

Regular Dirichlet extensions of one-dimensional Brownian motion¹

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Abstract. The regular Dirichlet extension is the dual concept of regular Dirichlet subspace. The main purpose of this paper is to characterize all the regular Dirichlet extensions of one-dimensional Brownian motion and to explore their structures. It is shown that every regular Dirichlet extension of one-dimensional Brownian motion may essentially decomposed into at most countable disjoint invariant intervals and an \mathcal{E} -polar set relative to this regular Dirichlet extension. On each invariant interval the regular Dirichlet extension is characterized uniquely by a scale function in a given class. To explore the structure of regular Dirichlet extension we apply the idea introduced in (*Ann. Probab.* **45** (2017) 857–872), we formulate the trace Dirichlet forms and attain the darning process associated with the restriction to each invariant interval of the orthogonal complement of $H_e^1(\mathbb{R})$ in the extended Dirichlet space of the regular Dirichlet extension. As a result, we find an answer to a long-standing problem whether a pure jump Dirichlet form has proper regular Dirichlet subspaces.

Résumé. L'extension régulière de Dirichlet est la notion duale de celle de sous-espace de Dirichlet régulier. Le but principal de cet article est de caractériser toutes les extensions régulières de Dirichlet du mouvement brownien unidimensionnel et d'explorer leurs structures. On montre que chaque extension régulière du mouvement brownien unidimensionnel de Dirichlet peut essentiellement se décomposer en intervalles invariants disjoints au plus dénombrables et en un ensemble \mathcal{E} -polaire relatif à cette extension régulière de Dirichlet. Sur chaque intervalle invariant, l'extension régulière de Dirichlet est caractérisée de manière unique par une fonction d'échelle dans une classe donnée. Pour explorer la structure de l'extension régulière de Dirichlet, on applique l'idée introduite dans (*Ann. Probab.* **45** (2017) 857–872), on formule les formes de Dirichlet de trace et atteint le processus de reprise associé à la restriction à chaque intervalle invariant du complément orthogonal de $H_e^1(\mathbb{R})$ dans l'espace de Dirichlet étendu de l'extension régulière de Dirichlet. En conséquence, nous trouvons une réponse à un problème de longue date, à savoir si une forme pure de Dirichlet avec sauts comporte des sous-espaces propres de Dirichlet réguliers.

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1. Introduction

A Dirichlet form is a closed symmetric form with Markovian property on $L^2(E, m)$ space, where E is a nice topological space and m is a fully supported Radon measure on E . Due to a series of important works by M. Fukushima and M. L. Silverstein in 1970', it is now well known that every regular Dirichlet form is associated with a symmetric

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Markov process. (The notions and terminologies in the theory of Dirichlet forms are referred to [1] and [7].) This link provides an analytic approach to study the Markov processes and their related notions. For example, the Beurling–Deny formula of Dirichlet form gives insight to the behaviours of trajectories of associated Markov process, and the so-called local property of Dirichlet form implies the continuity of trajectories, in other words, the associated Markov process is a diffusion process. In this paper, we shall focus on Dirichlet forms on $L^2(\mathbb{R})$. A fundamental and important example is $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$, where $H^1(\mathbb{R})$ is the 1-Sobolev space and \mathbf{D} is the Dirichlet integral, i.e., for any $u, v \in H^1(\mathbb{R})$,

$$\mathbf{D}(u, v) = \int_{\mathbb{R}} u'(x)v'(x) dx.$$

It is a regular Dirichlet form associated with the one-dimensional Brownian motion, which is denoted by $B = (B_t)_{t \geq 0}$ hereafter.

The notion of regular Dirichlet subspace (or simply regular subspace) was first raised by the second author and his co-authors in [3]. Roughly speaking, it is a subspace of a Dirichlet space, but also a regular Dirichlet form on the same state space. In what follows, we shall introduce the dual notion, the so-called regular Dirichlet extension, of regular Dirichlet subspace. Note tacitly when we say a regular Dirichlet subspace or extension of a symmetric Markov process, it means a regular Dirichlet subspace or extension of its associated Dirichlet form. The formal definition of regular Dirichlet extension, as well as that of regular Dirichlet subspace, is as follows.

Definition 1.1. Let E be a locally compact separable metric space and m a fully supported Radon measure on E . Given two regular Dirichlet forms $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^2, \mathcal{F}^2)$ on $L^2(E, m)$, $(\mathcal{E}^2, \mathcal{F}^2)$ is said to be a *regular Dirichlet extension* (or simply regular extension) of $(\mathcal{E}^1, \mathcal{F}^1)$ if

$$\mathcal{F}^1 \subset \mathcal{F}^2, \quad \mathcal{E}^2(u, v) = \mathcal{E}^1(u, v), \quad \forall u, v \in \mathcal{F}^1. \quad (1.1)$$

It is called a *proper* one provided $\mathcal{F}^1 \neq \mathcal{F}^2$. Conversely, $(\mathcal{E}^1, \mathcal{F}^1)$ is called a *regular Dirichlet subspace* (or simply regular subspace) of $(\mathcal{E}^2, \mathcal{F}^2)$.

Let us review some results about regular subspaces of one-dimensional Brownian motion before moving on to our main purpose, its regular extensions. Recall that an irreducible diffusion process $((X_t)_{t \geq 0}, \mathbf{P}_x)$ on \mathbb{R} , (also ‘regular’ in the terminology of [17, (45.2)]), i.e.,

$$\mathbf{P}_x(\sigma_y < \infty) > 0, \quad \forall x, y \in \mathbb{R},$$

where σ_y the first hitting time of $\{y\}$, can be characterized by a scale function (i.e. a strictly increasing and continuous function), a speed measure and a killing measure (cf. [9]) completely. Since one-dimensional Brownian motion is irreducible, it follows from Proposition 2.2(3) that any regular Dirichlet subspace is also irreducible, so that the characterization of regular Dirichlet subspaces of one-dimensional Brownian motion has been completed in [3]. It turns out that any regular Dirichlet subspace $(\mathcal{E}^1, \mathcal{F}^1)$ of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ corresponds to an irreducible diffusion process on \mathbb{R} with a scale function s in the following class,

$$\mathbf{S}(\mathbb{R}) = \{s : \mathbb{R} \rightarrow \mathbb{R} | s \text{ is strictly increasing and absolutely continuous, } s' = 0 \text{ or } 1\}. \quad (1.2)$$

In other words, $(\mathcal{E}^1, \mathcal{F}^1)$ could be written as

$$\begin{aligned} \mathcal{F}^1 &= \{u \in L^2(\mathbb{R}) : u \ll s, du/ds \in L^2(\mathbb{R}, ds)\}, \\ \mathcal{E}^1(u, v) &= \frac{1}{2} \int_{\mathbb{R}} \frac{du}{ds} \frac{dv}{ds} ds, \quad u, v \in \mathcal{F}^1, \end{aligned} \quad (1.3)$$

where the notation $u \ll s$ means that u is absolutely continuous with respect to s . A further study on structure of regular subspaces was performed in a recent paper [14]. It is evident that any scale function s in (1.2) could induce a measure-dense set (i.e., for any $a < b$, $m((a, b) \cap G_s) > 0$)

$$G_s := \{x : s'(x) = 1\}$$

and vice versa. The main result of [14] states that $(\mathcal{E}^1, \mathcal{F}^1)$ and one-dimensional Brownian motion differ only on the nowhere dense set $G_{\mathcal{E}}^c$, and the difference is characterized by their trace. Though this study is named as ‘the structure of regular Dirichlet subspaces’, it also manifests the links between the diffusion associated with $(\mathcal{E}^1, \mathcal{F}^1)$ and one-dimensional Brownian motion.

Similarly, we are interested in the following problems related to regular extensions for a given Dirichlet form:

- (Q.1) whether the proper regular Dirichlet extensions exist;
- (Q.2) if exist, how to characterize all of them; and
- (Q.3) how to describe their structure.

As mentioned above, we are also interested in the connection between their associated Markov processes.

However, these problems look trivial for one-dimensional Brownian motion because we thought at first that its regular Dirichlet extensions should be irreducible. An irreducible one-dimensional diffusion must be symmetric with respect to the speed measure and its Dirichlet form has representation given in [4, Theorem 3.1]. As an analogue of regular subspaces, we then conclude by applying [4, Theorem 4.1] that an (irreducible) regular Dirichlet form is a regular extension of one-dimensional Brownian motion, if and only if the scale function of its associated diffusion is singular, or precisely speaking, belongs to the following class:

$$\begin{aligned} \mathbf{T}(\mathbb{R}) &:= \left\{ \tau : \mathbb{R} \rightarrow \mathbb{R} \mid \tau \text{ is strictly increasing and continuous, } dx \ll d\tau, \frac{dx}{d\tau} = 1 \text{ or } 0, d\tau\text{-a.e.} \right\} \\ &= \left\{ \tau : \mathbb{R} \rightarrow \mathbb{R} \mid s := \tau^{-1} \in \mathbf{S}(\mathbb{R}) \right\} \quad (\text{cf. [3, Theorem 4.1]}), \end{aligned} \quad (1.4)$$

where τ^{-1} is the inverse function of τ . We tacitly use this superscript to stand for the inverse function of a suitable function unless otherwise specified. Note that $\mathbf{T}(\mathbb{R}) \neq \{\tau = s^{-1} \mid s \in \mathbf{S}(\mathbb{R})\}$, since the range $s(\mathbb{R})$ of s may be a proper subset of \mathbb{R} (such as the example at the end of [3]). At least we have proper examples, such as Example 3.1, for the problem (Q.1).

Looking at the definition, it is appropriate to call a function in $\mathbf{S}(\mathbb{R})/\mathbf{T}(\mathbb{R})$ a shrink/stretch (of natural scale function) respectively, and then it seems perfect that a regular Dirichlet subspace/extension is equivalent to a shrink/stretch. However we realized that the statement above was far from the truth when we found the following example of regular Dirichlet extension for Brownian motion which is not irreducible. This example was a surprise for us indeed and initiated this article.

Example 1.1. Let a linear diffusion process X on \mathbb{R} , having Lebesgue measure as speed measure, consist of two irreducible parts: a reflected Brownian motion on $I_1 := (-\infty, 0]$ and a linear diffusion on $I_2 := (0, \infty)$ with scale function τ^2 where the range of τ is \mathbb{R} (i.e. $\tau(0+) := \lim_{x \downarrow 0} \tau(x) = -\infty$, $\tau(\infty) := \lim_{x \uparrow \infty} \tau(x) = \infty$) and τ satisfies that $dx \ll d\tau$ and $dx/d\tau = 0$ or 1 . Referring to [4], the Dirichlet form of X on $L^2(\mathbb{R})$ is given by

$$\begin{aligned} \mathcal{F} &= \left\{ f \in L^2(\mathbb{R}) : f|_{I_1} \in H^1(I_1), f|_{I_2} \in H^1(I_2, d\tau) \right\}, \\ \mathcal{E}(f, f) &= \frac{1}{2} \int_{I_1} \left(\frac{df}{dx} \right)^2 dx + \frac{1}{2} \int_{I_2} \left(\frac{df}{d\tau} \right)^2 d\tau, \end{aligned} \quad (1.6)$$

²A concrete example of such τ . Consider the standard Cantor function c with $c(x) = 0$ for any $x \leq 0$ and $c(x) = 1$ for any $x \geq 1$. It induces a singular probability measure λ_c (with respect to the Lebesgue measure) supported in $[0, 1]$. For each $n \geq 1$, the image measure of λ_c , denoted by λ_c^n , under the map

$$[0, 1] \rightarrow \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right], \quad x \mapsto \frac{x+1}{2^n} \quad (1.5)$$

is another singular probability measure supported in $[\frac{1}{2^n}, \frac{1}{2^{n-1}}]$. Set

$$\tau(x) := x + \sum_{n \geq 1} \int_1^x d\lambda_c^n, \quad x \in I_2 = (0, \infty).$$

It is easy to find that τ is a scale function on I_2 such that $dx \ll d\tau$ and $dx/d\tau = 0$ or 1 , $d\tau$ -a.e. Moreover, $\tau(0+) = -\infty$, $\tau(\infty) = \infty$.

where $H^1(I_2, d\mathfrak{t}) = \{f \in L^2(I_2) : f \ll \mathfrak{t}, df/d\mathfrak{t} \in L^2(I_2; d\mathfrak{t})\}$. It is easy to check that $(\mathcal{E}, \mathcal{F})$ is a Dirichlet extension of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$. We need only to verify that it is regular, or precisely $\mathcal{F} \cap C_c(\mathbb{R})$ is dense in \mathcal{F} . It amounts to prove that for a function f on \mathbb{R} with $f|_{I_1} \in C_c^\infty(I_1)$ and $f|_{I_2} \in H^1(I_2, d\mathfrak{t}) \cap C_c(I_2)$, and any $\epsilon > 0$, there exists $f_\epsilon \in \mathcal{F} \cap C_c(\mathbb{R})$ such that

$$\mathcal{E}_1(f - f_\epsilon, f - f_\epsilon) < 2\epsilon.$$

We would like to spend a few lines to explain the proof because the idea inspires this paper. For simplicity we assume that $f(0) = 1$. We may let ϵ small enough such that $f(\epsilon) = 0$. Since $\mathfrak{t}(0+) = -\infty$, we may have $\epsilon' \in (0, \epsilon)$ so that $\mathfrak{t}(\epsilon) - \mathfrak{t}(\epsilon') > 2/\epsilon$. Define $\varphi \in C(\mathbb{R})$

$$\varphi(x) := \begin{cases} 1, & x \leq \mathfrak{t}(\epsilon'); \\ \frac{\mathfrak{t}(\epsilon) - x}{\mathfrak{t}(\epsilon) - \mathfrak{t}(\epsilon')}, & x \in (\mathfrak{t}(\epsilon'), \mathfrak{t}(\epsilon)), \\ 0, & x \geq \mathfrak{t}(\epsilon), \end{cases}$$

and

$$f_\epsilon := f \cdot 1_{\mathbb{R} \setminus (0, \epsilon]} + \varphi \circ \mathfrak{t} \cdot 1_{(0, \epsilon]}.$$

Then $f_\epsilon \in C_c(\mathbb{R})$, $f_\epsilon - f = \varphi \circ \mathfrak{t}$ and

$$\mathcal{E}_1(\varphi \circ \mathfrak{t}, \varphi \circ \mathfrak{t}) = \int_0^\epsilon (\varphi \circ \mathfrak{t}(x))^2 dx + \frac{1}{2} \int_0^\epsilon \left(\frac{d\varphi \circ \mathfrak{t}}{d\mathfrak{t}} \right)^2 d\mathfrak{t} \leq \epsilon + \frac{1}{2} \int_{\mathfrak{t}(\epsilon')}^{\mathfrak{t}(\epsilon)} (\varphi'(x))^2 dx \leq 2\epsilon.$$

This example tells that a regular extension comes from at least two aspects: *stretch and split* of scale function. The first aspect is what we mentioned above, but the second is new and leads to violence of irreducibility, so that we shall turn to the general theory of one-dimensional diffusions in classical textbooks [8] and [9] for help. The principal result phrased in Section 3 gives a complete characterization for regular extensions of one-dimensional Brownian motion and makes the rough observation above rigorous. More precisely, it turns out that every regular Dirichlet extension of one-dimensional Brownian motion has at most countable intervals as irreducible components and on each interval, the regular Dirichlet extension is determined uniquely by a scale function in the class (3.2). Heuristically speaking, the real line is split into at most countable components by impenetrable ‘walls’ (in Example 1.1, ‘0+’ is such a ‘wall’ in the sense that X reflects at 0 from left side and cannot reach 0 from right side) and an irreducible diffusion lives on each interval according to a stretched scale function. After having characterization theorem, we are interested in taking the structure of regular extensions into account. This contributes to a deeper understanding of associated diffusion processes. The idea of the approach stems from [14]. Needless to say, a difference appears now due to the presence of impenetrable ‘walls’. Nevertheless, we could still apply the trace and darning transforms by considering each component separately. The eventual result shows that on each invariant interval, the regular extension is identified with Brownian motion outside an (almost) nowhere dense set, on which their ‘shadows’ differ significantly. In Example 1.1, the positive component of X , i.e. the diffusion process with scale function \mathfrak{t} on $(0, \infty)$, has $F := \bigcup_{n \geq 1} K_n$, where K_n is the image of standard Cantor set under the map (1.5) for each $n \geq 1$, as this (almost) nowhere dense set. That is to say, X is nothing but the one-dimensional Brownian motion on F^c , but their traces on F behave significantly different as we shall characterize in Theorem 5.1. There is also a sense in which formulating the traces of regular extension and one-dimensional Brownian motion together on the sum of such (almost) nowhere dense sets of all the components sheds light on more links and differences between them. A typical example is based on Example 3.5, whose associated diffusion process X consists of a countable set of separate reflected Brownian motions. This process is identified with one-dimensional Brownian motion outside the standard Cantor set K . We shall conclude in Corollary 5.1 that the traces of X and B share the same form but have different Dirichlet spaces. Particularly, the former one is not irreducible, while the latter one is irreducible.

At a heuristic level, we could interpret one-dimensional Brownian motion and its regular extensions (as well as subspaces) by means of electric networks. We may view a diffusion on \mathbb{R} as a random motion on a ‘continuum’ electric network, and the measure induced by scale function plays the role of resistance, which reflects the ability

of material to resist the motion. As we know, one-dimensional Brownian motion is on the natural scale, i.e. the induced measure is translation invariant. This indicates that its related ‘continuum’ electronic network consists of normal material with constant resistance everywhere. A regular subspace has a scale function s in (1.2) as mentioned earlier. As a consequence, the principal part, i.e. $G_s := \{x : s'(x) = 1\}$, of \mathbb{R} consists of normal material, and the remainder, being (almost) nowhere dense, is superconductive (i.e. of zero resistance). In the case of regular extension, the principal part consists also of normal material, and the remainder, which is also (almost) nowhere dense, has two kinds of singular materials. One is insulator like the impenetrable ‘wall’ $0+$ in Example 1.1, which splits the line (as well as the diffusion), and another has singular but finite resistance like $\{x > 0 : dx/d\tau = 0\}$ in Example 1.1, which stretches the constant resistance. Intuitively, while moving like Brownian motion on the normal material, the diffusion associated with the regular subspace places super-conductance on a small subset of \mathbb{R} and flies like lights over it, and conversely, the one associated with the regular extension is split by insulators and moves hard on the material with stretched resistance.

This paper is organized as follows. In Section 2, we summarize some basic properties concerning regular Dirichlet extensions in the general setting. Particularly, a regular Dirichlet extension of one-dimensional Brownian motion must be strongly local and recurrent. Thus the associated Hunt process is a conservative diffusion process on \mathbb{R} . In Section 3, we treat the problem (Q.2) for one-dimensional Brownian motion. The main theorem, i.e. Theorem 3.1, characterizes all the regular Dirichlet extensions of one-dimensional Brownian motion. Several examples of proper regular Dirichlet extensions are presented in Section 3.3. In Section 4 and Section 5, we consider the problem (Q.3) for one-dimensional Brownian motion and describe the structures of regular Dirichlet extensions via the trace method introduced in [14]. We attain the expression of the orthogonal complement \mathcal{G} of $H_c^1(\mathbb{R})$ in \mathcal{F}_e in Theorem 4.2 and the regular representation of the restriction of \mathcal{G} on each invariant interval via the darning method in Theorem 4.3. The darning process turns out to be a Brownian motion being time changed by a Radon smooth measure. The trace formulae of regular Dirichlet extension and the one-dimensional Brownian motion are formulated in Theorem 5.1. In Corollary 5.1, a special case of Theorem 5.1 is emphasized, in which the trace Dirichlet forms of one-dimensional Brownian motion and its regular Dirichlet extension are both pure-jump type and have the same jumping measure but different Dirichlet spaces. The essential differences between them are illustrated in Corollary 5.3.

Notations and terminologies

Let us put some often used notations here for handy reference, though we may restate their definitions when they appear.

For $a < b$, $\langle a, b \rangle$ is an interval where a or b may or may not be contained $\langle a, b \rangle$. Notations m , dx and $|\cdot|$ stand for the Lebesgue measure on \mathbb{R} throughout the paper if no confusion caused. The restrictions of a measure μ and a function f on I are denoted by $\mu|_I$ and $f|_I$ respectively. The notation ‘:=’ is read as ‘to be defined as’.

For a scale function τ (i.e. a continuous and strictly increasing function) on some interval I , $d\tau$ represents its associated Lebesgue-Stieltjes measure on I and τ^{-1} stands for its inverse function. Set $\tau(I) := \{\tau(x) : x \in I\}$. For two measures μ and ν , $\mu \ll \nu$ means μ is absolutely continuous with respect to ν . Given a scale function τ on I and another function f on I , $f \ll \tau$ means $f = g \circ \tau$ for some absolutely continuous function g and $\frac{df}{d\tau} := g' \circ \tau$. Given an interval I , the classes $C_c(I)$, $C_c^1(I)$ and $C_c^\infty(I)$ denote the spaces of all continuous functions on I with compact support, all continuously differentiable functions with compact support and all infinitely differentiable functions with compact support, respectively.

For a Markov process $X = (X_t)_{t \geq 0}$ associated with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$, $(P_t)_{t \geq 0}$ represents its probability transition semigroup, i.e. $P_t f(x) := \mathbf{E}_x f(X_t)$ for any $t \geq 0$, $f \in b\mathcal{B}(E)$ and $x \in E$, where $b\mathcal{B}(E)$ is all bounded Borel measurable functions on E . The semigroup $(P_t)_{t \geq 0}$ is a strongly continuous contraction semigroup on $L^2(E, m)$ associated with $(\mathcal{E}, \mathcal{F})$. If A is an invariant set of X (see Section 3.1), then the restriction of $(\mathcal{E}, \mathcal{F})$ to A is denoted by $(\mathcal{E}^A, \mathcal{F}^A)$ and the restriction of X to A is denoted by X^A . If U is an open subset of E , then the part Dirichlet form of $(\mathcal{E}, \mathcal{F})$ on U is denoted by $(\mathcal{E}_U, \mathcal{F}_U)$ and the part process of X on U is denoted by X_U . All terminologies about Dirichlet forms are standard and we refer them to [1, 7].

2. Basic properties of regular Dirichlet extensions

In this section, we summarize several basic properties of regular Dirichlet extensions or subspaces, which are contained in [10–13, 15, 18]. We always fix two regular Dirichlet forms $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^2, \mathcal{F}^2)$ on $L^2(E, m)$ and assume

that $(\mathcal{E}^1, \mathcal{F}^1)$ is a regular Dirichlet subspace of $(\mathcal{E}^2, \mathcal{F}^2)$, equivalently $(\mathcal{E}^2, \mathcal{F}^2)$ is a regular Dirichlet extension of $(\mathcal{E}^1, \mathcal{F}^1)$.

The first theorem is taken from [12], and it characterizes Beurling–Deny decompositions of regular Dirichlet subspaces or extensions.

Theorem 2.1 (Theorem 2.1, [12]). *Let (J_1, k_1) and (J_2, k_2) be the jumping and killing measures in the Beurling–Deny decompositions of $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^2, \mathcal{F}^2)$ respectively. Then $J_1 = J_2$ and $k_1 = k_2$.*

As a corollary of this result, if one of $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^2, \mathcal{F}^2)$ is strongly local or local, then the other one has to be strongly local or local. Particularly, both regular Dirichlet subspaces and extensions of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ must be a strongly local Dirichlet form.

The following proposition describes the quasi notions of $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^2, \mathcal{F}^2)$. Its proof is obvious from the fact $\text{Cap}^1(A) \geq \text{Cap}^2(A)$ for any appropriate set A , where Cap^1 and Cap^2 are the 1-Capacities of $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^2, \mathcal{F}^2)$ respectively.

Proposition 2.1 (Remark 1.1, [12]). *The following assertions hold.*

- (1) *An \mathcal{E}^1 -polar set is \mathcal{E}^2 -polar.*
- (2) *An \mathcal{E}^1 -nest is also an \mathcal{E}^2 -nest.*
- (3) *An \mathcal{E}^1 -quasi continuous function is also \mathcal{E}^2 -quasi continuous.*

Another proposition states the relation of their global properties.

Proposition 2.2 (Remark 3.5, [13]). *The following assertions hold.*

- (1) *If a Dirichlet form is transient, then its regular Dirichlet subspace is also transient.*
- (2) *If a Dirichlet form is recurrent, then its regular Dirichlet extension is also recurrent.*
- (3) *If a Dirichlet form is strongly local and irreducible, then its regular Dirichlet subspace is also irreducible.*

Proof. The first and second assertions are the direct corollaries of [7, Theorem 1.6.4]. The third assertion follows from Proposition 2.1(3) and [7, Theorem 4.6.4]. \square

The following characterization via the extended Dirichlet spaces is very simple but sometimes very useful.

Proposition 2.3. *Let \mathcal{F}_e^1 and \mathcal{F}_e^2 be the extended Dirichlet spaces of $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^2, \mathcal{F}^2)$ respectively. Then $(\mathcal{E}^1, \mathcal{F}^1)$ is a regular Dirichlet subspace of $(\mathcal{E}^2, \mathcal{F}^2)$ if and only if*

$$\mathcal{F}_e^1 \subset \mathcal{F}_e^2, \quad \mathcal{E}^2(f, g) = \mathcal{E}^1(f, g), \quad f, g \in \mathcal{F}_e^1.$$

Furthermore, if $(\mathcal{E}^1, \mathcal{F}^1)$ is a proper one in addition, $\mathcal{F}_e^1 \neq \mathcal{F}_e^2$.

The next proposition will be frequently used in Section 3.2.4. The proof is direct from the definition of part Dirichlet form (cf. [7, Section 4.4]).

Proposition 2.4. *Let U be an open subset of E . The part Dirichlet forms of $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^2, \mathcal{F}^2)$ on U are denoted by $(\mathcal{E}_U^1, \mathcal{F}_U^1)$ and $(\mathcal{E}_U^2, \mathcal{F}_U^2)$. Then $(\mathcal{E}_U^1, \mathcal{F}_U^1)$ is a regular Dirichlet subspace of $(\mathcal{E}_U^2, \mathcal{F}_U^2)$.*

3. Representation of regular Dirichlet extensions

3.1. Main result

The existence problem (Q.1) for the one-dimensional Brownian motion is already answered in the next paragraph after this problem in Section 1. Indeed, the one-dimensional Brownian motion has proper regular Dirichlet extensions

such as those with the scale functions in the class (1.4). Particularly, they are all irreducible. In this section, we shall treat the second problem (Q.2).

Before presenting the main theorem, we need to do some preparatory works. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E, m)$ associated with a symmetric Hunt process X . A Borel subset $A \subset E$ is called an invariant set of $(X_t)_{t \geq 0}$ provided for any $x \in A$,

$$\mathbf{P}_x(X_t \in A, \forall t) = 1.$$

Clearly, the restriction denoted by X^A or $(X_t^A)_{t \geq 0}$ of X to A is still a Hunt process and symmetric with respect to $m|_A$. Its associated Dirichlet form on $L^2(A, m|_A)$ is (see [1, Section 2.1])

$$\mathcal{F}^A := \{f|_A : f \in \mathcal{F}\}, \quad \mathcal{E}^A(f|_A, g|_A) := \mathcal{E}(1_A f, 1_A g), \quad f, g \in \mathcal{F}.$$

We call $(\mathcal{E}^A, \mathcal{F}^A)$ the restriction of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ to the invariant set A .

Another preparatory work is to introduce a few classes of scale functions. Let $a < b$ and $I = \langle a, b \rangle$ be an interval such that a or b may or may not be in I . In other words, $I = (a, b)$, $(a, b]$, $[a, b)$ or $[a, b]$. Particularly, a or b may be infinity if a or $b \notin I$. The interior of I is denoted by $\overset{\circ}{I} := (a, b)$. A scale function \mathfrak{t} on I is a strictly increasing and continuous function on I . Thus we can always define its limit at boundary

$$\mathfrak{t}(a) := \lim_{x \downarrow a} \mathfrak{t}(x) \geq -\infty, \quad \left(\text{resp. } \mathfrak{t}(b) := \lim_{x \uparrow b} \mathfrak{t}(x) \leq \infty \right)$$

no matter a (resp. b) belongs to I or not. Denote all the scale functions \mathfrak{t} on I satisfying

$$dx \ll d\mathfrak{t}, \quad \frac{dx}{d\mathfrak{t}} = 0 \quad \text{or } 1, d\mathfrak{t}\text{-a.e.}$$

by $\mathbf{T}(I)$ (see (1.4)). A subset of $\mathbf{T}(I)$ is defined as

$$\mathbf{T}_\infty(I) := \{ \mathfrak{t} \in \mathbf{T}(I) \mid \mathfrak{t}(a) = -\infty \text{ iff } a \notin I, \mathfrak{t}(b) = \infty \text{ iff } b \notin I \}, \quad (3.1)$$

where ‘iff’ stands for ‘if and only if’.

Remark 3.1. Note that in any case the class $\mathbf{T}_\infty(I)$ of scale functions is not empty. For example let us treat the case $I = [a, b)$ with $b < \infty$. The other cases can be treated similarly. By [3, Theorem 4.1], we need only to find a scale function

$$s : [0, \infty) \rightarrow [a, b)$$

such that $ds \ll dx$, $s' = 0$ or 1 . Then $\mathfrak{t} := s^{-1} \in \mathbf{T}_\infty(I)$.

Take a measure-dense subset $G \subset [0, \infty)$. For example, assume $\{q_n : n \geq 1\}$ is the total of positive rational numbers and let

$$G := \left\{ \bigcup_{n \geq 1} B\left(q_n, \frac{1}{2^n}\right) \right\} \cap [0, \infty),$$

where $B(x, r) := \{y : |y - x| < r\}$. Clearly, the Lebesgue measure of G is positive, i.e. $|G| > 0$. Set $k := |G|/|b - a|$ and $G' := \{x : kx \in G\}$. Note that G' is still measure-dense. In fact, take any open interval $(c, d) \subset I$, we have

$$|G' \cap (c, d)| = \frac{1}{k} |G \cap (kc, kd)| > 0.$$

Let

$$s(x) := \int_0^x 1_{G'}(y) dy + a, \quad x \geq 0.$$

Then s is strictly increasing and absolutely continuous, $s' = 1_{G'}$, $s(0) = a$ and

$$s(\infty) = \int_0^\infty 1_{G'}(y) dy + a = \frac{1}{k} \cdot |G| + a = b.$$

Remark 3.2. Similar to [3, Theorem 4.1], we may also deduce that any scale function $\mathfrak{t} \in \mathbf{T}_\infty(I)$ can be written as

$$\mathfrak{t}(x) = x + c(x)$$

for a non-decreasing singular continuous function c on I .

Note that the scale functions of an irreducible diffusion process are not unique and may differ by a constant if its speed measure is fixed. To avoid this uncertainty, we make the following restriction on $\mathbf{T}_\infty(I)$:

$$\mathbf{T}_\infty^0(I) := \{\mathfrak{t} \in \mathbf{T}_\infty(I) : \mathfrak{t}(e) = 0\}, \quad (3.2)$$

where e is a fixed point in (a, b) : $e := (a + b)/2$ if $a > -\infty, b < \infty$; $e := b - 1$ if $a = -\infty, b < \infty$; $e := a + 1$ if $a > -\infty, b = \infty$ and $e := 0$ if $a = -\infty, b = \infty$. The choice of e is not essential.

Now we are in a position to state the main result of this section.

Theorem 3.1. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet extension of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ on $L^2(\mathbb{R})$ if and only if there exist a set of at most countable disjoint intervals $\{I_n = \langle a_n, b_n \rangle : n \geq 1\}$, satisfying that $(\bigcup_{n \geq 1} I_n)^c$ has Lebesgue measure zero, and a scale function $\mathfrak{t}_n \in \mathbf{T}_\infty^0(I_n)$ for each $n \geq 1$ such that*

$$\begin{aligned} \mathcal{F} &= \left\{ f \in L^2(\mathbb{R}) : f|_{I_n} \in \mathcal{F}^n, \sum_{n \geq 1} \mathcal{E}^n(f|_{I_n}, f|_{I_n}) < \infty \right\}, \\ \mathcal{E}(f, g) &= \sum_{n \geq 1} \mathcal{E}^n(f|_{I_n}, g|_{I_n}), \quad f, g \in \mathcal{F}, \end{aligned} \quad (3.3)$$

where for each $n \geq 1$, $(\mathcal{E}^n, \mathcal{F}^n)$ is expressed as

$$\begin{aligned} \mathcal{F}^n &= \left\{ f \in L^2(I_n) : f \ll \mathfrak{t}_n, \int_{I_n} \left(\frac{df}{d\mathfrak{t}_n} \right)^2 d\mathfrak{t}_n < \infty \right\}, \\ \mathcal{E}^n(f, g) &= \frac{1}{2} \int_{I_n} \frac{df}{d\mathfrak{t}_n} \frac{dg}{d\mathfrak{t}_n} d\mathfrak{t}_n, \quad f, g \in \mathcal{F}^n. \end{aligned} \quad (3.4)$$

Moreover, the intervals $\{I_n : n \geq 1\}$ and scale functions $\{\mathfrak{t}_n \in \mathbf{T}_\infty^0(I_n) : n \geq 1\}$ are uniquely determined, if the difference of order is ignored.

Remark 3.3. Denote the associated Hunt process of $(\mathcal{E}, \mathcal{F})$ by $X = (X_t)_{t \geq 0}$. Set $G := \bigcup_{n \geq 1} \overset{\circ}{I}_n$ and $F := G^c$. Note that G is an open set. We would like to make a few remarks for the theorem above.

(1) Though the intervals are mutually disjoint, they may have common endpoints. For example, $I_n = (a_n, b_n]$ and $I_m = (a_m, b_m)$ with $b_n = a_m$.

(2) Let $\tilde{\Lambda}_{pr} := \{a_n : a_n \in I_n\}$ and $\tilde{\Lambda}_{pl} := \{b_n : b_n \in I_n\}$. Further set $\tilde{\Lambda}_r := F \setminus \tilde{\Lambda}_{pl}$ and $\tilde{\Lambda}_l := F \setminus \tilde{\Lambda}_{pr}$. Note that neither $\tilde{\Lambda}_l$ nor $\tilde{\Lambda}_r$ is necessarily closed. For example, let K be the standard Cantor set in $[0, 1]$ and set

$$\bigcup_n I_n := K^c \cup (-\infty, 0] \cup [1, \infty).$$

Then $\tilde{\Lambda}_r = K \setminus \{0\}$ and $\tilde{\Lambda}_l = K \setminus \{1\}$. Neither of them is closed. Nevertheless, $\tilde{\Lambda}_l$ (resp. $\tilde{\Lambda}_r$) is closed from the right (resp. left), i.e. if $x_n \in \tilde{\Lambda}_l$ (resp. $\tilde{\Lambda}_r$) and $x_n \downarrow x$ (resp. $x_n \uparrow x$), then $x \in \tilde{\Lambda}_l$ (resp. $\tilde{\Lambda}_r$). This fact can be proved as

follows. Assume that $x_n \in \tilde{\Lambda}_l$ and $x_n \downarrow x$. Clearly $x \notin G$ since G is open. If $x \in \tilde{\Lambda}_{pr}$, then there exists an interval I_n such that $I_n = [x, b_n]$ or $[x, b_n]$ with $x < b_n$. Note that $(x, b_n) \subset G$. This leads to a contradiction with $x_n \downarrow x$ and $x_n \in \tilde{\Lambda}_l$.

It is also worth noting that $\tilde{\Lambda}_{pr}$, $\tilde{\Lambda}_{pl}$, $\tilde{\Lambda}_r$, $\tilde{\Lambda}_l$ coincide with the classes of right shunt points, left shunt points, right singular points and left singular points of X as defined in Definition 3.1 respectively. Moreover, G and F are identified with the classes of regular points and singular points of X .

(3) For each n , I_n is an invariant set of X and X^{I_n} is an irreducible and recurrent diffusion process with the scale function τ_n , the speed measure $m|_{I_n}$ and no killing inside (cf. [1, Theorem 2.2.11]). In other words,

$$\mathbf{P}_x(X_t^{I_n} = y, \exists t > 0) = 1, \quad \forall x, y \in I_n.$$

Furthermore, if $a_n \in I_n$, then X^{I_n} is reflected at the left endpoint a_n . If $a_n \notin I_n$, then X^{I_n} never reaches it in finite time (cf. [8] and [1, Example 3.5.7]). This also implies that any single point subset $\{x\} \subset I_n$ is not an m -polar set relative to X .

(4) The set $\tilde{\Lambda}_l \cap \tilde{\Lambda}_r = (\bigcup_{n \geq 1} I_n)^c$ is an m -polar set relative to X . Indeed, $m(\tilde{\Lambda}_r \cap \tilde{\Lambda}_l) = 0$, and for any $x \notin \tilde{\Lambda}_r \cap \tilde{\Lambda}_l$, there exists an interval I_n such that $x \in I_n$. Since I_n is an invariant set of X , we can conclude

$$\mathbf{P}_x(\sigma_{\tilde{\Lambda}_r \cap \tilde{\Lambda}_l} < \infty) = 0.$$

Note that any regular Dirichlet form corresponds to a Hunt process uniquely up to an m -polar set. The most convenient way to treat the part of X on $\tilde{\Lambda}_r \cap \tilde{\Lambda}_l$ is to enforce the process X starting from a point $x \in \tilde{\Lambda}_r \cap \tilde{\Lambda}_l$ to stay at x forever.

(5) The fact that $(\bigcup_{n \geq 1} I_n)^c$ has Lebesgue measure zero implies that it is nowhere dense.

Corollary 3.1. *An irreducible regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R})$ is a regular Dirichlet extension of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ if and only if there exists a unique scale function $\tau \in \mathbf{T}_\infty^0(\mathbb{R})$ such that*

$$\mathcal{F} = \left\{ f \in L^2(\mathbb{R}) : f \ll \tau, \int_{\mathbb{R}} \left(\frac{df}{d\tau} \right)^2 d\tau < \infty \right\},$$

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}} \frac{df}{d\tau} \frac{dg}{d\tau} d\tau, \quad f, g \in \mathcal{F}.$$

3.2. Proof of Theorem 3.1

The proof of Theorem 3.1 will be divided into several parts. We note here that the last assertion about the uniqueness is obvious from Remark 3.3(3). We shall prove necessity first and then sufficiency. To prove the necessity, we need to review one-dimensional or linear diffusions.

3.2.1. Classification of points for linear diffusions

In this part, we recall some results on the classification of points for linear diffusion. For those results which may be known to experts but not on standard references [8, Section 5] and [9, Section 3], we will give a proof.

Let $X = (X_t)$ be a diffusion process on \mathbb{R} , i.e. a strong Markov process with continuous paths. Without loss of generality, we always assume that X is conservative, in other words, the lifetime ζ of X is infinite \mathbf{P}_x -a.s. for any $x \in \mathbb{R}$. Now fix a point $x \in \mathbb{R}$. Note that

$$e^\pm := \mathbf{P}_x(\sigma_{x\pm} = 0) = 0 \quad \text{or} \quad 1$$

by Blumenthal's 0-1 law, where $\sigma_{x+} := \inf\{t > 0 : X_t > x\}$, $\sigma_{x-} := \inf\{t > 0 : X_t < x\}$.

Definition 3.1. A point $x \in \mathbb{R}$ is called

- (1) regular, denoted by $x \in \Lambda_2$, if $e^+ = e^- = 1$;
- (2) singular, if $e^+e^- = 0$;
- (3) left singular, denoted by $x \in \Lambda_l$, if $e^+ = 0$; right singular, denoted by $x \in \Lambda_r$, if $e^- = 0$;

- (4) left shunt, denoted by $x \in \Lambda_{pl}$, if $e^+ = 0, e^- = 1$; right shunt, denoted by $x \in \Lambda_{pr}$, if $e^- = 0, e^+ = 1$;
 (5) a trap, denoted by $x \in \Lambda_t$, if $e^+ = e^- = 0$.

The sets $\Lambda_2, \Lambda_l, \Lambda_r, \Lambda_{pl}, \Lambda_{pr}$ and Λ_t stand for the subsets of \mathbb{R} containing all the regular points, left singular points, right singular points, left shunt points, right shunt points and traps of X respectively. The set of all singular points is $\Lambda_l \cup \Lambda_r$.

Clearly, $\Lambda_2 = (\Lambda_r \cup \Lambda_l)^c$, $\Lambda_{pr} \cap \Lambda_l = \emptyset$, $\Lambda_{pl} \cap \Lambda_r = \emptyset$ and $\Lambda_r \cap \Lambda_l = \Lambda_t$. The following facts will be very useful in proving Theorem 3.1.

Lemma 3.1.

- (1) Assume $a < b < c$. Then

$$\mathbf{P}_a(\sigma_c < \infty) = \mathbf{P}_a(\sigma_b < \infty)\mathbf{P}_b(\sigma_c < \infty),$$

$$\mathbf{P}_c(\sigma_a < \infty) = \mathbf{P}_c(\sigma_b < \infty)\mathbf{P}_b(\sigma_a < \infty).$$

- (2) A point $b \in \Lambda_r$ (resp. $b \in \Lambda_l$) if and only if $\mathbf{P}_b(X_t \geq b, \forall t) = 1$ (resp. $\mathbf{P}_b(X_t \leq b, \forall t) = 1$). Thus $b \in \Lambda_t$ if and only if $\mathbf{P}_b(X_t = b, \forall t) = 1$.

- (3) Fix $b \in \Lambda_r$ (resp. $b \in \Lambda_l$). Then for any $a > b$ (resp. $a < b$),

$$\mathbf{P}_a(X_t \geq b, \forall t) = 1, \quad (\text{resp. } \mathbf{P}_a(X_t \leq b, \forall t) = 1).$$

- (4) Fix $b \in \Lambda_{pr}$ (resp. $b \in \Lambda_{pl}$). Then there exists a point $a > b$ (resp. $a < b$) such that

$$\mathbf{P}_b(\sigma_a < \infty) > 0.$$

- (5) The left singular set Λ_l is closed from the right, i.e. if $x_n \in \Lambda_l$ and $x_n \downarrow x, x \in \Lambda_l$. The right singular set Λ_r is closed from the left, i.e. if $x_n \in \Lambda_r$ and $x_n \uparrow x, x \in \Lambda_r$.

- (6) The regular set Λ_2 is open. Thus the singular set $\Lambda_r \cup \Lambda_l$ is closed.

- (7) If each point in an open interval (a, b) is regular, i.e. $(a, b) \subset \Lambda_2$, then for any $x, y \in (a, b)$,

$$\mathbf{P}_x(\sigma_y < \infty)\mathbf{P}_y(\sigma_x < \infty) > 0.$$

Proof. For the first fact, since in the sense of \mathbf{P}_a -a.s., $\sigma_c > \sigma_b$, it follows that $\sigma_c = \sigma_b + \sigma_c \circ \theta_{\sigma_b}$ where (θ_t) are the shift operators of X , i.e. $X_{t+s} = X_t \circ \theta_s$ for any $t, s \geq 0$. By the strong Markovian property of X , we have

$$\begin{aligned} \mathbf{P}_a(\sigma_c < \infty) &= \mathbf{P}_a(\sigma_b < \infty, \sigma_c \circ \theta_{\sigma_b} < \infty) \\ &= \mathbf{P}_a(\sigma_b < \infty, \mathbf{P}_{X_{\sigma_b}}(\sigma_c < \infty)) \\ &= \mathbf{P}_a(\sigma_b < \infty)\mathbf{P}_b(\sigma_c < \infty). \end{aligned}$$

Another assertion can be deduced similarly.

For the second fact, we need only to remark that

$$\mathbf{P}_x(\sigma_{x\pm} = 0) = \mathbf{P}_x(\sigma_{x\pm} < \infty)$$

for any $x \in \mathbb{R}$ (cf. [9, Section 3.3, 10a]).

For the third fact, fix $b \in \Lambda_r$ and $a > b$. For any point $y < b$, it follows from (2) that

$$\mathbf{P}_b(\sigma_y < \infty) = \mathbf{P}_b(X_t = y, \exists t) = 0.$$

Thus from (1) we may deduce that $\mathbf{P}_a(\sigma_y < \infty) = 0$ for any $y < b$. Take a sequence $y_n \uparrow b$. Then $\mathbf{P}_a(\sigma_{y_n} < \infty) = 0$ implies

$$\mathbf{P}_a\left(\bigcup_t \{X_t \leq y_n\}\right) = 0.$$

Hence

$$0 = \mathbf{P}_a\left(\bigcup_n \bigcup_t \{X_t \leq y_n\}\right) = \mathbf{P}_a\left(\bigcup_t \{X_t < b\}\right) = \mathbf{P}_a(\{X_t \geq b, \forall t\}^c).$$

Another assertion is similar.

For the forth fact, fix $b \in \Lambda_{pr}$. Suppose that for any $y > b$,

$$\mathbf{P}_b(\sigma_y < \infty) = \mathbf{P}_b\left(\bigcup_t \{X_t \geq y\}\right) = 0.$$

Take a sequence $y_n \downarrow b$ and then

$$0 = \mathbf{P}_b\left(\bigcup_n \bigcup_t \{X_t \geq y_n\}\right) = \mathbf{P}_b\left(\bigcup_t \{X_t > b\}\right).$$

This implies $\mathbf{P}_b(X_t \leq b, \forall t) = 1$ and thus $b \in \Lambda_l$ by (2), which contradicts with $\Lambda_{pr} \cap \Lambda_l = \emptyset$.

The fifth and sixth facts can be found in [9, Section 3.4].

For the final fact, note that for any regular point ξ , there exist two points c, d close enough to ξ such that $c < \xi < d$ and $\mathbf{P}_c(\sigma_d < \infty)\mathbf{P}_d(\sigma_c < \infty) > 0$ (cf. [9, Section 3.4]). Now fix a regular interval $(a, b) \subset \Lambda_2$ and assume that $x, y \in (a, b)$, $x < y$ and $\mathbf{P}_x(\sigma_y < \infty) = 0$. Set

$$A_x := \{z > x : \mathbf{P}_x(\sigma_z < \infty) = 0\}.$$

Clearly, $y \in A_x$. Moreover, if $z \in A_x$ and $z' > z$ then $z' \in A_x$. Let $w := \inf A_x$. If $w = x$, then $\mathbf{P}_x(\sigma_{x+} < \infty) = 0$ and $x \in \Lambda_l$, which contradicts with $x \in \Lambda_2$. If $w > x$, note that $w \in (a, b)$ is a regular point. It follows that there exist two points w_1, w_2 with $x < w_1 < w < w_2 < y$ such that

$$\mathbf{P}_{w_1}(\sigma_{w_2} < \infty)\mathbf{P}_{w_2}(\sigma_{w_1} < \infty) > 0.$$

Since $w_1 \notin A_x$ and $w_2 \in A_x$, we have $\mathbf{P}_x(\sigma_{w_1} < \infty) > 0$ and $\mathbf{P}_x(\sigma_{w_2} < \infty) = 0$. However, from (1) we can deduce that

$$\mathbf{P}_x(\sigma_{w_2} < \infty) = \mathbf{P}_x(\sigma_{w_1} < \infty)\mathbf{P}_{w_1}(\sigma_{w_2} < \infty) > 0,$$

which leads to a contradiction. That completes the proof. \square

Intuitively, a left (resp. right) singular point looks like a ‘wall’ to the left (resp. right), and no trajectory can run through it from its left (resp. right) side to the right (resp. left). The left (resp. right) shunt point means more: the trajectories starting from this point must enter its left (resp. right) side in finite time.

We need to point out X possibly admits a left or right shunt interval (a, b) , i.e. $(a, b) \subset \Lambda_{pr}$ or Λ_{pl} . For example, let $X_t = X_0 + t$. Then $\Lambda_{pr} = \mathbb{R}$. This example also indicates that for a right shunt point b , there may exist another point $a < b$ such that the trajectory starting from a can run through b to its right side. We shall see in the next part that these behaviors are not allowed under the symmetry assumption.

3.2.2. Linear diffusion under the symmetry

In this part, we further assume that X is symmetric with respect to the Lebesgue (or any fully supported Radon) measure m on \mathbb{R} . In other words, the semigroup $(P_t)_{t \geq 0}$ of X satisfies

$$(P_t f, g)_m = (f, P_t g)_m, \quad f, g \in \mathcal{B}_+(\mathbb{R}), t \geq 0, \quad (3.5)$$

where $(f, g)_m$ and $\mathcal{B}_+(\mathbb{R})$ stand for the inner product of $L^2(\mathbb{R}, m)$ and the set of all the non-negative Borel measurable functions on \mathbb{R} , respectively.

Lemma 3.2. Fix a right (resp. left) singular point $b \in \Lambda_r$ (resp. Λ_l). Under the symmetry, for any $a < b$ (resp. $a > b$), it holds that

$$\mathbf{P}_a(\sigma_b < \infty) = 0.$$

Particularly, under the symmetry if $b \in \Lambda_t$ then for any $a \neq b$,

$$\mathbf{P}_a(\sigma_b < \infty) = 0.$$

Proof. We first consider $b \in \Lambda_{pr}$. The case of $b \in \Lambda_{pl}$ is similar. It follows from Lemma 3.1(2, 3) that for any $x \geq b$,

$$\mathbf{P}_x(X_t \geq b, \forall t) = 1.$$

Set $f(x) := 1_{(-\infty, b)}(x)$, $g(x) := 1_{[b, \infty)}(x)$. Since for any $x \geq b$,

$$P_t f(x) = \mathbf{P}_x(X_t < b) = 0,$$

the left side of (3.5) equals 0. Thus for m -a.e. $x \in (-\infty, b)$,

$$0 = P_t g(x) = \mathbf{P}_x(X_t \geq b). \quad (3.6)$$

It follows that for m -a.e. $x < b$, (3.6) holds for any $t \in \mathbb{Q} \cap (0, \infty)$, where \mathbb{Q} is the set of all rational numbers. Take a point $x < b$ such that (3.6) holds for any $t \in \mathbb{Q} \cap (0, \infty)$. We have

$$\mathbf{P}_x\left(\bigcup_{t \in \mathbb{Q}, t > 0} \{X_t \geq b\}\right) = 0.$$

It follows that

$$\mathbf{P}_x\left(\bigcap_{t \in \mathbb{Q}, t > 0} \{X_t < b\}\right) = \mathbf{P}_x\left(\left\{\bigcup_{t \in \mathbb{Q}, t > 0} \{X_t \geq b\}\right\}^c\right) = 1.$$

Since X is continuous, we may conclude that

$$\mathbf{P}_x\left(\bigcap_t \{X_t \leq b\}\right) = 1. \quad (3.7)$$

As a result,

$$\mathbf{P}_x(\sigma_y < \infty) = 0 \quad (3.8)$$

for any $y > b$. Note that m has full support and thus we may take a sequence $x_n \uparrow b$ such that (3.8) holds for $x = x_n$. For any $z < x_n$, it follows from Lemma 3.2(1) that $\mathbf{P}_z(\sigma_y < \infty) = 0$ for any $y > b$. Hence $\mathbf{P}_z(\sigma_y < \infty) = 0$ for any $z < b < y$. Therefore, from Lemma 3.1(1) and (4), we assert that $\mathbf{P}_x(\sigma_b < \infty) = 0$ for any $x < b$.

Now assume that $b \in \Lambda_t$. Let $f = 1_{[b]}$. Clearly, the right side of (3.5) equals 0 for any $g \in \mathcal{B}_+(\mathbb{R})$. By the symmetry, $P_t f(x) = 0$ for m -a.e. $x \in \mathbb{R}$. On the other hand, for any $x \neq b$,

$$P_t f(x) = \mathbf{E}_x(f(X_t); t \geq \sigma_b) = \mathbf{P}_x(t \geq \sigma_b).$$

This implies for m -a.e. $x \in \mathbb{R}$ with $x \neq b$,

$$\mathbf{P}_x(\sigma_b \leq n) = 0, \quad \forall n \in \mathbb{N}.$$

Thus $\mathbf{P}_x(\sigma_b < \infty) = 0$ for m -a.e. $x \in \mathbb{R}$ with $x \neq b$. By the similar arguments as above, we can conclude that $\mathbf{P}_a(\sigma_b < \infty) = 0$ for any $a \neq b$. \square

The following lemma implies that the diffusion X is non-decreasing (resp. non-increasing) in the right (resp. left) singular interval. However, if X is symmetric, then any point in a right or left singular interval must be a trap.

Lemma 3.3.

(1) If an open interval $(a, b) \subset \Lambda_r$ (resp. Λ_l), then for any $x \in (a, b)$,

$$\mathbf{P}_x(X_t \geq X_s, \forall s < t \leq \sigma_b) = 1$$

(resp. $\mathbf{P}_x(X_t \leq X_s, \forall s < t \leq \sigma_a) = 1$).

(2) Under the symmetry, if $(a, b) \subset \Lambda_r$ (resp. Λ_l), then $(a, b) \subset \Lambda_t$.

Proof. We first prove (1) and only consider the case $(a, b) \subset \Lambda_r$. Since any point $x \in (a, b)$ is right singular, we have

$$\mathbf{P}_x(X_t \geq X_0, \forall t) = 1$$

by Lemma 3.1(2). From the Markovian property of X , we can deduce that for fixed $s < t$,

$$\begin{aligned} \mathbf{P}_x(X_t < X_s, s < t \leq \sigma_b) &= \mathbf{P}_x(\{X_{t-s} < X_0\} \circ \theta_s, t-s \leq \sigma_b \circ \theta_s, s < \sigma_b) \\ &= \mathbf{P}_x(\mathbf{P}_{X_s}(X_{t-s} < X_0, t-s \leq \sigma_b); s < \sigma_b) \\ &= 0. \end{aligned}$$

The last equality above follows from the fact that, \mathbf{P}_x -a.s. on $\{s < \sigma_b\}$, $X_s \in (a, b)$. It is then clear that

$$\mathbf{P}_x\left(\bigcup_{s < t \leq \sigma_b, s, t \in \mathbb{Q}} \{X_t < X_s\}\right) = 0.$$

Thus

$$1 = \mathbf{P}_x\left(\bigcap_{s < t \leq \sigma_b, s, t \in \mathbb{Q}} \{X_t \geq X_s\}\right) = \mathbf{P}_x\left(\bigcap_{s < t \leq \sigma_b} \{X_t \geq X_s\}\right).$$

For the second assertion (2), fix $x \in (a, b) \subset \Lambda_r$. Take another point w in (a, b) such that $x < w$. Mimicking the proof of (3.7), we deduce that

$$\mathbf{P}_x(\sigma_{w+} < \infty) = 0.$$

Take a sequence $w_n \downarrow x$ and we then have

$$0 = \mathbf{P}_x\left(\bigcup_n \bigcup_t \{X_t > w_n\}\right) = \mathbf{P}_x\left(\bigcup_t \{X_t > x\}\right).$$

Thus

$$\mathbf{P}_x(X_t \leq x, \forall t) = 1.$$

It follows from Lemma 3.1(2) that $x \in \Lambda_l$ and then $x \in \Lambda_r \cap \Lambda_l = \Lambda_t$. It concludes that $(a, b) \subset \Lambda_t$. □

3.2.3. A merging theorem

Before proving Theorem 3.1, we need a result to merge a sequence of Dirichlet forms into a new one. Because it holds in general and may have independent interest, we state it as a theorem.

Theorem 3.2. Let $E := \bigcup_{n \geq 1} E_n$ with $\{E_n : n \geq 1\}$ disjoint be a measurable space and m a σ -finite measure on it. Denote the restriction of m to E_n by m_n . Assume that $(\mathcal{E}^n, \mathcal{F}^n)$ is a Dirichlet form on $L^2(E_n, m_n)$. Then

$$\mathcal{F} := \left\{ f \in L^2(E, m) : f|_{E_n} \in \mathcal{F}^n, \sum_{n \geq 1} \mathcal{E}^n(f|_{E_n}, f|_{E_n}) < \infty \right\},$$

$$\mathcal{E}(f, g) := \sum_{n \geq 1} \mathcal{E}^n(f|_{E_n}, f|_{E_n}), \quad f, g \in \mathcal{F}$$
(3.9)

is a Dirichlet form on $L^2(E, m)$.

Proof. Let (T_t^n) be the semigroup of $(\mathcal{E}^n, \mathcal{F}^n)$ on $L^2(E_n, m_n)$. For any $f \in L^2(E, m)$, $t \geq 0$, define

$$(T_t f)|_{E_n}(x) := \sum_{n \geq 1} T_t^n(f|_{E_n})(x), \quad n \geq 1. \quad (3.10)$$

Set $f^n := f|_{E_n}$ for convenience. We assert that (T_t) is a strongly continuous and symmetric contraction semigroup on $L^2(E, m)$. The semigroup property is clear from those of $\{(T_t^n) : n \geq 1\}$. For the contraction property, fix $f \in L^2(E, m)$. The L^2 -norm of $L^2(E_n, m_n)$, $L^2(E, m)$ are denoted by $\|\cdot\|_{E_n}$, $\|\cdot\|_E$ for short. Note that $\|f\|_E^2 = \sum_{n \geq 1} \|f^n\|_{E_n}^2$. Then we have

$$\|T_t f\|_E = \sum_{n \geq 1} \|T_t^n f^n\|_{E_n}^2 \leq \sum_{n \geq 1} \|f^n\|_{E_n}^2 = \|f\|_E^2.$$

To prove strong continuity, we fix $f \in L^2(E, m)$ and $\epsilon > 0$, and take an integer n large enough such that $\sum_{k > n} \|f^k\|_{E_k}^2 < \epsilon/4$. By the strong continuity of $\{(T_t^k) : 1 \leq k \leq n\}$, we may take $t_\epsilon > 0$ such that for any $t < t_\epsilon$,

$$\|T_t^k f^k - f^k\|_{E_k}^2 < \frac{\epsilon}{2^k}, \quad 1 \leq k \leq n.$$

Then we have for any $t < t_\epsilon$,

$$\|T_t f - f\|_E^2 = \sum_{1 \leq k \leq n} \|T_t^k f^k - f^k\|_{E_k}^2 + \sum_{k > n} \|T_t^k f^k - f^k\|_{E_k}^2 < \sum_{1 \leq k \leq n} \frac{\epsilon}{2^k} + 4 \sum_{k > n} \|f^k\|_{E_k}^2 < 2\epsilon.$$

Therefore, (T_t) corresponds uniquely to a closed form $(\mathcal{E}', \mathcal{F}')$ on $L^2(E, m)$. Precisely,

$$\mathcal{F}' = \left\{ f \in L^2(E, m) : \uparrow \lim_{t \downarrow 0} \frac{1}{t} (f - T_t f, f)_m < \infty \right\},$$

$$\mathcal{E}'(f, f) = \lim_{t \downarrow 0} \frac{1}{t} (f - T_t f, f)_m, \quad f \in \mathcal{F}'.$$

Note that the limit above is an increasing limit as $t \downarrow 0$. On the other hand,

$$\lim_{t \downarrow 0} \frac{1}{t} (f - T_t f, f)_m = \lim_{t \downarrow 0} \sum_{n \geq 1} \frac{1}{t} (f^n - T_t^n f^n, f^n)_{m_n} = \sum_{n \geq 1} \lim_{t \downarrow 0} \frac{1}{t} (f^n - T_t^n f^n, f^n)_{m_n}.$$

Thus $f \in \mathcal{F}'$ if and only if $f^n \in \mathcal{F}^n$ and $\sum_{n \geq 1} \mathcal{E}^n(f^n, f^n) < \infty$. In other words,

$$(\mathcal{E}', \mathcal{F}') = (\mathcal{E}, \mathcal{F}).$$

The Markovian property of $(\mathcal{E}, \mathcal{F})$ may be deduced as follows. Let φ be a normal contraction on \mathbb{R} and $f \in \mathcal{F}$. Note that $(\varphi \circ f)|_{E_n} = \varphi(f|_{E_n}) \in \mathcal{F}^n$ and

$$\mathcal{E}^n(\varphi(f|_{E_n}), \varphi(f|_{E_n})) \leq \mathcal{E}^n(f|_{E_n}, f|_{E_n})$$

since $(\mathcal{E}^n, \mathcal{F}^n)$ satisfies the Markovian property. Hence $\varphi \circ f \in \mathcal{F}$ and $\mathcal{E}(\varphi \circ f, \varphi \circ f) \leq \mathcal{E}(f, f)$. That completes the proof. \square

Note that the semigroup of $(\mathcal{E}, \mathcal{F})$ in Theorem 3.2 is characterized by (3.10). From this fact, we have the following corollary.

Corollary 3.2. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(E, m)$ associated with a symmetric Markov process X . Suppose that $\{E_n : n \geq 1\}$ is a sequence of disjoint invariant sets of X and*

$$E = \bigcup_{n \geq 1} E_n \quad m\text{-a.e.}$$

Denote $(\mathcal{E}^n, \mathcal{F}^n) := (\mathcal{E}^{E_n}, \mathcal{F}^{E_n})$. Then $(\mathcal{E}, \mathcal{F})$ can be expressed as (3.9).

3.2.4. Proof of necessity

In this part, we prove the necessity of Theorem 3.1. Note that m stands for the Lebesgue measure on \mathbb{R} in this part. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet extension of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ on $L^2(\mathbb{R})$ associated with a Hunt process X . It follows from Theorem 2.1 and Proposition 2.2(2) that $(\mathcal{E}, \mathcal{F})$ is strongly local and recurrent. Without loss of generality, by [7, Theorem 4.5.1 (3)], we may assume that X is a recurrent (hence conservative, see [7, Lemma 1.6.5]) diffusion process on \mathbb{R} .

We use the same notations as Section 3.2.1 to denote the classes of points for X . Let

$$G := \Lambda_2$$

be the set of regular points and which is open by Lemma 3.1(6). Thus G may be written as a union of countable disjoint open intervals:

$$G = \bigcup_{n \geq 1} (a_n, b_n). \quad (3.11)$$

We assert $F := G^c$ is nowhere dense, and the shunt point must be an endpoint of some interval in (3.11).

Lemma 3.4. *The singular set $F = \Lambda_r \cup \Lambda_l$ is nowhere dense. Furthermore,*

$$F \setminus \{a_n, b_n : n \geq 1\} \subset \Lambda_t.$$

Proof. We first prove Λ_r has empty interior. Assume that $(a, b) \subset \Lambda_r$, it follows from Lemma 3.3 that $(a, b) \subset \Lambda_t$. The part Dirichlet forms of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ and $(\mathcal{E}, \mathcal{F})$ on (a, b) are denoted by $(\frac{1}{2}\mathbf{D}_{(a,b)}, H_0^1((a, b)))$ and $(\mathcal{E}_{(a,b)}, \mathcal{F}_{(a,b)})$. Clearly, $(\frac{1}{2}\mathbf{D}_{(a,b)}, H_0^1((a, b)))$ is still a regular Dirichlet subspace of $(\mathcal{E}_{(a,b)}, \mathcal{F}_{(a,b)})$ by Proposition 2.4. However, since X stays at the starting point in (a, b) forever (cf. Lemma 3.1(2)), it follows that $\mathcal{E}_{(a,b)}(f, f) = 0$ for any $f \in C_c^\infty((a, b)) \subset H_0^1((a, b)) \subset \mathcal{F}_{(a,b)}$, whereas

$$\frac{1}{2}\mathbf{D}_{(a,b)}(f, f) = \frac{1}{2} \int_a^b f'(x)^2 dx.$$

This leads to a contradiction. Thus Λ_r has empty interior. Similarly, Λ_l also has empty interior.

Suppose that $(a, b) \subset F = \Lambda_r \cup \Lambda_l$. We also assert that $(a, b) \subset \Lambda_t$, which leads to the same contradiction. In fact, it is enough to check that $(a, b) \cap \Lambda_{pr} = \emptyset$. Suppose that $x \in \Lambda_{pr} \cap (a, b)$. Since Λ_r has empty interior, we have for any n large enough, $(x, x + 1/n)$ must contain a point in Λ_{pl} . Then we can take a sequence $x_n \downarrow x$ in Λ_{pl} . By Lemma 3.1(5), $x \in \Lambda_l$, which contradicts to $x \in \Lambda_{pr}$. Therefore, any point in (a, b) must be a trap.

For the second assertion, fix any point $x \in F \setminus \{a_n, b_n : n \geq 1\}$. Suppose that $x \in \Lambda_{pr}$. Since F is nowhere dense and x is not an endpoint of some (a_n, b_n) , there exists a subsequence of intervals (a_{n_k}, b_{n_k}) in (3.11) such that $a_{n_k}, b_{n_k} \downarrow x$ as $k \uparrow \infty$. Note that the left singular set Λ_l is closed from the right and $x \in \Lambda_{pr}$. Hence for k large enough,

$a_{n_k}, b_{n_k} \in \Lambda_{pr}$. By Lemma 3.1(4), there exists a point $y > x$ such that $\mathbf{P}_x(\sigma_y < \infty) > 0$. Take k large enough with $x < a_{n_k} < y$ and $a_{n_k} \in \Lambda_{pr}$. Particularly,

$$\mathbf{P}_x(\sigma_{a_{n_k}} < \infty) \geq \mathbf{P}_x(\sigma_y < \infty) > 0.$$

However, Lemma 3.2 implies $\mathbf{P}_x(\sigma_{a_{n_k}} < \infty) = 0$ since $a_{n_k} \in \Lambda_{pr}$. This leads to a contradiction, and we conclude that $x \notin \Lambda_{pr}$. The same reasoning shows $x \notin \Lambda_{pl}$. Hence $x \in \Lambda_t$. That completes the proof. \square

Now we deal with X on an interval (a_n, b_n) of (3.11) with its endpoints. For convenience, we get rid of the subscript n and write (a_n, b_n) as (a, b) . Since (a, b) is a regular interval, it follows from Lemma 3.1(7) that

$$\mathbf{P}_x(\sigma_y < \infty)\mathbf{P}_y(\sigma_x < \infty) > 0, \quad \forall x, y \in (a, b).$$

Thus $\mathbf{P}_x(\sigma_b < \infty) = 0$ (resp. $\mathbf{P}_x(\sigma_a < \infty) = 0$) for some $x \in (a, b)$ if and only if it holds for any $x \in (a, b)$.

Consider the right endpoint b . If $b = \infty$, take the part process of X on (a, ∞) . It is an irreducible minimal diffusion process on (a, ∞) (cf. [1, Example 3.5.7]). Denote its scale function by \mathfrak{t} . The Brownian motion on (a, ∞) (a is the absorbing boundary) is its regular Dirichlet subspace. Thus from [4, Theorem 4.1], we know that $\mathfrak{t}(\infty) = \infty$. Particularly, ∞ is not approachable and $\mathbf{P}_x(X_t < \infty, \forall t) = 1$ for any $x \in (a, \infty)$. Hereafter assume $b < \infty$. It has the following cases.

(1) $b \in \Lambda_r$. By Lemma 3.2, for any $x < b$, $\mathbf{P}_x(\sigma_b < \infty) = 0$.

(2) $b \in \Lambda_{pl}$. There are two cases.

(2i) For some (equivalently, all) $x \in (a, b)$, $\mathbf{P}_x(\sigma_b < \infty) > 0$. By Lemma 3.1(4), we also have $\mathbf{P}_b(\sigma_x < \infty) > 0$ for any $x \in (a, b)$. Furthermore, $\mathbf{P}_x(X_t \leq b, \forall t) = 1$ for any $x \in (a, b]$ by Lemma 3.1(2, 3).

(2ii) For some (equivalently, all) $x \in (a, b)$, $\mathbf{P}_x(\sigma_b < \infty) = 0$.

We can also classify another endpoint a as above. When b (or a) is in the case (2i), we add b (or a) to (a, b) and attain a new interval $\langle a, b \rangle$. Clearly, $\langle a, b \rangle$ is an invariant set of X in the sense that

$$\mathbf{P}_x(X_t \in \langle a, b \rangle, \forall t) = 1, \quad \forall x \in \langle a, b \rangle.$$

Moreover, $X^{\langle a, b \rangle}$ is an irreducible diffusion process with no killing inside on $\langle a, b \rangle$ in the sense that

$$\mathbf{P}_x(\sigma_y < \infty) = 0, \quad x, y \in \langle a, b \rangle.$$

Then $X^{\langle a, b \rangle}$ is characterized by a scale function \mathfrak{t} and the speed measure $m|_{\langle a, b \rangle}$. Note that $b \in \langle a, b \rangle$ if and only if $\mathbf{P}_x(\sigma_b < \infty) > 0$. In other words, b is approachable in finite time. From [1, (3.5.13)], we can deduce that $b \in \langle a, b \rangle$ if and only if $\mathfrak{t}(b) < \infty$. Similarly we have $a \in \langle a, b \rangle$ if and only if $\mathfrak{t}(a) > -\infty$. On the other hand, the part process of $X^{\langle a, b \rangle}$ on (a, b) is a minimal diffusion with the scale function \mathfrak{t} . Clearly, $(\frac{1}{2}\mathbf{D}_{(a,b)}, H_0^1((a, b)))$ is its regular Dirichlet subspace. By using [4, Theorem 4.1] again, we have

$$dx \ll d\mathfrak{t}, \quad \frac{dx}{d\mathfrak{t}} = 0 \quad \text{or} \quad 1, \quad d\mathfrak{t}\text{-a.e.}$$

Therefore, after adjusting the value of \mathfrak{t} up to a constant, we can conclude that $\mathfrak{t} \in \mathbf{T}_\infty^0(\langle a, b \rangle)$. The associated Dirichlet form of $X^{\langle a, b \rangle}$ is expressed as (3.4) by [4, Theorem 3.1].

When we treat any interval (a_n, b_n) in (3.11), we obtain an invariant set $I_n := \langle a_n, b_n \rangle$ of X and $X^{\langle a_n, b_n \rangle}$ is an irreducible diffusion process on I_n with a unique scale function $\mathfrak{t}_n \in \mathbf{T}_\infty^0(\langle a_n, b_n \rangle)$ and the speed measure $m|_{\langle a_n, b_n \rangle}$. Finally any two intervals are disjoint. In fact suppose two intervals $\langle a_n, b_n \rangle, \langle a_m, b_m \rangle$ ($b_n \leq a_m$) have common point. Then $a_m = b_n \in \langle a_n, b_n \rangle \cap \langle a_m, b_m \rangle$. However $b_n \in \langle a_n, b_n \rangle$ implies $b_n \in \Lambda_{pl}$ and $a_m \in \langle a_m, b_m \rangle$ implies $a_m \in \Lambda_{pr}$, which contradicts the fact that $\Lambda_{pr} \cap \Lambda_{pl} = \emptyset$.

Note that any $x \in F \setminus \{a_n, b_n\}$ is a trap by Lemma 3.4. This implies $T_t f(x) = f(x)$ for any $f \in L^2(\mathbb{R})$ and m -a.e. $x \in F \setminus \{a_n, b_n\}$. Thus from Theorem 3.2 and Corollary 3.2, we can deduce that $(\mathcal{E}, \mathcal{F})$ is expressed as (3.3).

Finally, we assert $m(F) = 0$. Then the proof of the necessity of Theorem 3.1 is complete.

Lemma 3.5. $m(F) = 0$.

Proof. Note that for an absolutely continuous function $f \in \mathcal{F}^n$,

$$\mathcal{E}^n(f, f) = \frac{1}{2} \int_{a_n}^{b_n} \left(\frac{df}{d\tau_n} \right)^2 d\tau_n = \frac{1}{2} \int_{a_n}^{b_n} \left(\frac{df}{dx} \right)^2 \left(\frac{dx}{d\tau_n} \right)^2 d\tau_n = \frac{1}{2} \int_{a_n}^{b_n} \left(\frac{df}{dx} \right)^2 dx. \quad (3.12)$$

Since $C_c^\infty(\mathbb{R}) \subset H^1(\mathbb{R}) \subset \mathcal{F}$, we have for any $f \in C_c^\infty(\mathbb{R})$,

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{n \geq 1} \int_{a_n}^{b_n} \left(\frac{df}{dx} \right)^2 dx = \frac{1}{2} \int_G \left(\frac{df}{dx} \right)^2 dx.$$

On the other hand,

$$\mathcal{E}(f, f) = \frac{1}{2} \mathbf{D}(f, f) = \frac{1}{2} \int_{\mathbb{R}} \left(\frac{df}{dx} \right)^2 dx.$$

It follows that

$$\int_F \left(\frac{df}{dx} \right)^2 dx = 0, \quad \forall f \in C_c^\infty(\mathbb{R}).$$

This implies $m(F) = 0$. □

3.2.5. Proof of sufficiency

In this part, we shall prove the sufficiency of Theorem 3.1. Note that $(\mathcal{E}, \mathcal{F})$ given by (3.3), with invariant intervals $\{I_n : n \geq 1\}$ and scale function $\tau_n \in \mathbf{T}_\infty^0(I_n)$, is a Dirichlet form on $L^2(\mathbb{R})$ by Theorem 3.2. For convenience, an endpoint of I_n which is included in I_n is called a closed endpoint, and otherwise an open endpoint. For any function $f \in C_c^\infty(\mathbb{R})$, it follows from $\tau_n \in \mathbf{T}_\infty^0(I_n)$ and (3.12) that $f|_{I_n} \ll \tau_n$ and

$$\mathcal{E}^n(f|_{I_n}, f|_{I_n}) = \frac{1}{2} \int_{I_n} f'(x)^2 dx.$$

Thus from $m(F) = 0$, we can deduce that

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{n \geq 1} \int_{I_n} f'(x)^2 dx = \frac{1}{2} \int_{\mathbb{R}} f'(x)^2 dx = \frac{1}{2} \mathbf{D}(f, f).$$

This implies

$$H^1(\mathbb{R}) \subset \mathcal{F}, \quad \mathcal{E}(f, f) = \frac{1}{2} \mathbf{D}(f, f), \quad f \in H^1(\mathbb{R}).$$

Finally, we need only to prove the Dirichlet form $(\mathcal{E}, \mathcal{F})$ given by (3.3) is regular on $L^2(\mathbb{R})$.

Lemma 3.6. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ given by (3.3) is regular on $L^2(\mathbb{R})$.*

Proof. Clearly, $C_c^\infty(\mathbb{R}) \subset \mathcal{F} \cap C_c(\mathbb{R})$. So $\mathcal{F} \cap C_c(\mathbb{R})$ is dense in $C_c(\mathbb{R})$ with the uniform norm. It suffices to prove $\mathcal{F} \cap C_c(\mathbb{R})$ is dense in \mathcal{F} with the norm $\|\cdot\|_{\mathcal{E}_1}$.

We first note that $(\mathcal{E}^n, \mathcal{F}^n)$ is regular on $L^2(I_n)$. Set $\mathcal{C}_n := \mathcal{F}^n \cap C_c(I_n)$. Define the following class

$$\mathcal{C} := \{f \in \mathcal{F} : f^n \in \mathcal{C}_n\},$$

where $f^n := f|_{I_n}$. Then \mathcal{C} is dense in \mathcal{F} with the norm $\|\cdot\|_{\mathcal{E}_1}$. In fact, fix $f \in \mathcal{F}$ and $\epsilon > 0$. For each n , take a function $g_n \in \mathcal{C}_n$ such that $\|f^n - g_n\|_{\mathcal{E}_1^n}^2 < \epsilon/2^n$. Let g be the function: $g|_{I_n} = g_n$, $g = 0$ outside $\bigcup_{n \geq 1} I_n$. Clearly $g \in \mathcal{F}$ and hence $g \in \mathcal{C}$. Furthermore,

$$\mathcal{E}_1(f - g, f - g) = \sum_{n \geq 1} \mathcal{E}_1^n(f^n - g_n, f^n - g_n) < \epsilon.$$

Therefore, we need only to prove $\mathcal{F} \cap C_c(\mathbb{R})$ is dense in \mathcal{C} with the norm $\|\cdot\|_{\mathcal{E}_1}$.

Fix $f \in \mathcal{C}$ and a constant $\epsilon > 0$. There exists an integer n large enough such that $\sum_{k > n} \mathcal{E}_1^k(f^k, f^k) < \epsilon$. Let $g := f \cdot 1_{\bigcup_{k=1}^n \langle a_k, b_k \rangle}$. Then $g \in \mathcal{C} \subset \mathcal{F}$ and

$$\mathcal{E}_1(f - g, f - g) = \sum_{k > n} \mathcal{E}_1^k(f^k, f^k) < \epsilon.$$

We need now to find a function in $\mathcal{F} \cap C_c(\mathbb{R})$ which is \mathcal{E}_1 -close enough to g .

Note that g is continuous on $\langle a_k, b_k \rangle$. The discontinuous points of g are those closed endpoints of $\{I_k : 1 \leq k \leq n\}$. Particularly, the discontinuous points of g are finite. Take such a discontinuous point c of g . Without loss of generality, assume that c is the right endpoint of some interval $\langle a_k, b_k \rangle$ with $b_k \in \langle a_k, b_k \rangle$. Set $h := g(c)$. There are two different situations.

(1) For any $\beta > 0$, there exists an open endpoint of $\{I_n : n \geq 1\}$ in $[c, c + \beta)$, i.e., c is a limit point of open endpoints of $\{I_n : n \geq 1\}$.

(1a) Let us start from a simple case where c is an open endpoint of some $I_{k'}$, which is essentially the same as the example given in introduction.

In this case $c = a_{k'} \notin \langle a_{k'}, b_{k'} \rangle$ and $\tau_{k'}(c) = -\infty$. Since $g^{k'} := g|_{\langle a_{k'}, b_{k'} \rangle} \in C_c(\langle a_{k'}, b_{k'} \rangle)$, we can take $d \in \langle a_{k'}, b_{k'} \rangle$ such that $g = 0$ on $(c, d]$. We shall construct a continuous function $\varphi = \varphi_c^\epsilon$ on $[c, d]$ (set $\varphi(x) = 0$ if $x \notin [c, d]$) such that

$$\varphi(c) = h, \quad \varphi(d) = 0, \quad \varphi \in \mathcal{F}, \quad \mathcal{E}_1(\varphi, \varphi) < \frac{\epsilon}{2n}. \quad (3.13)$$

Obviously $g + \varphi$ will be continuous at c and its \mathcal{E}_1 -distance to g is small.

Take a constant $\delta > c$ such that $h^2 \cdot (\delta - c) < \epsilon/4n$. Let $\tilde{\delta} := \tau_{k'}(\delta) > -\infty$. Take another constant $\tilde{\delta}' < \tilde{\delta} - (8h^2n)/\epsilon$ and let $\delta' := \tau_{k'}^{-1}(\tilde{\delta}')$. Clearly there exists a C^1 -function ϕ on $[\tilde{\delta}', \tilde{\delta}]$ such that

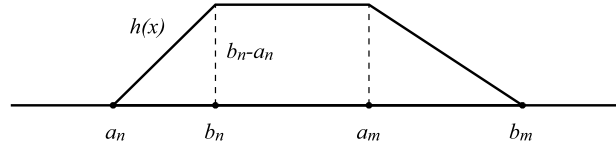
$$0 \leq \phi \leq h, \quad \phi(\tilde{\delta}') = h, \quad \phi(\tilde{\delta}) = 0, \quad |\phi'| \leq \frac{2h}{\tilde{\delta} - \tilde{\delta}'}.$$

Define

$$\varphi(x) := \begin{cases} h, & x \in [c, \delta'], \\ \phi(\tau_{k'}(x)), & x \in [\delta', \delta], \\ 0, & x > \delta. \end{cases}$$

Clearly, φ is continuous on $[c, d]$ and $\varphi \in \mathcal{F}$. Furthermore,

$$\begin{aligned} \mathcal{E}_1(\varphi, \varphi) &= \frac{1}{2} \int_{\delta'}^{\delta} \left(\frac{d\varphi}{d\tau_{k'}} \right)^2 d\tau_{k'} + \int_c^{\delta} \varphi_c(x)^2 dx \\ &= \frac{1}{2} \int_{\tilde{\delta}'}^{\tilde{\delta}} (\phi'(x))^2 dx + \int_c^{\delta} \varphi(x)^2 dx \\ &\leq \frac{1}{2} \frac{4h^2}{\tilde{\delta} - \tilde{\delta}'} + h^2 \cdot (\delta - c) \\ &< \frac{\epsilon}{2n}. \end{aligned}$$

Fig. 1. Compensate function φ .

Therefore, φ satisfies (3.13). See Figure 1.

(1b) c is a limit point of open endpoints of $\{I_n : n \geq 1\}$.

We see that the main reason that φ above can be constructed is that there is a non-closed interval I_k close to c , because in this case it follows from $g|_{I_k} \in C_c(I_k)$ that g vanishes on an interval contained in I_k . More precisely we can take a non-closed interval $\langle a_q, b_q \rangle$, where b_q is an open endpoint, such that

$$b_q - c < \epsilon / (4nh^2)$$

and $g = 0$ on $(c, b_q]$. Similarly to the first case, we can also construct a continuous function $\varphi = \varphi_c^\epsilon$ on $[c, b_q]$ ($\varphi := 0$ outside $[c, b_q]$) such that

$$\varphi(c) = h, \quad \varphi(b_q) = 0, \quad \varphi \in \mathcal{F}, \quad \mathcal{E}_1(\varphi, \varphi) < \frac{\epsilon}{2n}.$$

(2) For some $\beta > 0$, the endpoints of $\{I_n : n \geq 1\}$ located between c and $c + \beta$ are all closed.

Let β be small enough so that $g = 0$ on $(c, c + \beta]$ and

$$\beta < \frac{\epsilon}{2nh^2}.$$

Denote the intervals of $\{\langle a_k, b_k \rangle : k > n\}$ in $(c, c + \beta)$ by $\{[a_{q_j}, b_{q_j}] : j \geq 1\}$. Note that they are disjoint and dense in $[c, c + \beta]$. Hence $\bigcup_j [a_{q_j}, b_{q_j}]$ is a Cantor-type set. We may construct a Cantor-type function ϕ on $[c, c + \beta]$ (for its existence, see Remark 3.4), such that ϕ is decreasing and continuous, $\phi(c) = 1$, $\phi(c + \beta) = 0$ and ϕ is a constant on each interval $[a_{q_j}, b_{q_j}]$. Define $\varphi = \varphi_c^\epsilon(x) := h \cdot \phi(x)$ for $x \in [c, c + \beta]$ and vanishes elsewhere. Clearly, $\varphi \in \mathcal{F}$ and

$$\mathcal{E}_1(\varphi, \varphi) = \int \varphi(x)^2 dx < \epsilon / (2n).$$

Thus φ satisfies (3.13) if d is replaced by $c + \beta$.

From the above discussions, we can always construct a compensate function φ_c^ϵ which depends on discontinuous point c of g and ϵ . The construction above may guarantee that for any $c \neq c'$, φ_c^ϵ and $\varphi_{c'}^\epsilon$ have disjoint supports. Define

$$f_\epsilon := g + \sum_c \varphi_c^\epsilon,$$

where c in the sum takes all possible discontinuous points of g . The number of the terms in this sum is less than $2n$. One may easily check that $f_\epsilon \in \mathcal{F} \cap C_c(\mathbb{R})$. Therefore

$$\mathcal{E}_1(f_\epsilon - g, f_\epsilon - g) = \mathcal{E}_1\left(\sum_c \varphi_c^\epsilon, \sum_c \varphi_c^\epsilon\right) = \sum_c \mathcal{E}_1(\varphi_c^\epsilon, \varphi_c^\epsilon) \leq 2n \cdot \frac{\epsilon}{2n} = \epsilon.$$

That completes the proof. □

Remark 3.4. In this remark, we give a Cantor-type function ϕ on $[c, c + \beta]$ which is used in the proof of Lemma 3.6, though it may be found in some textbook. Without loss of generality, assume that $[c, c + \beta] = [0, 1]$, $\{I_n = [a_n, b_n] :$

$n \geq 1$ are disjoint closed intervals in $(0, 1)$ and

$$m\left([0, 1] \setminus \bigcup_{n \geq 1} I_n\right) = 0.$$

The continuous function ϕ on $[0, 1]$ is desired to satisfy $\phi(0) = 1$, $\phi(1) = 0$ and ϕ is a constant on each I_n .

Rearrange the positive integers as the following way:

$$K_1 := \{k_1 := 1\},$$

$$K_2 := \{k_{2,1} := \min\{n : a_n < a_{k_1}\}, k_{2,2} := \min\{n : a_n > a_{k_1}\}\},$$

Assume that the sets K_1, \dots, K_{m-1} have been defined. Then

$$(0, 1) \setminus \bigcup \left\{ I_n : n \in \bigcup_{j=1}^{m-1} K_j \right\}$$

is separated into 2^{m-1} disjoint and connected open intervals. Let $k_{m,i}$ be the smallest integer n of I_n in the i th interval from left to right for $1 \leq i \leq 2^{m-1}$. Define inductively $K_m = \{k_{m,i} : 1 \leq i \leq 2^{m-1}\}$. Clearly

$$a_{k_{m,1}} < a_{k_{m,2}} < \dots < a_{k_{m,2^{m-1}}},$$

and $\mathbb{N} = \bigcup_{m \geq 1} K_m$. Define the function ϕ as follows: $\phi(0) := 1$, $\phi(1) := 0$ and for any $m \geq 1$, $1 \leq j \leq 2^{m-1}$,

$$\phi(x) := \frac{2^{m-1} - j}{2^{m-1}}, \quad \forall x \in I_{k_{m,j}}.$$

One may prove that ϕ can be extended to a decreasing and continuous function on $[0, 1]$ similar to the standard Cantor function on $[0, 1]$.

3.3. More examples of extension

In this section, we give several examples for the regular Dirichlet extensions of one-dimensional Brownian motion. Recall that the regular Dirichlet extension $(\mathcal{E}, \mathcal{F})$ is characterized by $\{I_n : n \geq 1\}$ and $\{\mathfrak{t}_n \in \mathbf{T}_\infty^0(I_n) : n \geq 1\}$ in Theorem 3.1. It is evident that the extension $(\mathcal{E}, \mathcal{F})$ is same as Brownian motion if and only if only invariant interval is \mathbb{R} and the scale function $\mathfrak{t}(x) = x$.

Example 3.1. Let $I_1 = \mathbb{R}$ and $\mathfrak{t}_1(x) = x + c(x)$, where $c(x)$ is the standard Cantor function on $[0, 1]$ and we set $c(x) := 0$ for $x \leq 0$ and $c(x) := 1$ for $x \geq 1$. Then the corresponding regular Dirichlet extension is irreducible.

Example 3.2. Let $I_1 := (-\infty, 0)$ and $I_2 := (0, \infty)$. Take $\mathfrak{t}_1 \in \mathbf{T}_\infty^0(I_1)$ and $\mathfrak{t}_2 \in \mathbf{T}_\infty^0(I_2)$. Then I_1 and I_2 are two invariant sets of X . The single point set $\{0\}$ is an m -polar set relative to X . Formally, we may assume 0 is a trap of X , i.e. $\mathbf{P}_0(X_t = 0, \forall t) = 1$.

Example 3.3. Let $I_1 := (-\infty, -1]$, $I_2 := [1, \infty)$, $I_{2k+1} := (-\frac{1}{k}, -\frac{1}{k+1}]$ and $I_{2k+2} := [\frac{1}{k+1}, \frac{1}{k})$ for any $k \geq 1$. Take $\mathfrak{t}_n \in \mathbf{T}_\infty^0(I_n)$ for each n . Then $\{0\}$ is an m -polar set relative to X and any other single point set is not m -polar.

Example 3.4. Let $I_1 := (-\infty, 0]$, $I_2 = (1, \infty)$, and $I_n := (\frac{1}{n-1}, \frac{1}{n-2})$ for any $n \geq 3$. Take $\mathfrak{t}_n \in \mathbf{T}_\infty^0(I_n)$ for each n . Then $\{\frac{1}{k} : k \geq 1\}$ is an m -polar set relative to X .

Example 3.5. Let K be the standard Cantor set in $[0, 1]$. Set $U := K^c$ and write U as a union of disjoint open intervals:

$$U = \bigcup_{n \geq 1} (a_n, b_n),$$

where $(a_1, b_1) = (-\infty, 0)$, $(a_2, b_2) = (1, \infty)$. Let $I_1 := (-\infty, 0]$, $I_2 := [1, \infty)$, $I_n := [a_n, b_n]$ for any $n \geq 3$. For each n , let $\tau_n(x) = x$ on I_n . Then the associated diffusion process X of this regular Dirichlet extension is a reflected Brownian motion on each interval I_n . Moreover,

$$K \setminus \{a_n, b_n : n \geq 1\}$$

is m -polar.

4. Structures of regular Dirichlet extensions: Orthogonal complements and darning processes

In [14], the structures of regular Dirichlet subspaces for one-dimensional Brownian motion were investigated by using the trace method and a darning transform. As we have seen, ‘trace method’ could efficiently trace the different behavior of regular subspace from Brownian motion. In this section, we shall apply the same approach to investigate the behavior of regular Dirichlet extensions of one-dimensional Brownian motion. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ always stands for a proper regular Dirichlet extension of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ on $L^2(\mathbb{R})$, which is characterized by Theorem 3.1.

4.1. Orthogonal complement of Brownian motion

Let us characterize the orthogonal complement of one-dimensional Brownian motion in extension space. We need first to formulate extended Dirichlet space. The extended Dirichlet space of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ is

$$H_e^1(\mathbb{R}) := \{f : f \text{ is absolutely continuous and } f' \in L^2(\mathbb{R})\}.$$

The extended Dirichlet space of $(\mathcal{E}^n, \mathcal{F}^n)$ given by (3.4) is expressed as (cf. [1, Theorem 2.2.11])

$$\mathcal{F}_e^n = \left\{ f \text{ on } I_n : f \ll \tau_n, \int_{I_n} \left(\frac{df}{d\tau_n} \right)^2 d\tau_n < \infty \right\}. \quad (4.1)$$

We formulate the extended Dirichlet space of the regular Dirichlet extension (3.3) in the following theorem.

Theorem 4.1. *The extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$ given by (3.3) is*

$$\mathcal{F}_e = \left\{ f : |f| < \infty \text{ } m\text{-a.e. on } \mathbb{R}, f|_{I_n} \in \mathcal{F}_e^n, \sum_{n \geq 1} \mathcal{E}^n(f|_{I_n}, f|_{I_n}) < \infty \right\}. \quad (4.2)$$

Proof. Take an arbitrary $f \in \mathcal{F}_e$. Clearly, $|f| < \infty$ m -a.e. on \mathbb{R} . By the definition of extended Dirichlet space (cf. [1, Definition 1.1.4]), there exists an \mathcal{E} -Cauchy sequence $\{f_l\} \subset \mathcal{F}$ such that $\lim_{l \rightarrow \infty} f_l = f$ m -a.e. on \mathbb{R} . Particularly, for each n , $\{f_l|_{I_n}\} \subset \mathcal{F}^n$ is \mathcal{E}^n -Cauchy and $\lim_{l \rightarrow \infty} f_l|_{I_n} = f|_{I_n}$ $m|_{I_n}$ -a.e. on I_n . This implies $f|_{I_n} \in \mathcal{F}_e^n$ and

$$\mathcal{E}^n(f|_{I_n}, f|_{I_n}) = \lim_{l \rightarrow \infty} \mathcal{E}^n(f_l|_{I_n}, f_l|_{I_n}).$$

On the other hand, since $\{f_l\}$ is \mathcal{E} -Cauchy, we may take an integer M large enough such that for any $l > M$,

$$\mathcal{E}(f_M - f_l, f_M - f_l) < 1.$$

Then we have

$$\begin{aligned}
 \sum_{n \geq 1} \mathcal{E}^n(f|_{I_n}, f|_{I_n}) &= \sum_{n \geq 1} \lim_{l \rightarrow \infty} \mathcal{E}^n(f_l|_{I_n}, f_l|_{I_n}) \\
 &\leq \liminf_{l \rightarrow \infty} \sum_{n \geq 1} \mathcal{E}^n(f_l|_{I_n}, f_l|_{I_n}) \\
 &= \liminf_{l \rightarrow \infty} \mathcal{E}(f_l, f_l) \\
 &\leq \liminf_{l \rightarrow \infty} 2(\mathcal{E}(f_l - f_M, f_l - f_M) + \mathcal{E}(f_M, f_M)) \\
 &\leq 2(1 + \mathcal{E}(f_M, f_M)) \\
 &< \infty.
 \end{aligned}$$

This indicates f is in the right side of (4.2).

On the contrary, let f be a function in the right side of (4.2). Since $f|_{I_n} \in \mathcal{F}_c^n$, we may take an \mathcal{E}^n -Cauchy sequence $\{g_l^n : l \geq 1\} \subset \mathcal{F}^n$ such that $g_l^n \rightarrow f|_{I_n}$ m -a.e. as $l \rightarrow \infty$. Particularly,

$$\lim_{l \rightarrow \infty} \mathcal{E}^n(g_l^n - f|_{I_n}, g_l^n - f|_{I_n}) = 0.$$

Thus for each positive integer k , there are two integers l_k^n and N_k such that

$$\mathcal{E}^n(g_{l_k^n}^n - f|_{I_n}, g_{l_k^n}^n - f|_{I_n}) < \frac{1}{k} \cdot \frac{1}{2^n}, \quad \sum_{n > N_k} \mathcal{E}^n(f|_{I_n}, f|_{I_n}) < \frac{1}{k}.$$

Without loss of generality, we may assume $l_k^n, N_k \uparrow \infty$ as $k \rightarrow \infty$. Define a function h_k m -a.e. on \mathbb{R} : $h_k|_{I_n} := g_{l_k^n}^n$ for any $1 \leq n \leq N_k$ and $h_k := 0$ elsewhere. Clearly, $h_k \in L^2(\mathbb{R})$ and h_k converges to f m -a.e. as $k \rightarrow \infty$. Note that $g_{l_k^n}^n \in \mathcal{F}^n$ and

$$\begin{aligned}
 \sum_{n=1}^{N_k} \mathcal{E}^n(g_{l_k^n}^n, g_{l_k^n}^n) &\leq 2 \sum_{n=1}^{N_k} (\mathcal{E}^n(g_{l_k^n}^n - f|_{I_n}, g_{l_k^n}^n - f|_{I_n}) + \mathcal{E}^n(f|_{I_n}, f|_{I_n})) \\
 &< 2 \sum_{n=1} \frac{1}{k} \cdot \frac{1}{2^n} + 2 \sum_{n \geq 1} \mathcal{E}^n(f|_{I_n}, f|_{I_n}) \\
 &< \infty.
 \end{aligned}$$

This implies $h_k \in \mathcal{F}$. Finally, we show that $\{h_k : k \geq 1\}$ is \mathcal{E} -Cauchy in \mathcal{F} . In fact, for any $\epsilon > 0$, take an integer K satisfying $8/K < \epsilon$. Then for any $k, k' > K$, we have

$$\begin{aligned}
 &\mathcal{E}(h_k - h_{k'}, h_k - h_{k'}) \\
 &\leq 2(\mathcal{E}(h_k - f, h_k - f) + \mathcal{E}(h_{k'} - f, h_{k'} - f)) \\
 &\leq 2 \left(\sum_{n=1}^{N_k} \mathcal{E}^n(g_{l_k^n}^n - f|_{I_n}, g_{l_k^n}^n - f|_{I_n}) + \sum_{n > N_k} \mathcal{E}^n(f|_{I_n}, f|_{I_n}) \right) \\
 &\quad + 2 \left(\sum_{n=1}^{N_{k'}} \mathcal{E}^n(g_{l_{k'}^n}^n - f|_{I_n}, g_{l_{k'}^n}^n - f|_{I_n}) + \sum_{n > N_{k'}} \mathcal{E}^n(f|_{I_n}, f|_{I_n}) \right)
 \end{aligned}$$

$$\leq 4 \sum_{n \geq 1} \frac{1}{K} \cdot \frac{1}{2^n} + \frac{4}{K} < \epsilon.$$

That completes the proof. \square

The purpose of the next part is to formulate the orthogonal complement of $H_e^1(\mathbb{R})$ in \mathcal{F}_e in $(\mathcal{E}, \mathcal{F})$. For each $n \geq 1$, denote

$$U_n := \left\{ x \in I_n : \frac{dx}{d\tau_n}(x) = 1 \right\}, \quad (4.3)$$

$$W_n := I_n \setminus U_n.$$

Then U_n, W_n are defined in the sense of $d\tau_n$ -a.e. Since the natural scale is strictly increasing and continuous, it follows that U_n is measurable dense in I_n in the sense that

$$d\tau_n(U_n \cap (c, d)) > 0, \quad \forall (c, d) \subset I_n.$$

Particularly,

$$m|_{I_n} = 1_{U_n} d\tau_n, \quad d\tau_n(U_n) = m(U_n) = |b_n - a_n|, \quad m(W_n) = 0.$$

Define

$$\mathcal{G} := \{ f \in \mathcal{F}_e : \mathcal{E}(f, g) = 0, \forall g \in H_e^1(\mathbb{R}) \}. \quad (4.4)$$

We write $\mathcal{F}_e = H_e^1(\mathbb{R}) \oplus \mathcal{G}$ or $\mathcal{G} = \mathcal{F}_e \ominus H_e^1(\mathbb{R})$.

Theorem 4.2. *The orthogonal complement \mathcal{G} of $H_e^1(\mathbb{R})$ in \mathcal{F}_e is expressed as*

$$\mathcal{G} = \left\{ f \in \mathcal{F}_e : \frac{df|_{I_n}}{d\tau_n} = 0, d\tau_n\text{-a.e. on } U_n \text{ for any } n \geq 1 \right\}. \quad (4.5)$$

Furthermore, any $f \in \mathcal{F}_e$ can be decomposed into

$$f = f_1 + f_2, \quad f_1 \in H_e^1(\mathbb{R}), f_2 \in \mathcal{G}. \quad (4.6)$$

This decomposition is unique up to a constant. In other words, $H_e^1(\mathbb{R}) \cap \mathcal{G}$ only contains the constant functions.

Proof. We first prove the expression (4.5) of \mathcal{G} . Fix a function f in the right side of (4.5) and take another function g in $H_e^1(\mathbb{R})$. We have

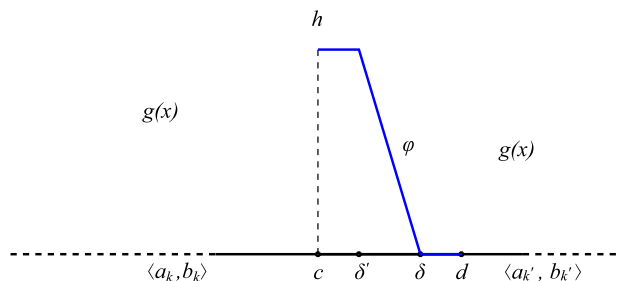
$$\mathcal{E}(f, g) = \sum_{n \geq 1} \mathcal{E}(f|_{I_n}, g|_{I_n}) = \frac{1}{2} \sum_{n \geq 1} \int_{I_n} \frac{df|_{I_n}}{d\tau_n}(x) g'(x) 1_{U_n}(x) d\tau_n(x) = 0.$$

It follows that $f \in \mathcal{G}$. On the contrary, take an arbitrary function $f \in \mathcal{G}$. Note that $I_n = \langle a_n, b_n \rangle$. Any function in $C_c^\infty((a_n, b_n))$ is treated as a function on \mathbb{R} and clearly $C_c^\infty((a_n, b_n)) \subset H_e^1(\mathbb{R})$. From (4.4), we have

$$\mathcal{E}(f, g) = 0, \quad \forall g \in C_c^\infty((a_n, b_n)).$$

It follows that

$$\int_{a_n}^{b_n} \left(\frac{df|_{I_n}}{d\tau_n}(x) 1_{U_n}(x) \right) g'(x) dx = 0, \quad \forall g \in C_c^\infty((a_n, b_n)).$$

Fig. 2. The function h .

This implies that $df|_{I_n}/d\mathfrak{t}_n \cdot 1_{U_n}$ is a constant a.e. on (a_n, b_n) , or equivalently $df|_{I_n}/d\mathfrak{t}_n$ is constant $d\mathfrak{t}_n$ -a.e. on U_n . Denote this constant by c_n . Take two integers m, n so that $a_n < a_m$. Define a function h on \mathbb{R} :

$$h(x) := (x - a_n)1_{I_n}(x) + |I_n| \cdot 1_{[b_n, a_m]}(x) + \left[|I_n| - \frac{|I_n|}{|I_m|} \cdot (x - a_m) \right] \cdot 1_{[a_m, b_m]}(x)$$

(see Figure 2). Clearly, $h \in H_e^1(\mathbb{R})$. Hence we have

$$\begin{aligned} 0 &= \mathcal{E}(f, h) \\ &= \frac{1}{2} \left(\int_{I_n} \frac{df|_{I_n}}{d\mathfrak{t}_n}(x) h'(x) 1_{U_n}(x) d\mathfrak{t}_n(x) + \int_{I_m} \frac{df|_{I_m}}{d\mathfrak{t}_m}(x) h'(x) 1_{U_m}(x) d\mathfrak{t}_m(x) \right) \\ &= \frac{1}{2} \left(c_n \int_{I_n} h'(x) dx + c_m \int_{I_m} h'(x) dx \right) \\ &= \frac{b_n - a_n}{2} \cdot (c_n - c_m). \end{aligned}$$

It follows that $c_n = c$ for some constant c and any $n \geq 1$. On the other hand, the fact $f \in \mathcal{F}_e$ implies

$$\mathcal{E}(f, f) < \infty.$$

However,

$$\mathcal{E}(f, f) = \sum_{n \geq 1} \mathcal{E}^n(f|_{I_n}, f|_{I_n}) \geq \frac{c^2}{2} \sum_{n \geq 1} \int_{I_n} 1_{U_n}(x) d\mathfrak{t}_n(x) = \frac{c^2}{2} m(\mathbb{R}).$$

Therefore, $c = 0$ and f is in the right side of (4.5).

Next, we prove the decomposition (4.6). Fix a function $f \in \mathcal{F}_e$. We decompose $f|_{I_n}$ for any $n \geq 1$ as

$$f|_{I_n} = g_1^n + g_2^n,$$

where $g_1^n \in H_e^1(\mathbb{R})$ is supported on I_n and $g_1^n(a_n) = 0$ (resp. $g_1^n(b_n) = 0$) if $a_n > -\infty$ (resp. $b_n < \infty$), $dg_2^n/d\mathfrak{t}_n$ is a constant $d\mathfrak{t}_n$ -a.e. on U_n . In fact, let e_n be a fixed point in (a_n, b_n) . If $(a_n, b_n) = (-\infty, \infty)$, set

$$\begin{aligned} g_1^n(x) &:= \int_{e_n}^x \frac{df}{d\mathfrak{t}_n}(x) 1_{U_n}(x) d\mathfrak{t}_n(x), \\ g_2^n(x) &:= f|_{I_n}(x) - g_1^n(x) = \int_{e_n}^x \frac{df}{d\mathfrak{t}_n}(x) 1_{W_n}(x) d\mathfrak{t}_n(x) + f(e_n) \end{aligned}$$

for any $x \in I_n$. If a_n is finite but $b_n = \infty$ (the case $a_n = -\infty$ and $b_n < \infty$ is similar), set

$$g_1^n(x) := \int_{e_n}^x \frac{df}{d\mathfrak{t}_n}(x) 1_{U_n}(x) d\mathfrak{t}_n(x) + C,$$

$$g_2^n(x) := f|_{I_n}(x) - g_1^n(x) = \int_{e_n}^x \frac{df}{d\mathfrak{t}_n}(x) 1_{W_n}(x) d\mathfrak{t}_n(x) + f(e_n) - C$$

for any $x \in I_n$, where

$$C := \int_{a_n}^{e_n} \frac{df}{d\mathfrak{t}_n}(x) 1_{U_n}(x) d\mathfrak{t}_n(x).$$

Note that

$$|C| \leq \left(\int_{a_n}^{e_n} \left(\frac{df}{d\mathfrak{t}_n}(x) \right)^2 d\mathfrak{t}_n(x) \right)^{1/2} \cdot |e_n - a_n|^{1/2} < \infty$$

and $g_1^n(a_n) = 0$. When I_n is not finite, let $C_1^n := 0$. If a_n and b_n are both finite, then

$$M := \int_{I_n} \frac{df}{d\mathfrak{t}_n}(x) 1_{U_n}(x) d\mathfrak{t}_n(x)$$

is finite. Set $C_1^n := M/(b_n - a_n)$ and

$$C_2^n := \int_{a_n}^{e_n} \left(\frac{df}{d\mathfrak{t}_n}(x) - C_1^n \right) 1_{U_n}(x) d\mathfrak{t}_n(x).$$

Clearly, C_2^n is also finite. Define

$$g_1^n(x) := \int_{e_n}^x \left(\frac{df}{d\mathfrak{t}_n}(y) - C_1^n \right) 1_{U_n}(y) d\mathfrak{t}_n(y) + C_2^n,$$

$$g_2^n(x) := f|_{I_n}(x) - g_1^n(x).$$

It is easily seen that $\lim_{x \downarrow a_n} g_1^n(x) = \lim_{x \uparrow b_n} g_1^n(x) = 0$. For all three cases above, we may easily deduce that $g_1^n \in H_e^1(\mathbb{R})$ and $dg_2^n/d\mathfrak{t}_n = C_1^n d\mathfrak{t}_n$ -a.e. on U_n . Then we define a function f_0 on \mathbb{R} as follows: $f_0(x) := g_1^n(x)$ for any $x \in I_n$ and $n \geq 1$ and $f_0(x) := 0$ elsewhere. It follows that $f_0 \in H_e^1(\mathbb{R})$. Next define $h|_{I_n} := C_1^n$ for any $n \geq 1$. Since

$$\int_{\mathbb{R}} h^2(x) dx = \sum_{n \geq 1} \left(\frac{M}{b_n - a_n} \right)^2 \cdot (b_n - a_n) \leq \sum_{n \geq 1} \int_{I_n} \left(\frac{df}{d\mathfrak{t}_n} \right)^2 d\mathfrak{t}_n < \infty,$$

we conclude that $h \in L^2(\mathbb{R})$ is locally integrable. Let

$$f_1(x) := f_0(x) + \int_0^x h(y) dy, \quad x \in \mathbb{R},$$

$$f_2 := f - f_1.$$

Then we have $f_1 \in H_e^1(\mathbb{R})$ and thus $f_2 \in \mathcal{F}_e$. On the other hand,

$$\frac{df_2|_{I_n}}{d\mathfrak{t}_n} = \frac{df|_{I_n}}{d\mathfrak{t}_n} - \frac{df_0|_{I_n}}{d\mathfrak{t}_n} - C_1^n = \frac{dg_2^n}{d\mathfrak{t}_n} - C_1^n = 0,$$

$d\mathfrak{t}_n$ -a.e. on U_n . Hence $f_2 \in \mathcal{G}$ by (4.5).

Finally, assume that $f \in H^1_e(\mathbb{R}) \cap \mathcal{G}$. It follows from (4.4) that

$$0 = \mathcal{E}(f, f) = \sum_{n \geq 1} \mathcal{E}^n(f|_{I_n}, f|_{I_n}).$$

This implies that $\mathcal{E}^n(f|_{I_n}, f|_{I_n}) = 0$. Since $f|_{I_n} \in \mathcal{F}^n_e$ and $(\mathcal{E}^n, \mathcal{F}^n)$ is irreducible, we conclude from [1, Theorem 5.2.16] that $f|_{I_n}$ is a constant on I_n . Then we have $f' = 0$ on $\bigcup_{n \geq 1} I_n$ and hence m -a.e. on \mathbb{R} . Therefore, f is a constant on \mathbb{R} . That completes the proof. \square

Remark 4.1. One may feel that the decomposition (4.6) is obvious by applying the orthogonal decomposition theorem in Hilbert space. However, though the terminology ‘orthogonal complement’ is used here, we should notice that $(\mathcal{E}, \mathcal{F}_e)$ is not a Hilbert space. For $f \in \mathcal{F}_e$ with $\mathcal{E}(f, f) = 0$, f may not be necessarily a constant. Hence the decomposition (4.6) can not be deduced simply from the orthogonal decomposition of Hilbert space.

Example 4.1. In this example, let us consider the regular Dirichlet extension $(\mathcal{E}, \mathcal{F})$ stated in Example 3.5. Note that the associated diffusion process X is a reflected Brownian motion on each interval I_n and $U_n = I_n$. Then the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$ is expressed as

$$\mathcal{F}_e = \left\{ f : f \text{ is absolutely continuous on each interval } I_n \text{ and } \sum_{n \geq 1} \int_{I_n} f'(x)^2 dx < \infty \right\}.$$

Moreover,

$$\mathcal{G} = \mathcal{F}_e \ominus H^1_e(\mathbb{R}) = \{f : f \text{ is a constant on each interval } I_n\}.$$

The orthogonal complement \mathcal{G} contains continuous functions as well as discontinuous functions. For example, the Cantor-type function introduced in Remark 3.4 belongs to \mathcal{G} .

4.2. Darning processes

Though the orthogonal complement is not a Dirichlet space, it is in wide sense. We shall give its regular representation in this section so that its structure is shown clearly. Recall the definitions of U_n and W_n in (4.3). From now on, we impose the following assumptions on U_n :

- (H1) U_n has (and is taken as) a $d\tau_n$ -a.e. open version;
- (H2) for any $x \in W_n \cap (a_n, b_n)$ and $\epsilon > 0$, $d\tau_n((x - \epsilon, x + \epsilon) \cap W_n) > 0$.

The first assumption is not always right and the second one is not essential as we remarked in [14, Section 1]. In fact, if (H1) is satisfied, we can always find an open $d\tau_n$ -version of U_n that satisfies (H2), see also [14, Section 1]. Write

$$U_n = \bigcup_{m \geq 1} (a_m^n, b_m^n) \tag{4.7}$$

as a union of disjoint open intervals and set

$$U := \bigcup_{n \geq 1} U_n = \bigcup_{n \geq 1} \bigcup_{m \geq 1} (a_m^n, b_m^n),$$

$$K := U^c.$$

Remark 4.2. We need to give some explanation for the structures of U_n and K . Now we only consider the right endpoint b_n of I_n (the case of the left endpoint a_n is similar). We first note that $b_n \notin U_n$ if $b_n \in I_n$, since U_n is assumed to be open in \mathbb{R} . If $b_n \in I_n$, then it may happen that $b_n = b_m^n$ for some integer m in (4.7). For instance, in

Example 3.5, we have $U_n = \overset{\circ}{I}_n = (a_n, b_n)$. If $b_n \notin I_n$ and $b_n < \infty$, then W_n is not trivial and $d\tau_n(W_n) = \infty$. This follows from $\tau_n(b_n) = \infty$ and $d\tau_n(U_n \cap (e_n, b_n)) = m((e_n, b_n)) < \infty$. Particularly, for any $\epsilon > 0$,

$$W_n \cap (b_n - \epsilon, b_n) \neq \emptyset.$$

In other words, it will not happen that $b_n = b_m^n$ for some integer m in (4.7). If $b_n = \infty$, then W_n may be trivial as in Example 3.5, i.e. $W_2 = \{1\}$. Also possibly as in [14, Remark 3.2], W_n is not trivial in the sense that for any $L > a_n$,

$$W_n \cap (L, \infty) \neq \emptyset.$$

Finally we note that

$$K = \left(\bigcup_{n \geq 1} W_n \right) \cup \left(\bigcup_{n \geq 1} I_n \right)^c,$$

where $(\bigcup_{n \geq 1} I_n)^c$ is an m -polar set relative to X by Theorem 3.1. Since $m(W_n) = 0$ and $m((\bigcup_{n \geq 1} I_n)^c) = 0$, we obtain $m(K) = 0$.

The darning method introduced in [14, Section 3.2] may also be applied to investigate the behavior of $(\mathcal{E}, \mathcal{G})$, where \mathcal{G} is the orthogonal complement (4.5) of $H_c^1(\mathbb{R})$ in \mathcal{F}_c . Let

$$\begin{aligned} \mathcal{G}^n &:= \mathcal{G}|_{I_n} = \left\{ f \in \mathcal{F}_c^n : \frac{df}{d\tau_n} = 0, d\tau_n\text{-a.e. on } U_n \right\}, \\ \mathcal{E}^n(f, f) &= \mathcal{E}^{I_n}(f, f) = \frac{1}{2} \int_{I_n} \left(\frac{df}{d\tau_n} \right)^2 d\tau_n, \quad f \in \mathcal{G}^n, \end{aligned}$$

where \mathcal{F}_c^n is given by (4.1). Further denote

$$\mathcal{G}_0^n := \mathcal{G}^n \cap L^2(I_n) = \left\{ f \in \mathcal{F}^n : \frac{df}{d\tau_n} = 0, d\tau_n\text{-a.e. on } U_n \right\}.$$

Note that U_n is open and expressed as (4.7). Thus the function $f \in \mathcal{G}^n$ is a constant on $[a_m^n, b_m^n]$ for any integer $m \geq 1$. We need to exclude the case $d\tau_n(W_n) = 0$, which gives us a trivial darning process. Thus we would make the following assumption in this section:

(H3) $d\tau_n(W_n) > 0$.

Define $r_n^- := \inf\{x : x \in W_n\}$, $r_n^+ := \sup\{x : x \in W_n\}$ and

$$J_n := \langle r_n^-, r_n^+ \rangle,$$

where $r_n^- \in J_n$ (resp. $r_n^+ \in J_n$) if and only if $a_n \in I_n$ (resp. $b_n \in I_n$). Note that if b_n (resp. a_n) is finite, then $r_n^+ = b_n$ (resp. $r_n^- = a_n$) by Remark 4.2. If $b_n = \infty$ (resp. $a_n = -\infty$), then r_n^+ (resp. r_n^-) may be finite and meanwhile $f = 0$ on $[r_n^+, \infty)$ (resp. $(-\infty, r_n^-]$) for any $f \in \mathcal{G}_0^n$. Thus $(\mathcal{E}^n, \mathcal{G}_0^n)$ on $L^2(I_n)$ can be identified with the one on $L^2(J_n)$. Then we have the following lemma. The proof is similar to [14, Lemma 3.2] and we omit it.

Lemma 4.1. *The quadratic form $(\mathcal{E}^n, \mathcal{G}_0^n)$ is a Dirichlet form on $L^2(J_n)$ in the wide sense, i.e. it satisfies all conditions of Dirichlet form except for the denseness of \mathcal{G}_0^n in $L^2(J_n)$.*

As stated in [14, Section 3.2], $(J_n, m|_{J_n}, \mathcal{G}_0^n, \mathcal{E}^n)$ is a D-space that named by Fukushima in [5]. We introduced the darning method to find the regular representations of the D-spaces we explored in [14]. In what follows, we shall describe the road map to attain the regular representation of $(J_n, m|_{J_n}, \mathcal{G}_0^n, \mathcal{E}^n)$ via the darning method, but omit most details of the proof, since it is indeed similar to [14].

Recall that e_n is a fixed point in (a_n, b_n) . We introduce the following transform on J_n that collapses each open component (a_m^n, b_m^n) with its endpoints of U_n into a new point:

$$j_n(x) := \int_{e_n}^x 1_{W_n}(y) d\mathfrak{t}_n(y), \quad x \in J_n.$$

If $a_n \in I_n$ (resp. $b_n \in I_n$), then $r_n^- = a_n$ and $r_n^{-*} := j_n(r_n^-) > -\infty$ (resp. $r_n^+ = b_n$ and $r_n^{+*} := j_n(r_n^+) < \infty$). If $a_n \notin I_n$ and $a_n > -\infty$ (resp. $b_n \notin I_n$ and $b_n < \infty$), then $r_n^- = a_n$ and $r_n^{-*} = -\infty$ (resp. $r_n^+ = b_n$ and $r_n^{+*} = \infty$). If $a_n = -\infty$ (resp. $b_n = \infty$), then r_n^{-*} (resp. r_n^{+*}) may be finite or infinite. Denote

$$J_n^* := \langle r_n^{-*}, r_n^{+*} \rangle,$$

where $r_n^{-*} \in J_n^*$ (resp. $r_n^{+*} \in J_n^*$) if and only if $a_n \in I_n$ (resp. $b_n \in I_n$). Clearly, $j_n(J_n) = J_n^*$, j_n is non-decreasing, and $j_n(x) = j_n(y)$ if and only if $x, y \in [a_m^n, b_m^n]$ for some integer m . The assumption (H3) guarantees that J_n^* is a nontrivial interval.

We further introduce the image measure of $m|_{J_n}$ relative to j_n on J_n^* :

$$m_n^* := m|_{J_n} \circ j_n^{-1}.$$

Note that m_n^* is a Radon measure on J_n^* . Moreover, when $r_n^{-*} \in J_n^*$ (resp. $r_n^{+*} \in J_n^*$), it probably holds $m_n^*({r_n^{-*}}) > 0$ (resp. $m_n^*({r_n^{+*}}) > 0$). This situation only happens when a_n (resp. b_n) is the left (resp. right) endpoint of (a_m^n, b_m^n) for some integer m .

Since $f \in \mathcal{G}_0^n$ is a constant on $[a_m^n, b_m^n]$ for any $m \geq 1$, this function determines a unique function \hat{f} on J_n^* through a darning method:

$$\hat{f} \circ j_n = f.$$

For any function $f \in \mathcal{G}_0^n \subset \mathcal{F}^n$, it may be written as $f = g \circ \mathfrak{t}_n$ for some absolutely continuous function g with $\int_{\mathfrak{t}_n(I_n)} g'(x)^2 dx < \infty$. Particularly, g is a constant on $[\mathfrak{t}_n(a_m^n), \mathfrak{t}_n(b_m^n)]$. Then g determines a unique function \hat{g} on J_n^* via $\hat{g} \circ j_n' = g$, where

$$j_n' : \mathfrak{t}_n(J_n) \rightarrow J_n^*, \quad x \mapsto \int_{\mathfrak{t}_n(e_n)}^x 1_{\mathfrak{t}_n(W_n)}(y) dy.$$

Clearly, $\hat{f} = \hat{g}$. Hence

$$\mathcal{E}^n(f, f) = \frac{1}{2} \int_{I_n} \left(\frac{df}{d\mathfrak{t}_n} \right)^2 d\mathfrak{t}_n = \frac{1}{2} \int_{\mathfrak{t}_n(I_n)} g'(x)^2 dx = \frac{1}{2} \int_{J_n^*} \hat{g}'(x)^2 dx = \frac{1}{2} \int_{J_n^*} \hat{f}'(x)^2 dx.$$

On the other hand, when $r_n^{-*} \notin J_n^*$ but $r_n^{-*} > -\infty$ (resp. $r_n^{+*} \notin J_n^*$ but $r_n^{+*} < \infty$), $\mathcal{E}^n(f, f) < \infty$ implies $\hat{f}(r_n^{-*}) := \lim_{x \downarrow r_n^{-*}} \hat{f}(x)$ (resp. $\hat{f}(r_n^{+*}) := \lim_{x \uparrow r_n^{+*}} \hat{f}(x)$) exists. We assert that $\hat{f}(r_n^{-*}) = 0$ (resp. $\hat{f}(r_n^{+*}) = 0$). We only treat the left endpoint r_n^{-*} . Note that $r_n^{-*} \notin J_n^*$ and $r_n^{-*} > -\infty$ indicate $a_n = -\infty$. If $r_n^- > -\infty$, we pointed out $f = 0$ on $(-\infty, r_n^-]$ and thus $\hat{f}(r_n^{-*}) = 0$. If $r_n^- = -\infty$, it follows that $f(-\infty) := \lim_{x \downarrow -\infty} f(x)$ exists, whereas $f \in L^2(I_n)$. Hence it holds $f(-\infty) = 0$, which implies $\hat{f}(r_n^{-*}) = 0$. Therefore, we are lead to define the quadratic form on $L^2(J_n^*, m_n^*)$:

$$\begin{aligned} \mathcal{G}_0^{n*} &:= \{ \hat{f} : f \in \mathcal{G}_0^n \}, \\ \mathcal{E}^{n*}(\hat{f}, \hat{g}) &:= \frac{1}{2} \int_{J_n^*} \hat{f}'(x) \hat{g}'(x) dx, \quad \hat{f}, \hat{g} \in \mathcal{G}_0^{n*}. \end{aligned} \tag{4.8}$$

The following theorem is an analogue of [14, Theorem 3.2].

Theorem 4.3. *The quadratic form $(\mathcal{E}^{n*}, \mathcal{G}_0^{n*})$ defined by (4.8) can be expressed as*

$$\mathcal{G}_0^{n*} = H_{0,e}^1(J_n^*) \cap L^2(J_n^*, m_n^*),$$

$$\mathcal{E}^{n*}(\hat{f}, \hat{g}) := \frac{1}{2} \int_{J_n^*} \hat{f}'(x) \hat{g}'(x) dx, \quad \hat{f}, \hat{g} \in \mathcal{G}_0^{n*},$$

where

$$H_{0,e}^1(J_n^*) = \{ \hat{f} \in H_e^1(J_n^*) : \hat{f}(r_n^{\pm*}) = 0 \text{ whenever } r_n^{\pm*} \notin J_n^* \text{ and } |r_n^{\pm*}| < \infty \}.$$

Furthermore, a regular representation of D -space $(J_n, m|_{J_n}, \mathcal{G}_0^n, \mathcal{E}^n)$ can be realized by the regular local Dirichlet form $(\mathcal{E}^{n*}, \mathcal{G}_0^{n*})$ on $L^2(J_n^*, m_n^*)$. Its associated diffusion process X^{n*} is a Brownian motion B^* on J_n^* being time changed by its positive continuous additive functional with the Revuz measure m_n^* , where B^* reflects at the finite endpoints $r_n^{\pm*} \in J_n^*$ and absorbs at the finite endpoints $r_n^{\pm*} \notin J_n^*$.

At the finite endpoints $r_n^{\pm*} \in J_n^*$, X^{n*} is said to be slowly reflecting if $m_n^*({r_n^{\pm*}}) > 0$ and instantaneously reflecting if $m_n^*({r_n^{\pm*}}) = 0$ by [16, Chapter VII (3.11)]. The former case occurs if and only if a_n (resp. b_n) is finite and a_n (resp. b_n) is the left (resp. right) endpoint of (a_m^n, b_m^n) for some integer m . At this time, X^{n*} is also called a diffusion with sojourn in [6].

We end this section with two examples of darning processes.

Example 4.2. We first consider the regular Dirichlet extension of one-dimensional Brownian motion in Example 3.1. Note that it is irreducible and thus only has one invariant interval $I_1 = \mathbb{R}$. Hereafter, we get rid of the subscript '1' for convenience and write $I = \mathbb{R}$. Moreover, $U = K^c$, $W = K$, where K is the standard Cantor set in $[0, 1]$. Clearly, U and W satisfy (H1), (H2) and (H3). Recall that $\mathfrak{t}(x) = x + c(x)$, where c is the standard Cantor function and

$$\mathcal{G}_0 = \left\{ f \in \mathcal{F} : \frac{df}{d\mathfrak{t}} = 0, d\mathfrak{t}\text{-a.e. on } U \right\},$$

$$\mathcal{E}(f, f) = \frac{1}{2} \int_{\mathbb{R}} \left(\frac{df}{d\mathfrak{t}} \right)^2 d\mathfrak{t}, \quad f \in \mathcal{G}_0,$$

where $\mathcal{F} = \{ f \in L^2(\mathbb{R}) : f \ll \mathfrak{t}, \mathcal{E}(f, f) < \infty \}$. Clearly, for any $f \in \mathcal{G}_0$, $f = 0$ on $(-\infty, 0]$ and $[1, \infty)$.

Since I is open, we have

$$J = (0, 1).$$

Take the fixed point $e = 0$, and the darning transform j is

$$j(x) = \int_0^x 1_K(y) d\mathfrak{t}(y) = \int_0^x 1_K(y) dc(y) = c(x), \quad x \in (0, 1).$$

Then $J^* = j(J) = (0, 1)$ and $m^* = m|_{(0,1)} \circ j^{-1}$ is a fully supported Radon measure on J^* with $m^*(J^*) = m(J) = 1$. Note that the single point set of J^* may be of positive m^* -measure. For example, $m^*({1/2}) = m([1/3, 2/3]) = 1/3$. Furthermore,

$$\mathcal{G}_0^* = H_{0,e}^1((0, 1)) \cap L^2((0, 1), m^*),$$

$$\mathcal{E}^*(f, g) = \frac{1}{2} \int_0^1 f'(x) g'(x) dx, \quad f, g \in \mathcal{G}_0^*,$$

where $H_{0,e}^1((0, 1)) = \{ f \in H_e^1((0, 1)) : f(0) = f(1) = 0 \}$. The associated darning process is a time-changed absorbing Brownian motion by m^* on $(0, 1)$.

Example 4.3. In this example, we show a darning process with sojourn at the boundary. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet extension of one-dimensional Brownian motion:

$$I_1 = (-\infty, -1), \quad I_2 = [-1, \infty)$$

and $\mathfrak{t}_2(x) = x + c(x)$, where $c(x)$ is still the standard Cantor function with $c(x) := 0$ for $x \leq 0$ and $c(x) := 1$ for $x \geq 1$.

We only consider the restriction to I_2 of the orthogonal complement \mathcal{G} . Let K be the standard Cantor set in $[0, 1]$. Then

$$U_2 = (-1, 0) \cup ([0, 1] \setminus K) \cup (1, \infty), \quad W_2 = \{-1\} \cup K.$$

Clearly, U_2 and W_2 satisfy (H1), (H2) and (H3). Since $-1 \in I_2$, we have

$$J_2 = [-1, 1).$$

Take the fixed point $e_2 = 0$ and the darning transform is

$$j_2(x) = \int_0^x 1_K(y) d\mathfrak{t}_2(y) = \int_0^x 1_K(y) dc(y), \quad x \in [-1, 1).$$

Thus $J_2^* = j_2(J_2) = [0, 1)$ and $m_2^* = m|_{[-1, 1)} \circ j_2^{-1}$ is a fully supported Radon measure on $[0, 1)$. Particularly, $m_2^*({0}) = m([-1, 0]) = 1$. Furthermore,

$$\mathcal{G}_0^{2*} = H_{0,e}^1([0, 1)) \cap L^2([0, 1), m_2^*),$$

$$\mathcal{E}^{2*}(f, g) = \frac{1}{2} \int_0^1 f'(x)g'(x) dx, \quad f, g \in \mathcal{G}_0^{2*},$$

where $H_{0,e}^1([0, 1)) = \{f \in H_e^1([0, 1)) : f(1) = 0\}$. The associated darning process X^{2*} is a Brownian motion B^* on $[0, 1)$ being time-changed by m^* , where B^* reflects at 0 and absorbs at 1. Since $m_2^*({0}) = 1 > 0$, X^{2*} is a diffusion process with sojourn and slowly reflecting at 0.

5. Structures of regular Dirichlet extensions: Trace Dirichlet forms

In previous section we discuss the orthogonal complement \mathcal{G} of one-dimensional Brownian motion in extension space $(\mathcal{E}, \mathcal{F}_e)$. In this section we shall only impose (H1) and (H2) of Section 4.2 and discuss the orthogonal complement of the part Dirichlet form of $(\mathcal{E}, \mathcal{F})$ on the open set U , or intuitively the biggest Brownian motion contained in $(\mathcal{E}, \mathcal{F})$. The later complement is called the trace of $(\mathcal{E}, \mathcal{F})$ on U^c , which may be orthogonally decomposed into the former complement \mathcal{G} and the trace of Brownian motion on U^c . The trace will show us the structure of extension in another approach.

The following lemma is similar to [14, Lemma 2.2], which indicates that X is a Brownian motion before leaving the open set U .

Lemma 5.1. Let $(\frac{1}{2}\mathbf{D}_U, H_0^1(U))$ and $(\mathcal{E}_U, \mathcal{F}_U)$ be the part Dirichlet forms of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ and $(\mathcal{E}, \mathcal{F})$ on U respectively. Then it holds that $(\mathcal{E}_U, \mathcal{F}_U) = (\frac{1}{2}\mathbf{D}_U, H_0^1(U))$.

Proof. Note that $H_0^1(U) \subset \mathcal{F}_U$ and $\mathcal{E}_U(f, f) = \frac{1}{2}\mathbf{D}_U(f, f)$ for any $f \in H_0^1(U)$ by Proposition 2.4. Thus it suffices to prove $\mathcal{F}_U \subset H_0^1(U)$. Note that \mathfrak{t}_n is a natural scale (i.e. $\mathfrak{t}_n(x) = x + c$ for some constant c) on (a_m^n, b_m^n) . This implies any function $f \in \mathcal{F}_U$ is absolutely continuous on (a_m^n, b_m^n) and $f(a_m^n) = f(b_m^n) = 0$. It follows from (3.3) that

$$\sum_{n,m \geq 1} \int_{(a_m^n, b_m^n)} f'(x)^2 dx < \infty.$$

Then we can conclude that f is absolutely continuous on \mathbb{R} and hence $f \in H_0^1(U)$. □

We now turn to the trace Dirichlet forms of $(\mathcal{E}, \mathcal{F})$ and $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ on K . To do that, we have to find a smooth measure supported on K . For each n , $d\tau_n$ is a Radon measure on I_n but not necessarily finite. Nevertheless, we can always take a finite measure $d\tau'_n$ equivalent to $d\tau_n$ if I_n is finite. For example,

$$d\tau'_n = \sum_{k \geq 1} \frac{C_n}{2^k \cdot d\tau_n([a_n + 1/k, b_n - 1/k])} \cdot d\tau_n|_{[a_n + 1/k, b_n - 1/k]},$$

where C_n is some positive constant and we make the convention $0/0 = 0$. Particularly, we may choose C_n so that $d\tau'_n(I_n) = b_n - a_n$. If I_n is infinite, i.e. $I_n = \langle a_n, \infty \rangle$, $(-\infty, b_n)$ or \mathbb{R} , we write $d\tau'_n := d\tau_n$. Define a measure

$$\mu := \sum_{n \geq 1} d\tau'_n|_{W_n} + \sum_{n \geq 1} (b_n - a_n) \cdot (\delta_{a_n} \cdot 1_{\{a_n \in I_n\}} + \delta_{b_n} \cdot 1_{\{b_n \in I_n\}}), \quad (5.1)$$

where $W_n = I_n \setminus U_n$ and δ_{a_n} is the Dirac measure at a_n . It can be seen from the following lemma that μ might be a suitable choice.

Lemma 5.2. *The measure μ given by (5.1) is a Radon smooth measure with the topological support K relative to $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ and $(\mathcal{E}, \mathcal{F})$ respectively. Hence the quasi support of μ relative to $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ is K . Furthermore, the quasi support of μ relative to $(\mathcal{E}, \mathcal{F})$ can be taken as a finely closed q.e. version \bar{K} .*

Proof. Clearly, μ is a Radon measure on \mathbb{R} . Since the m -polar set relative to $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ must be the empty set, it follows that μ is smooth relative to $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$. It is also smooth relative to $(\mathcal{E}, \mathcal{F})$ since the m -polar sets relative to $(\mathcal{E}, \mathcal{F})$ must be the subsets of $(\bigcup_{n \geq 1} I_n)^c$ and clearly $\mu((\bigcup_{n \geq 1} I_n)^c) = 0$.

Next, we prove the topological support of μ is K . Note that K is closed and $\mu(K^c) = \mu(U) = 0$. If K' is another closed set and $\mu(K'^c) = 0$, we assert that $K \subset K'$. Suppose that $x \in K \setminus K'$. Then $(x - \epsilon, x + \epsilon) \cap K' = \emptyset$ for some constant $\epsilon > 0$. If $x \in W_n \cap (a_n, b_n)$ for some n , then it follows from (H2) that $\mu((x - \epsilon, x + \epsilon)) \geq \mu((x - \epsilon, x + \epsilon) \cap W_n) > 0$, which contradicts the fact $\mu(K'^c) = 0$. Otherwise $(x - \epsilon, x]$ must contain a part with an endpoint of some interval I_n . When this endpoint belongs to I_n , clearly $\mu((x - \epsilon, x]) > 0$. When this endpoint does not belong to I_n , we have $d\tau_n(I_n \cap (x - \epsilon, x]) = \infty$ whereas $d\tau_n(U_n \cap (x - \epsilon, x]) \leq \epsilon$. This implies $d\tau'_n|_{W_n}((x - \epsilon, x]) > 0$ and thus $\mu((x - \epsilon, x + \epsilon)) > 0$, which also contradicts the fact $\mu(K'^c) = 0$.

Since the fine topology relative to the one-dimensional Brownian motion is the same as the usual topology, we conclude that the quasi-support of μ relative to $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ is also K . For the last assertion, we need only to prove [1, Theorem 3.3.5 (b)] for $F = K$. If $u \in \mathcal{F}$ and $u = 0$ q.e. on K , then $u(x) = 0$ for any $x \in \bigcup_{n \geq 1} W_n$. This implies $u = 0$ μ -a.e. On the contrary, let $u \in \mathcal{F}$ and $u = 0$ μ -a.e. We assert that $u(x) = 0$ for any $x \in I_n \cap K = W_n$, which implies $u = 0$ q.e. on K . In fact, assume $u(x) \neq 0$ for some $x \in W_n$. Since $u = 0$ μ -a.e., x is not the endpoint of I_n . Note that $u|_{I_n}$ is continuous. Thus $u(y) \neq 0$ for any $y \in (x - \epsilon, x + \epsilon)$ with some constant $\epsilon > 0$. However, $\mu((x - \epsilon, x + \epsilon)) > 0$ by (H2), which contradicts the fact $u = 0$ μ -a.e. That completes the proof. \square

Remark 5.1. From Lemma 5.2 and [1, Lemma 5.2.9 (iii)], we know that any Radon smooth measure μ' , for example μ given by (5.1), with the quasi support K relative to $(\mathcal{E}, \mathcal{F})$ or $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ always has the topological support K . The trace Dirichlet form induced by μ' is a regular Dirichlet form on $L^2(K, \mu')$ as asserted by [1, Corollary 5.2.10]. The choice of μ' is not essential in the sense of [1, Theorem 5.2.15].

Trace Dirichlet form on some appropriate set F characterizes the ‘trace’ of the associated Markov process left on F . Precisely, given a symmetric Markov process Y with the regular Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ on the state space E and $F \subset E$ a closed subset with the positive capacity, let ν be a Radon smooth measure on E with the same topological and quasi support F and σ_F^Y be the hitting time of F relative to Y . Set for any $f \in \mathcal{F}_e^Y$,

$$\mathbf{H}_F^Y f(x) := \mathbf{E}_x f(Y_{\sigma_F^Y}), \quad x \in E.$$

Then

$$\check{\mathcal{F}}^Y := \{\varphi \in L^2(F, \nu) : \varphi = f \nu\text{-a.e. on } F \text{ for some } f \in \mathcal{F}_e^Y\},$$

$$\check{\mathcal{E}}^Y(\varphi, \varphi) := \mathcal{E}^Y(\mathbf{H}_F^Y f, \mathbf{H}_F^Y f), \quad \varphi \in \check{\mathcal{F}}^Y, \varphi = f \nu\text{-a.e. on } F, f \in \mathcal{F}_e$$

is called the trace Dirichlet form of $(\mathcal{E}^Y, \mathcal{F}^Y)$ induced by ν . It is a regular Dirichlet form on $L^2(F, \nu)$ as in Remark 5.1. We refer the details of trace Dirichlet forms and their Feller measures to [2], [1, Section 5.5] and [14].

Denote the trace Dirichlet forms of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ and $(\mathcal{E}, \mathcal{F})$ induced by the measure μ , given by (5.1), by $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ and $(\check{\mathcal{E}}, \check{\mathcal{F}})$, respectively. They are both regular and recurrent (cf. [1, Theorem 5.2.5]) Dirichlet forms on $L^2(K, \mu)$. The associated Hunt processes are denoted by $\check{B} = (\check{B}_t)_{t \geq 0}$ and $\check{X} = (\check{X}_t)_{t \geq 0}$. Their extended Dirichlet spaces are naturally denoted by \check{H}_e^1 and $\check{\mathcal{F}}_e$. We have (cf. [1, Theorem 5.2.15])

$$\check{H}_e^1 = H_e^1(\mathbb{R})|_K = \{f|_K : f \in H_e^1(\mathbb{R})\},$$

$$\check{\mathcal{F}}_e = \mathcal{F}_e|_K = \{f|_K : f \in \mathcal{F}_e\}.$$

Recall that U_n, W_n are defined by (4.3) and U_n is expressed as (4.7). We now state the main result in this section, which is similar to [14, Theorem 2.1] in the sense that they both give an example that a pure jump Dirichlet form is a proper regular Dirichlet subspace of a Dirichlet form with strongly local part.

Theorem 5.1. *Let $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ and $(\check{\mathcal{E}}, \check{\mathcal{F}})$ be given above. Then $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ is a proper regular Dirichlet subspace of $(\check{\mathcal{E}}, \check{\mathcal{F}})$, i.e.*

$$\check{H}^1 \subset \check{\mathcal{F}}, \quad \check{\mathcal{E}}(\varphi, \varphi) = \frac{1}{2}\check{\mathbf{D}}(\varphi, \varphi), \quad \varphi \in \check{H}^1.$$

Furthermore for any $\varphi \in \check{\mathcal{F}}_e = \mathcal{F}_e|_K$,

$$\check{\mathcal{E}}(\varphi, \varphi) = \frac{1}{2} \sum_{n \geq 1} \int_{W_n} \left(\frac{d\varphi}{d\tau_n} \right)^2 d\tau_n + \frac{1}{2} \sum_{n \geq 1} \sum_{m \geq 1} \frac{(\varphi(a_m^n) - \varphi(b_m^n))^2}{|a_m^n - b_m^n|}, \quad (5.2)$$

and for any $\varphi \in \check{H}_e^1 = H_e^1(\mathbb{R})|_K$,

$$\frac{1}{2}\check{\mathbf{D}}(\varphi, \varphi) = \frac{1}{2} \sum_{n \geq 1} \sum_{m \geq 1} \frac{(\varphi(a_m^n) - \varphi(b_m^n))^2}{|a_m^n - b_m^n|}. \quad (5.3)$$

Proof. The first assertion is similar to [14, Theorem 2.1 (1)] by the above Lemma 5.1. The trace formula (5.3) of one-dimensional Brownian motion on K can be formulated as in the proof of [14, Theorem 2.1 (2)] and we further remark that $m(K) = 0$.

Now we prove the trace formula (5.2). Note that the trace Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ corresponds to a time-changed Markov process \check{X} of X . Precisely, let $(A_t)_{t \geq 0}$ be the associated positive continuous additive functional of μ relative to X and τ_t be its right continuous inverse, i.e. $\tau_t := \inf\{s > 0 : A_s > t\}$ for any $t \geq 0$. Then

$$\check{X}_t = X_{\tau_t}, \quad t \geq 0.$$

On the other hand, a subset $F \subset K$ is $\check{\mathcal{E}}$ -polar if and only if F is \mathcal{E} -polar as a subset of \mathbb{R} (cf. [1, Theorem 5.2.8]). This implies that $(\bigcup_{n \geq 1} I_n)^c$ is an $\check{\mathcal{E}}$ -polar set. Since for each n , I_n is an invariant set of X , it follows that $I_n \cap K = W_n$ is an invariant set of \check{X} in the sense that

$$\mathbf{P}_x^{\check{X}}(\check{X}_t \in W_n, \forall t) = 1, \quad x \in W_n,$$

where $\mathbf{P}_x^{\check{X}}$ is the probability measure of \check{X} starting from x . Particularly, $(\check{\mathcal{E}}^{W_n}, \check{\mathcal{F}}^{W_n})$ is the trace Dirichlet form of $(\mathcal{E}^{I_n}, \mathcal{F}^{I_n})$ induced by $\mu|_{W_n}$. It suffices to prove that for any $\varphi \in \mathcal{F}_e^{I_n}|_{W_n}$,

$$\check{\mathcal{E}}^{W_n}(\varphi, \varphi) = \frac{1}{2} \int_{W_n} \left(\frac{d\varphi}{d\tau_n} \right)^2 d\tau_n + \frac{1}{2} \sum_{m \geq 1} \frac{(\varphi(a_m^n) - \varphi(b_m^n))^2}{|a_m^n - b_m^n|}, \quad (5.4)$$

since the trace formula (5.2) may then be attained from Corollary 3.2. Indeed, note that for any $f \in \mathcal{F}_e^{I_n}$, the energy measure (cf. [1, (4.3.8)]) of f is equal to

$$\mu_{\langle f \rangle} = \left(\frac{df}{d\tau_n} \right)^2 d\tau_n,$$

which can be formulated by an approach similar to [14, (2.2)]. The Feller measure corresponding to the Dirichlet form $(\check{\mathcal{E}}^{W_n}, \check{\mathcal{F}}^{W_n})$ is deduced by the same idea as in the proof of [14, Theorem 2.1] since X^{I_n} is a Brownian motion before leaving U_n by Lemma 5.1. Therefore we obtain (5.4) which is similar to [14, Theorem 2.1]. That completes the proof. \square

Although the main ideas to prove the above theorem come from [14, Theorem 2.1], we still need to point out the different significance of Theorem 5.1. In [14, Theorem 2.1], the state space F of the trace Dirichlet forms must be of positive Lebesgue measure to guarantee that the associated regular Dirichlet subspace $(\mathcal{E}^{(s)}, \mathcal{F}^{(s)})$ of the one-dimensional Brownian motion is a proper subspace. This fact causes that the strong local part of one of the trace Dirichlet forms in [14, Theorem 2.1] never disappears. When coming back to the above theorem, we find that the strongly local part may disappear in some special situation (i.e. W_n is of zero $d\tau_n$ -measure) and then an interesting phenomena shows up.

Corollary 5.1. *Let $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ and $(\check{\mathcal{E}}, \check{\mathcal{F}})$ be given in Theorem 5.1. Assume that $d\tau_n(W_n) = 0$, in other words, I_n is closed and τ_n is the natural scale function on I_n , for each $n \geq 1$. Then $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ is a proper regular Dirichlet subspace of $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(K, \mu)$. Furthermore, for any $\varphi \in \check{\mathcal{F}}_e$,*

$$\check{\mathcal{E}}(\varphi, \varphi) = \frac{1}{2} \sum_{n \geq 1} \frac{(\varphi(a_n) - \varphi(b_n))^2}{|a_n - b_n|}, \quad (5.5)$$

and for any $\varphi \in \check{H}_e^1$,

$$\frac{1}{2}\check{\mathbf{D}}(\varphi, \varphi) = \frac{1}{2} \sum_{n \geq 1} \frac{(\varphi(a_n) - \varphi(b_n))^2}{|a_n - b_n|}. \quad (5.6)$$

The proof of Corollary 5.1 is trivial by Theorem 5.1. Note that if $(\mathcal{E}, \mathcal{F})$ is such a regular Dirichlet extension in this corollary, then its associated diffusion process is a reflected Brownian motion on each closed interval I_n . An example is given in Example 3.5, in which K is the standard Cantor set in $[0, 1]$.

The above corollary partially answers a problem in which we have been interested and studied for years. We know from Theorem 2.1 that if $(\mathcal{E}^1, \mathcal{F}^1)$ is a regular Dirichlet subspace of $(\mathcal{E}^2, \mathcal{F}^2)$, then the jumping and killing measures in their Beurling–Deny decompositions are the same. Moreover, given a regular Dirichlet form, its killing part and ‘big jump’ part do not play a role in producing a proper regular Dirichlet subspace as described in [12, Section 2.2.3] and [15] respectively. For a strongly local Dirichlet form, many examples including [3, 4, 13] hint that it should always have proper regular Dirichlet subspaces. However we have not found any result to illustrate how the ‘small jump’ part plays a role when concerning the regular Dirichlet subspaces. For the first time Corollary 5.1 gives us an example that a pure jump Dirichlet form has a proper regular Dirichlet subspace. This encourages us to keep going in this direction.

On the other hand, the jumps of a Hunt process are described by its Lévy system denoted by (N, H) in [19], where $N(x, dy)$ is a kernel on the state space and H is a positive continuous additive functional. We know that all Lévy systems of a symmetric Hunt process are equivalent in the sense that if (N', H') is another Lévy system, then $N(x, dy)\mu_H(dx) = N'(x, dy)\mu_{H'}(dx)$, where μ_H and $\mu_{H'}$ are the Revuz measures of H and H' respectively. Therefore, Corollary 5.1 also conduces to the following.

Corollary 5.2. *There exist two different symmetric pure jump Hunt processes that have the equivalent Lévy systems.*

The following corollary gives us an intuitive understanding of the differences between the two regular Dirichlet forms in Corollary 5.1.

Corollary 5.3. Let $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ and $(\check{\mathcal{E}}, \check{\mathcal{F}})$ be the regular Dirichlet forms on $L^2(K, \mu)$ in Corollary 5.1. Then the following assertions hold.

(1) $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ is irreducible and for any $x, y \in K$,

$$\mathbf{P}_x^{\check{B}}(\sigma_y < \infty) > 0, \quad (5.7)$$

where $\mathbf{P}_x^{\check{B}}$ is the probability measure of \check{B} starting from x and σ_y is the first hitting time of $\{y\}$ relative to \check{B} .

(2) $(\check{\mathcal{E}}, \check{\mathcal{F}})$ is not irreducible. For each n such that a_n and b_n are finite, $\{a_n, b_n\}$ is an invariant set of $(\check{\mathcal{E}}, \check{\mathcal{F}})$ and the associated Hunt process \check{X} only jumps between a_n and b_n . Furthermore, $K \setminus \{a_n, b_n : a_n > -\infty, b_n < \infty, n \geq 1\}$ is $\check{\mathcal{E}}$ -polar.

Proof. The second assertion is obvious from the proof of Theorem 5.1. We only prove (1). Note that $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ is recurrent by [1, Theorem 5.2.5]. Let $\varphi = f|_K \in \check{H}_e^1$ such that $\check{\mathbf{D}}(\varphi, \varphi) = 0$. It follows that

$$\mathbf{D}(\mathbf{H}_K^B f, \mathbf{H}_K^B f) = 0.$$

Thus $\mathbf{H}_K^B f \equiv C$ for some constant C and $\varphi = f|_K = \mathbf{H}_K^B f|_K \equiv C$. From [1, Theorem 5.2.16] we obtain that $(\frac{1}{2}\check{\mathbf{D}}, \check{H}^1)$ is irreducible. On the other hand, [1, Theorem 5.2.8] implies the $\frac{1}{2}\check{\mathbf{D}}$ -polar set must be the empty set. Thus (5.7) follows from [1, Theorem 3.5.6 (1)]. \square

Corollary 5.3 shows us some interesting behavior of a Markov process associated with a Dirichlet form. Since the Feller measures in (5.5) and (5.6) are supported on

$$\{(a_n, b_n), (b_n, a_n) : a_n > -\infty, b_n < \infty, n \geq 1\},$$

it seems that the Markov processes \check{B} and \check{X} only jump between a_n and b_n . Actually \check{X} does jump this way. However, the trace \check{B} of Brownian motion will hit any point in K with positive probability. In other words, the motions of \check{B} happen at where its potential energy is zero. These motions are not reflected in the energy form (i.e. Dirichlet form) but in its Dirichlet space. Recall that \check{H}_e^1 is the restriction of $H_e^1(\mathbb{R})$ to K which are composed all by continuous functions, whereas $\check{\mathcal{F}}_e$ is the restriction of \mathcal{F}_e to K which is much bigger and contains many discontinuous functions.

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