\|_\xi^\xi$ or $f(x) = \mathbf{1}$, we have

$$\sum_{\substack{A \subset [n] \\ |A| = \lceil n/2 \rceil}} c_n \cdot \int_{\mathbb{R}^n} f(x) e^{-\|x\|_p^p} \cdot \prod_{i,j \in A; i<j} \big|x_i^2 - x_j^2\big|^2 \cdot \prod_{l,k \notin A; l<k} \big|x_l^2 - x_k^2\big|^2 \cdot \prod_{l \notin A} |x_l|^2\, dx$$
$$= \binom{n}{\lceil n/2 \rceil} c_n \cdot \int_{\mathbb{R}^n} f(x) e^{-\|x\|_p^p} \cdot \prod_{i,j \in I_1; i<j} \big|x_i^2 - x_j^2\big|^2 \cdot \prod_{l,k \notin I_1; l<k} \big|x_l^2 - x_k^2\big|^2 \cdot \prod_{l \notin I_1} |x_l|^2\, dx$$

where I_1 is the subset of the first $n_1 = \lceil n/2 \rceil$ indices from $\{1, 2, \dots, n\}$, and where we will write I_2 for its complement. Let us denote by N_{p,I_1} integration over $\mathbb{R}^{I_1}$ with respect to the density $\prod_{i,j \in I_1; i<j} |x_i^2 - x_j^2|^2 e^{-\|x\|_{p,I_1}^p}$, where $\|x\|_{p,I_1}$ denotes the p-norm of the coordinates of x with indices in I_1 only, and let us denote by N_{p,I_2} integration over $\mathbb{R}^{I_2}$ with respect to the density $\prod_{l,k \in I_2; l<k} |x_l^2 - x_k^2|^2 \cdot \prod_{l \in I_2} |x_l|^2 e^{-\|x\|_{p,I_2}^p}$. Let us finally denote by N_{p,I_1,I_2} integration over $\mathbb{R}^n$ with respect to the product of both densities. Then, by the independent nature of these two densities and by the above relations, we see that

$$\frac{M_{1,2,0,p}(\|x\|_\xi^\xi)}{M_{1,2,0,p}(1)} = \frac{N_{p,I_1,I_2}(\|x\|_\xi^\xi)}{N_{p,I_1,I_2}(1)} = \frac{N_{p,I_1}(\|x\|_{\xi,I_1}^\xi)}{N_{p,I_1}(1)} + \frac{N_{p,I_2}(\|x\|_{\xi,I_2}^\xi)}{N_{p,I_2}(1)}.$$

Similarly we have that

$$\frac{M_{1,2,0,p}(\|x\|_2^4)}{M_{1,2,0,p}(1)} = \frac{N_{p,I_1,I_2}(\|x\|_2^4)}{N_{p,I_1,I_2}(1)}$$
$$= \frac{N_{p,I_1,I_2}(\|x\|_{2,I_1}^4 + \|x\|_{2,I_2}^4 + 2\|x\|_{2,I_1}^2 \|x\|_{2,I_2}^2)}{N_{p,I_1,I_2}(1)}$$
$$= \frac{N_{p,I_1}(\|x\|_{2,I_1}^4)}{N_{p,I_1}(1)} + \frac{N_{p,I_2}(\|x\|_{2,I_2}^4)}{N_{p,I_2}(1)} + 2\frac{N_{p,I_1}(\|x\|_{2,I_1}^2)}{N_{p,I_1}(1)} \frac{N_{p,I_2}(\|x\|_{2,I_2}^2)}{N_{p,I_2}(1)}.$$

Therefore, to show that

$$\frac{M_{1,2,0,p}(\|x\|_2^4)}{M_{1,2,0,p}(1)} = \left(1 + O\left(\frac{1}{n^2}\right)\right)\left(\frac{M_{1,2,0,p}(\|x\|_2^2)}{M_{1,2,0,p}(1)}\right)^2$$

for some index p, we only need to establish that

$$\frac{N_{p,I_1}(\|x\|_{2,I_1}^4)}{N_{p,I_1}(1)} = \left(1 + O\left(\frac{1}{n^2}\right)\right)\left(\frac{N_{p,I_1}(\|x\|_{2,I_1}^2)}{N_{p,I_1}(1)}\right)^2$$

and that

$$\frac{N_{p,I_2}(\|x\|_{2,I_2}^4)}{N_{p,I_2}(1)} = \left(1 + O\left(\frac{1}{n^2}\right)\right)\left(\frac{N_{p,I_2}(\|x\|_{2,I_2}^2)}{N_{p,I_2}(1)}\right)^2.$$

But we have already seen the latter are true when $p \geq n^2 \log n$ or when $p = 2$ (since N_{p,I_1} is exactly $M_{2,2,0,p}$ over $\mathbb{R}^{I_1}$, while N_{p,I_2} stands for $M_{2,2,2,p}$ over $\mathbb{R}^{I_2}$).

At the same time, for $p \gtrsim n^2 \log n$, we can show as before that

$$\frac{N_{p,I_r}(\|x\|_{2,I_r}^4)}{N_{p,I_r}(1)} - \left(\frac{N_{p,I_r}(\|x\|_{2,I_r}^2)}{N_{p,I_r}(1)}\right)^2 \gtrsim \frac{1}{n^2}\left(\frac{N_{p,I_r}(\|x\|_{2,I_r}^2)}{N_{p,I_r}(1)}\right)^2$$

for $r = 1, 2$. Therefore,

$$\begin{aligned}
\mathrm{Var}_{M_p}\big(\|x\|_2^2\big) &= \frac{M_{1,2,0,p}(\|x\|_2^4)}{M_{1,2,0,p}(1)} - \left(\frac{M_{1,2,0,p}(\|x\|_2^2)}{M_{1,2,0,p}(1)}\right)^2 \\
&\gtrsim \frac{1}{n^2}\left[\left(\frac{N_{p,I_1}(\|x\|_{2,I_1}^2)}{N_{p,I_1}(1)}\right)^2 + \left(\frac{N_{p,I_2}(\|x\|_{2,I_2}^2)}{N_{p,I_2}(1)}\right)^2\right] \\
&\gtrsim \frac{1}{n^2}\left(\frac{M_{1,2,0,p}(\|x\|_2^2)}{M_{1,2,0,p}(1)}\right)^2 \simeq 1.
\end{aligned}$$

Finally, for large p we can also obtain that

$$\frac{N_{p,I_1}(x_1^4)}{N_{p,I_1}(1)} \geq \left(\frac{3}{2} + o(1)\right)\left(\frac{N_{p,I_1}(x_1^2)}{N_{p,I_1}(1)}\right)^2$$

and

$$\frac{N_{p,I_2}(x_n^4)}{N_{p,I_2}(1)} \geq \left(\frac{3}{2} + o(1)\right)\left(\frac{N_{p,I_2}(x_n^2)}{N_{p,I_2}(1)}\right)^2,$$

whence inequality (5) of Theorem 3 follows:

$$\begin{aligned}
\frac{M_{1,2,0,p}(\|x\|_4^4)}{M_{1,2,0,p}(1)} &= \frac{N_{p,I_1}(\|x\|_{4,I_1}^4)}{N_{p,I_1}(1)} + \frac{N_{p,I_2}(\|x\|_{4,I_2}^4)}{N_{p,I_2}(1)} \\
&\geq \left(\frac{3}{2} + o(1)\right)\left[\frac{1}{n_1}\left(\frac{N_{p,I_1}(\|x\|_{2,I_1}^2)}{N_{p,I_1}(1)}\right)^2 + \frac{1}{n_2}\left(\frac{N_{p,I_2}(\|x\|_{2,I_2}^2)}{N_{p,I_2}(1)}\right)^2\right] \\
&\geq \left(\frac{3}{2} + o(1)\right)\frac{1}{n}\left(\frac{N_{p,I_1}(\|x\|_{2,I_1}^2)}{N_{p,I_1}(1)} + \frac{N_{p,I_2}(\|x\|_{2,I_2}^2)}{N_{p,I_2}(1)}\right)^2 \\
&= \left(\frac{3}{2} + o(1)\right)\frac{1}{n}\left(\frac{M_{1,2,0,p}(\|x\|_2^2)}{M_{1,2,0,p}(1)}\right)^2.
\end{aligned}$$

Remark 18. Note that for the subspaces of anti-symmetric Hermitian and of Hermitian matrices, as well as for the subspace of complex symmetric spaces, inequality (5) holds only conditionally, depending on whether we have

$$\frac{M_p(x_1^2 x_2^2)}{M_p(1)} = \big(1 + o(1)\big)\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2,$$

or equivalently whether we have $\sigma^2_{K_{p,E}} = o(n^2)$, which we a priori do not know for these balls $K_{p,E}$. Nevertheless, since through the arguments for Theorem 1 we can conclude that $\sigma^2_{K_{p,E}} = o(n^2)$ for all $p \gg \log n$ for these subspaces too, inequality (5) holds unconditionally in this range of p.

At any rate, the final conclusion of Theorem 3 remains unaffected: given any $p \geq 1$, for the thin-shell conjecture to hold true for $K_{p,E}$, where E is any of the three subspaces mentioned here, or even for $\sigma^2_{K_{p,E}}$ to be $o(n)$, we need the density $f_{a,b,c,p}$ corresponding to $K_{p,E}$ to possess a negative correlation property.

6. More on the negative correlation property when $E = \mathcal{M}_n(\mathbb{F})$

The purpose of this final section is to establish a type of negative correlation property for the original, uniform measures on $K_{p,\mathcal{M}_n(\mathbb{F})}$ as well. We start with the following lemma which allows us to relate terms that appear when we expand $\mathrm{Var}_{M_p}(\|x\|_2^2)$ and $\mathrm{Var}_{\overline{K}_{p,E}}(\|T\|_{\mathrm{HS}}^2)$ respectively.

Lemma 19. *For every $n \times n$ matrix $T \in \mathcal{M}_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, we have that, if $T = (a_{i,j})_{1\leq i,j\leq n}$ and if $(s_i(T))_{1\leq i\leq n} = (s_i)_{1\leq i\leq n}$ is the non-increasing rearrangement of the singular values of T, then*

$$\sum_{i=1}^n s_i^4 = \sum_{1\leq i,j\leq n} |a_{i,j}|^4 + \sum_{i=1}^n \sum_{j\neq l} \big(|a_{i,j}|^2|a_{i,l}|^2 + |a_{j,i}|^2|a_{l,i}|^2\big) + \sum_{i\neq l}\sum_{j\neq k} a_{i,j}\overline{a_{l,j}}a_{l,k}\overline{a_{i,k}}, \tag{35}$$

while

$$\sum_{i\neq j} s_i^2 s_j^2 = \sum_{i\neq l}\sum_{j\neq k} |a_{i,j}|^2|a_{l,k}|^2 - \sum_{i\neq l}\sum_{j\neq k} a_{i,j}\overline{a_{l,j}}a_{l,k}\overline{a_{i,k}}. \tag{36}$$

Remark 20. When the entries of the matrix T are real or complex numbers, we have that multiplication between different entries is commutative, hence we can rewrite (36) as

$$\begin{aligned}\sum_{i\neq j} s_i^2 s_j^2 &= \sum_{i\neq l}\sum_{j\neq k} |a_{i,j}|^2|a_{l,k}|^2 - \sum_{i\neq l}\sum_{j\neq k} a_{i,j}\overline{a_{l,j}a_{i,k}}a_{l,k} \\ &= \sum_{i\neq l}\sum_{j\neq k} a_{i,j}\overline{a_{i,j}}a_{l,k}\overline{a_{l,k}} - \sum_{i\neq l}\sum_{j\neq k} a_{i,j}\overline{a_{l,j}a_{i,k}}a_{l,k} \\ &= 2\sum_{i<l}\sum_{j\neq k} a_{i,j}a_{l,k}\big(\overline{a_{i,j}a_{l,k}} - \overline{a_{i,k}a_{l,j}}\big) \\ &= 2\sum_{i<l}\sum_{j<k} (a_{i,j}a_{l,k} - a_{i,k}a_{l,j})\cdot\big(\overline{a_{i,j}a_{l,k}} - \overline{a_{i,k}a_{l,j}}\big) \\ &= 2\sum_{i<l}\sum_{j<k} |a_{i,j}a_{l,k} - a_{i,k}a_{l,j}|^2. \end{aligned} \tag{37}$$

This is of course not necessarily true when $T \in \mathcal{M}_n(\mathbb{H})$, since $\mathbb{H}$ is a skew field. Note however that the last sum in both (35) and (36) is a real number in all cases.

Proof. Note that $\sum_{i=1}^n s_i^4 = \mathrm{tr}((T^*T)^2) = \mathrm{tr}((TT^*)^2)$. We also have that

$$TT^* = \left(\sum_{j=1}^n a_{i,j}\overline{a_{l,j}}\right)_{1\leq i,l\leq n},$$

thus the (i,i)th entry of $(TT^*)^2$ is equal to

$$\big((TT^*)^2\big)_{i,i} = \sum_{l=1}^n \left(\sum_{j=1}^n a_{i,j}\overline{a_{l,j}}\right)\left(\sum_{k=1}^n a_{l,k}\overline{a_{i,k}}\right) = \sum_{l=1}^n \sum_{1\leq j,k\leq n} a_{i,j}\overline{a_{l,j}}a_{l,k}\overline{a_{i,k}}.$$

Summing over all $i \in \{1,\dots,n\}$ we get (35).

To also obtain (36), we recall that

$$\sum_{i=1}^{n} s_i^2 = \|T\|_{\mathrm{HS}}^2 = \sum_{1\le i,j\le n} |a_{i,j}|^2,$$

and also that

$$\left(\sum_{i=1}^{n} s_i^2\right)^2 = \sum_{i=1}^{n} s_i^4 + \sum_{i\neq j} s_i^2 s_j^2.$$

Thus

$$\sum_{i=1}^{n} s_i^4 + \sum_{i\neq j} s_i^2 s_j^2 = \Big(\sum_{1\le i,j\le n} |a_{i,j}|^2\Big)^2$$

$$= \sum_{1\le i,j\le n} |a_{i,j}|^4 + \sum_{i=1}^{n}\sum_{j\neq l}\big(|a_{i,j}|^2|a_{i,l}|^2 + |a_{j,i}|^2|a_{l,i}|^2\big) + \sum_{i\neq l}\sum_{j\neq k} |a_{i,j}|^2|a_{l,k}|^2,$$

which combined with (35) leads to (36). □

We now need to study the orders of magnitude of the terms appearing in (35)–(36). This will be done through the study of symmetries of the balls $K_{p,\mathcal{M}_n(\mathbb{F})}$, one immediate consequence of which is the isotropicity of these convex bodies.

Lemma 21. *Suppose $p \ge 1$ and let $E = \mathcal{M}_n(\mathbb{R})$ or $\mathcal{M}_n(\mathbb{C})$ or $\mathcal{M}_n(\mathbb{H})$. If $A : E \to E$ is an invertible transformation that can be realised as left or right multiplication by an orthogonal or unitary or symplectic matrix respectively, then $A(K_{p,E}) = K_{p,E}$. The same conclusion is true if A takes a matrix in E to its conjugate transpose, or simply to its transpose. Immediate consequences are the following*:

1. *For every $p \ge 1$, the normalised unit ball $\overline{K}_{p,E}$ is in isotropic position.*
2. *For all $i, j \in \{1, 2, \dots, n\}$, and for every power $s > 0$,*

$$\int_{K_{p,E}} |a_{i,j}|^s \, dT = \int_{K_{p,E}} |a_{1,1}|^s \, dT.$$

3. *For all $i, j, l, k \in \{1, 2, \dots, n\}$ with $i \neq l$, $j \neq k$,*

$$\int_{K_{p,E}} |a_{i,j}|^2|a_{i,k}|^2 \, dT = \int_{K_{p,E}} |a_{j,i}|^2|a_{k,i}|^2 \, dT = \int_{K_{p,E}} |a_{1,1}|^2|a_{1,2}|^2 \, dT = \int_{K_{p,E}} |a_{1,1}|^2|a_{2,1}|^2 \, dT,$$

$$\int_{K_{p,E}} |a_{i,j}|^2|a_{l,k}|^2 \, dT = \int_{K_{p,E}} |a_{1,1}|^2|a_{2,2}|^2 \, dT,$$

$$\int_{K_{p,E}} a_{i,j}\overline{a_{l,j}}a_{l,k}\overline{a_{i,k}} \, dT = \int_{K_{p,E}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}} \, dT.$$

Proof. Let U be an orthogonal (or unitary) matrix. Then for every $n \times n$ matrix M, we have that the singular values of UM or of MU are the same as those of M: indeed, $(UM)^*(UM) = M^*(U^*U)M = M^*M$, while $(MU)^*(MU) = U^*(M^*M)U$, so it has the same eigenvalues as M^*M. This implies that $\{UM : M \in K_{p,E}\}$ or $\{MU : M \in K_{p,E}\}$ coincide with $K_{p,E}$.

On the other hand, if $A(M) = M^*$, then $(A(M)^*A(M)) = MM^*$, which has the same eigenvalues as M^*M. The latter is true even if $A(M)$ is just the transpose of M.

Finally, if A is a linear transformation on $E = \mathcal{M}_n(\mathbb{F})$ of one of the above forms, then, since $A(K_{p,E}) = K_{p,E}$, we must have that $|\det(A)| = 1$. This shows that for every integrable function F on $K_{p,E}$,

$$\int_{K_{p,E}} F(T)\, dT = \int_{A(K_{p,E})} F(T)\, dT = \int_{K_{p,E}} \big|\det(A)\big| \cdot F\big(A(T)\big)\, dT = \int_{K_{p,E}} F\big(A(T)\big)\, dT. \tag{38}$$

It is now easy to establish statements 1, 2 and 3 of the lemma: we apply (38) with F being suitable functions of the entries of $T \in K_{p,E}$, and the linear transformation A being either multiplication from the left or from the right (or from both sides) by a permutation matrix $P_{i,j}$ (formed by permuting the ith and the jth row of the identity matrix, and leaving all other rows unchanged) or by its transpose, or A being the transformation that sends each matrix T to its (conjugate) transpose.

To also show that $\overline{K}_{p,E}$ is isotropic, we need to prove in addition that all integrals of products of pairs of different entries (or of pairs of real and imaginary parts of them) are 0. In the real case, all such integrals must be equal

$$\text{either to } \int_{K_{p,E}} a_{1,1}a_{1,2}\,dT = \int_{K_{p,E}} a_{1,1}a_{2,1}\,dT, \quad \text{or to} \quad \int_{K_{p,E}} a_{1,1}a_{2,2}\,dT, \tag{39}$$

so we just have to show that the latter integrals are 0. For the first one, consider the rotation matrix

$$U = \begin{pmatrix} \cos(\theta) & \sin(\theta) & \mathbf{0} \\ -\sin(\theta) & \cos(\theta) & \\ \mathbf{0} & & \mathrm{Id}_{n-2} \end{pmatrix}, \tag{40}$$

and apply (38) with A being multiplication from the left by U and F being the absolute value of the first entry squared (or simply the first entry squared): since

$$\int_{K_{p,E}} \left|a_{1,1}(T)\right|^2 dT = \int_{K_{p,E}} \left|a_{1,1}(UT)\right|^2 dT \quad \text{and} \quad \int_{K_{p,E}} a_{1,1}(T)^2\,dT = \int_{K_{p,E}} a_{1,1}(UT)^2\,dT,$$

we must have

$$\int_{K_{p,E}} 2\cos(\theta)\sin(\theta)\,\mathrm{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\,dT = \int_{K_{p,E}} 2\cos(\theta)\sin(\theta)a_{1,1}(T)a_{2,1}(T)\,dT = 0,$$

which in the real case is one and the same thing and shows that the first integral in (39) is 0. In the complex and quaternion cases, we should also use as A linear combinations of permutation matrices with coefficients from $\{1, i, j, k\} \cap \mathbb{F}$ to deduce first that

$$\begin{aligned} &\int_{K_{p,E}} \mathrm{Re}(a_{1,1})\,\mathrm{Re}(a_{2,1})\,dT = \int_{K_{p,E}} \mathrm{Im}_1(a_{1,1})\,\mathrm{Im}_1(a_{2,1})\,dT = \cdots, \\ &\int_{K_{p,E}} \mathrm{Re}(a_{1,1})\,\mathrm{Im}_1(a_{2,1})\,dT = \int_{K_{p,E}} \mathrm{Im}_1(a_{1,1})\,\mathrm{Re}(a_{2,1})\,dT, \\ &\int_{K_{p,E}} 2\,\mathrm{Re}(a_{1,1})^2\,dT = \int_{K_{p,E}} 2\,\mathrm{Im}_1(a_{1,1})^2\,dT = \int_{K_{p,E}} \mathrm{Re}\big((1+i)a_{1,1}\big)^2\,dT = \cdots, \end{aligned} \tag{41}$$

and so on.

Finally, to also show that the second integral in (39) is 0, note that

$$\begin{aligned} 0 &= \int_{K_{p,E}} a_{1,1}(T)a_{2,1}(T)\,dT \\ &= \int_{K_{p,E}} a_{1,1}(T)a_{1,2}(T)\,dT \\ &= \int_{K_{p,E}} a_{1,1}(UT)a_{1,2}(UT)\,dT \\ &= \int_{K_{p,E}} \big(\cos(\theta)a_{1,1}(T) + \sin(\theta)a_{2,1}(T)\big)\big(\cos(\theta)a_{1,2}(T) + \sin(\theta)a_{2,2}(T)\big)\,dT \\ &= \int_{K_{p,E}} \cos^2(\theta)a_{1,1}(T)a_{1,2}(T)\,dT + \int_{K_{p,E}} \sin^2(\theta)a_{2,1}(T)a_{2,2}(T)\,dT \\ &\quad + \cos(\theta)\sin(\theta)\left[\int_{K_{p,E}} a_{1,1}(T)a_{2,2}(T)\,dT + \int_{K_{p,E}} a_{2,1}(T)a_{1,2}(T)\,dT\right] \\ &= 2\cos(\theta)\sin(\theta)\int_{K_{p,E}} a_{1,1}(T)a_{2,2}(T)\,dT. \end{aligned}$$

This shows that $\int_{K_{p,E}} a_{1,1}(T)a_{2,2}(T)\,dT = 0$ and completes the proof (again, in the complex and quaternion cases, if we combine it with equalities from (41)). □

The next proposition is about how the integrals appearing in substatement 3 of Lemma 21 relate to each other.

Proposition 22. *Suppose $p \geq 1$ and $E = \mathcal{M}_n(\mathbb{F})$ with $\mathcal{M}_n(\mathbb{F}) = \mathcal{M}_n(\mathbb{R})$ or $\mathcal{M}_n(\mathbb{C})$ or $\mathcal{M}_n(\mathbb{H})$. Then*

$$\int_{K_{p,E}} |a_{1,1}|^2|a_{1,2}|^2\,dT = \int_{K_{p,E}} |a_{1,1}|^2|a_{2,2}|^2\,dT + \frac{2}{\beta}\int_{K_{p,E}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\,dT,$$

where $\beta = 1$ if $\mathbb{F} = \mathbb{R}$, $\beta = 2$ if $\mathbb{F} = \mathbb{C}$, and $\beta = 4$ if $\mathbb{F} = \mathbb{H}$.

Proof. We apply (38) again with A being multiplication from the left by the rotation matrix U in (40)): we obtain

$$\begin{aligned}
&\int_{K_{p,E}} \big|a_{1,1}(T)\big|^2\big|a_{2,2}(T)\big|^2\,dT \\
&\quad= \int_{K_{p,E}} \big|a_{1,1}(UT)\big|^2\big|a_{2,2}(UT)\big|^2\,dT \\
&\quad= \int_{K_{p,E}} \big|\cos(\theta)a_{1,1}(T)+\sin(\theta)a_{2,1}(T)\big|^2\big|-\sin(\theta)a_{1,2}(T)+\cos(\theta)a_{2,2}(T)\big|^2\,dT \\
&\quad= \int_{K_{p,E}} \cos^2(\theta)\sin^2(\theta)\big(\big|a_{1,1}(T)\big|^2\big|a_{1,2}(T)\big|^2+\big|a_{2,1}(T)\big|^2\big|a_{2,2}(T)\big|^2\big)\,dT \\
&\qquad+ \int_{K_{p,E}} \big(\cos^4(\theta)\big|a_{1,1}(T)\big|^2\big|a_{2,2}(T)\big|^2+\sin^4(\theta)\big|a_{2,1}(T)\big|^2\big|a_{1,2}(T)\big|^2\big)\,dT \\
&\qquad+ \int_{K_{p,E}} \cos(\theta)\sin^3(\theta)\big(2\operatorname{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot\big|a_{1,2}(T)\big|^2-2\operatorname{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\cdot\big|a_{2,1}(T)\big|^2\big)\,dT \\
&\qquad+ \int_{K_{p,E}} \cos^3(\theta)\sin(\theta)\big(2\operatorname{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot\big|a_{2,2}(T)\big|^2-2\operatorname{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\cdot\big|a_{1,1}(T)\big|^2\big)\,dT \\
&\qquad- \int_{K_{p,E}} \cos^2(\theta)\sin^2(\theta)\big(2\operatorname{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot 2\operatorname{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\big)\,dT.
\end{aligned}$$

Given that

$$\begin{aligned}
&\int_{K_{p,E}} \operatorname{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot\big|a_{1,2}(T)\big|^2\,dT \\
&\quad= \int_{K_{p,E}} \operatorname{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\cdot\big|a_{1,1}(T)\big|^2\,dT \\
&\quad= \int_{K_{p,E}} \operatorname{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot\big|a_{1,2}(T)\big|^2\,dT \\
&\quad= \int_{K_{p,E}} \operatorname{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\cdot\big|a_{1,1}(T)\big|^2\,dT
\end{aligned}$$

(which follows if we use (38) with A given by suitable permutation matrices), we immediately see that

$$\int_{K_{p,E}} |a_{1,1}|^2|a_{1,2}|^2\,dT = \int_{K_{p,E}} |a_{1,1}|^2|a_{2,2}|^2\,dT + 2\int_{K_{p,E}} \operatorname{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot\operatorname{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\,dT.$$

Now, combining Lemmas 19 and 21, we have that

$$\int_{K_{p,E}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\,dT$$

is a real number, and hence

$$\int_{K_{p,E}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\,dT = \int_{K_{p,E}} \mathrm{Re}(a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}})\,dT = \int_{K_{p,E}} a_{1,1}a_{2,1}a_{2,2}a_{1,2}\,dT$$

if $E = \mathcal{M}_n(\mathbb{R})$, or

$$= \int_{K_{p,E}} \mathrm{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot \mathrm{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\,dT - \int_{K_{p,E}} \mathrm{Im}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot \mathrm{Im}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\,dT$$

if $E = \mathcal{M}_n(\mathbb{C})$, or

$$\begin{aligned} &= \int_{K_{p,E}} \mathrm{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot \mathrm{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\,dT - \int_{K_{p,E}} \mathrm{Im}_1\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot \mathrm{Im}_1\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\,dT \\ &\quad - \int_{K_{p,E}} \mathrm{Im}_2\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot \mathrm{Im}_2\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\,dT \\ &\quad - \int_{K_{p,E}} \mathrm{Im}_3\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot \mathrm{Im}_3\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\,dT \end{aligned}$$

if $E = \mathcal{M}_n(\mathbb{H})$. By applying (38) with suitable permutation matrices again (or linear combinations of such matrices with coefficients from $\{1, i, j, k\} \cap \mathbb{F}$), we conclude that

$$\int_{K_{p,E}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\,dT = \beta \int_{K_{p,E}} \mathrm{Re}\big(a_{1,1}(T)\overline{a_{2,1}(T)}\big)\cdot \mathrm{Re}\big(a_{2,2}(T)\overline{a_{1,2}(T)}\big)\,dT.$$

The conclusion of the proposition follows. □

Corollary 23. *We have that*

$$\left|\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\,dT\right| \lesssim \frac{1}{n}\left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT\right)^2 = \frac{1}{n}\beta^2 L^4_{K_{p,\mathcal{M}_n(\mathbb{F})}}, \tag{42}$$

and

$$\begin{aligned} &\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2|a_{1,2}|^2\,dT,\ \int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2|a_{2,2}|^2\,dT \\ &\quad = \big(1+o(1)\big)\left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT\right)^2 = \big(1+o(1)\big)\beta^2 L^4_{K_{p,\mathcal{M}_n(\mathbb{F})}}, \end{aligned} \tag{43}$$

where $\beta \in \{1, 2, 4\}$ *is as above.*

Proof. In Proposition 11 we saw that

$$\frac{M_p(\|x\|_4^4)}{M_p(1)} = n\frac{M_p(x_1^4)}{M_p(1)} \simeq n\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2 \simeq n\cdot n^{4/p}.$$

By reversing the identities that Fact A gives us, we can write

$$\begin{aligned} \frac{1}{|K_{p,\mathcal{M}_n(\mathbb{F})}|}\int_{K_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT &= \frac{1}{n^2}\frac{1}{|K_{p,\mathcal{M}_n(\mathbb{F})}|}\int_{K_{p,\mathcal{M}_n(\mathbb{F})}} \big\|s(T)\big\|_2^2\,dT \\ &\simeq \frac{d_n^{-2/p}}{n^2}\frac{M_p(\|x\|_2^2)}{M_p(1)} = \frac{d_n^{-2/p}}{n}\frac{M_p(x_1^2)}{M_p(1)} \simeq n^{-1-2/p}, \end{aligned}$$

as well as

$$\frac{1}{|K_{p,\mathcal{M}_n(\mathbb{F})}|}\int_{K_{p,\mathcal{M}_n(\mathbb{F})}} \|s(T)\|_4^4\,dT \simeq d_n^{-4/p}\frac{M_p(\|x\|_4^4)}{M_p(1)} \simeq \frac{n}{n^{4/p}}.$$

This implies that

$$\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} \|s(T)\|_4^4\,dT \simeq n^3\cdot\left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT\right)^2.$$

But by Lemmas 19 and 21, we know that

$$\begin{aligned}\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} \|s(T)\|_4^4\,dT &= n^2\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{i,j}|^4\,dT + 2n^2(n-1)\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2|a_{1,2}|^2\,dT\\ &\quad + n^2(n-1)^2\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\\ &\geq n^2(n-1)^2\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}.\end{aligned}$$

Moreover, since $\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} \|s(T)\|_4^4\,dT > 0$, we also have that

$$\begin{aligned}-n^2(n-1)^2\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}} &< n^2\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^4\,dT + 2n^2(n-1)\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2|a_{1,2}|^2\,dT\\ &\leq Cn^2(2n-1)\cdot\left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT\right)^2,\end{aligned}$$

where the last inequality is a consequence of the Cauchy–Schwarz inequality and of standard properties of convex bodies (see e.g. [9, Theorem 2.4.6]). Inequality (42) follows.

To also establish (23), we recall that

$$\begin{aligned}\sigma^2_{K_{p,\mathcal{M}_n(\mathbb{F})}}\cdot d_n &\simeq \mathrm{Var}_{\overline{K}_{p,E}}\big(\|T\|_{\mathrm{HS}}^2\big)\\ &= n^2\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{i,j}|^4\,dT + 2n^2(n-1)\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2|a_{1,2}|^2\,dT\\ &\quad + n^2(n-1)^2\cdot\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2|a_{2,2}|^2\,dT - n^4\cdot\left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT\right)^2.\end{aligned}$$

Since by [7] we know that $\sigma^2_{K_{p,\mathcal{M}_n(\mathbb{F})}} = O(n)$ for all $p \geq 1$, we can infer that

$$\left|\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2|a_{2,2}|^2\,dT - \left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT\right)^2\right| = O\left(\frac{1}{n}\right)\left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT\right)^2.$$

Furthermore, combining this with Proposition 22 and (42), we get the same conclusion for the difference $\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2|a_{2,2}|^2\,dT - (\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2\,dT)^2$, as claimed. □

We are finally in a position to establish a type of negative correlation property for the original, uniform measures on $K_{p,\mathcal{M}_n(\mathbb{F})}$ as well: this can be done for p for which the estimate in (5) is accurate, or close to it.

Theorem 24. *Let p be such that*

$$\frac{M_{2,\beta,\beta-1,p}(x_1^4)}{M_{2,\beta,\beta-1,p}(1)} < \big(2+o(1)\big)\left(\frac{M_{2,\beta,\beta-1,p}(x_1^2)}{M_{2,\beta,\beta-1,p}(1)}\right)^2, \tag{44}$$

where $\beta = 1$ if $\mathbb{F} = \mathbb{R}$, $\beta = 2$ if $\mathbb{F} = \mathbb{C}$, and $\beta = 4$ if $\mathbb{F} = \mathbb{H}$, and suppose in addition that $K_{p,\mathcal{M}_n(\mathbb{F})}$ satisfies the thin-shell conjecture, or at least that $\sigma^2_{K_{p,\mathcal{M}_n(\mathbb{F})}} = o(n)$. Then for every $i, j, k \in \{1, \dots, n\}$, $j \neq k$, we have

$$\begin{aligned}\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{i,j}|^2 |a_{i,k}|^2 \, dT &= \int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{j,i}|^2 |a_{k,i}|^2 \, dT \\ &< \left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{i,j}|^2 \, dT\right)\left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |a_{i,k}|^2 \, dT\right) \\ &= \beta^2 L^4_{K_{p,\mathcal{M}_n(\mathbb{F})}}.\end{aligned}$$

Proof. Let us write

$$\frac{M_{2,\beta,\beta-1,p}(x_1^4)}{M_p(1)} = c_4 \left(\frac{M_{2,\beta,\beta-1,p}(x_1^2)}{M_p(1)}\right)^2 \tag{45}$$

where, by the previous section and the assumption of the theorem, we know that $\frac{3}{2} + o(1) \leq c_4 < 2 + o(1)$. We start by recalling that

$$\frac{M_p(x_1^4)}{M_p(1)} = \frac{1}{n} \cdot \frac{\Gamma(1 + \frac{d_n+4}{p})}{\Gamma(1 + \frac{d_n}{p})} \frac{1}{|K_{p,\mathcal{M}_n(\mathbb{F})}|} \int_{K_{p,\mathcal{M}_n(\mathbb{F})}} \|s(T)\|_4^4 \, dT,$$

and that

$$\begin{aligned}\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2 &= \frac{1}{n^2} \cdot \left(\frac{\Gamma(1 + \frac{d_n+2}{p})}{\Gamma(1 + \frac{d_n}{p})}\right)^2 \left(\frac{1}{|K_{p,\mathcal{M}_n(\mathbb{F})}|} \int_{K_{p,\mathcal{M}_n(\mathbb{F})}} \|s(T)\|_2^2 \, dT\right)^2 \\ &= n^2 \cdot \left(\frac{\Gamma(1 + \frac{d_n+2}{p})}{\Gamma(1 + \frac{d_n}{p})}\right)^2 \left(\frac{1}{|K_{p,\mathcal{M}_n(\mathbb{F})}|} \int_{K_{p,\mathcal{M}_n(\mathbb{F})}} |a_{1,1}|^2 \, dT\right)^2.\end{aligned}$$

Similarly,

$$\frac{M_p(x_1^2 x_2^2)}{M_p(1)} = \frac{1}{n(n-1)} \cdot \frac{\Gamma(1 + \frac{d_n+4}{p})}{\Gamma(1 + \frac{d_n}{p})} \frac{1}{|K_{p,\mathcal{M}_n(\mathbb{F})}|} \int_{K_{p,\mathcal{M}_n(\mathbb{F})}} \sum_{i \neq j} s_i^2 s_j^2 \, dT. \tag{46}$$

We now combine these identities with Proposition 22, identities (35)–(36), and the assumptions that $c_4 < 2 + o(1)$ and $\sigma^2_{K_{p,\mathcal{M}_n(\mathbb{F})}} = o(n)$: we first see that, because of (45), we must have

$$\begin{aligned}&n \cdot \int_{\overline{K}_p} |a_{i,j}|^4 \, dT + 2n(n-1) \cdot \int_{\overline{K}_p} |a_{1,1}|^2 |a_{1,2}|^2 \, dT + n(n-1)^2 \cdot \int_{\overline{K}_p} a_{1,1}\overline{a_{2,1}} a_{2,2} \overline{a_{1,2}} \\ &= \big(c_4 + O\big(1/n^2\big)\big) n^2 \cdot \left(\int_{\overline{K}_p} |a_{1,1}|^2 \, dT\right)^2.\end{aligned}$$

Given that $\int_{\overline{K}_p} |a_{1,1}|^2 |a_{1,2}|^2 \, dT = (1 + O(1/n))(\int_{\overline{K}_p} |a_{1,1}|^2 \, dT)^2$, it follows that

$$\int_{\overline{K}_p} a_{1,1}\overline{a_{2,1}} a_{2,2} \overline{a_{1,2}} \, dT \leq \left(\frac{c_4 - 2}{n} + O\left(\frac{1}{n^2}\right)\right) \cdot \left(\int_{\overline{K}_p} |a_{1,1}|^2 \, dT\right)^2. \tag{47}$$

But now recall that, because of Proposition 2, the assumption $\sigma^2_{K_{p,\mathcal{M}_n(\mathbb{F})}} = o(n)$ implies that

$$\frac{M_p(x_1^2 x_2^2)}{M_p(1)} \leq \left(1 - \frac{c_4 - 1}{n} + o\left(\frac{1}{n}\right)\right)\left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2$$

(where $o(1/n)$ is at least $O(1/n^2)$ here, but may be larger if $\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}$ does not satisfy the thin-shell conjecture). This, through Proposition 22, and equations (36) and (46), translates into

$$\begin{aligned} n(n-1)&\bigg(\int_{\overline{K}_p}|a_{1,1}|^2|a_{1,2}|^2\,dT-(1+2/\beta)\int_{\overline{K}_p}a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\bigg)\\ &=n(n-1)\bigg(\int_{\overline{K}_p}|a_{1,1}|^2|a_{2,2}|^2\,dT-\int_{\overline{K}_p}a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\bigg)\\ &\le\bigg(1+\frac{1-c_4}{n}+o\Big(\frac{1}{n}\Big)\bigg)n^2\cdot\bigg(\int_{\overline{K}_p}|a_{1,1}|^2\,dT\bigg)^2, \end{aligned}$$

which combined with (47) (and Lemma 21) implies the conclusion of the theorem. □

Here are some concluding remarks concerning this theorem:

- Note that this negative correlation property is again a necessary condition for the thin-shell conjecture to be true for p for which (44) is true. These include all $p\gtrsim\log n$ (in fact, it is not difficult to see that c_4 can be as close to $3/2+o(1)$ in these cases if the implied absolute constant in the latter inequality is sufficiently large). We should clarify however that we cannot expect (44) to be true for all p: for example, for the Euclidean ball ($p=2$) we know that all cross terms are equal, that is

$$\int_{K_{2,\mathcal{M}_n(\mathbb{F})}}|a_{1,1}|^2|a_{1,2}|^2\,dT=\int_{K_{2,\mathcal{M}_n(\mathbb{F})}}|a_{1,1}|^2|a_{2,1}|^2\,dT=\int_{K_{2,\mathcal{M}_n(\mathbb{F})}}|a_{1,1}|^2|a_{2,2}|^2\,dT;$$

  then, by Proposition 22, we see that $\int_{K_{2,\mathcal{M}_n(\mathbb{F})}}a_{1,1}\overline{a_{2,1}}a_{2,2}\overline{a_{1,2}}\,dT=0$, and hence

$$\frac{M_{2,\beta,\beta-1,2}(x_1^4)}{M_{2,\beta,\beta-1,2}(1)}=\big(2+o(1)\big)\bigg(\frac{M_{2,\beta,\beta-1,2}(x_1^2)}{M_{2,\beta,\beta-1,p}(1)}\bigg)^2.$$

  Recall that for the Euclidean ball we know that all cross terms are

$$<\bigg(\int_{K_{2,\mathcal{M}_n(\mathbb{F})}}|a_{1,1}|^2|a_{1,2}|^2\,dT\bigg)^2,$$

  simply because $\sigma^2_{K_{2,\mathcal{M}_n(\mathbb{F})}}=O(1/n^2)$, and not $O(1)$.
  Another case for which (44) is not true is the case of $p=1$: we have that

$$\frac{M_1(x_1^4)}{M_1(1)}\ge\bigg(\frac{17}{8}+o(1)\bigg)\bigg(\frac{M_1(x_1^2)}{M_1(1)}\bigg)^2,$$

  which moreover implies that in this case it is the cross terms $\int_{\overline{K}_1}|a_{i,j}|^2|a_{l,k}|^2\,dT$ with $i\neq l,\ j\neq k$, which are the smallest ones.
- The assumption $\sigma^2_{K_{p,\mathcal{M}_n(\mathbb{F})}}=o(n)$ can be relaxed a little, and replaced by the assumption $\sigma^2_{K_{p,\mathcal{M}_n(\mathbb{F})}}\le c_0n$ (with a constant that may be smaller than the one guaranteed by [7] or [24] however): for example, we can have the same conclusion to the theorem if we take c_0 to be sufficiently small and we also assume

$$\frac{M_{2,\beta,\beta-1,p}(x_1^4)}{M_{2,\beta,\beta-1,p}(1)}\le\bigg(\frac{9}{5}+o(1)\bigg)\bigg(\frac{M_{2,\beta,\beta-1,p}(x_1^2)}{M_{2,\beta,\beta-1,p}(1)}\bigg)^2 \tag{48}$$

  say. Since the latter estimate is satisfied anyway when $p\ge c_1n\log n$, and since we also saw in the previous section that $\sigma^2_{K_{p,\mathcal{M}_n(\mathbb{F})}}\lesssim n$ for such p (and the implied constant can be made as small as we want as long as c_1 is sufficiently large), this gives us the range of p for which we already know that the theorem can be applied, and that the stated negative correlation property holds true anyway.
- As mentioned earlier, this negative correlation property is a necessary condition for the thin-shell conjecture to be true for some of the balls $K_{p,\mathcal{M}_n(\mathbb{F})}$, but does not seem to be a sufficient one too. In fact, our arguments do not appear to

allow us to distinguish between the cases

$$\int_{\overline{K}_p} |a_{1,1}|^2 |a_{2,2}|^2 \, dT < \left(\int_{\overline{K}_p} |a_{1,1}|^2 \, dT \right)^2 \quad \text{or}$$

$$\int_{\overline{K}_p} |a_{1,1}|^2 |a_{2,2}|^2 \, dT = \left(1 + \frac{c}{n^2}\right) \left(\int_{\overline{K}_p} |a_{1,1}|^2 \, dT \right)^2.$$

Nevertheless it still seems like a question of independent interest to study for which other indices p, if any, we have some sort of negative correlation property as above, or even to try to re-establish the property for the known cases in a more direct manner, that is, without having to go through estimates for σ_{K_p} (if the latter turns out to be possible, it would immediately give us one more proof of the estimate $\sigma_{K_p}^2 = O(n)$ which we have from [7] and indirectly from the results of [24]).

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