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Optimal survival strategy for branching Brownian motion in a Poissonian trap field

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Abstract. We study a branching Brownian motion Z with a generic branching law, evolving in $\mathbb{R}^d$, where a field of Poissonian traps is present. Each trap is a ball with constant radius. The traps are hard in the sense that the process is killed instantly once it enters the trap field. We focus on two cases of Poissonian fields, a uniform field and a radially decaying field, and consider an annealed environment. Using classical results on the convergence of the speed of branching Brownian motion, we establish precise annealed results on the population size of Z, given that it avoids the trap field, while staying alive up to time t. The results are stated so that each gives an 'optimal survival strategy' for Z. As corollaries of the results concerning the population size, we prove several other optimal survival strategies concerning the range of Z, and the size and position of clearings in $\mathbb{R}^d$. We also prove a result about the hitting time of a single trap by a branching system (Lemma 1), which may be useful in a completely generic setting too.

Inter alia, we answer some open problems raised in (*Markov Process. Related Fields* **9** (2003) 363–389).

Résumé. Nous étudions un mouvement brownien branchant Z ayant une loi de branchement générique et évoluant dans $\mathbb{R}^d$, où se trouve un champ de pièges poissonniens. Chaque piège est constitué d'une boule de rayon constant. Les pièges sont durs, au sens où le processus est tué instantanément dès qu'il pénètre dans l'un des pièges. Nous nous concentrons sur deux cas particuliers de champs poissonniens, un champ uniforme et un champ décroissant radialement, et nous considérons un environnement *annealed*. En utilisant des résultats classiques sur la convergence de la vitesse du mouvement brownien branchant, nous établissons des résultats *annealed* précis sur la taille de la population décrite par Z, conditionnellement à ce qu'il évite l'ensemble des pièges et reste en vie jusqu'au temps t. Les résultats sont formulés de sorte que chacun d'entre eux donne une 'stratégie optimale de survie' pour Z. En corollaires de ces résultats, nous démontrons l'optimalité de plusieurs autres stratégies concernant le support de Z jusqu'au temps t et la taille et la position de clairières dans $\mathbb{R}^d$. Nous démontrons également un résultat sur le temps d'atteinte d'un seul piège par un système branchant (Lemme 1), qui pourra aussi être utile dans un cadre totalement générique.

Au passage, nous apportons une réponse à plusieurs questions ouvertes formulées dans (*Markov Process. Related Fields* **9** (2003) 363–389).

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1. Introduction

Branching Brownian motion (BBM) in Poissonian trap fields has been studied recently in [6,9,10,16,17]. The most classical problem on this model is the large time asymptotics of the survival probability of the BBM, where one defines survival up to time t to be the event that none of the particles of the BBM has hit the trap field until that time.

Another classical problem is that of the optimal survival strategies: How must have the system behaved (what strategy must it have followed) given that it has avoided the traps up to time t? In this work, we study the optimal survival for a BBM that evolves in $\mathbb{R}^d$, where a Poissonian trap field is present. Our focus is on the population size. Conditioned on survival among traps, we expect the system to suppress branching and produce fewer particles than it otherwise would (had it not been conditioned on survival). Here, we quantify how much the branching would be suppressed. Investigation of this problem leads us to proving an important lemma of independent interest, which provides an upper bound that is valid for large t on the survival probability of a BBM in a large class of random (or deterministic) trap fields, not restricted to Poissonian fields.

Next, we describe the two sources of randomness.

1. Branching Brownian motion: Let $Z = (Z(t))_{t\geq 0}$ be a d-dimensional BBM with initial distribution δ_0, branching rate $\beta > 0$, and offspring distribution $(p_k)_{k\in\mathbb{N}_0}$, where t represents time. The process starts with a single particle at the origin, which performs a Brownian motion in $\mathbb{R}^d$ for a random time which is distributed exponentially with constant parameter β. Then, the particle dies and simultaneously gives birth to a random number of particles distributed according to the offspring distribution $(p_k)_{k\in\mathbb{N}_0}$, where $p_k \geq 0$ for each $k \in \mathbb{N}_0$ and $\sum_{k=0}^{\infty} p_k = 1$. Similarly, each offspring particle repeats the same procedure independently of all others and the parent, starting from the position of her parent. In this way, one obtains a measure-valued Markov process $Z = (Z(t))_{t\geq 0}$, where for each $t \geq 0$, $Z(t)$ can be viewed as a particle configuration on $\mathbb{R}^d$. By assumption, $Z(0) = \delta_0$. Define the process $|Z| = (|Z(t)|)_{t\geq 0}$, where $|Z(t)|$ represents the population size of Z at time t. The number of particles in generation n of $|Z|$ is denoted by $N(n)$ and the process $N = (N(n))_{n\in\mathbb{N}_0}$ is a Galton-Watson process. The initial particle present at $t = 0$ constitutes the 0th generation, the offspring of the initial particle constitute the 1st generation, and so forth. We denote the extinction time of the process $|Z|$ by τ, which is formally defined as $\tau = \inf\{t \geq 0 : |Z(t)| = 0\}$, where we set $\inf\varnothing = \infty$. We then denote the event of extinction of the process $|Z|$ by $\mathcal{E}$, and formally write $\mathcal{E} = \{\tau < \infty\}$. We use the term *non-extinction* for the event $\mathcal{E}^c$. In this work, P and E denote respectively the probability law and corresponding expectation for the BBM. Finally, for $t \geq 0$, let

$$R(t) := \bigcup_{s\in[0,t]} \operatorname{supp}\big(Z(s)\big)$$

be the range of Z up to time t.

2. Trap field: The branching Brownian motion is assumed to live in $\mathbb{R}^d$, to which a 'random trap field' is attached. That is, besides the process Z, on some additional space $(\Omega, \mathbb{P})$ (with expectation $\mathbb{E}$), we also consider a d-dimensional Poisson random measure Π, with mean measure ν. We assume that ν is boundedly finite, that is, $\nu(B)$ is finite for each bounded Borel set B of $\mathbb{R}^d$. By a 'trap' associated to a trap point at $x \in \mathbb{R}^d$, we mean a closed ball of fixed radius $r > 0$ centered at x; by a (random) 'trap field,' we mean the set

$$K := \bigcup_{x_i\in\operatorname{supp}(\Pi)} \bar{B}(x_i, r),$$

where $\bar{B}(x, r)$ denotes the closed ball centered at $x \in \mathbb{R}^d$ with radius r. By a 'clearing,' we mean a region in $\mathbb{R}^d$ that is free of traps, that is disjoint from K.

Definition 1 (Survival). We define $T := \inf\{t \geq 0 : R(t) \cap K \neq \varnothing\}$ to be the 'first trapping time' of the BBM. By 'survival up to time $t > 0$,' we mean the event $\mathsf{S_t} := \{T > t\} \cap \mathcal{E}^c$, which, in case of $p_0 = 0$, reduces of course to $\{T > t\}$. This event is contained in the event that neither the internal branching mechanism nor the trap field has killed the process by t; thus, *survival is a subset of non-extinction*, according to our terminology. Conditioning Z on $\mathsf{S_t}$ can be achieved in two steps: consider first the process Z conditioned on non-extinction for all times, and then condition this new immortal process on trap-avoiding up to t.

Let dx denote the Lebesgue measure. In this paper, we consider two types of Poissonian trap fields: a uniform field where $\mathrm{d}\nu/\mathrm{d}x = v$, $v > 0$, and a trap field where the intensity is radially decaying as

$$\frac{\mathrm{d}\nu}{\mathrm{d}x} \sim \frac{l}{|x|^{d-1}}, \quad |x| \to \infty, l > 0. \tag{1}$$

The reason the decay rate given in (1) is the 'interesting' one is that it is in fact the 'borderline' one. This is explained in Theorem 1.3 in [10], which describes the optimal survival strategy, as it depends on the 'fine tuning constant' ℓ (we use l instead of ℓ in the present paper). Namely, it was shown that

- In the low intensity regime $\ell < \ell_{\mathrm{cr}}$, the system clears a ball of radius $\sqrt{2\beta}t$ from traps, and until time t stays inside this ball and branches at rate β.
- In the high intensity regime $\ell > \ell_{\mathrm{cr}}$:
  - $d = 1$: The system clears an $o(t)$-ball (i.e., a ball with radius greater than r but of lower order than t as $t \to \infty$), and until time t suppresses the branching (i.e., produces a polynomial number of particles) and stays inside this ball.
  - $d \geq 2$: The system clears a ball of radius $\sqrt{2\beta}(1-\eta^*)t$ around a point at distance c^*t from the origin, suppresses the branching until time η^*t, and during the remaining time $(1-\eta^*)t$ branches at rate β.

  (The constant $0 \leq \eta^* \leq 1$ appears as one of the minimizers of a certain relevant variational problem, which is given in (6).)

(See Theorem 1.3 in [10] for the precise statements.[1])

Hence, the decay considered is indeed the 'borderline' one, where the behavior of the system depends only on the constant ℓ, and exhibits a change of behavior at the crossover. If one considers a larger (smaller) decay order, the optimal strategy will simply follow the one exhibited when the decay is as in (1) and $\ell > \ell_{\mathrm{cr}}$ ($\ell < \ell_{\mathrm{cr}}$); although if the decay order is very large, then $\eta^* = 1$ (complete suppression of branching) may occur even for $d \geq 2$, while $0 < \eta^* < 1$ is always the case in the high intensity regime studied in [10].

Let $\mathbf{P}$ denote the annealed law, i.e., $\mathbf{P} := P \otimes \mathbb{P}$. In this paper, the probability measure of interest is $\mathbf{P}(\cdot \mid \mathsf{S_t})$, the annealed probability conditioned on survival up to t. As explained above, the conditioning can be achieved in two steps: first considering an immortal process (Z conditioned on never becoming extinct) and then conditioning on the event that this immortal process avoids traps up to t.

Definition 2 (Optimal survival strategy). By an (annealed) 'optimal survival strategy,' we mean a collection of events $\{A_t\}_{t>0}$ indexed by time, such that

$$\lim_{t\to\infty} \mathbf{P}(A_t \mid \mathsf{S_t}) = 1.$$

We look for optimal survival strategies concerning the population size mainly.

The problem of trap-avoiding asymptotics for BBM among Poissonian traps has been first studied by Engländer in [6], where a uniform field was considered in $d \geq 2$. Then, in search for an extension to the case $d = 1$, Engländer and den Hollander [10] studied the more interesting case where the trap intensity was radially decaying as given in (1). In both [6] and [10], the main result was the exponential asymptotic decay rate of the annealed survival probability as $t \to \infty$, and the branching was taken to be strictly dyadic, i.e., $p_2 = 1$. In addition, in [10], optimal survival strategies of the type we consider here were proved (see Theorem 1.3(i)–(iv) therein). Part of the work in this paper could be regarded as a refinement and generalization of the corresponding work in [10].

In [6], optimal survival strategies were not studied. In Theorem 1, we consider a uniform field in $\mathbb{R}^d$, $d \geq 2$, as in [6], and prove that conditioned on survival up to time t, for any $0 < \widehat{\varepsilon} < 1$, with overwhelming probability, there is only 1 particle present at time $t(1-\widehat{\varepsilon})$ as $t \to \infty$, which means complete suppression of branching occurs with overwhelming probability. In Theorem 1.3(iii) in [10], where the setting was that of the radially decaying trap intensity in (1), it was shown that conditioned on survival up to time t, the population size at time $(\eta^* - \widehat{\varepsilon})t$ is at most $\lfloor t^d \rfloor$ for large t with overwhelming probability. (The constant η^* appears as one of the minimizers in (6).) Here, we improve this bound to just a *single particle* in Theorem 2.

When the BBM is supercritical and $p_0 > 0$, we have to take into account that extinction for the underlying Galton-Watson process has positive probability, and hence condition the process on non-extinction. In this case, the particles are grouped into those with infinite or finite line of descent, so-called 'skeleton' and 'doomed' particles, respectively, and in this way a 'skeleton decomposition' is performed to analyze the problem. In [16] and [15], the work in [6]

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[1] It does not include another, intuitively plausible ingredient though, namely, that in the last case a particle is moved to distance c^*t (into the clearing) from the origin in time η^*t.

was extended to a BBM with a general offspring distribution, where the possibility of $p_0 > 0$ was allowed. Likewise in [17], a general offspring distribution is considered for the BBM, and the work in [10] on the radially decaying trap field is extended to cover the case $p_0 > 0$. Here we allow for $p_0 > 0$, and extend Theorem 1 and Theorem 2 to Theorem 3 and Theorem 4, respectively, to obtain optimal survival strategies on the population size of both the skeleton and doomed particles.

In the final section, we use our optimal survival results on the population size to prove several others of different types in the same spirit as in [10, Thm. 1.3], concerning the range of the BBM, and the size and position of trap-free regions (i.e., clearings) in $\mathbb{R}^d$.

We refer the reader to [8] for a survey on the topic of BBM among Poissonian traps, and to [9,12] for various related problems. Analogous questions in the discrete setting could also be asked, where the continuum $\mathbb{R}^d$ is replaced by the integer lattice $\mathbb{Z}^d$, and the BBM is replaced by the branching random walk. In [3], a random walk among a randomly moving field of traps on $\mathbb{Z}^d$ was studied, and it was shown that conditioned on survival up to time t, the random walk is sub-diffusive. In the discrete setting, we note that the survival asymptotics of the random walk was studied earlier in [5] for both the annealed and quenched cases.

We conclude this section with an often used terminology and the outline of the paper.

Definition 3 (SES). A generic function $g : \mathbb{R}_+ \to \mathbb{R}$ is called *super-exponentially small* (SES) if $\lim_{t\to\infty} \log g(t)/t = -\infty$.

Outline: The rest of the paper is organized as follows. In Section 2, we state the main results. Section 3 is devoted to the central lemma of this work, on which the main results are built. This lemma is a general result that applies to a much broader class of trap fields on $\mathbb{R}^d$ than the ones considered in this work. In Section 4, we give the proofs of the main results. In Section 5, we state and prove a lemma of independent interest about the decomposition of a supercritical continuous-time branching process, which is used in the subsequent section to generalize the main results to the case $p_0 > 0$. In Section 6, we extend the main results to the case $p_0 > 0$. The first six sections study the optimal survival strategies on the population size of the branching system. Finally, in Section 7, we provide optimal survival results on the range of the branching system, and the size and position of the clearings in $\mathbb{R}^d$, as corollaries of the results on population size.

2. Results

Our main results will be stated so that each gives an optimal survival strategy. Let us now introduce further notation in order to state the results. Let f be the probability generating function (p.g.f.) of the offspring distribution and μ be the mean number of offspring:

$$f(s) := \sum_{j=0}^{\infty} p_j s^j; \qquad \mu := \sum_{j=0}^{\infty} j p_j,$$

and define

$$m := \mu - 1.$$

Note the significance of m: it is the net average growth per particle since by assumption a particle dies at the moment it gives birth to offspring.

Throughout this work, we assume that $\mu < \infty$, and without loss of generality that $p_1 = 0$ (as nonzero p_1 can be absorbed into the branching rate β). Also, from the elementary theory of branching processes (see for example [1,2]), recall the fact that $P(\mathcal{E}) = 1 \Leftrightarrow \mu \leq 1$. Processes for which $\mu > 1$ are called supercritical. It is clear that if $p_0 = 0$, then $P(\mathcal{E}) = 0$.

The following two theorems constitute the main results of this paper. They both give the population size of the branching system given survival among traps, and hence quantify how much branching is suppressed under survival. The setting in the first theorem is a uniform trap field, whereas the second one is concerned with a radially decaying field. Lemma 1 in Section 3 is central to the proof of both theorems. One should keep in mind that without the conditioning on survival, the expected population size of a ‘free’ BBM at time t is $\exp[\beta mt]$ for $t \geq 0$. Moreover, by a well known limit theorem, $\lim_{t\to\infty} |Z(t)|e^{-\beta mt}$ exists almost surely (see for example [2, Thm. III.7.1]).

As before, $\mathrm{d}x$ denotes the Lebesgue measure.

Theorem 1 (Survival in a uniform field; $d \geq 2$). *Let* $p_0 = 0$. *Suppose that* $\mathrm{d}\nu/\mathrm{d}x = v$, $v > 0$. *Then for* $d \geq 2$ *and* $0 < \widehat{\varepsilon} < 1$,

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z\big((1-\widehat{\varepsilon})t\big)\big| = 1 \mid \mathsf{S_t}\big) = 1.$$

Remark. Theorem 1 says that for large t, conditioned on survival up to time t, with overwhelming probability, the population size at the earlier time $(1-\widehat{\varepsilon})t$ is 1. In other words, with overwhelming probability the population doesn't grow at all up to time $(1-\widehat{\varepsilon})t$; branching is completely suppressed. We stress that this is not an almost sure pathwise statement, so there could be realizations where the population grows.

The following theorem has the setting of a trap field where the intensity is radially decaying as in (1):

$$\frac{\mathrm{d}\nu}{\mathrm{d}x} \sim \frac{l}{|x|^{d-1}}, \quad |x| \to \infty, l > 0.$$

In this case, as emphasized in the introduction, there is a critical intensity l_{cr} at which the switching of regime occurs. The survival strategy of the system depends on whether l is above or below this critical intensity, and in particular for $l > l_{\mathrm{cr}}$, the system suppresses the branching until time $\eta^* t$, where η^* is one of the minimizers in (6). For a definition of and a formula for l_{cr}, and for details on η^*, we refer the reader to [17, Thm. 2] and its proof. Here, we only note that $0 < \eta^* < 1$ when $d \geq 2$, and $\eta^* = 1$ when $d = 1$. Also, it is clear that when $d = 1$, the trap intensity in (1) gives a uniform field as a special case, hence covering the missing case of $d = 1$ in Theorem 1.

The next result answers some of the open problems raised in Section 1.3 in [10].

Theorem 2 (Survival in a radially decaying field; $d \geq 1$). *Let* $p_0 = 0$. *Suppose that* $\mathrm{d}\nu/\mathrm{d}x$ *exists*, *is continuous on* $\mathbb{R}^d$, *and satisfies* (1). *Let* l_{cr} *be the constant in the critical trap intensity. Then*, *in the high-intensity regime* $l > l_{\mathrm{cr}}$, *for* $d \geq 1$ *and* $0 < \widehat{\varepsilon} < \eta^*$,

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z\big((\eta^*-\widehat{\varepsilon})t\big)\big| = 1 \mid \mathsf{S_t}\big) = 1.$$

Remark. (i) It is easy to see, that on the other hand,

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z\big((\eta^*+\widehat{\varepsilon})t\big)\big| = 1 \mid \mathsf{S_t}\big) = 0.$$

(Here we assume that $d \geq 2$, as $\eta^* = 1$ for $d = 1$, and then the claim follows from the time-homogeneous branching Markov property.)

Intuitively, suppressing the branching up to $\eta^* t$ already gives the probabilistic cost that (together with the other costs) results in the known total cost. If the branching were also suppressed for an additional $\widehat{\varepsilon}t$ time, then the total cost would be higher than what is derived. The computation is that, as $t \to \infty$,

$$\begin{aligned}\mathbf{P}\big(\big|Z\big((\eta^*+\widehat{\varepsilon})t\big)\big| = 1 \mid \mathsf{S_t}\big) &= \frac{\mathbf{P}(\{|Z((\eta^*+\widehat{\varepsilon})t)| = 1\} \cap \mathsf{S_t})}{\mathbf{P}(\mathsf{S_t})} \\ &= \mathbf{P}\big(\big\{\big|Z\big((\eta^*+\widehat{\varepsilon})t\big)\big| = 1\big\} \cap \mathsf{S_t}\big)e^{I(l,f,\beta,d)t(1+o(1))},\end{aligned}$$

where $I(l, f, \beta, d)$ is the rate function in (5), given by (6), and so it is enough to check that

$$t\bigg(-I(l, f, \beta, d) - \frac{1}{t}\log \mathbf{P}\big(\big\{\big|Z\big((\eta^*+\widehat{\varepsilon})t\big)\big| = 1\big\} \cap \mathsf{S_t}\big)\bigg) \to \infty.$$

And indeed,

$$\liminf_{t\to\infty}(-1/t)\log \mathbf{P}\big(\big\{\big|Z\big(\big(\eta^*+\widehat{\varepsilon}\big)t\big)\big|=1\big\}\cap \mathsf{S_t}\big)>I(l,f,\beta,d). \tag{2}$$

To see why (2) is true, the reader should skip forward to the proof of Theorem 2, and consider the estimate (35) there. For $\mathbf{P}(\{|Z((\eta^*+\widehat{\varepsilon})t)|=1\}\cap \mathsf{S_t})$, we get a very similar estimate, but now the summation in (35) starts at $i=\lfloor(\eta^*+\widehat{\varepsilon})n\rfloor$, which cannot be optimal for survival, since the unique minimizer for η in (37) is η^*. This verifies (2).

(ii) Optimal survival strategies about the population size arise when the branching is suppressed for at least part of the time interval in question in order to realize the event of survival. Therefore, the strategy in the theorem above applies only when $l>l_{\mathrm{cr}}$, where the branching is suppressed in the time interval $[0,\eta^*t]$. When $l<l_{\mathrm{cr}}$, the system undergoes ‘free’ branching.

For the two types of Poissonian fields that we consider here, the following asymptotics for the annealed trap-avoiding probabilities have been derived in [15] and [17], respectively. Define $\alpha:=1-f'(q)$, where $q:=P(\mathcal{E})$. When $p_0=0$, it is clear that $P(\mathcal{E})=0$ so that the conditioning on $\mathcal{E}^c$ is redundant, and since $p_1=0$ by assumption, $\alpha=1$.

Theorem A (Survival asymptotics in a uniform field; $d\geq 2$; [15]). *Let $\mu>1$. Suppose that $\mathrm{d}\nu/\mathrm{d}x=v$, $v>0$. Then, for $d\geq 2$,*

$$\lim_{t\to\infty}\frac{1}{t}\log \mathbf{P}\big(T>t\mid \mathcal{E}^c\big)=-\beta\alpha. \tag{3}$$

Theorem B (Survival asymptotics in a radially decaying field; $d\geq 1$; [17]). *Let $\mu>1$. Suppose that $\mathrm{d}\nu/\mathrm{d}x$ exists, is continuous on $\mathbb{R}^d$, and satisfies (1). For $r,b\geq 0$, define*

$$g_d(r,b)=\int_{B(0,r)}\frac{\mathrm{d}x}{|x+b\mathbf{e}|^{d-1}}, \tag{4}$$

where $\mathbf{e}=(1,0,\dots,0)$ is the unit vector in the direction of the first coordinate. Then, for $d\geq 1$,

$$\lim_{t\to\infty}\frac{1}{t}\log \mathbf{P}\big(T>t\mid \mathcal{E}^c\big)=-I(l,f,\beta,d), \tag{5}$$

where

$$I(l,f,\beta,d)=\min_{\eta\in[0,1],c\in[0,\sqrt{2\beta}]}\bigg\{\beta\alpha\eta+\frac{c^2}{2\eta}+lg_d\big(\sqrt{2\beta m}(1-\eta),c\big)\bigg\}. \tag{6}$$

(For $\eta=0$, $c=0$, set $c^2/2\eta=0$, and for $\eta=0$, $c>0$, set $c^2/2\eta=\infty$.)

3. A trap in a subcritical ball

In this section, we state and prove the central lemma of this work, on which the main results are built. The following lemma is of independent interest, because it applies to a much more general class of trap fields (random or deterministic) on $\mathbb{R}^d$ as opposed to only Poissonian fields.

Lemma 1 (Survival among traps in a subcritical ball). *Let $p_0=0=p_1$. Let $0<\varepsilon<1$ and define $\rho_t=\sqrt{2\beta m}(1-\varepsilon)t$. Suppose that $\operatorname{supp}(\Pi)\cap\bar{B}(0,\rho_t)\neq\varnothing$. Then, the probability that the BBM avoids the trap field up to time t satisfies the following asymptotical bound:*

$$\limsup_{t\to\infty}\frac{1}{t}\log P(\mathsf{S_t})\leq -\beta\varepsilon\big(\sqrt{m^2+m}-m\big).$$

Remark. We call $B(0,\rho_t)$, where $\rho_t=\rho_t(\varepsilon)$, a *subcritical ball* since the 'speed' of a BBM is equal to $\sqrt{2\beta m}$, and for any $0<\varepsilon<1$, a BBM that starts with a single particle at the origin will escape this ball with a probability tending to one as $t\to\infty$ (see [13,14]).

Proof. The strategy is to divide the time interval $[0,t]$ into two pieces: $[0,\delta t]$ and $[\delta t,t]$, and then to condition on the number of particles and the radius of the range at time δt. Here, $0<\delta<1$ is a number, which will later depend on ε.

Let A_t be the event that Z avoids the trap field in the time interval $[0,t]$, and let $p(t):=P(A_t)$. For an upper bound on $p(t)$, we may suppose[2] that $\bar{B}(0,\rho_t)$ contains precisely 1 point from $\operatorname{supp}(\Pi)$, which is on the boundary of $\bar{B}(0,\rho_t)$.

For $0<\delta'<\delta$ and $\delta''>0$, let B_t be the event that at least $\lfloor e^{\beta\delta' t}\rfloor$ particles are produced in the time interval $[0,\delta t]$ and C_t be the event that the BBM remains inside $B(0,(\delta+\delta'')t\sqrt{2\beta m})$ throughout $[0,\delta t]$. Use the estimate

$$P(A)\le P(A\mid B\cap C)+P\big(B^c\big)+P\big(C^c\big)$$

to obtain

$$p(t)\le P(A_t\mid B_t\cap C_t)+P\big(B_t^c\big)+P\big(C_t^c\big). \tag{7}$$

Let $p_3(t):=P(B_t^c)$ and $N(t):=|Z(t)|$. From [11], for strictly dyadic branching (denote this process by $\tilde{N}$), we have

$$P\big(\tilde{N}(t)>k\big)=\big(1-e^{-\beta t}\big)^k\quad\text{for } k=0,1,2,\dots. \tag{8}$$

Then, for nonnegative k, by convexity of the function g given by $x\mapsto x^k$ on $[0,\infty)$, for any t,

$$P\big(\tilde{N}(\delta t)\le k\big)=1-\big(1-e^{-\beta\delta t}\big)^k\le ke^{-\beta\delta t}\quad\text{for } k=0,1,2,\dots. \tag{9}$$

Indeed, taking $x_1=1$ and $x_2=1-e^{-\beta\delta t}$, by convexity of g, we obtain $(g(x_1)-g(x_2))/(x_1-x_2)\le g'(x_1)=k$, from which (9) follows. Setting $k=\lfloor e^{\delta'\beta t}\rfloor$, and comparing a BBM having $p_0=p_1=0$ (which holds by hypothesis) with a strictly dyadic BBM, we have for all $t>0$,

$$p_3(t)\le\exp\big[-\big(\delta-\delta'\big)\beta t+o(t)\big]. \tag{10}$$

Let $p_4(t):=P(C_t^c)$. Define $M(t):=\inf\{r\ge 0: R(t)\subseteq B(0,r)\}$ to be the radius of the minimal ball containing the range of the BBM up to time t. Observe that

$$p_4(t)=P\left(M(\delta t)>\sqrt{2\beta m}\left(1+\frac{\delta''}{\delta}\right)\delta t\right). \tag{11}$$

We now find an upper bound for $p_4(t)$. Let N_t denote the set of particles that are alive at t and for $1\le u\le|N_t|$, $X_u(t)$ denote the position of particle u at time t. Then, using the union bound, for $\gamma>0$,

$$\begin{aligned} P\big(M(t)>\gamma t\big)&=P\Big(\exists u\in N_t:\sup_{0\le s\le t}\big|X_u(s)\big|>\gamma t\Big)\\ &\le E\big[N(t)\big]\mathbf{P}_0\Big(\sup_{0\le s\le t}\big|B(s)\big|>\gamma t\Big), \end{aligned} \tag{12}$$

where $B=(B(t))_{t\ge 0}$ represents standard Brownian motion starting at the origin, with probability $\mathbf{P}_0$. It is a standard result that $E[N(t)]=\exp(\beta m t)$ (see for example [11, Section 8.11]). Moreover, we know from [17, Lemma 5] that

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[2] By Brownian scaling, changing the distance of the trap is equivalent to speeding up or slowing down time.

$\mathbf{P}_0(\sup_{0\le s\le t}|B(s)|>\gamma t)=\exp[-\gamma^2 t/2+o(t)]$. Then, by choosing $\gamma=\sqrt{2\beta m}(1+\frac{\delta''}{\delta})$ and replacing t by δt in (12), it follows from (11) and (12) that

$$p_4(t)\le \exp\left[-\beta m\delta t\left(\frac{\delta''^2}{\delta^2}+2\frac{\delta''}{\delta}\right)+o(t)\right]. \tag{13}$$

Now let $p_2(t):=P(A_t\mid B_t\cap C_t)$. Note that conditioned on the event $B_t\cap C_t$, there are at least $\lfloor e^{\beta\delta' t}\rfloor$ particles within the ball $B(0,(\delta+\delta'')t\sqrt{2\beta m})$ at time δt, each of which is *at most* at a distance

$$\sqrt{2\beta m}t\left(1-\varepsilon+\delta+\delta''\right)=:r(t)$$

away from the trap point. (Recall that $\rho_t=\sqrt{2\beta m}(1-\varepsilon)t$.) Focus on one such particle. The probability that the sub-BBM emerging from this particle avoids the trap in the remaining time $(1-\delta)t$ is at most the sum of the probability that it remains in its $r(t)$-ball (call this $p_5(t)$) and the probability that it avoids the trap given that it escapes its $r(t)$-ball (call this $p_6(t)$). By the $r(t)$-ball, we mean the ball with radius $r(t)$ that is centered at the position of the particle at time δt. Hence, by the Markov property and independence of particles, we have

$$p_2(t)\le\left[p_5(t)+p_6(t)\right]^{\lfloor e^{\delta'\beta t}\rfloor}. \tag{14}$$

Consider $p_5(t)$. Choose $\delta>0$ and $\delta''>0$ so that $1-\varepsilon+\delta+\delta''<1-\delta$, which is equivalent to

$$2\delta+\delta''<\varepsilon. \tag{15}$$

Then, [7, Prop. 5] implies[3] that there exists a constant $c=c(\varepsilon,\beta,m)>0$ such that

$$p_5(t)\le e^{-ct}\quad\text{for all large } t. \tag{16}$$

Now consider $p_6(t)$. For a lower bound on the probability that the sub-BBM hits the trap within the remaining time $(1-\delta)t$ conditional on exiting the $r(t)$-ball, we consider a single Brownian particle and look for the probability that this particle hits the trap, which is contained in the $r(t)$-ball, conditional on exiting the $r(t)$-ball. Since the particle is conditioned to exit the $r(t)$-ball by time $(1-\delta)t$, by that time it must also exit the $\widehat{r}(t)$-ball, where $\widehat{r}(t)$ is the distance between the starting point of the sub-BBM and the center of the trap. Therefore, since the Brownian exit distribution out of a ball centered at the starting point has rotational invariance (even under the conditioning), by comparing the surface area of the $\widehat{r}(t)$-ball that intersects the trap to the total surface area of the $\widehat{r}(t)$-ball, and since $\widehat{r}(t)\le r(t)$ for each $t>0$, we obtain

$$p_6(t)\le 1-\frac{\gamma_{r,d}}{r(t)^{d-1}}\quad\text{for all } t>0, \tag{17}$$

where $\gamma_{r,d}$ is a constant that depends on the dimension d and the trap radius r.

From (14)–(17), it is clear that there exists a constant $c=c(\varepsilon,r,d,\beta,m)>0$ such that for all large t, we have

$$p_2(t)\le\left(1-\frac{\gamma_{r,d}/2}{r(t)^{d-1}}\right)^{\lfloor e^{\delta'\beta t}\rfloor}\le\left[\exp(-ct)\right]^{\lfloor e^{\delta'\beta t}\rfloor/t^d}, \tag{18}$$

which is super-exponentially small (SES) in t. Note that we have used the estimate $1+x\le e^x$ in the second inequality above, and that the factor $1/2$ in the numerator in the second expression makes up for $p_5(t)$.

Now, putting everything together, from (7), (10), (13) and (18), we have

$$p(t)\le\exp\left[-(\delta-\delta')\beta t+o(t)\right]+\exp\left[-\beta m\delta t\left(\frac{\delta''^2}{\delta^2}+2\frac{\delta''}{\delta}\right)+o(t)\right]+g(t),$$

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[3] In [7] the branching was strictly dyadic but the proof can be adapted easily to our more general setting.

subject to the constraint $2\delta+\delta''<\varepsilon$ for some function g that is SES in t, i.e., $(1/t)\log g(t)\to-\infty$ as $t\to\infty$. In the equation above, $g(t)$ comes from (18). First, let $\delta'\to 0$ to obtain

$$\limsup_{t\to\infty}\frac{1}{t}\log p(t)\le-\beta\delta\min\left\{1,m\left(\frac{\delta''^2}{\delta^2}+2\frac{\delta''}{\delta}\right)\right\}. \tag{19}$$

Next, find the sharpest bound on $p(t)$ by optimizing over the parameters δ and δ'', respecting the condition (15).

It is clear from (19) that we need to maximize

$$f\left(\delta,\delta''\right):=\min\left\{\delta,m\delta\left(\frac{\delta''^2}{\delta^2}+2\frac{\delta''}{\delta}\right)\right\}\quad\text{subject to } 2\delta+\delta''<\varepsilon.$$

Let $\delta''=\delta/k$, $k>0$ so that $f(\delta,\delta'')=f(\delta,k)=\min\{\delta,m\delta(\frac{1}{k^2}+\frac{2}{k})\}$, and the constraint becomes $\delta<\varepsilon k/(2k+1)$. In order to maximize f, we solve

$$1=m\left(\frac{1}{k^2}+\frac{2}{k}\right)$$

for positive k. This gives $k=m+\sqrt{m^2+m}$ as the optimal value for k, and the constraint becomes $\delta<\varepsilon(\sqrt{m^2+m}-m)$. By letting $\delta\to\varepsilon(\sqrt{m^2+m}-m)$, it follows from (19) that

$$\limsup_{t\to\infty}\frac{1}{t}\log p(t)\le-\beta\varepsilon\left(\sqrt{m^2+m}-m\right).$$

Indeed, by choosing k differently, one obtains a weaker bound for $p(t)$. If $k>m+\sqrt{m^2+m}$, then $1>m(\frac{1}{k^2}+\frac{2}{k})$ so that $f(\delta,k)=m\delta(\frac{1}{k^2}+\frac{2}{k})$. In view of $\delta<\varepsilon k/(2k+1)$, we then have $p(t)\le\exp[-\beta t\varepsilon\frac{m}{k}+o(t)]$, where $m/k<m/(m+\sqrt{m^2+m})$. Similarly, if $k<m+\sqrt{m^2+m}$, then $1<m(\frac{1}{k^2}+\frac{2}{k})$ so that $f(\delta,k)=\delta$, and we have $p(t)\le\exp[-\beta t\varepsilon\frac{k}{2k+1}+o(t)]$, where $k/(2k+1)<m/(m+\sqrt{m^2+m})$. □

Remark. Intuitively, what we are using in the proof of Lemma 1 is that there are exponentially many particles at the frontier of a BBM instead of just one particle. In our proof, this appears as the factor $\lfloor e^{\beta\delta' t}\rfloor$ in (18). Even though the BBM on average has $e^{\beta mt}$ particles at time t, and the ones on the frontier (meaning the ones that have escaped out of $B(0,\sqrt{2\beta m}t(1-\varepsilon))$) are not "too many," they are not "too few" either, there are still exponentially many ($\lfloor e^{\beta\delta' t}\rfloor$) on the frontier.

Remark. Lemma 1 enables us to easily conclude the following: Let $0<\varepsilon<1$ and $\rho_t:=\sqrt{2\beta m}(1-\varepsilon)t$. Let Π denote any Poisson random measure on $\mathbb{R}^d$ with mean measure ν such that the probability that $B(0,\rho_t)$ is trap-free is exponentially small in t. (For example, any ν that yields $\mathbb{P}(B(0,r)$ is trap-free$)\le e^{-\mathrm{cr}}$ for all $r>0$ for some $c>0$.) Then, for all large t, the annealed probability that the system avoids the trap field up to time t is at least exponentially small in t, that is,

$$\mathbf{P}(\mathsf{S}_t)\le e^{-kt}$$

for some constant $k>0$ that possibly depends on ε, β, m and ν. Indeed, one easily obtains this result by conditioning on the event that $B(0,\rho_t)$ is trap-free, and applying Lemma 1 on its complement.

4. Proof of main theorems

We now give the proof of the main theorems: Theorem 1 and Theorem 2. The central ingredient in both proofs is Lemma 1. We give a bootstrap argument in each proof. Namely, in the proof of Theorem 1, we first show that for a given $\widehat{\varepsilon}>0$, with overwhelming probability under S_t, there is at most $k(\widehat{\varepsilon})$ particles present at time $(1-\widehat{\varepsilon})t$, where k doesn't depend on time. Then, using this, we show that there is actually just one particle present. In the proof of Theorem 2, the same strategy is followed with the only difference being the replacement of $k(\widehat{\varepsilon})$ by $\lfloor t\rfloor$.

Next, we summarize the argument we will use, before turning to the proof of Theorem 1.

A general 'bootstrap argument'

Let $\{A_t\}_{t>0}$ and $\{B_t\}_{t>0}$ be two families of events. Furthermore, let $E_t \subset A_t$, $t > 0$. We are going to apply the general argument that, since

$$\frac{P(B_t \mid \mathsf{S}_\mathsf{t})}{P(A_t \mid \mathsf{S}_\mathsf{t})} = \frac{P(B_t)}{P(A_t)} \frac{P(\mathsf{S}_\mathsf{t} \mid B_t)}{P(\mathsf{S}_\mathsf{t} \mid A_t)} \leq \frac{P(B_t)}{P(E_t)} \frac{P(\mathsf{S}_\mathsf{t} \mid B_t)}{P(\mathsf{S}_\mathsf{t} \mid E_t)},$$

it follows that if

$$\lim_{t\to\infty} \frac{P(\mathsf{S}_\mathsf{t} \mid B_t)}{P(\mathsf{S}_\mathsf{t} \mid E_t)} = 0,$$

$\frac{P(B_t)}{P(E_t)}$ remains bounded from above and $\lim_{t\to\infty} P(A_t \mid \mathsf{S}_\mathsf{t}) = 1$, then $\lim_{t\to\infty} P(B_t \mid \mathsf{S}_\mathsf{t}) = 0$. In other words, in this situation, if we know that $\{A_t\}_{t>0}$ is an optimal strategy for S_t, then so is $\{A_t \cap B_t^c\}_{t>0}$.

This enables a 'bootstrap' argument, namely, one first checks that $\{A_t\}_{t>0}$ is an optimal strategy, and then strengthens the argument by replacing $\{A_t\}_{t>0}$ with $\{A_t \cap B_t^c\}_{t>0}$.

Proof of Theorem 1. Fix $0 < \widehat{\varepsilon} < 1$. Let $K_t := \{|Z((1-\widehat{\varepsilon})t)| \leq \lfloor(\sqrt{2}+1)/\widehat{\varepsilon}\rfloor\}$. We first show that

$$\lim_{t\to\infty} \mathbf{P}(K_t \mid \mathsf{S}_\mathsf{t}) = 1. \tag{20}$$

By (3), noting that $\alpha = 1$ when $p_0 = 0$, it is enough to verify that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{P}\big(K_t^c \cap \mathsf{S}_\mathsf{t}\big) < -\beta. \tag{21}$$

Split the time interval $[0, t]$ into two pieces: $[0, (1-\widehat{\varepsilon})t]$ and $[(1-\widehat{\varepsilon})t, t]$. For $0 < \varepsilon < 1$, let $\widehat{\rho}_t = (1-\varepsilon)\sqrt{2\beta m \widehat{\varepsilon}}t$, which is $(1-\varepsilon)$ times the radius of the 'typical ball' for the remaining time $\widehat{\varepsilon}t$. Define A_t to be the event that among the particles alive at time $(1-\widehat{\varepsilon})t$, there is at least one such that the ball with radius $\widehat{\rho}_t$ centered around it is trap-free. Estimate

$$\mathbf{P}\big(K_t^c \cap \mathsf{S}_\mathsf{t}\big) \leq \mathbf{P}(A_t) + \mathbf{P}\big(K_t^c \cap \mathsf{S}_\mathsf{t} \mid A_t^c\big). \tag{22}$$

By the definition of the Poisson random measure, and using the union bound, having a uniform intensity yields that for $t > 0$, $\mathbf{P}(A_t) \leq u(t) \exp[-ct^d]$, where

$$u(t) := E\big|Z\big((1-\widehat{\varepsilon})t\big)\big| = \exp\big[\beta m(1-\widehat{\varepsilon})t\big],$$

and $c = c(\varepsilon, \widehat{\varepsilon}, d, \beta, m) > 0$ is some constant. Since $d \geq 2$ by assumption, it follows that the first term on the right-hand side of (22) is SES in t. By Lemma 1, the second term on the right-hand side of (22) is at most

$$\exp\big[-\big(\lfloor(\sqrt{2}+1)/\widehat{\varepsilon}\rfloor + 1\big)\beta\varepsilon\big(\sqrt{m^2+m} - m\big)\widehat{\varepsilon}t + o(t)\big].$$

Since $m \geq 1$, observe that $\sqrt{2} - 1 \leq \sqrt{m^2+m} - m$. Finally, use (22) and let $\varepsilon \to 1$ to obtain (21), which completes the proof of (20). (Note that the $\widehat{\varepsilon}$ appearing in the statement of the theorem is different from the ε appearing in the definition of $\widehat{\rho}_t$.)

Next, using (20), we reduce the bound on the number of particles to 1. Namely, we show that

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z\big((1-\widehat{\varepsilon})t\big)\big| = 1 \mid \mathsf{S}_\mathsf{t}\big) = 1.$$

This is done by using the 'bootstrap argument' (explained at the beginning of this section) with the following casting. Define

$$F_{t,i} := \big\{\big|Z\big((1-\widehat{\varepsilon})t\big)\big| = i\big\}$$

for $1 \le i \le \lfloor(\sqrt{2}+1)/\widehat{\varepsilon}\rfloor$. Now let $A_t := \{|Z((1-\widehat{\varepsilon})t)| \le \lfloor(\sqrt{2}+1)/\widehat{\varepsilon}\rfloor\}$, $B_t = B_{t,k} := F_{t,k}$ with some $2 \le k \le \lfloor(\sqrt{2}+1)/\widehat{\varepsilon}\rfloor$ fixed, and $E_t := F_{t,1}$. We first note that

$$0 < \liminf_{t\to\infty} \frac{P(B_t)}{P(E_t)} \le \limsup_{t\to\infty} \frac{P(B_t)}{P(E_t)} \le k \le \lfloor(\sqrt{2}+1)/\widehat{\varepsilon}\rfloor, \quad k \ge 2, \tag{23}$$

and so, in particular, $\frac{P(B_t)}{P(E_t)}$ remains bounded from above, as required. Now (23) is clearly true for a Yule process (corresponding to $p_2 = 1$), as we have the explicit formula for the distribution of $|Z(t)|$ as $P(|Z(t)| = k) = e^{-\beta t}(1-e^{-\beta t})^{k-1}$ for $t > 0$ and $k \ge 1$. For a general supercritical process with $p_0 = p_1 = 0$, (23) follows by (9) and comparison with a Yule process.

Next, we have to show that (suppressing the time dependence in the notation for F)

$$\lim_{t\to\infty} \frac{\mathbf{P}(\mathsf{S_t} \mid F_2)}{\mathbf{P}(\mathsf{S_t} \mid F_1)} = 0, \tag{24}$$

as required in the bootstrap argument. For this we need to define the 'branching Brownian sausage.' For $0 \le t_1 \le t_2$, let

$$R(t_1, t_2) := \bigcup_{s=t_1}^{t_2} \operatorname{supp}\bigl(Z(s)\bigr),$$

that is $R(t_1, t_2)$ is the (accumulated) support of Z on $[t_1, t_2]$. For $t \ge 0$ and $r > 0$, let us call

$$Z^r_{[t_1,t_2]} := \bigcup_{x \in R(t_1,t_2)} \bar{B}(x, r) \tag{25}$$

the *branching Brownian sausage* corresponding to the branching Brownian motion Z. This notion is useful since Fubini's theorem implies that

$$\mathbf{P}(\mathsf{S_t}) = E\bigl[e^{-\nu \mathsf{vol}(Z^r_{[0,t]})}\bigr], \tag{26}$$

where $\nu > 0$ is the uniform intensity of the Poisson random measure Π, and vol denotes volume (Lebesgue measure).

Let F be an event concerning the BBM only (i.e. $F \in \sigma(Z(s); 0 \le s \le t)$). Using that $\mathbf{P}(F) = P(F)$, it follows easily that (26) remains valid if we condition both sides on F. Choosing $F = F_1$ and using the shorthand $u_t := (1-\widehat{\varepsilon})t$, if W^r_t denotes the classical Wiener sausage with radius r up to time t with expectation $\widehat{E}$, then

$$\begin{aligned}
\mathbf{P}(\mathsf{S_t} \mid F_1) &= E\bigl[e^{-\nu \mathsf{vol}(Z^r_{[0,t]})} \mid F_1\bigr] \\
&\ge E\bigl[e^{-\{\nu \mathsf{vol}(Z^r_{[0,u_t]}) + \nu \mathsf{vol}(Z^r_{[u_t,t]})\}} \mid F_1\bigr] \\
&= E\bigl[e^{-\nu \mathsf{vol}(Z^r_{[0,u_t]})} \mid F_1\bigr] \cdot E\bigl[e^{-\nu \mathsf{vol}(Z^r_{[u_t,t]})} \mid F_1\bigr] \\
&= \widehat{E}\bigl[e^{-\nu \mathsf{vol}(W^r_{u_t})}\bigr] E\bigl[e^{-\nu \mathsf{vol}(Z^r_{[u_t,t]})} \mid F_1\bigr].
\end{aligned} \tag{27}$$

In passing to the last line of (27), we have used the Markov property of Z at time u_t, along with the translation invariance of the sausages, which together imply that the random variables $\mathsf{vol}(Z^r_{[0,u_t]})$ and $\mathsf{vol}(Z^r_{[u_t,t]})$ are conditionally independent, given $|Z(u_t)|$. (Note that for an inhomogeneous Poisson random measure with mean measure ν, this reasoning would break down.[4]) The first factor on the right-hand side of (27) is equal to $\mathbf{P}(\mathsf{S}^{\mathrm{BM}}_{\mathsf{t}})$, where we write $\mathsf{S}^{\mathrm{BM}}_{\mathsf{t}}$ to denote the event of survival for a single Brownian particle, which in the case of a uniform field decays sub-exponentially fast[5] by the well-known result of Donsker and Varadhan [4].

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[4] The random variables $\nu(Z^r_{[0,u_t]})$ and $\nu(Z^r_{[u_t,t]})$ would be dependent, even conditionally on $|Z(u_t)|$.

[5] The decay is $\exp[-ct^{d/(d+2)}(1+o(1))]$ as $t \to \infty$, a.k.a. 'stretched exponential decay.'

For $0 \le t_1 \le t_2$, let

$$\mathsf{S}_{[\mathsf{t}_1,\mathsf{t}_2]} := \{R(t_1,t_2) \cap K = \varnothing\}$$

denote the event of trap-avoiding over $[t_1,t_2]$. Then the second factor on the right-hand side of (27) is equal to

$$\mathbf{P}(\mathsf{S}_{[\mathsf{u}_t,\mathsf{t}]} \mid F_1)$$

so that from (27) it follows that

$$\frac{\mathbf{P}(\mathsf{S}_\mathsf{t} \mid F_2)}{\mathbf{P}(\mathsf{S}_\mathsf{t} \mid F_1)} \le e^{o(t)} \frac{\mathbf{P}(\mathsf{S}_{[\mathsf{u}_t,\mathsf{t}]} \mid F_2)}{\mathbf{P}(\mathsf{S}_{[\mathsf{u}_t,\mathsf{t}]} \mid F_1)}, \tag{28}$$

where $e^{o(t)}$ is written for the first term on the right-hand side of (27).

To show (24), now it suffices to show that there exists $c > 0$ such that for all large t

$$\frac{\mathbf{P}(\mathsf{S}_{[\mathsf{u}_t,\mathsf{t}]} \mid F_2)}{\mathbf{P}(\mathsf{S}_{[\mathsf{u}_t,\mathsf{t}]} \mid F_1)} \le e^{-ct}. \tag{29}$$

To this end, let $Y_t^x = Y_t^x(\omega)$ be the probability of survival up to t for a BBM that starts with a single particle at x. (Here ω represents a realization of the trap field.) By the Cauchy-Schwarz inequality and spatial homogeneity, for any $x, y \in \mathbb{R}^d$,

$$\mathbb{E}[Y_t^x Y_t^y] \le \sqrt{\mathbb{E}[(Y_t^x)^2]\mathbb{E}[(Y_t^y)^2]} = \mathbb{E}[(Y_t^0)^2]. \tag{30}$$

Now let $p_t = Y_t^0$. Since $t - u_t = \widehat{\varepsilon}t$, to verify (29), it is enough to show that $\mathbb{E}[p^2_{\widehat{\varepsilon}t}]/\mathbb{E}[p_{\widehat{\varepsilon}t}] \le e^{-ct}$ for some c for all large t.

For simplicity, suppose that the remaining time is t instead of $\widehat{\varepsilon}t$ (write t in place of $\widehat{\varepsilon}t$ and use a different c). Let $\varepsilon > 0$ and $\rho_t := (1-\varepsilon)\sqrt{2\beta m t}$ be a subcritical radius. Denote

$$\Omega_t := \{\omega \in \Omega \mid \operatorname{supp}(\Pi(\omega)) \cap \bar{B}(0,\rho_t) \ne \varnothing\}.$$

Then for all large t,

$$\mathbb{E}[p_t^2] \le \mathbb{E}[p_t^2 \mathbb{1}_{\Omega_t}] + \mathbb{P}(\Omega_t^c) \le e^{-ct}\mathbb{E}[p_t \mathbb{1}_{\Omega_t}] + e^{-ct^d} \le e^{-ct}\mathbb{E}[p_t] + e^{-ct^d}, \tag{31}$$

where all constants are denoted generically, as before, by c, and Lemma 1 is used in passing to the second inequality: write $p_t^2 = p_t p_t$ and on Ω_t, bound the second p_t from above by e^{-ct}, according to Lemma 1. Now, since $d \ge 2$, we have for all large t,

$$\frac{\mathbb{E}[p_t^2]}{\mathbb{E}[p_t]} \le e^{-ct} + \frac{e^{-ct^d}}{\mathbb{E}[p_t]} \to 0 \quad \text{as } t \to \infty, \tag{32}$$

since we know from (3) that for a BBM with $p_0 = p_1 = 0$ in $d \ge 2$, we have $\mathbb{E}[p_t] = \exp[-\beta t + o(t)]$. This proves (29), completing the proof of (24). Since the presence of more particles reduces the probability of survival, it follows from (24) that

$$\lim_{t\to\infty} \frac{\mathbf{P}(\mathsf{S}_\mathsf{t} \mid F_i)}{\mathbf{P}(\mathsf{S}_\mathsf{t} \mid F_1)} = 0$$

for any $i \ge 2$ as well. Then, by the 'general bootstrap argument' and (23), for each fixed $2 \le i \le \lfloor(\sqrt{2}+1)/\widehat{\varepsilon}\rfloor$, we have

$$\lim_{t\to\infty} P(|Z((1-\widehat{\varepsilon})t)| = i \mid \mathsf{S}_\mathsf{t}) = 0.$$

To finish the proof, apply the union bound over $2 \le i \le \lfloor(\sqrt{2}+1)/\widehat{\varepsilon}\rfloor$. This completes the bootstrap argument and shows that (20) can be improved to

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z\big((1-\widehat{\varepsilon})t\big)\big| = 1 \mid \mathsf{S}_\mathsf{t}\big) = 1. \qquad \square$$

Proof of Theorem 2. Fix $0 < \widehat{\varepsilon} < \eta^*$. Let $K_t = K_t^{f,\widehat{\varepsilon}} := \{|Z((\eta^* - \widehat{\varepsilon})t)| \le f(t)\}$ for a function $f : \mathbb{R}_+ \to [1, \infty)$ such that $\lim_{t\to\infty} f(t) = \infty$. We first show that for any such function,

$$\lim_{t\to\infty} \mathbf{P}(K_t \mid \mathsf{S}_\mathsf{t}) = 1. \tag{33}$$

To show (33), in view of (5), it suffices to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{P}\big(K_t^c \cap \mathsf{S}_\mathsf{t}\big) < -I(l, f, \beta, d). \tag{34}$$

Obviously, we may (and will) assume that $\lim_{t\to\infty} \frac{f(t)}{t^n} = 0$ for some $n \in \mathbb{N}$.

We follow an argument similar to the one in Section 3.2 in [10]. For $t \ge 0$, let

$$\eta_t := \sup\big\{\eta \in [0,1] : \big|Z(\eta t)\big| \le f(t)\big\} \ge 0,$$

and notice that

$$K_t^c = \big\{\eta_t \le \eta^* - \widehat{\varepsilon}\big\};$$
$$\{\eta_t > x\} = \big\{\big|Z(xt)\big| \le f(t)\big\} \quad \text{and} \quad \{\eta_t < x\} \subseteq \big\{\big|Z(xt)\big| > f(t)\big\}, \quad \text{for } x \in (0,1).$$

Introducing the conditional probabilities

$$\mathbf{P}_t^{(i,n)}(\cdot) = \mathbf{P}\bigg(\cdot \mid \frac{i}{n} \le \eta_t < \frac{i+1}{n}\bigg), \quad i = 0, 1, \ldots, n-1,$$

we have that, for every $n \in \{1, 2, 3, \ldots\}$,

$$\begin{aligned}
&\mathbf{P}\big(K_t^c \cap \mathsf{S}_\mathsf{t}\big) \\
&\quad \le \sum_{i=0}^{\lceil(\eta^*-\widehat{\varepsilon})n\rceil-1} \mathbf{P}\bigg(\mathsf{S}_\mathsf{t} \cap \bigg\{\frac{i}{n} \le \eta_t < \frac{i+1}{n}\bigg\}\bigg) + \mathbf{P}\big(\mathsf{S}_\mathsf{t} \cap \big\{\eta_t = \eta^* - \widehat{\varepsilon}\big\}\big) \\
&\quad \le \sum_{i=0}^{\lceil(\eta^*-\widehat{\varepsilon})n\rceil-1} \exp\bigg[-\beta\frac{i}{n}t + o(t)\bigg]\mathbf{P}_t^{(i,n)}(\mathsf{S}_\mathsf{t}) + \exp\big[-\beta\big(\eta^* - \widehat{\varepsilon}\big)t + o(t)\big]\mathbf{P}\big(\mathsf{S}_\mathsf{t} \mid \eta_t = \eta^* - \widehat{\varepsilon}\big),
\end{aligned} \tag{35}$$

where we have used (9) with $k = f(t)$ to control $P(i/n \le \eta_t < (i+1)/n)$ in passing to the second inequality. Concerning the last term on the right-hand side of (35), note that either $\{\eta_t = \eta^* - \widehat{\varepsilon}\}$ is a zero event and we can omit the last term, or it is a positive event, and then we condition on it. Therefore, in the first case the argument is even simpler than what follows, as the last term simply vanishes. Consider the particles alive at time $t(i+1)/n$ (resp. $(\eta^* - \widehat{\varepsilon})t$), and the balls with radius[6]

$$\rho_t^{(i,n)} := (1-\varepsilon)\sqrt{2\beta m}\bigg(1 - \frac{i+1}{n}\bigg)t, \quad \text{resp.} \quad \rho_t^* := (1-\varepsilon)\sqrt{2\beta m}\big(1 - \eta^* + \widehat{\varepsilon}\big)t$$

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[6] I.e. a ball of subcritical radius for the remaining time.

around them, and finally, let $\mathsf{TF}_t^{(i,n)}$ (resp. TF_t) be the number of trap-free[7] balls among these. Define the events

$$A_t^{(i,n)} := \{\mathsf{TF}_t^{(i,n)} \ge 1 \vee (|Z(t(i+1)/n)| - f(t))\}; \qquad A_t := \{\mathsf{TF}_t \ge 1 \vee (|Z((\eta^* - \widehat{\varepsilon})t)| - f(t))\}.$$

(So on $[A_t^{(i,n)}]^c \cap \{|Z(t(i+1)/n)| > f(t)\} \supseteq [A_t^{(i,n)}]^c \cap \{\eta_t < (i+1)/n\}$, there are more than $f(t)$ balls receiving at least one point from Π, while on $[A_t^{(i,n)}]^c \cap \{|Z(t(i+1)/n)| \le f(t)\} = [A_t^{(i,n)}]^c \cap \{\eta_t > (i+1)/n\}$, none of the balls are trap-free.) Use the trivial estimate

$$\mathbf{P}_t^{(i,n)}(\mathsf{S_t}) \le \mathbf{P}_t^{(i,n)}(A_t^{(i,n)}) + \mathbf{P}_t^{(i,n)}(\mathsf{S_t} \mid [A_t^{(i,n)}]^c), \tag{36}$$

and a similar estimate for $\mathbf{P}(\mathsf{S_t} \mid \eta_t = \eta^* - \widehat{\varepsilon})$. Letting $\eta = i/n$, it is shown in the proof of [17, Thm. 1] that

$$\begin{aligned} &\exp\left[-\beta\frac{i}{n}t + o(t)\right]\mathbf{P}_t^{(i,n)}(A_t^{(i,n)}) \\ &\quad \le \exp\left[-\min_{\eta\in[0,(\eta^*-\widehat{\varepsilon})],c\in[0,\sqrt{2\beta}]}\left\{\beta\eta + \frac{c^2}{2\eta} + lg_d(\sqrt{2\beta m}(1-\eta), c)\right\}t + o(t)\right] \end{aligned} \tag{37}$$

(with a similar bound for $\mathbf{P}(A_t)$). Heuristically, the second and third terms of the function to be minimized on the right-hand side of (37) arise as follows. If $A_t^{(i,n)}$ occurs, then there is at least one particle at time ηt positioned at some distance ct away from the origin, with the $\rho_t^{(i,n)}$-ball centered at it trap-free. The probabilistic cost of moving a Brownian particle to a distance of ct away from the origin over a time of ηt is $\exp[-c^2/(2\eta)t + o(t)]$, which gives the second term. The probabilistic cost of clearing from traps a ball with radius $\rho_t^{(i,n)}$ centered at a point ct away from the origin is $\exp[-lg_d(\sqrt{2\beta m}(1-\eta), c)t + o(t)]$ by (1), and this gives the third term. Optimization over c is performed to catch the lowest cost on an exponential scale. We know from [17, Thm. 2] that (η^*, c^*) is the unique pair of minimizers for the variational problem in (6), whereas the parameter η on the right-hand side of (37) is bounded away from η^*. Therefore, putting (36) and (37) together with (35), to obtain (34), it suffices to show that $\mathbf{P}_t^{(i,n)}(\mathsf{S_t} \mid [A_t^{(i,n)}]^c)$ and $\mathbf{P}(\mathsf{S_t} \mid [A_t]^c)$ are SES, that is, that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbf{P}_t^{(i,n)}(\mathsf{S_t} \mid [A_t^{(i,n)}]^c) = -\infty \quad \text{and} \quad \lim_{t\to\infty}\frac{1}{t}\log\mathbf{P}(\mathsf{S_t} \mid [A_t]^c) = -\infty \tag{38}$$

for $i = 0, 1, 2, \ldots, \lceil(\eta^* - \widehat{\varepsilon})n\rceil - 1$ for some large enough n.

We now verify the first statement in (38); the second could be verified similarly. Let $p^{i,n}(t)$ be the probability that a BBM, which starts its life at time $\frac{i+1}{n}t$ with a single particle at a point $x \in \mathbb{R}^d$, and whose $\rho_t^{(i,n)}$-ball (centered at x) receives a point from $\mathrm{supp}(\Pi)$, avoids the trap field in the time interval $[\frac{i+1}{n}t, t]$. It is enough to show that $[p^{i,n}(t)]^{\lfloor f(t)\rfloor}$ is SES in t for $i = 0, 1, 2, \ldots, \lceil(\eta^* - \widehat{\varepsilon})n\rceil - 1$. We may drop the floor function and work with $f(t)$ directly. Furthermore, we may work with the entire interval $[0, t]$ instead of $[\frac{i+1}{n}t, t]$. (It is enough to consider $[0, t]$ instead of the smaller interval $[\frac{i+1}{n}t, t]$ as this will not affect the final probabilistic cost being SES in t. In more detail, we show that for all large t, $p^{-1,n}(t)$ is bounded from above by $e^{-\kappa t}$ for some $\kappa > 0$. If we consider the smaller interval $[\frac{i+1}{n}t, t]$, then $p^{i,n}(t)$ will be bounded by $e^{-\kappa' t}$, where $\kappa' = \kappa(1 - (i+1)/n)$, which is still positive since it is at least $\kappa(1 - \eta^*)$ for large enough n.) Now let $p(t) := p^{-1,n}(t)$ and $\rho_t := \rho_t^{(-1,n)}$. Note that since we are conditioning only on the event that the ρ_t-ball around the particle contains a point from $\mathrm{supp}(\Pi)$, we may suppose that $x = 0$, that is, our problem becomes the trap-avoiding probability of a BBM, starting with a single particle at the origin, presuming that $\bar{B}(0, \rho_t) \cap \mathrm{supp}(\Pi) \ne \varnothing$. Now, by Lemma 1, $p(t)$ is at least exponentially small in t, and since by assumption, $\lim_{t\to\infty} f(t) = \infty$, it follows that

$$\lim_{t\to}\frac{1}{t}\log(p(t))^{f(t)} = -\infty.$$

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[7] In the weak sense that they do not receive points from Π.

Recall that under $\mathbf{P}_t^{(i,n)}$, there are more than $f(t)$ particles at time $(i+1)/n$. When conditioning on $[A_t^{(i,n)}]^c$, there are at least $f(t)$ particles whose $\rho_t^{(i,n)}$-balls around them are not trap-free, which is why we raise $p(t)$ to the power $f(t)$.) This completes the proof of (33).

Note: For the second part of the bootstrap argument, we could have used [10, Thm. 1.3(iii)], which is the same as (33) with $f(t)=t^{d+\varepsilon}$, but for the sake of completeness, we decided to include the proof of (33) with a general f satisfying $\lim_{t\to\infty} f(t)=\infty$.

Next, following a similar strategy as in the proof of Theorem 1, we reduce the number of particles to one, i.e., we show that

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z\big((1-\widehat{\varepsilon})t\big)\big|=1 \mid \mathsf{S}_\mathsf{t}\big)=1.$$

We consider the cases $d=1$ and $d\geq 2$ separately, since for $d\geq 2$, the trap field is not spatially homogeneous due to (1), whereas for $d=1$, it is. In what follows, we use the notation from the proof of Theorem 1.

The case $d=1$ and $l>l_{\mathrm{cr}}$.

If $d=1$, then (30) holds. In view of this, we first show that $\mathbb{E}[p_t^2]=o(\mathbb{E}[p_t])$ as $t\to\infty$. Let $\varepsilon>0$ and $\rho_t:=(1-\varepsilon)\sqrt{2\beta m t}$ be a subcritical radius. Then, the same calculation as in (31) yields that for all large t,

$$\mathbb{E}\big[p_t^2\big]\leq e^{-ct}\mathbb{E}[p_t]+\exp\big[-2l(1-\varepsilon)\sqrt{2\beta m t}\big],$$

where l is the constant in the trap intensity. Now, since $l>l_{\mathrm{cr}}$, we put $l-l_{\mathrm{cr}}=:\delta>0$. From [17], we know that, when $d=1$, the variational problem in (6) exhibits a crossover at $l_{\mathrm{cr}}=\frac{1}{2}\sqrt{\beta/(2m)}$. Therefore, we choose ε small enough ($0<\varepsilon<\min\{1/2,\delta\sqrt{2\beta m}\}$ will suffice) so that

$$\begin{aligned}2l(1-\varepsilon)\sqrt{2\beta m}&=2(\delta+l_{\mathrm{cr}})(1-\varepsilon)\sqrt{2\beta m}\\&=2\delta(1-\varepsilon)\sqrt{2\beta m}+\beta(1-\varepsilon)>\beta.\end{aligned}$$

When $d=1$ and $l>l_{\mathrm{cr}}$, for a BBM with $p_0=p_1=0$, we know from [17, Thm. 2.2] that (6) becomes $I=\beta$, meaning that $\mathbb{E}[p_t]=\exp[-\beta t+o(t)]$. Therefore,

$$\frac{\mathbb{E}[p_t^2]}{\mathbb{E}[p_t]}\leq e^{-ct}+\frac{\exp[-2l(1-\varepsilon)\sqrt{2\beta m t}]}{\mathbb{E}[p_t]}\to 0 \quad \text{as } t\to\infty,$$

where the convergence of the ratio $\frac{\mathbb{E}[p_t^2]}{\mathbb{E}[p_t]}$ to 0 is at least exponentially fast in t. Hence, we conclude that for all large t, we have

$$\frac{\mathbb{E}[p_t^2]}{\mathbb{E}[p_t]}\leq e^{-kt} \tag{39}$$

for some $k>0$. Now let $f(t)=\lfloor t\rfloor$. Then, (33) gives:

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z\big((1-\widehat{\varepsilon})t\big)\big|\leq\lfloor t\rfloor \mid \mathsf{S}_\mathsf{t}\big)=1. \tag{40}$$

Next, by (9) and comparison with a Yule process, we see that for any $s>0$ and $t>1$

$$\frac{P(|Z(s)|\leq\lfloor t\rfloor)}{P(|Z(s)|=1)}\leq\lfloor t\rfloor. \tag{41}$$

Finally, in view of the bootstrap argument given at the beginning of this section, we write

$$\begin{aligned}\frac{\mathbf{P}(2\leq|Z((1-\widehat{\varepsilon})t)|\leq\lfloor t\rfloor\mid\mathsf{S}_\mathsf{t})}{\mathbf{P}(|Z((1-\widehat{\varepsilon})t)|=1\mid\mathsf{S}_\mathsf{t})}&=\frac{\mathbf{P}(\mathsf{S}_\mathsf{t}\mid 2\leq|Z((1-\widehat{\varepsilon})t)|\leq\lfloor t\rfloor)}{\mathbf{P}(\mathsf{S}_\mathsf{t}\mid|Z((1-\widehat{\varepsilon})t)|=1)}\cdot\frac{P(2\leq|Z((1-\widehat{\varepsilon})t)|\leq\lfloor t\rfloor)}{P(|Z((1-\widehat{\varepsilon})t)|=1)}\\&\leq\frac{\mathbf{P}(\mathsf{S}_\mathsf{t}\mid|Z((1-\widehat{\varepsilon})t)|=2)}{\mathbf{P}(\mathsf{S}_\mathsf{t}\mid|Z((1-\widehat{\varepsilon})t)|=1)}\cdot\frac{P(|Z((1-\widehat{\varepsilon})t)|\leq\lfloor t\rfloor)}{P(|Z((1-\widehat{\varepsilon})t)|=1)}\end{aligned} \tag{42}$$

and it then follows from (39), (41), and the proof of Theorem 1 that (40) can be improved to

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z\big((1-\widehat{\varepsilon})t\big)\big| = 1 \mid \mathsf{S_t}\big) = 1.$$

The case $d \ge 2$ and $l > l_{\mathrm{cr}}$.

When $d \ge 2$, the trap intensity is no longer uniform; instead, it is radially decaying. Therefore, (27) and (30) do not hold, and so we are going to follow a different approach. Once we have the first part of the 'bootstrap,' that is, (33) with $f(t) = \lfloor t \rfloor$, we continue as follows. Throughout this proof, we use c to denote a generic positive constant whose value may change from line to line.

Let $k \ge 2$, and for fixed $0 < \widehat{\varepsilon} < \eta^*$, let $u_t := (\eta^* - \widehat{\varepsilon})t$. Using the 'general bootstrap argument,' we want to show that

$$\frac{\mathbf{P}(\mathsf{S_t} \mid |Z(u_t)| = k)}{\mathbf{P}(\mathsf{S_t} \mid |Z(u_t)| = 1)}$$

tends to zero as $t \to \infty$, exponentially fast. For an upper bound on $\mathbf{P}(\mathsf{S_t} \mid |Z(u_t)| = k)$, we may take $k = 2$, since more particles would give a smaller probability of survival. Note that

$$P\big(\big|Z(u_t)\big| = 2\big) / P\big(\big|Z(u_t)\big| = 1\big)$$

remains bounded between two positive constants. Therefore, it suffices to show that

$$\lim_{t\to\infty} e^{ct} \frac{\mathbf{P}(\mathsf{S_t} \cap \{|Z(u_t)| = 2\})}{\mathbf{P}(\mathsf{S_t} \cap \{|Z(u_t)| = 1\})} = 0, \tag{43}$$

with some $c > 0$. The bootstrap argument does not require the exponential pre-factor in (43), and only requires that the quotient on the left-hand side converges to zero (since the function f in the definition of the event K_t can be as slowly growing as we like, making the union bound always work), but in fact we are even going to show that it is exponentially small.

To do so, on the event $\{|Z(u_t)| = 2\}$, let us pick randomly (independently from everything else) the 'nice' offspring at the first splitting time. This time must be less than or equal to u_t, and for an upper bound on $\mathbf{P}(\mathsf{S_t} \cap \{|Z(u_t)| = 2\})$, we may and do assume that the splitting occurs at u_t. (Indeed, if the first splitting occurs at an earlier time than u_t, say at $\tilde{t}$, then there would be more than one particle in the system over the period $[\tilde{t}, u_t]$, which would decrease $\mathbf{P}(\mathsf{S_t} \cap \{|Z(u_t)| = 2\})$, and it would be even simpler to show (43).) Let us call the other offspring 'ugly,' and call the branch emanating from the nice (ugly) particle concatenated with the single Brownian path over $[0, u_t]$ the 'first branch' ('the second branch'). Let S_t^1 and S_t^2 denote respectively the events that the first branch resp. the second branch survives up to time t.

Let $X = (X_s)_{s\ge 0}$ represent a standard Brownian path starting at the origin. The position where the splitting occurs will be denoted by $\widehat{X}_t$, and again, we may and will pretend that it is distributed as X_{u_t}. Let $\rho_t^* := \sqrt{2\beta m}(1-\varepsilon)(t-u_t)$ and define the event

$$\widehat{\Omega}_t := \big\{\mathrm{supp}(\Pi) \cap \bar{B}\big(\widehat{X}_t, \rho_t^*\big) \ne \varnothing\big\}.$$

Then,

$$\begin{aligned}
\mathbf{P}\big(\mathsf{S_t} \cap \big\{\big|Z(u_t)\big| = 2\big\}\big) &= \mathbf{P}\big(\mathsf{S_t} \cap \widehat{\Omega}_t \cap \big\{\big|Z(u_t)\big| = 2\big\}\big) + \mathbf{P}\big(\mathsf{S_t} \cap \widehat{\Omega}_t^c \cap \big\{\big|Z(u_t)\big| = 2\big\}\big)\\
&= \mathbf{P}\big(\mathsf{S}_\mathsf{t}^1 \cap \mathsf{S}_\mathsf{t}^2 \cap \widehat{\Omega}_t \cap \big\{\big|Z(u_t)\big| = 2\big\}\big) + \mathbf{P}\big(\mathsf{S}_\mathsf{t}^1 \cap \mathsf{S}_\mathsf{t}^2 \cap \widehat{\Omega}_t^c \cap \big\{\big|Z(u_t)\big| = 2\big\}\big)\\
&\le \mathbf{P}\big(\mathsf{S}_\mathsf{t}^2 \cap \widehat{\Omega}_t \mid \mathsf{S}_\mathsf{t}^1 \cap \big\{\big|Z(u_t)\big| = 2\big\}\big)\mathbf{P}\big(\mathsf{S}_\mathsf{t}^1 \cap \big\{\big|Z(u_t)\big| = 2\big\}\big) + \mathbf{P}\big(\widehat{\Omega}_t^c \cap \big\{\big|Z(u_t)\big| = 2\big\}\big)\\
&\le e^{-ct}\mathbf{P}\big(\mathsf{S}_\mathsf{t}^1 \cap \big\{\big|Z(u_t)\big| = 2\big\}\big) + \mathbf{P}\big(\widehat{\Omega}_t^c\big) P\big(\big|Z(u_t)\big| = 2\big)
\end{aligned} \tag{44}$$

for all large t. In passing to the last inequality, we have used the branching Markov property at time u_t and Lemma 1, which together imply that even under the conditioning,

$$\mathbf{P}\big(\mathsf{S}_\mathsf{t}^2 \cap \widehat{\Omega}_t \mid \mathsf{S}_\mathsf{t}^1 \cap \big\{\big|Z(u_t)\big| = 2\big\}\big) \le e^{-ct}$$

for some $c > 0$ for all large t. Moreover, by Fubini's theorem,

$$\mathbf{P}\big(\widehat{\Omega}_t^c\big) = E\big[\mathbb{P}\big(\widehat{\Omega}_t^c\big)\big].$$

Let $|x|$ denote the Euclidean norm of a point $x \in \mathbb{R}^d$. By conditioning $\widehat{X}_t$ on the events $\{\frac{i-1}{n}\sqrt{2\beta}t \leq |\widehat{X}_t| \leq \frac{i}{n}\sqrt{2\beta}t\}$ for $i = 1, 2, \dots, n$ and on $\{|\widehat{X}_t| > \sqrt{2\beta}t\}$, and following an argument similar to the proof of the upper bound of Theorem 1 in [17], it is not hard to show that

$$\begin{aligned} &E\big[\mathbb{P}\big(\widehat{\Omega}_t^c\big)\big] \\ &\quad = \exp\bigg[-\min_{x\in[0,\sqrt{2\beta}]}\bigg\{\frac{x^2}{2(\eta^* - \widehat{\varepsilon})} + lg_d\big(\rho_t^*/t, x\big)\bigg\}t + o(t)\bigg] + \exp\bigg[-\frac{\beta}{\eta^* - \widehat{\varepsilon}}t + o(t)\bigg], \end{aligned} \tag{45}$$

where [17, Lemma 5] was used to control the probabilistic cost of linear Brownian displacements. Also, (9) implies that

$$P\big(\big|Z(u_t)\big| = 2\big) \leq \exp[-\beta u_t] = \exp\big[-\beta\big(\eta^* - \widehat{\varepsilon}\big)t\big]. \tag{46}$$

To prove (43), in view of (44), we need to show that there exists a constant $c > 0$ such that for all large t,

$$\frac{\mathbf{P}(\mathsf{S}_\mathsf{t}^1 \cap \{|Z(u_t)| = 2\})}{\mathbf{P}(\mathsf{S}_\mathsf{t} \cap \{|Z(u_t)| = 1\})} \leq c \tag{47}$$

and

$$\frac{\mathbf{P}(\widehat{\Omega}_t^c)P(|Z(u_t)| = 2)}{\mathbf{P}(\mathsf{S}_\mathsf{t} \cap \{|Z(u_t)| = 1\})} \leq e^{-ct}. \tag{48}$$

Note that

$$\frac{\mathbf{P}(\mathsf{S}_\mathsf{t}^1 \cap \{|Z(u_t)| = 2\})}{\mathbf{P}(\mathsf{S}_\mathsf{t} \cap \{|Z(u_t)| = 1\})} = \frac{\mathbf{P}(\mathsf{S}_\mathsf{t}^1 \mid |Z(u_t)| = 2)}{\mathbf{P}(\mathsf{S}_\mathsf{t} \mid |Z(u_t)| = 1)}\,\frac{P(|Z(u_t)| = 2)}{P(|Z(u_t)| = 1)},$$

where on the right-hand side, the first quotient is one, and the second quotient remains bounded, which proves (47).

To prove (48), consider the following survival strategy, which is shown to be optimal in [10] and [17], and can be used to obtain a lower bound for $\mathbf{P}(\mathsf{S}_\mathsf{t} \cap \{|Z(u_t)| = 1\})$ as well. (This means that for large t, the events $\mathsf{S}_\mathsf{t} \cap \{|Z(u_t)| = 1\}$ and S_t have the same probabilistic cost on a logarithmic scale.)

1. Suppress the branching of the BBM up to time $\eta^* t$;
2. move the single particle to a distance of $c^* t + o(t)$ from the origin in the time interval $[0, \eta^* t]$;
3. make the ball $B(c^* t\mathbf{e}, \sqrt{2\beta m}(1 - \eta^*)t + \delta t)$ trap-free, where $\delta > 0$ and $\mathbf{e}$ is the unit vector in the direction of the position vector of the single particle at time $\eta^* t$;
4. let the BBM branch freely in the remaining time interval $[\eta^* t, t]$ inside this ball.

(For details regarding this type of survival strategy, please see [17, Section 5.1].) Finally, let $\delta \to 0$ to obtain

$$\mathbf{P}\big(\mathsf{S}_\mathsf{t} \cap \big\{\big|Z(u_t)\big| = 1\big\}\big) \geq \exp\bigg[-\bigg\{\beta\eta^* + \frac{(c^*)^2}{2\eta^*} + lg_d\big(\sqrt{2\beta m}\big(1 - \eta^*\big), c^*\big)\bigg\}t + o(t)\bigg].$$

Since the factor $\exp[-\beta(\eta^* - \widehat{\varepsilon})t]$ does not depend on the minimizing parameter in (45), by (45) and (46), the statement under (48) will follow if we show that

$$\begin{aligned} &\min_{x\in[0,\sqrt{2\beta}]}\bigg\{\beta\big(\eta^* - \widehat{\varepsilon}\big) + \frac{x^2}{2(\eta^* - \widehat{\varepsilon})} + lg_d\big(\sqrt{2\beta m}\big(1 - \eta^* + \widehat{\varepsilon}\big), x\big)\bigg\} \\ &\quad > \beta\eta^* + \frac{(c^*)^2}{2\eta^*} + lg_d\big(\sqrt{2\beta m}\big(1 - \eta^*\big), c^*\big) \end{aligned} \tag{49}$$

and

$$\frac{\beta}{\eta^*-\widehat{\varepsilon}}+\beta\big(\eta^*-\widehat{\varepsilon}\big)>\beta\eta^*+\frac{(c^*)^2}{2\eta^*}+lg_d\big(\sqrt{2\beta m}\big(1-\eta^*\big),c^*\big). \tag{50}$$

(Above, in writing the function g_d from (45), we have used that $\rho_t^*=\sqrt{2\beta m}(1-\varepsilon)(1-\eta^*+\widehat{\varepsilon})t$ and then let $\varepsilon\to 0$.) Now (49) follows, because we know from [17, Thm. 2] that the pair (η^*,c^*) is the unique pair of minimizers for the variational problem

$$\min_{\eta\in[0,1],c\in[0,\sqrt{2\beta}]}\left\{\beta\eta+\frac{c^2}{2\eta}+lg_d\big(\sqrt{2\beta m}(1-\eta),c\big)\right\}. \tag{51}$$

Indeed, on the left-hand side of (49), the function in (51) is evaluated at $(\eta,c)=(\eta^*-\widehat{\varepsilon},x)$ for some $x\in[0,\sqrt{2\beta}]$, and (49) then follows by the uniqueness of minimizers. Similarly, (50) follows since the expression on its left-hand side is bounded from below by β (recall that $\eta^*<1$) and the value of the expression on its right-hand side becomes β if we replace (η^*,c^*) by $(1,0)$. This shows (43). To complete the proof, use (42) with $Z((1-\widehat{\varepsilon})t)$ replaced by $Z((\eta^*-\widehat{\varepsilon})t)$, (41), and (43). □

Remark. (i) As we have noted in the proof of Theorem 1, for a 'free' BBM, the probabilistic cost of having 1 particle and at most k particles are asymptotically similar up to a constant as $t\to\infty$. What Theorem 1 and Theorem 2 say is that, for the trap fields considered here, for large t, whenever the system has to suppress branching in order to survive from traps up to t, with overwhelming probability, it must do so completely up to time $(1-\widehat{\varepsilon})t$ (resp. $(\eta^*-\widehat{\varepsilon})t$). Furthermore, the proofs reveal that having even 2 particles instead of 1 at $(1-\widehat{\varepsilon})t$ (resp. $(\eta^*-\widehat{\varepsilon})t$) is exponentially unlikely in t. This shows that conditioning a BBM on survival among traps has a drastic effect on its population size.

(ii) The proofs of Theorem 1 and Theorem 2 reveal something stronger than the statement of the theorems; namely, that conditional on survival up to time t, the probability of the respective complement events $\{|Z((1-\widehat{\varepsilon})t)|>1\}$ and $\{|Z((\eta^*-\widehat{\varepsilon})t)|>1\}$ converge to zero exponentially fast in t.

5. Particle production along a skeletal line

Theorem 1 and Theorem 2 are stated for $p_0=0$. In Section 6, they will be extended to the case where $p_0>0$ (see Theorem 3 and Theorem 4), which yields a positive probability of extinction for the BBM. In this case, we condition the BBM on non-extinction for meaningful results on optimal survival strategies. A detailed treatment of a BBM conditioned on non-extinction is given in [17] (see in particular Lemma 4 and Proposition 2 therein). Here, in preparation for Section 6, we briefly mention the development needed, followed by the statement and proof of Lemma 2. Conditioned on the event of non-extinction (denoted by $\mathcal{E}^c$), recall that the BBM has the following two-type decomposition:

$$\big(Z(t)\big)_{t\geq 0}=\big(Z^1(t),Z^2(t)\big)_{t\geq 0},$$

where Z^1 is the process consisting of the 'skeleton' particles, and Z^2 is the one consisting of the 'doomed' particles. Skeleton particles are those with infinite lines of descent, whereas the doomed particles have finite lines of descent. We refer to the totality of all skeleton particles as the 'skeleton' so that the tree of $|Z|$ conditioned on non-extinction can be described as an infinite skeleton decorated with infinitely many finite 'bushes' composed of doomed particles.

The process $Z^1=(Z^1(t))_{t\geq 0}$ is itself a BBM, with the same offspring mean μ as the original process Z, branching rate $\beta\alpha$, and $p_0^*=p_1^*=0$ where we use $(p_k^*)_{k\in\mathbb{N}_0}$ for the offspring distribution of Z^1. (Recall that $\alpha:=1-f'(q)$, where $q:=P(\mathcal{E})$, and f is the p.g.f. of the offspring distribution of Z.) The process $Z^2=(Z^2(t))_{t\geq 0}$ is not a BBM, but it is formed by a collection of BBMs in the following sense: each time a doomed particle is born from a skeleton particle, it initiates an independent subcritical BBM, whose mean offspring number is less than one. It is clear that conditioning Z on $\mathcal{E}^c$ is equivalent to the initial condition $(|Z^1(0)|,|Z^2(0)|)=(1,0)$.

By a 'skeletal ancestral line up to time t,' we mean the continuous trajectory traversed up to time t by a skeleton particle present at time t, concatenated with the trajectories of all its ancestors including the one traversed by the initial particle. We use the term ‘skeletal line’ in short to mean a skeletal ancestral line up to time t. We say that a doomed particle is *produced* by a skeletal line if the most recent skeleton ancestor of the doomed particle is a part of this skeletal line. Note that by this definition, a doomed particle may be produced by more than one skeletal line, but it has to be produced by at least one skeletal line. The following lemma gives an upper bound on the number of doomed particles, all alive at the present time, which are produced by a given single skeletal line.

Lemma 2 (Very few doomed particles). *Let* $\log^{(0)}(t) := t$ *and* $\log^{(n)}(t) := \log(\log(\dots(\log t)\dots))$ *for* $n \in \mathbb{N}$ *be the logarithm function iterated n times. Then, for any* $n \in \mathbb{N}_0$, *for a fixed skeletal line, the probability that this line has produced more than* $\log^{(n)}(t)$ *doomed particles in* $[0, t]$, *which are all alive at time t, goes to zero at least at the rate* $1/\log^{(n)}(t)$ *as* $t \to \infty$.

Remark. Note that in particular Lemma 2 implies that

$$\lim_{t\to\infty} P\bigl(\bigl|Z^2(t)\bigr| > \log^{(n)}(t) \mid \bigl|Z^1(t)\bigr| = 1\bigr) = 0$$

since the event in the condition implies that there is exactly one skeletal ancestral line for all $0 \le s \le t$.

In order to prove Lemma 2, we first present two preparatory propositions. The first provides an upper bound on the non-extinction probability of a subcritical BBM up to time t, and follows from the trivial estimate $P(|Z(t)| > 0) \le E[|Z(t)|]$; the second follows directly from a standard Poissonian tail bound.

Proposition 1. *Let Z be a subcritical BBM with rate* $\beta > 0$ *and offspring p.g.f. f, and* $|Z|$ *be the associated total-mass process. Let* $\mu = f'(1)$ *be the mean number of offspring so that* $m := \mu - 1 < 0$. *Then, for any* $t \ge 0$,

$$P\bigl(\bigl|Z(t)\bigr| > 0\bigr) \le e^{\beta m t}.$$

Remark. For precise results on $P(|Z(t)| > 0)$, please see [1, Thm. 2.4].

Proposition 2 (Tail estimate). *Let Y be a Poisson random variable with parameter* λ. *Then for* $x > \lambda$,

$$P(Y \ge x) \le e^{-k\lambda},$$

where $k = k(x/\lambda)$ *is a positive number.*

Proof. Let $z := \frac{\lambda}{x} \in (0, 1)$. By the standard Poissonian tail estimate,

$$\log P(Y \ge x) \le \log\left[\frac{e^{-\lambda}(e\lambda)^x}{x^x}\right] = \lambda\left(-1 + x\frac{1 + \log\lambda}{\lambda} - x \log x/\lambda\right) = -\lambda k(z),$$

where

$$k(z) := 1 - z^{-1}(1 + \log z) > 1 - z^{-1}z = 0,$$

as $z > 1 + \log z$. □

Proof of Lemma 2. We prove the statement by an inductive argument as follows. Fix a single skeletal ancestral line. In this proof, by a doomed particle born ‘directly’ along this skeletal line, we refer to a doomed particle whose direct ancestor is a skeleton particle of this line and by a ‘doomed subtree,’ we refer to a subtree that is initiated by a doomed particle born directly along this fixed skeletal line. Let μ^* be the mean number of offspring for a doomed particle, define $m^* = \mu^* - 1$, and let E_1 be the event that the doomed subtrees created in the interval $I_1 := [0, t - \frac{4}{\beta|m^*|}\log t]$ do not all go extinct by time t (recall that $\mu^* < 1$ for a doomed particle which implies $m^* < 0$) and let $P_1 := P(E_1)$.

Let F_1 be the event that at most $2\beta t$ occurrences of branching occur along the skeletal line in the time interval I_1. Estimate

$$P_1 \leq P\bigl(F_1^c\bigr) + P(E_1 \mid F_1). \tag{52}$$

By Proposition 2, since the number of occurrences of branching up to time t along a single skeletal line is a Poisson process with mean βt, we have

$$P\bigl(F_1^c\bigr) \leq e^{-k_1(1)t}, \tag{53}$$

where $k_1(1) > 0$ is a constant that depends on β. Now focus on $P(E_1 \mid F_1)$. Let G_1 be the event that at most t^3 doomed subtrees are born in the interval I_1. Estimate

$$P(E_1 \mid F_1) \leq P\bigl(G_1^c \mid F_1\bigr) + P(E_1 \mid G_1, F_1). \tag{54}$$

Let ρ be the expected number of doomed offspring for a skeleton particle. (From [17], we know that $\rho = [f'(1) - f'(q)]q/(1-q)$, where q is the probability of extinction for Z.) The first term on the right-hand side of (54) is bounded from above by the probability that at least one skeletal branching among $2\beta t$ many gives at least $t^3/(2\beta t) = t^2/(2\beta)$ doomed offspring, which, by the union bound and Markov inequality, is bounded from above to yield

$$P\bigl(G_1^c \mid F_1\bigr) \leq 2\beta t \frac{\rho}{t^2/(2\beta)} = k_2(1)/t, \tag{55}$$

where $k_2(1)$ is a constant that depends on β and f. The second term on the right-hand side of (54) is bounded from above by the probability that the doomed subtrees created in the interval I_1, of which there are at most t^3 many, do not all go extinct by t, which, by the union bound and Proposition 1 (recall that each doomed subtree is a subcritical BBM), is bounded from above to yield

$$P(E_1 \mid G_1, F_1) \leq t^3 \exp\biggl(\beta m^* \frac{-4}{\beta m^*} \log t\biggr) = 1/t. \tag{56}$$

Putting the pieces together, from (52)–(56), we obtain

$$P_1 \leq e^{-k_1(1)t} + k_2(1)/t + 1/t, \tag{57}$$

which implies that the doomed subtrees created in $I_1 = [0, t - \frac{4}{\beta|m^*|} \log t]$ all go extinct by time t with a probability tending to 1 as $t \to \infty$.

We now extend the argument above to the doomed subtrees created in the interval $I_n := [t - \frac{4}{\beta|m^*|} \log^{(n-1)} t, t - \frac{4}{\beta|m^*|} \log^{(n)} t]$ for $n \geq 2$. For $n \geq 2$, let E_n be the event that the doomed subtrees created in the interval I_n do not all go extinct by time t and let $P_n := P(E_n)$. Let F_n be the event that at most $2\beta \frac{4}{\beta|m^*|} \log^{(n-1)} t = (8/|m^*|) \log^{(n-1)} t$ occurrences of branching occur along the skeletal line in the time interval I_n. Estimate

$$P_n \leq P\bigl(F_n^c\bigr) + P(E_n \mid F_n). \tag{58}$$

By Proposition 2, since the number of occurrences of branching in I_n along a single skeletal line is a Poisson process with mean at most $(4/|m^*|) \log^{(n-1)} t$, we have

$$P\bigl(F_n^c\bigr) \leq \begin{cases} 1/t^{k_1(2)}, & n = 2, \\ 1/(\log^{(n-2)} t)^{k_1(n)}, & n \geq 3, \end{cases} \tag{59}$$

where $k_1(n) > 0$ is a constant that depends on β. Now focus on $P(E_n \mid F_n)$. Let G_n be the event that at most $(\log^{(n-1)} t)^3$ doomed subtrees are born in the interval I_n. Estimate

$$P(E_n \mid F_n) \leq P\bigl(G_n^c \mid F_n\bigr) + P(E_n \mid G_n \cap F_n). \tag{60}$$

The first term on the right-hand side of (60) is bounded from above by the probability that at least one skeletal branching among $(8/|m^*|)\log^{(n-1)} t$ many gives at least $(\log^{(n-1)} t)^3/((8/|m^*|)\log^{(n-1)} t) = (|m^*|/8)(\log^{(n-1)} t)^2$ doomed offspring, which, by the union bound and Markov inequality, is bounded from above to yield

$$P\left(G_n^c \mid F_n\right) \le \left(8/\left|m^*\right|\right)\log^{(n-1)} t \frac{\rho}{(|m^*|/8)(\log^{(n-1)} t)^2} = k_2(n)/\left(\log^{(n-1)} t\right), \tag{61}$$

where $k_2(n)$ is a constant that depends on f. The second term on the right-hand side of (60) is bounded from above by the probability that the doomed subtrees created in the interval I_n, of which there are at most $(\log^{(n-1)} t)^3$ many, do not all go extinct by t, which, by the union bound and Proposition 1, is bounded from above to yield

$$P(E_n \mid G_n \cap F_n) \le \left(\log^{(n-1)} t\right)^3 \exp\left(\beta m^* \frac{-4}{\beta m^*} \log^{(n)} t\right) = 1/\left(\log^{(n-1)} t\right). \tag{62}$$

Then, from (58)–(62), we obtain

$$P_n \le \begin{cases} 1/t^{k_1(2)} + k_2(2)/\log t + 1/\log t, & n = 2, \\ 1/(\log^{(n-2)} t)^{k_1(n)} + k_2(n)/\log^{(n-1)} t + 1/\log^{(n-1)} t, & n \ge 3. \end{cases} \tag{63}$$

This implies that the doomed subtrees produced by the skeletal line in I_n have all gone extinct by time t with a probability tending to 1 as $t \to \infty$. We recall that $I_1 := [0, t - \frac{4}{\beta|m^*|}\log t]$ and $I_n = [t - \frac{4}{\beta|m^*|}\log^{(n-1)} t, t - \frac{4}{\beta|m^*|}\log^{(n)} t]$ for $n \ge 2$ to conclude the following: for any $n \ge 1$, as $t \to \infty$,

$$P\left(\text{doomed subtrees born in } \left[0, t - \log^{(n)} t\right] \text{ have all gone extinct by time } t\right) \to 1. \tag{64}$$

The convergence in (64) can easily be seen from (57) and (63) to be at least at the rate $1/t$ for $n = 1$, and $1/(\log^{(n-1)} t)$ for $n \ge 2$. In view of (64), since each doomed particle that is produced by the skeletal line is a member of a doomed subtree, each doomed particle present at time t is a member of a doomed subtree that is created in the interval $[t - \log^{(n+1)} t, t]$ with probability tending to 1 as $t \to \infty$. The result follows by applying similar bounds as above on the total progeny generated by the doomed subtrees produced along the skeletal line in the interval $[t - \log^{(n+1)} t, t]$; one just needs to multiply ρ by the expected total progeny of a doomed subtree, which is finite as well. □

6. Extension to the case $p_0 > 0$

In this section, Theorem 1 and Theorem 2 are extended to the case $p_0 > 0$, where the probability of extinction for the BBM is positive. We condition the BBM on non-extinction $\mathcal{E}^c$ for meaningful results on optimal survival strategies. Recall that Z has the offspring p.g.f. f, where $f(s) = \sum_{j=0}^{\infty} p_k s^k$ for $s \in [0, 1]$. Suppose that $p_0 > 0$ and $\mu = f'(1) > 1$. Let $Z = (Z^1, Z^2)$ be the decomposition of Z into skeleton and doomed particles. Define $\alpha := 1 - f'(q)$, which is the factor by which the branching rate is reduced for the skeleton, giving an effective branching rate of $\beta\alpha$. It is easy to see that if $p_0 > 0$ and $\mu > 1$, then $0 < \alpha < 1$ (see [17, Lemma 4]).

Theorem 3 (Survival in a uniform field; $d \ge 2$ and $p_0 > 0$). *Let $p_0 > 0$ and $\mu > 1$. Suppose that $\mathrm{d}\nu/\mathrm{d}x = v$, $v > 0$. Then, for $d \ge 2$, $0 < \widehat{\varepsilon} < 1$ and any $n \in \mathbb{N}$,*

$$\lim_{t \to \infty} \mathbf{P}\left(\left|Z^1\left((1-\widehat{\varepsilon})t\right)\right| = 1 \mid \mathsf{S}_t\right) = 1, \tag{65}$$

$$\lim_{t \to \infty} \mathbf{P}\left(\left|Z^2\left((1-\widehat{\varepsilon})t\right)\right| \le \log^{(n)} t \mid \mathsf{S}_t\right) = 1. \tag{66}$$

Proof. Let $(p_k^*)_{k \in \mathbb{N}_0}$ be the offspring probabilities for Z^1, i.e., the skeleton process. Then, (65) follows from Theorem 1, since $p_0^* = 0$, and one can adjust the branching rate of the skeleton (from β to $\beta\alpha$) in order to make $p_1^* = 0$.

To prove (66), let $\widehat{\varepsilon} > 0$, fix $n \in \mathbb{N}$, and define the events indexed by t as

$$K_t := \left\{\left|Z^1\left((1-\widehat{\varepsilon})t\right)\right| = 1\right\}, \qquad L_t := \left\{\left|Z^2\left((1-\widehat{\varepsilon})t\right)\right| \le \log^{(n)} t\right\}.$$

Estimate

$$\mathbf{P}\big(L_t^c \mid \mathsf{S_t}\big) \le \mathbf{P}\big(L_t^c \mid \mathsf{S_t} \cap K_t\big) + \mathbf{P}\big(K_t^c \mid \mathsf{S_t}\big). \tag{67}$$

Note that the second term on the right-hand side of (67) tends to zero by (65). Now consider the first term. Using the shorthand $t^* := (1-\widehat{\varepsilon})t$, we need to show that

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z^2\big(t^*\big)\big| > \log^{(n)} t \mid \mathsf{S_t} \cap K_t\big) = 0.$$

Note that the event in the conditioning is the same as the event

$$\{T > t\} \cap \big\{\big|Z^1\big(t^*\big)\big| = 1\big\}$$

since the presence of a skeleton particle at any time implies non-extinction (recall that a skeleton particle by definition has an infinite line of descent.) Therefore, in the presence of the event K_t, the event $\mathsf{S_t}$ boils down to $\{T > t\}$. Now it is not difficult to see that dropping $\{T > t\}$ in the condition makes the event $\{|Z^2(t^*)| > \log^{(n)}(t)\}$ more likely (the random variable $Z^2(t^*)$ becomes stochastically larger) since conditioning on trap-avoiding tends to reduce the number of particles. Then, the first term on the right-hand side of (67) can be estimated from above by

$$\mathbf{P}\big(\big|Z^2\big(t^*\big)\big| > \log^{(n)}(t) \mid \big|Z^1\big(t^*\big)\big| = 1\big).$$

In the condition above, there is exactly one skeleton particle at time t^*, which implies that there is exactly one skeletal ancestral line up to that time. Hence, Lemma 2 (see the remark that follows it) gives (66). □

The proof of the following theorem is identical to that of the former; one only needs to replace $(1-\varepsilon)t$ by $(\eta^* - \varepsilon)t$ in the theorem statement and its proof.

Theorem 4 (Survival in a radially decaying field; $d \ge 1$ **and** $p_0 > 0$**).** *Let* $p_0 > 0$ *and* $\mu > 1$. *Let the trap intensity be radially decaying as in Theorem 2. For* $n \in \mathbb{N}$ *let* $\log^{(n)} t$ *be defined as before. Then for* $d \ge 1$, $l > l_{\mathrm{cr}}$, $0 < \widehat{\varepsilon} < \eta^*$ *and any* $n \in \mathbb{N}$,

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z^1\big(\big(\eta^* - \widehat{\varepsilon}\big)t\big)\big| = 1 \mid \mathsf{S_t}\big) = 1,$$

$$\lim_{t\to\infty} \mathbf{P}\big(\big|Z^2\big(\big(\eta^* - \widehat{\varepsilon}\big)t\big)\big| \le \log^{(n)} t \mid \mathsf{S_t}\big) = 1.$$

7. Corollaries: Different types of optimal survival strategies

In this section, using our results on the optimal survival strategies regarding the population size, namely Theorem 3 and Theorem 4, we prove optimal survival results regarding the range of the BBM, and the size and position of the clearings in $\mathbb{R}^d$ as corollaries. Our proofs are in the same spirit as the ones for [10, Thm. 1.3(i)–(iv)]. We emphasize that our results concerning the population size were all about suppressing the branching given survival among traps up to time t. Hence, the corollaries below arise in cases where there is some suppression of branching. For instance, when the trap intensity is uniform and $d = 1$, in the case $l < l_{\mathrm{cr}}$, the system does not need to suppress branching in order to avoid traps; hence this case is not studied below. Recall that $R = (R(t))_{t\ge 0}$ is the range process for the BBM.

Corollary 1 ($d = 1$)**.** *Let the trap intensity be uniform. If* Z *is supercritical, then for* $d = 1$, $l > l_{\mathrm{cr}}$ *and* $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathbf{P}\big(R(t) \subseteq B(0, \varepsilon t) \mid \mathsf{S_t}\big) = 1, \tag{68}$$

$$\lim_{t\to\infty} \mathbf{P}\big(B(0, \varepsilon t) \cap K \ne \varnothing \mid \mathsf{S_t}\big) = 1. \tag{69}$$

Note: Regarding (69), at the first sight, it may seem counterintuitive that trap avoidance implies the presence (and not the lack) of traps anywhere. However, for example in the $p_0 = 0$ case, the correct intuition is as follows: by Theorem 2, given survival, the system only produces a single particle with overwhelming probability, and this single particle will most likely be close to the origin. Therefore, creating clearings further away from the origin would result in an unnecessary probabilistic cost.

Proof. We prove the two displayed formulas separately.

(a) *Proof of* (68)*:* Let $\varepsilon > 0$ be fixed and let $|\mathcal{Z}(t)| = (|\mathcal{Z}^1(t)|, |\mathcal{Z}^2(t)|)$ be the decomposition of the *total progeny* for the BBM up to time t for $t \geq 0$. Let $Z = (Z^1, Z^2)$ be the decomposition of Z as before. From [17, Thm. 2.2], we know that $\eta^* = 1$ when $d = 1$. Let $0 < \varepsilon' < 1$ and $\delta > 0$, which both will depend on ε later. Define the events indexed by t as

$$\mathcal{L}_t := \{|\mathcal{Z}^2((1-\varepsilon')t)| \leq e^{\delta t}\}, \qquad K_t := \{|Z^1((1-\varepsilon')t)| = 1\},$$

and

$$F_t := \{R(t) \subseteq B(0, \varepsilon t)\}.$$

It is enough to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{P}(F_t^c \cap \mathsf{S}_t) < -I,$$

where $I = \beta\alpha$ (see [15, Thm. 1]). Estimate

$$\mathbf{P}(F_t^c \cap \mathsf{S}_t) \leq P(F_t^c \cap \mathcal{L}_t \cap K_t) + \mathbf{P}(K_t^c \cap \mathsf{S}_t) + P(\mathcal{L}_t^c \cap K_t). \tag{70}$$

The second term on the right-hand side of (70) is lower order than $\exp[-It]$ on an exponential scale as $t \to \infty$ by the proof of Theorem 1. The third term can be written as $P(\mathcal{L}_t^c \mid K_t)P(K_t)$. Similarly to the argument leading to (8)–(10), one can show that $P(K_t) = \exp[-\beta\alpha t + o(t)]$ since the effective branching rate for the skeleton is $\beta\alpha$. Now consider $P(\mathcal{L}_t^c \mid K_t)$. Conditioned on K_t, since there is only one skeleton particle present at time $(1-\varepsilon')t$ and the expected number of occurrences of branching along its skeletal line is $\beta(1-\varepsilon')t$ up to time $(1-\varepsilon')t$, we have

$$E[|\mathcal{Z}^2((1-\varepsilon')t)| \mid K_t] = \kappa\beta(1-\varepsilon')t,$$

where $\kappa > 0$ is the product of the expected total progeny of a doomed subtree and the expected doomed offspring of a skeleton particle, which are both finite and don't depend on t. Then, Markov inequality implies that $P(\mathcal{L}_t^c \mid K_t) \leq \exp[-\delta t + o(t)]$ so that $P(\mathcal{L}_t^c \mid K_t)P(K_t)$ is lower order than $\exp[-It]$ on an exponential scale. It remains to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log P(F_t^c \cap \mathcal{L}_t \cap K_t) < -I.$$

Define the following events:

$$F_t^1 := \{R((1-\varepsilon')t) \subseteq B(0, \varepsilon t/2)\},$$

$$F_t^2 := \{\text{each sub-BBM emanating from one of the ‘parent’ particles at time } (1-\varepsilon')t \text{ is contained in an } \varepsilon t/2\text{-ball around the position of the parent particle}\}.$$

It is clear that $F_t^1 \cap F_t^2 \subseteq F_t$. Therefore, using de Morgan's law, followed by the union bound, it suffices to show the following two inequalities:

$$\limsup_{t\to\infty} \frac{1}{t} \log P((F_t^1)^c \cap \mathcal{L}_t \cap K_t) < -I, \tag{71}$$

$$\limsup_{t\to\infty} \frac{1}{t} \log P((F_t^2)^c \cap \mathcal{L}_t \cap K_t) < -I. \tag{72}$$

On the event $(F_t^1)^c \cap \mathcal{L}_t \cap K_t$, the following probabilistic costs arise: The system has only 1 skeleton particle throughout the time interval $[0, (1-\varepsilon')t]$, which has probability $\exp[-\beta\alpha(1-\varepsilon')t]$. Also, at least one Brownian path must go outside $B(0, \varepsilon t/2)$ for some $s \in [0, (1-\varepsilon')t]$, which has probability at most $\exp[-\varepsilon^2/[8(1-\varepsilon')t] + \delta t + o(t)]$ by [17, Lemma 5] and the union bound since on the event $\mathcal{L}_t \cap K_t$, there are at most $\exp[\delta t + o(t)]$ particles in the system at all times in the period $[0, (1-\varepsilon')t]$. Therefore, by independence of branching and motion mechanisms, we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log P\big(\big(F_t^1\big)^c \cap \mathcal{L}_t \cap K_t\big) \le -\beta\alpha\big(1-\varepsilon'\big) - \frac{\varepsilon^2}{8(1-\varepsilon')} + \delta.$$

Then, since $I = \beta\alpha$ when $d = 1$, $l > l_{\mathrm{cr}}$; to prove (71), it suffices to choose $\varepsilon' > 0$ such that the inequality

$$\beta\alpha\big(1-\varepsilon'\big) + \frac{\varepsilon^2}{8(1-\varepsilon')} > \beta\alpha + \delta \tag{73}$$

is satisfied.

Now consider the event $(F_t^2)^c \cap \mathcal{L}_t \cap K_t$. On the event $(F_t^2)^c$, at least one sub-BBM emanating from one of the 'parent' particles at time $(1-\varepsilon')t$ must escape its $\varepsilon t/2$-ball around the position of the parent particle. Fix one such sub-BBM. By the proof of Proposition 1 in [17], an argument similar to the one leading to (13) shows that if

$$\varepsilon/2 > 2\varepsilon'\sqrt{2\beta m}, \tag{74}$$

then the probability that this sub-BBM exits a $\varepsilon t/2$-ball around the position of the parent particle in the remaining time $\varepsilon' t$ is at most $\exp[-3\beta m\varepsilon' t + o(t)]$. Since $\mathcal{L}_t \cap K_t$ implies the existence of at most $\exp[\delta t + o(t)]$ many particles at time $(1-\varepsilon')t$, this introduces a factor of at most δt to the exponent in the latter estimate. Again, by independence of branching and motion, we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log P\big(\big(F_t^2\big)^c \cap \mathcal{L}_t \cap K_t\big) \le -\beta\alpha\big(1-\varepsilon'\big) - 3\beta m\varepsilon' + \delta < -\beta\alpha,$$

provided that δ is small enough, where the last inequality follows since $3m > \alpha$. (Recall that the BBM is supercritical, which means $m > 1$, whereas $\alpha \le 1$.) Finally, to satisfy (73) and (74), and hence to complete the proof of (68), choose δ small enough and $\varepsilon' = \min\{\varepsilon^2/(8\beta\alpha), \varepsilon/(4\sqrt{2\beta m})\}$.

(b) *Proof of* (69)*:* Let $0 < \varepsilon' < \varepsilon$, and define the events indexed by t as

$$D_t := \big\{R(t) \subseteq B\big(0, \varepsilon' t\big)\big\}, \qquad G_t^1 := \big\{B(0, \varepsilon t) \cap K \neq \varnothing\big\}, \qquad G_t^2 := \big\{B\big(0, \varepsilon' t + r\big) \cap K \neq \varnothing\big\}.$$

(Recall that r is the constant trap radius.) It is clear that $(G_t^1)^c \subset (G_t^2)^c$, and by the definition of Poisson random measure, the probabilities of $(G_t^2)^c$ and $(G_t^1)^c$ differ by $\varepsilon - \varepsilon'$ on an exponential scale. Estimate

$$\mathbf{P}\big(\big(G_t^1\big)^c \cap \mathsf{S}_\mathsf{t}\big) \le \mathbf{P}\big(\big(G_t^1\big)^c \cap D_t\big) + \mathbf{P}\big(\mathsf{S}_\mathsf{t} \cap D_t^c\big). \tag{75}$$

The second term on the right-hand side of (75) is lower order than $\exp[-It]$ on an exponential scale as $t \to \infty$, since it was shown previously that each term on the right-hand side of (70) is such. The first term has the following asymptotics:

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{P}\big(\big(G_t^1\big)^c \cap D_t\big) &= \limsup_{t\to\infty} \frac{1}{t} \log\big[\mathbb{P}\big(G_t^1\big)^c P(D_t)\big] \\
&< \limsup_{t\to\infty} \frac{1}{t} \log\big[\mathbb{P}\big(G_t^2\big)^c P(D_t)\big] \\
&= \limsup_{t\to\infty} \frac{1}{t} \log \mathbf{P}\big(\big(G_t^2\big)^c \cap D_t\big) \le -I,
\end{aligned}$$

where the first inequality follows from the fact that the probabilities of $(G_t^2)^c$ and $(G_t^1)^c$ differ by $\varepsilon - \varepsilon'$ on an exponential scale, and the last inequality follows since $(G_t^2)^c \cap D_t \subseteq \mathsf{S}_\mathsf{t}$. This completes the proof of (69). □

Corollary 2 **($d \geq 2$).** *Let the trap intensity be uniform. If* Z *is supercritical, then for* $d \geq 2$ *and* $\varepsilon > 0$,

$$\lim_{t \to \infty} \mathbf{P}\big(R(t) \subseteq B(0, \varepsilon t) \mid \mathsf{S}_\mathsf{t}\big) = 1, \tag{76}$$

$$\lim_{t \to \infty} \mathbf{P}\big(B\big(0, \varepsilon t^{1/d}\big) \cap K \neq \varnothing \mid \mathsf{S}_\mathsf{t}\big) = 1. \tag{77}$$

Proof. For the proof of (76), refer to the proof of Corollary 1.

To prove (77), let $0 < \varepsilon' < \varepsilon$, and define the events indexed by t as

$$D_t := \big\{R(t) \subseteq B\big(0, \varepsilon' t\big)\big\}, \qquad A_t := \big\{B\big(0, \varepsilon t^{1/d}\big) \cap K \neq \varnothing\big\}.$$

Estimate

$$\mathbf{P}\big(A_t^c \cap \mathsf{S}_\mathsf{t}\big) \leq \mathbf{P}\big(A_t^c \cap D_t\big) + \mathbf{P}\big(\mathsf{S}_\mathsf{t} \cap D_t^c\big) =: \mathbf{I} + \mathbf{II}. \tag{78}$$

Now, $\mathbf{II} = o(\mathbf{P}(\mathsf{S}_\mathsf{t}))$ as $t \to \infty$ by (76). By the independence of the BBM and the Poisson random measure,

$$\mathbf{I} = \mathbb{P}\big(A_t^c\big) P(D_t) \leq \exp\big(-v\omega_d \varepsilon^d t\big) \exp\big[-\big(\beta\alpha - \sqrt{\beta\alpha/(2m)}\varepsilon'\big)t + o(t)\big], \tag{79}$$

where ω_d is the volume of the d-dimensional unit ball and $v > 0$ is the constant trap intensity. In passing to the second equality of (79), we have used the definition of Poisson random measure and a more general version of [7, Prop. 5], which can be stated as:

Proposition 3. *For a BBM with branching rate* $\beta > 0$ *and* $p_0 = 0 = p_1$, *and for* $0 < a < \sqrt{2\beta m}$,

$$\limsup_{t \to \infty} \frac{1}{t} \log P\big(R(t) \subseteq B(0, at)\big) \leq -\beta + \sqrt{\frac{\beta}{2m}}a.$$

The proof of [7, Prop. 5] can easily be adapted to prove the proposition above. Recall that the process Z^1, which is generated by the skeleton, has the same offspring mean as the original process Z, and that $\beta\alpha$ is the branching rate for the skeleton, where $0 < \alpha \leq 1$ if the BBM is supercritical. We have applied Proposition 3 to the skeleton only for an upper bound on $P(D_t)$ since the presence of doomed particles can only decrease $P(D_t)$. Finally, since $I = \beta\alpha$ when $d \geq 2$ (see [15, Thm. 1]), to complete the proof, choose $\varepsilon' > 0$ sufficiently small to satisfy the inequality

$$v\omega_d \varepsilon^d + \beta\alpha - \sqrt{\beta\alpha/(2m)}\varepsilon' > \beta\alpha,$$

that is,

$$\varepsilon' < \frac{v\omega_d \varepsilon^d}{\sqrt{\beta\alpha/(2m)}},$$

and then $\mathbf{I}$ in (78) also satisfies $\mathbf{I} = o(\mathbf{P}(\mathsf{S}_\mathsf{t}))$ as $t \to \infty$. □

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