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Non-Gaussian limit theorem for non-linear Langevin equations driven by Lévy noise

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Abstract. In this paper, we study the small noise behaviour of solutions of a non-linear second order Langevin equation $\ddot{x}_t^\varepsilon + |\dot{x}_t^\varepsilon|^\beta = \dot{Z}_{\varepsilon t}^\varepsilon$, $\beta \in \mathbb{R}$, driven by symmetric non-Gaussian Lévy processes Z^ε. This equation describes the dynamics of a one-degree-of-freedom mechanical system subject to non-linear friction and noisy vibrations. For a compound Poisson noise, the process x^ε on the macroscopic time scale t/ε has a natural interpretation as a non-linear filter which responds to each single jump of the driving process. We prove that a system driven by a general symmetric Lévy noise exhibits essentially the same asymptotic behaviour under the principal condition $\alpha + 2\beta < 4$, where $\alpha \in [0, 2]$ is the "uniform" Blumenthal–Getoor index of the family $\{Z^\varepsilon\}_{\varepsilon>0}$.

Résumé. Dans cet article, nous étudions le comportement du bruit des solutions d'une équation de Langevin non linéaire du second ordre $\ddot{x}_t^\varepsilon + |\dot{x}_t^\varepsilon|^\beta = \dot{Z}_{\varepsilon t}^\varepsilon$, $\beta \in \mathbb{R}$, dirigée par un processus de Lévy Z^ε symétrique et non Gaussien. Cette équation décrit la dynamique d'un système mécanique à un degré de liberté soumis à un frottement non linéaire et à des vibrations aléatoires. Pour un bruit de Grenaille (quantique, shot noise), le processus x^ε sur l'échelle de temps macroscopique t/ε s'interprète naturellement comme un filtre non linéaire qui réagit à chaque saut du processus Z. Nous montrons qu'un système conduit par un bruit de Lévy symétrique présente le même comportement asymptotique sous la condition principale $\alpha + 2\beta < 4$, où $\alpha \in [0, 2]$ est l'indice Blumenthal–Getoor « uniforme » de la famille $\{Z^\varepsilon\}_{\varepsilon>0}$.

1. Introduction and motivation

In this paper we study a non-linear response of a one-dimensional system to both external stochastic excitation and non-linear friction. In the simplest mathematical setting in the absence of external forcing, one can assume that the friction force is proportional to a power ($\beta \in \mathbb{R}$) of the particle's velocity; that is, the equation of motion has the form

$$\ddot{x}_t = -|\dot{x}_t|^\beta \operatorname{sgn} \dot{x}_t. \tag{1}$$

This model covers such prominent particular cases as the linear viscous (Stokes) friction $\beta = 1$, the dry (Coulomb) friction $\beta = 0$, and the high-speed limit of the Rayleigh friction $\beta = 2$ (see [34,35,38]). As usual, the second-order equation (1) can be written as a first order system

$$\begin{aligned}
\dot{x}_t &= v_t, \\
\dot{v}_t &= -|v_t|^\beta \operatorname{sgn} v_t,
\end{aligned} \tag{2}$$

which is a particular case of a (non-linear) Langevin equation. The second equation in this system is autonomous, and the corresponding *velocity* component can be given explicitly, once its initial value v_0 is fixed:

$$v_t = \begin{cases} v_0 e^{-t}, & \beta = 1; \\ (|v_0|^{1-\beta} - (1-\beta)t)_+^{1/(1-\beta)} \operatorname{sgn} v_0, & \text{otherwise.} \end{cases} \tag{3}$$

Clearly, for any $\beta \in \mathbb{R}$ and $v_0 \in \mathbb{R}$ such a solution tends to 0 as $t \to \infty$; that is, in any case, the velocity component of the system *dissipates*. The complete picture which also involves the *position* component, is more sophisticated. Clearly,

$$x_t = x_0 + \int_0^t v_s \, \mathrm{d}s,$$

and one can easily observe that $v = (v_t)_{t \geq 0}$ is integrable on $\mathbb{R}_+$ if $\beta < 2$. In this case the position component $x = (x_t)_{t \geq 0}$ dissipates as well and tends to a limiting value

$$x_t \to x_\infty = x_0 + F(v_0), \quad t \to \infty, \qquad F(v) = \frac{1}{2-\beta} |v|^{2-\beta} \operatorname{sgn} v.$$

The function $F(v)$ has the meaning of a complete *response* of the system to the instant perturbation of its velocity by v. For $\beta \geq 2$, the integral of v_t over $\mathbb{R}_+$ diverges, and x_t tends to $\pm\infty$ depending on the sign of v_0. In other words, the friction in the system in the vicinity of zero is too weak to slow down the particle.

In this paper we consider the interplay between the non-linear dissipation and the weak random vibrations of the particle, namely we study perturbations of the velocity by a weak (symmetric) Lévy process Z,

$$\begin{aligned} \dot{x}_t^\varepsilon &= v_t^\varepsilon, \\ \dot{v}_t^\varepsilon &= -|v_t^\varepsilon|^\beta \operatorname{sgn} v_t^\varepsilon + \dot{Z}_{\varepsilon t} \end{aligned} \tag{4}$$

in the small noise limit $\varepsilon \to 0$. Often in the literature, a weak perturbation is chosen in the form εZ_t under the assumption that $Z = B$ is a Brownian motion or an α-stable Lévy process, $\alpha \in (0, 2)$. In this case, the self-similarity of these processes yields that $(\varepsilon Z_t)_{t \geq 0} \overset{\text{law}}{=} (Z_{\varepsilon^\alpha t})_{t \geq 0}, \alpha \in (0, 2]$. A mere renaming of ε^α into ε gives us the parametrization (4).

Heuristically, we consider a system, which consists of two different components acting on different *time scales*. The *microscopic* behaviour of the system is primarily determined by the non-linear model (2) under random perturbations of low intensity. It is clear that neither these perturbations themselves nor their impact on the system are visible on the microscopic time scale; that is on any finite time interval $[0, T]$, $Z_{\varepsilon t}$ tends to 0, and $(x_t^\varepsilon, v_t^\varepsilon)$ become close to (x_t, v_t) as $\varepsilon \to 0$.

The influence of random perturbations becomes significant on the *macroscopic* time scale $\varepsilon^{-1} t$ which suggests to focus our analysis on the limit behaviour of the pair

$$(X_t^\varepsilon, V_t^\varepsilon) := (x_{\varepsilon^{-1}t}^\varepsilon, v_{\varepsilon^{-1}t}^\varepsilon) \tag{5}$$

satisfying the system of SDEs

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= \frac{1}{\varepsilon} V_t^\varepsilon \, \mathrm{d}t, \\ \mathrm{d}V_t^\varepsilon &= -\frac{1}{\varepsilon} |V_t^\varepsilon|^\beta \operatorname{sgn} V_t^\varepsilon \, \mathrm{d}t + \mathrm{d}Z_t. \end{aligned} \tag{6}$$

We will actually study a slightly more general system

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= \frac{1}{\varepsilon} V_t^\varepsilon \, \mathrm{d}t, \\ \mathrm{d}V_t^\varepsilon &= -\frac{1}{\varepsilon} |V_t^\varepsilon|^\beta \operatorname{sgn} V_t^\varepsilon \, \mathrm{d}t + \mathrm{d}Z_t^\varepsilon \end{aligned} \tag{7}$$

with a *family* of Lévy processes $\{Z^\varepsilon\}$, and look for a non-trivial limit for the position process X^ε as $\varepsilon \to 0$, in dependence on the friction exponent β and the properties of the family $\{Z^\varepsilon\}$. It will be assumed that $Z^\varepsilon \overset{\text{f.d.d.}}{\to} Z$ as $\varepsilon \to 0$; that is, the system (7) includes a possibility of slight fluctuations in the characteristics of the noise. This may look as just a technical complication of (6); however, this seeming complication is a blessing in disguise, since it allows one to use a "truncation of small jumps" procedure in order to resolve a difficult question about existence and uniqueness of the corresponding SDE in the case $\beta < 0$; see Section 2.1 below. This will make the entire construction mathematically rigorous without any loss in the physical relevance; note that the friction models with negative values of β are quite common, see [5, Chapter 7.3].

The case of Stokes friction $\beta = 1$ is probably the simplest one: the system (6) is linear, and under zero initial conditions $X_0^\varepsilon = V_0^\varepsilon = 0$, its solution X^ε is found explicitly as a convolution integral

$$X_t^\varepsilon = \int_0^t \left(1 - e^{-(t-s)/\varepsilon}\right) dZ_s^\varepsilon.$$

It was shown in [23] that for a fixed Lévy forcing Z^ε, X^ε converges to Z in the sense of finite-dimensional distributions. It is worth noticing that although X^ε is an absolutely continuous process, the limit is in general a jump process. In that case, a functional limit theorem requires the convergence in non-standard Skorokhod topologies such as the M_1-Skorokhod topology.

Non-linear ($\beta \neq 1$) stochastic systems of the type (6) driven by Brownian motion, $Z = B$, have been studied in recent years both in physical and mathematical literature; see [28–31] for the analysis for $\beta = 1, 2, 3, 5$, see [4,11,22, 25,33,42] for the important case of dry (Coulomb) friction $\beta = 0$, and see [18] for experiments and simulations for the dry friction $\beta = 0$ and irregular friction $\beta = 0.4$. The main goal of these papers was to determine on the physical level of rigour how the so-called *effective diffusion coefficient*, which is roughly speaking the variance of the particle's position, depends on ε. In mathematical terms, the result from [23] gave convergence $X^\varepsilon \Rightarrow B$ for $\beta = 1$, whereas it was proven in [16] that for $\beta > -1$, the scaled process $\varepsilon^{2(\beta-1)/(\beta+1)} X^\varepsilon$ weakly converges in the uniform topology to a Brownian motion whose variance is calculated explicitly.

The limiting behaviour of (6) with a symmetric α-stable Lévy forcing was also the subject of the paper [16]. Under the condition $\alpha + 2\beta > 4$ the authors proved that the scaled process $\varepsilon^{\alpha(\alpha+2\beta-4)/2(\alpha+\beta-1)} X^\varepsilon$ weakly converges to a Brownian motion. The proof is based on the application of the central limit theorem for ergodic processes.

In the present paper, we establish a principally different type of the limit behaviour of the process X^ε. We specify a condition on the Lévy noises $\{Z^\varepsilon\}$, which ensures that X^ε, without any additional scaling, converges to a non-Gaussian limit. Such a behaviour is easy to understand once $Z^\varepsilon = Z$ is a compound Poisson process, which is the simplest model for mechanical or physical shocks. If $\beta < 2$, the position process X^ε is a composition of individual responses of the deterministic system (1) on a series of rare impulse perturbations. Since a general (say, symmetric) non-Gaussian Lévy process Z can be interpreted as limit of compound Poisson processes, one can naively guess that the same effect should be observed for (7) in the general case as well. This guess is not completely true, because now the "large jumps" part of the noise (being, of course, a compound Poisson process) now interferes with the "small jumps" via a non-linear drift $|v|^\beta \operatorname{sgn} v$. To guarantee that the "small jumps" are indeed negligible, we have to impose a balance condition between the non-linearity index β and the proper version of the Blumenthal–Getoor index $\alpha_{\text{BG}}(\{Z^\varepsilon\})$ (see (10)) of the family $\{Z^\varepsilon\}$, namely we require that

$$\alpha_{\text{BG}}\left(\{Z^\varepsilon\}\right) + 2\beta < 4. \tag{8}$$

Combined with the aforementioned analysis of the symmetric α-stable case considered in [16], this clearly separates two alternatives available for the system (7). Once (8) holds true, the small jumps are negligible, and X^ε converges to a non-Gaussian limit; otherwise, the small jumps dominate, and X^ε is subject to the central limit theorem, i.e. after a proper scaling one gets a Gaussian limit for it. Note that since (8) necessitate the bound $\beta < 2$, a non-Gaussian limit for X^ε can be observed only when both the velocity and the position components of (2) are dissipative.

Systems of the type (6) driven by non-Gaussian Lévy processes, especially α-stable Lévy processes (Lévy flights) attract constant attention in the physical literature. A linear case ($\beta = 1$) is especially well studied. In [9], the equation (6) was studied with $\varepsilon = 1$ in a two- and three-dimensional setting in a model of plasma in an external constant magnetic field and subject to an α-stable Lévy electric forcing. In the context of stochastic volatility models in financial

mathematics such processes were studied in [2,3]. Convergence of a linear system driven by an α-stable Lévy process was studied in [1] under a different scaling. A stochastic harmonic oscillator was studies in [13,40]. In the non-linear case, we mention works [6,8,12,14] where stationary distributions of the velocity process V^ε were studied and several closed form formulae for the stationary density were obtained. There are just a few works [7,32] devoted to the dynamics of *non-linear* Lévy driven systems of the type (6).

The rest of the paper is organized as follows. In Section 2, we introduce the setting and formulate the main results of the paper. To clarify the presentation, we separate two preparatory results: Theorem 2.1 for the system (7) with the compound Poisson noise, and Theorem 2.2, which describes the asymptotic properties of the velocity component of a general system. The proofs of the preparatory results are contained in Section 3. The proof of the main statement of the paper, Theorem 2.3, is given separately in the *regular* case and in the *non-regular/quasi-ergodic* case in Section 4 and Section 5, respectively; see discussion of the terminology therein. Some technical auxiliary results are postponed to the Appendix.

2. Main results

2.1. *Notation and preliminaries*

For $a \in \mathbb{R}$, we denote $a_+ = \max\{a, 0\}$, $a \wedge b = \min\{a, b\}$

$$\operatorname{sgn} x = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0, \end{cases}$$

$X^\varepsilon \overset{\text{f.d.d.}}{\to} X$ denotes convergence in the sense of finite dimensional distributions.

Throughout the paper, Z denotes a Lévy process without a Gaussian component, which has the Lévy measure μ. In what follows, Z either is a compound Poisson process with $\mu(\mathbb{R}) \in (0, \infty)$, or is a symmetric Lévy process. In both cases, the Lévy–Hinchin formula for Z reads

$$\mathbf{E}e^{i\lambda Z_t} = \exp\left(t \int \left(e^{i\lambda z} - 1\right)\mu(dz)\right), \quad \lambda \in \mathbb{R}, t \geq 0.$$

We always assume that $\mu(\{0\}) = 0$. If Z is a compound Poisson process, we write

$$Z_t = \sum_{k=1}^{\infty} J_k \mathbb{I}_{[\tau_k, \infty)}(t),$$

where $\{\tau_k\}_{k\geq 1}$ are jump arrival times of Z, and $\{J_k\}_{k\geq 1}$ are jump amplitudes. For Z with infinite Lévy measure, an analogue of this representation is given by the *Itô–Lévy decomposition*

$$Z_t = \int_0^t \int_{|z| \leq 1} z \widetilde{N}(dz\, ds) + \int_0^t \int_{|z| > 1} z N(dz\, ds),$$

where $N(dz\, dt)$ is the Poisson point measure associated with Z, $\widetilde{N}(dz\, dt) = N(dz\, dt) - \mu(dz)\, dt$ is corresponding compensated measure.

In what follows, we consider the system (7) where the noise $\{Z^\varepsilon\}$ will be assumed to satisfy at least one of the following assumptions:

$\mathbf{H_{CP}}$ Each Z^ε is a compound Poisson process.
$\mathbf{H_{sym}}$ Each Z^ε is a symmetric Lévy process without a Gaussian component.

Such a diversity is caused by the question of the existence and uniqueness of solutions to (7), which is solved quite differently for different values of $\beta \in \mathbb{R}$.

If $\beta \geq 1$, the friction term is smooth and satisfies the *dissipativity* condition $vb(v) \leq -v^2$ for $|v| \geq 1$. To construct a unique solution, one truncates the drift term at the levels $\pm n$ so that it becomes bounded and Lipschitz continuous, obtains a sequence of approximations $\{V^n\}_{n \geq 1}$ and shows that they converge to a solution which is well defined for all $t \geq 0$. Details of this standard argument can be found, e.g. in [37].

For $\beta \in (0, 1)$, the friction term is non-Lipschitz. However, the argument remains essentially the same as above, and is actually simpler because the truncation step is not needed. Namely, b is monotonous and satisfies now the *one-sided Lipschitz condition*

$$(u - v)\big(b(u) - b(v)\big) \leq L|u - v|^2, \quad u, v \in \mathbb{R},$$

which guarantees existence of the strong solution to (6); see [39], Theorem 170 and Example 171.

For $\beta = 0$, the solution to (7) is well defined by Theorem 4.1 and subsequent Corollary in [41], provided that either $\mathbf{H}_{\text{sym}}$ or $\mathbf{H}_{\text{CP}}$ holds.

The case $\beta < 0$ is more subtle, and existence and uniqueness of solutions of the equation with such a *singular* drift and arbitrary symmetric Lévy noise is an open question. For the symmetric α-stable noise with $\alpha \in (1, 2)$, it is known that the *weak* solution to (6) is uniquely defined when $\alpha + \beta > 1$; see [36]. This lower bound for β seems to be crucial, because for the Brownian noise (that is, for $\alpha = 2$) it is known that in case $\beta < -1$ the solution after it reaches zero can not be further extended; see the general theory presented in [10].

Note however, that the situation simplifies drastically if Z^ε are compound Poisson processes. In this case, the number of jumps for every Z^ε is finite on each finite interval, and thus the system (7) can be uniquely solved path-by-path for any $\beta \in \mathbb{R}$; see the explicit formulae in Section 3.1 below.

Let us summarize: if Z has an infinite jump measure μ, then for $\beta < 0$ with large $|\beta|$ the solution to (6) is hardly specified. On the other hand, a solution is well defined once Z is replaced by its compound Poisson approximation

$$Z_t^\varepsilon = \int_0^t \int_{|z| > \ell(\varepsilon)} z N(\mathrm{d}z\, \mathrm{d}s),$$

where all the jumps of Z with amplitudes smaller than some threshold $\ell(\varepsilon)$ are truncated. Since the cut-off level $\ell(\varepsilon)$ can be chosen arbitrary small, the intensity of compound Poisson approximations Z^ε is finite but can increase arbitrarily fast as $\varepsilon \to 0$, so that from the point of view of physical applications the processes Z and Z^ε are practically indistinguishable. Such a "truncation of small jumps" procedure makes the entire construction mathematically rigorous without any loss in the physical relevance. In particular, it allows us to treat the system with the α-stable noise without any lower bounds on β, which actually would not be relevant from the point of view of the limit behavior of the system; see Corollary 2.1 and Example 2.1 below.

The *Blumenthal–Getoor index* $\alpha_{\text{BG}}(Z)$ of a Lévy process Z is defined by

$$\alpha_{\text{BG}}(Z) = \inf\Big\{\alpha > 0 \colon \sup_{r \in (0,1]} r^\alpha \mu\big(z \colon |z| > r\big) < \infty\Big\}.$$

Note that for an arbitrary Lévy measure μ the following estimate holds true:

$$r^2 \mu\big(z \colon |z| > r\big) = r^2 \int_{|z| > r} \mu(\mathrm{d}z) \leq \int_{\mathbb{R}} \big(z^2 \wedge 1\big) \mu(\mathrm{d}z) < \infty; \tag{9}$$

that is, $\alpha_{\text{BG}}(Z) \in [0, 2]$. For a family of Lévy processes $\{Z^\varepsilon\}_{\varepsilon \in (0,1]}$ with the Lévy measures $\{\mu^\varepsilon\}_{\varepsilon \in (0,1]}$, we define its Blumenthal–Getoor index $\alpha_{\text{BG}}(\{Z^\varepsilon\})$ by

$$\alpha_{\text{BG}}\big(\{Z^\varepsilon\}\big) = \inf\Big\{\alpha > 0 \colon \sup_{r \in (0,1]} \sup_{\varepsilon \in (0,1]} r^\alpha \mu^\varepsilon\big(z \colon |z| > r\big) < \infty\Big\}. \tag{10}$$

We will consider families $\{Z^\varepsilon\}$ such that

$$Z^\varepsilon \xrightarrow{\text{f.d.d.}} Z, \quad \varepsilon \to 0. \tag{11}$$

Then by Theorem 2 in [17, Chapter XVII.2],

$$\sup_{\varepsilon\in(0,1]}\int_{\mathbb{R}}\left(z^2\wedge 1\right)\mu^\varepsilon(\mathrm{d}z)<\infty, \tag{12}$$

which again provides $\alpha_{\mathrm{BG}}(\{Z^\varepsilon\})\in[0,2]$.

2.2. *The simplest non-Gaussian case*: *Compound Poisson impulses*

In this section we consider the case where $\mathbf{H}_{\mathrm{CP}}$ and (11) hold true, and the limiting process Z is compound Poisson. To avoid inessential complications, we assume that the jump arrival times and jump amplitudes for $\{Z^\varepsilon\}$ converge a.s.:

$$\tau_k^\varepsilon\to\tau_k,\qquad J_k^\varepsilon\to J_k,\quad \varepsilon\to 0, k\geq 1, \tag{13}$$

where $\{\tau_k\}_{k\geq 1}$ $\{J_k\}_{k\geq 1}$ are corresponding are jump arrival times and jump amplitudes for Z. Denote by

$$N_t=\sum_{k=1}^\infty\mathbb{I}_{[\tau_k,\infty)}(t),\quad t\geq 0,$$

the counting process for Z, so that

$$Z_t=\sum_{k=1}^{N_t}J_k.$$

Let the initial position and velocity x_0, v_0 be fixed, and let $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ be the corresponding solution to the system (7).

Theorem 2.1. *For any $t>0$, we have the following convergence a.s. as $\varepsilon\to 0$:*

1. *for $\beta<2$,*

$$X_t^\varepsilon\to X_t=x_0+\frac{1}{2-\beta}|v_0|^{2-\beta}\operatorname{sgn}v_0+\frac{1}{2-\beta}\sum_{k=1}^{N_t}|J_k|^{2-\beta}\operatorname{sgn}J_k, \tag{14}$$

2. *for $\beta=2$,*

$$\left(\ln\frac{1}{\varepsilon}\right)^{-1}X_t^\varepsilon\to X_t=\operatorname{sgn}v_0+\sum_{k=1}^{N_t}\operatorname{sgn}J_k,$$

3. *for $\beta>2$,*

$$\varepsilon^{\frac{\beta-1}{\beta-2}}X_t^\varepsilon\to X_t=\frac{(\beta-1)^{\frac{\beta-2}{\beta-1}}}{\beta-2}\left[\tau_1^{\frac{\beta-1}{\beta-2}}\operatorname{sgn}v_0+\sum_{k=2}^{N_t}(\tau_k-\tau_{k-1})^{\frac{\beta-1}{\beta-2}}\operatorname{sgn}J_{k-1}+(t-\tau_{N_t})^{\frac{\beta-1}{\beta-2}}\operatorname{sgn}J_{N_t}\right].$$

The proof of this theorem is postponed to Section 3.1

In the above theorem, the considerably different limits in the case 1 and the cases 2, 3 are caused by the different dissipativity properties of the system (2) discussed in the Introduction. For $\beta<2$, the complete response to the perturbation of the velocity is finite, and is given by the function

$$F(v)=\frac{1}{2-\beta}|v|^{2-\beta}\operatorname{sgn}v. \tag{15}$$

Note that the right hand side in (14) is just the sum of the initial position x_0, the response which corresponds to the initial velocity v_0, and the responses to the random impulses which had arrived into the system up to the time t.

Similar additive structure remains true in the cases 2 and 3 as well, however for $\beta \geq 2$ the complete response of the system to every single perturbation is infinite, which explains the necessity to introduce a proper scaling. For $\beta > 2$, this also leads to necessity to take into account the jump arrival times. Note that in all three regimes, the initial value v_0 of the velocity has a natural interpretation as a single jump with the amplitude $J_0 = v_0$, which occurs at the initial time instant $\tau_0 = 0$.

2.3. *General setup*

This section contains the main results of the paper, which concerns the system with infinite jump intensity of the limiting Lévy noise. The first statement actually shows that the velocity component of (7), under very wide assumptions on the Lévy noise, has a dissipative behaviour similar to the one of v_t, discussed in the Introduction.

Theorem 2.2. *Assume* $\mathbf{H}_{\mathrm{sym}}$ *and* (11) *hold true. If* $\beta < 0$, *then assume in addition* $\mathbf{H}_{\mathrm{CP}}$. *Then the following statements hold true*:

(i) *for any $T > 0$ and any initial value v_0,*

$$\lim_{R \to \infty} \sup_{\varepsilon \in (0,1]} \mathbf{P}\left(\sup_{t \in [0,T]} \left| V_t^\varepsilon \right| > R \right) = 0; \tag{16}$$

(ii) *for any $t > 0$, any initial value v_0, and any $\delta > 0$,*

$$\lim_{\varepsilon \searrow 0} \mathbf{P}\left(\left| V_t^\varepsilon \right| > \delta \right) = 0. \tag{17}$$

The main result of the entire paper is presented in the following theorem.

Theorem 2.3. *Let conditions of Theorem 2.2 hold true. Assume* (8), *and in the case* $\alpha_{\mathrm{BG}}(\{Z^\varepsilon\}) = 2$ *assume in addition that*

$$\lim_{r \searrow 0} \sup_{\varepsilon \in (0,1]} \int_{|z| \leq r} z^2 \mu^\varepsilon(\mathrm{d}z) = 0. \tag{18}$$

Then $X^\varepsilon \overset{\mathrm{f.d.d.}}{\to} X$, $\varepsilon \to 0$, *where*

$$X_t = x_0 + \frac{|v_0|^{2-\beta}}{2 - \beta} \operatorname{sgn} v_0 + \frac{1}{2 - \beta} \int_0^t \int |z|^{2-\beta} \operatorname{sgn} z \widetilde{N}(\mathrm{d}z \, \mathrm{d}s), \quad t \geq 0, \tag{19}$$

and $\widetilde{N}$ is the compensated Poisson random measure, which corresponds to the Lévy process Z.

Condition (18) prevents accumulation of small jumps for the family $\{\mu^\varepsilon\}$. If Z^ε is obtained from one process Z by the truncation of small jumps procedure, explained above, then (18) holds true immediately, and (8) is actually the condition on the Blumenthal–Getoor index of Z. This leads to the following.

Corollary 2.1. *Let Z be a symmetric Lévy process without a Gaussian component, and let its Blumenthal–Getoor index satisfy $\alpha_{\mathrm{BG}}(Z) + 2\beta < 4$. Let either $Z^\varepsilon = Z$ (in this case $\beta \geq 0$), or Z^ε be a compound Poisson process, obtained from Z by truncations of the jumps with amplitudes smaller than $\ell(\varepsilon)$ (in this case $\beta \in \mathbb{R}$ can be arbitrary). Let*

$$\ell(\varepsilon) \to 0, \quad \varepsilon \to 0.$$

Then the position component X^ε of the system (7) *satisfies* (19).

Note that the right hand side in (19) is a Lévy process with the Lévy measure

$$\mu^X(B) = \mu\left(\left\{ z : \frac{|z|^{2-\beta} \operatorname{sgn} z}{2 - \beta} \in B \right\} \right), \quad B \in \mathcal{B}(\mathbb{R}). \tag{20}$$

Theorem 2.3 actually shows that the Langevin equation (4) with small Lévy noise, considered at the macroscopic time scale, performs a non-linear filter of the noise, with the transformation of the jump intensities given by (20). Since μ is symmetric and the response function $F(v) = \frac{1}{2-\beta}|v|^{2-\beta} \operatorname{sgn} v$ is odd,

$$\int_0^t \int F(z)\widetilde{N}(\mathrm{d}z\,\mathrm{d}s) = L^2\text{-}\lim_{\delta \to 0} \sum_{s \leq t} F\big(|\triangle Z_s|\big) \cdot \mathbb{I}\big(|\triangle Z_s| > \delta\big).$$

In other words, the right hand side in (19) has exactly the same form as (14). Note that the assumption (8) again requires $\beta < 2$, since $\alpha \geq 0$. Hence, the operation of the aforementioned non-linear filtering can be shortly described as follows: every jump z of the input process Z is transformed to the jump $F(z)$ of the output process. From this point of view, the assumption (8) can be interpreted as a condition for the jumps to arrive "sparsely" enough, for the system to be able to filter them independently. The following example, in particular, shows that this assumption is sharp, and once it fails, the asymptotic regime for (7) may change drastically.

Example 2.1. Let Z be a symmetric α-stable process with the Lévy measure

$$\mu(\mathrm{d}z) = c\frac{\mathrm{d}z}{|z|^{\alpha+1}}, \quad c > 0,$$

and the corresponding $\{Z^\varepsilon\}$ be the same as in Corollary 2.1. Note that $\alpha_{BG}(Z) = \alpha$, thus Theorem 2.3 requires $\alpha + 2\beta < 4$. The limiting process X in (19) is also a symmetric stable process with the Lévy measure

$$\mu^X(\mathrm{d}z) = c_X\frac{\mathrm{d}z}{|z|^{\alpha_X+1}},$$

where

$$\alpha_X = \frac{\alpha}{2-\beta}, \qquad c_X = \frac{c}{(2-\beta)^{\alpha+1}}.$$

Note that the new stability index α_X is positive, and $\alpha_X < 2$ exactly when $\alpha + 2\beta < 4$. On the one hand, this is not surprising because we know from [16] that, once $\alpha + 2\beta > 4$, the properly scaled process X^ε has a Gaussian limit. This example also shows one more aspect, at which the assumption $\beta \geq 0$ is too restrictive and non-natural. Namely, allowing β to be an arbitrary real number, we can interpret the system (7) as a non-linear Lévy filter which processes an incoming symmetric α-stable process Z into a symmetric $\frac{\alpha}{2-\beta}$-stable process X without any restriction on the stability indices.

The boundary case $\alpha + 2\beta = 4$ is yet open for a study.

Before proceeding with the proofs, let us give two more remarks. First, it will be seen from the proofs that for any $t > 0$

$$X_t^\varepsilon - x_0 - \frac{|v_0|^{2-\beta}}{2-\beta}\operatorname{sgn} v_0 - \frac{1}{2-\beta}\int_0^t \int |z|^{2-\beta}\operatorname{sgn} z\,\widetilde{N}^\varepsilon(\mathrm{d}z\,\mathrm{d}s) \to 0, \quad \varepsilon \to 0, \tag{21}$$

in probability, where $\widetilde{N}^\varepsilon$ denotes the compensated Poisson random measures for the processes Z^ε. This is a stronger feature than just the weak convergence stated in Theorem 2.3. Hence the non-linear filter, discussed above, actually operates with the trajectories of the noise rather than with its law.

Second, we consider the present paper as the first work devoted to the convergence of non-linear Lévy filters and restrict ourselves to the f.d.d. weak convergence (actually, the point-wise convergence in probability), rather than the functional convergence. In the compound Poisson case (Theorem 2.1), it can be easily verified with the help of explicit trajectory-wise calculations that the functional convergence holds true in the M_1-topology for $\beta \leq 2$, and in the uniform topology for $\beta > 2$. We believe that (19) holds true in the M_1-topology, similarly to the case $\beta = 1$ studied in [23] straightforwardly. For the sake of reader's convenience and readability of the paper we prefer to pursue this question in subsequent works, probably in a more general setting.

3. Proofs of preparatory results

3.1. *Proof of Theorem* 2.1

The solution of the system (6) can be written explicitly. Namely, denote

$$
\mathbf{V}_t^\varepsilon(v) =
\begin{cases}
v e^{-t/\varepsilon}, & \beta = 1; \\
(|v|^{1-\beta} - t(1-\beta)/\varepsilon)_+^{1/(1-\beta)} \operatorname{sgn} v, & \text{otherwise,}
\end{cases}
$$

which is just the velocity component of the system (2) with $v_0 = v$, taken at the macroscopic time scale $\varepsilon^{-1}t$; see (3). The integral of the velocity

$$
\mathbf{I}_t^\varepsilon(v) = \frac{1}{\varepsilon} \int_0^t \mathbf{V}_s^\varepsilon(v)\, ds,
$$

can be also easily computed:

$$
\mathbf{I}_t^\varepsilon(v) =
\begin{cases}
v(1 - e^{-t/\varepsilon}), & \beta = 1; \\
\ln(1 + \frac{|v|t}{\varepsilon}) \operatorname{sgn} v, & \beta = 2; \\
\frac{1}{\beta-2}[(|v|^{1-\beta} - (1-\beta)\frac{t}{\varepsilon})_+^{\frac{\beta-2}{\beta-1}} - |v|^{2-\beta}] \operatorname{sgn} v, & \text{otherwise.}
\end{cases}
$$

Then $(X_t^\varepsilon, V_t^\varepsilon)$, defined by (7), can be expressed as follows:

$$
V_t^\varepsilon = \sum_{k=0}^\infty \mathbf{V}_{(t-\tau_k) \wedge (\tau_{k+1}^\varepsilon - \tau_k^\varepsilon)}^\varepsilon \big(V_{\tau_k-}^\varepsilon + J_k^\varepsilon\big) \mathbb{I}_{[\tau_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t) \tag{22}
$$

and

$$
X_t^\varepsilon = x_0 + \sum_{k=0}^\infty \mathbf{I}_{(t-\tau_k^\varepsilon) \wedge (\tau_{k+1}^\varepsilon - \tau_k^\varepsilon)}^\varepsilon \big(V_{\tau_k^\varepsilon-}^\varepsilon + J_k^\varepsilon\big) \cdot \mathbb{I}_{[\tau_k^\varepsilon, \infty)}(t), \tag{23}
$$

where we adopt the notation

$$
\tau_0^\varepsilon = 0, \qquad J_0^\varepsilon = v_0, \qquad V_{\tau_0-}^\varepsilon = 0.
$$

Note that for any $\varepsilon > 0$, $t \mapsto X_t^\varepsilon$ is continuous. Since $\mathbf{V}_t^\varepsilon(v)$ and $\mathbf{I}_t^\varepsilon(v)$ are given explicitly, we now easily obtain the required statements. First, observe that for each $t > 0$ and $v \in \mathbb{R}$,

$$
\mathbf{V}_t^\varepsilon(v) \to 0, \quad \varepsilon \to 0,
$$

hence

$$
V_{\tau_k-}^\varepsilon \to 0, \quad \varepsilon \to 0, k \geq 0, \tag{24}
$$

almost surely. Next, we have for $\beta < 2$ for any $t > 0$, $v \in \mathbb{R}$

$$
\mathbf{I}_t^\varepsilon(v) \to F(v) = \frac{1}{2-\beta} |v|^{2-\beta} \operatorname{sgn} v, \quad \varepsilon \to 0.
$$

Since any fixed time instant $t > 0$ with probability 1 does not belong to the set $\{\tau_k\}_{k \geq 0}$, the latter relation combined with (24) gives

$$
X_t^\varepsilon \to x_0 + \frac{1}{2-\beta} \sum_{k=0}^{N_t} |J_k|^{2-\beta} \operatorname{sgn} J_k, \quad \varepsilon \to 0,
$$

almost surely. For $\beta = 2$, for any for $t > 0$, $v \in \mathbb{R}$ we have

$$\mathbf{I}_t^\varepsilon(v) - \left(\ln\frac{1}{\varepsilon}\right)\operatorname{sgn} v \to \ln\big(|v|t\big) \cdot \operatorname{sgn} v, \quad \varepsilon \to 0.$$

Combined with (13) and (24), this gives

$$\left(\ln\frac{1}{\varepsilon}\right)^{-1} X_t^\varepsilon \to \sum_{k=0}^{N_t} \operatorname{sgn} J_k, \quad \varepsilon \to 0,$$

almost surely. In the case $\beta > 2$ the argument is completely analogous, and is based on the relation

$$\left| \varepsilon^{\frac{\beta-2}{\beta-1}} \mathbf{I}_t^\varepsilon(v) - \frac{(\beta-1)^{\frac{\beta-2}{\beta-1}}}{\beta-2} t^{\frac{\beta-1}{\beta-2}} \operatorname{sgn} v \right| \to 0, \quad \varepsilon \to 0, t \geq 0.$$

Note that opposite to the previous cases, this convergence holds *uniformly* w.r.t. $t \in [0, T]$ for any $T > 0$. Recalling the exact formula (23) we get that for any path

$$\varepsilon^{\frac{\beta-1}{\beta-2}} X_t^\varepsilon \to \frac{\beta^{\frac{\beta-1}{\beta-2}}}{\beta-2} \sum_{k=0}^{\infty} \big((t - \tau_k) \wedge (\tau_{k+1} - \tau_k)\big)^{\frac{\beta-2}{\beta-1}} \operatorname{sgn} J_k \cdot \mathbb{I}_{[\tau_k,\infty)}(t)$$

pointwise for $t \geq 0$, and also uniformly on finite time intervals. Note that in this case, the limiting process X is continuous.

3.2. *Proof of Theorem* 2.2

1. In what follows, we assume that all the processes $\{Z^\varepsilon\}_{\varepsilon\in(0,1]}$ are defined on the same filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$. We will systematically use the following "truncation of large jumps" procedure. For $A > 1$, denote by $Z^{\varepsilon,A}$ the truncation of the Lévy process Z^ε at the level A, namely

$$Z_t^{\varepsilon,A} = \int_0^t \int_{|z|\leq 1} z\widetilde{N}^\varepsilon(\mathrm{d}z\,\mathrm{d}s) + \int_0^t \int_{1<|z|\leq A} zN^\varepsilon(\mathrm{d}z\,\mathrm{d}s).$$

For a given $T > 0$,

$$\mathbf{P}\big(Z_t^\varepsilon = Z_t^{\varepsilon,A}, t \in [0, T]\big) = \mathbf{P}\big(N^\varepsilon\big(\{z\colon |z| > A\} \times [0, T]\big) = 0\big) = 1 - \exp\left(-T\int_{|z|>A} \mu^\varepsilon(\mathrm{d}z)\right).$$

Recall that the convergence $Z^\varepsilon \overset{\text{f.d.d}}{\to} Z$, $\varepsilon \to 0$, of Lévy processes yields

$$\lim_{\varepsilon\downarrow 0} \int f(z)\mu^\varepsilon(\mathrm{d}z) = \int f(z)\mu(\mathrm{d}z) \tag{25}$$

for any $f \in C_b(\mathbb{R}, \mathbb{R})$ such that $f(z) = 0$ in a neighbourhood of the origin. This means that the tails of the Lévy measures μ^ε uniformly vanish at ∞:

$$\sup_{\varepsilon\in(0,1]} \mu^\varepsilon\big(z\colon |z| > A\big) \to 0, \quad A \to \infty.$$

That is, for any $T > 0$ and $\theta > 0$ we can fix $A > 0$ large enough such that

$$\inf_{\varepsilon\in(0,1]} \mathbf{P}\big(Z_t^\varepsilon = Z_t^{\varepsilon,A}, t \in [0, T]\big) \geq 1 - \theta.$$

Assume that for such A we manage to prove statements (i), (ii) of the Theorem for the system (7) driven by $Z^{\varepsilon,A}$ instead of Z^{ε}. Since this system coincides with the original one on a set of probability larger than $1 - \theta$, we immediately get the following weaker versions of (16) and (17):

$$\limsup_{N \to \infty} \sup_{\varepsilon \in (0,1]} \mathbf{P}\Big(\sup_{t \in [0,T]} \big| V_t^{\varepsilon} \big| > N \Big) \le \theta,$$

$$\limsup_{\varepsilon \searrow 0} \mathbf{P}\big(\big| V_t^{\varepsilon} \big| > \delta \big) \le \theta.$$

Taking A large enough, we can make θ arbitrarily small. Hence, in order to get the required statements, it is sufficient to prove the same statements under the additional assumption that, for some A,

$$\operatorname{supp} \mu^{\varepsilon} \subseteq [-A, A], \quad \varepsilon \in (0, 1]. \tag{26}$$

2. Let us proceed with the proof of (16). By (26) and the symmetry of μ^{ε}, we have that

$$Z_t^{\varepsilon} = Z_t^{\varepsilon,A} = \int_0^t \int_{-A}^A z \widetilde{N}^{\varepsilon}(\mathrm{d}s, \, \mathrm{d}z)$$

is a square integrable martingale. We have

$$\big| V_t^{\varepsilon} \big|^2 = v_0^2 - \frac{2}{\varepsilon} \int_0^t \big| V_s^{\varepsilon} \big|^{\beta+1} \mathbb{I}\big(\big| V_s^{\varepsilon} \big| \ne 0 \big) \, \mathrm{d}s + t \int_{-A}^A z^2 \mu^{\varepsilon}(\mathrm{d}z) + M_t^{\varepsilon}, \tag{27}$$

where

$$M_t^{\varepsilon} = 2 \int_0^t \int_{-A}^A z V_{s-}^{\varepsilon} \, \widetilde{N}^{\varepsilon}(\mathrm{d}s \, \mathrm{d}z) \tag{28}$$

is a local martingale. For $\beta \ge 0$, this follows by the Itô formula applied to the process V^{ε}; for $\beta < 0$, this can be derived directly from the representation (22) for V^{ε} (recall that for $\beta < 0$ each Z^{ε} is a compound Poisson process). The sequence

$$\tau_m^{\varepsilon} := \inf\big\{ t \ge 0 \colon \big| V_t^{\varepsilon} \big| > m \big\}, \quad m \ge 1,$$

is a localizing sequence for M^{ε} and thus

$$\big| V_{t \wedge \tau_m^{\varepsilon}}^{\varepsilon} \big|^2 \le v_0^2 + T \int_{-A}^A z^2 \mu^{\varepsilon}(\mathrm{d}z) + M_{t \wedge \tau_m^{\varepsilon}}^{\varepsilon}.$$

By the Doob maximal inequality,

$$\mathbf{E} \sup_{t \in [0,T]} \big| M_{t \wedge \tau_m^{\varepsilon}}^{\varepsilon} \big|^2 \le 4 \mathbf{E} \big| M_{T \wedge \tau_m^{\varepsilon}}^{\varepsilon} \big|^2 = 16 \cdot \mathbf{E} \int_0^{T \wedge \tau_m^{\varepsilon}} \big| V_s^{\varepsilon} \big|^2 \, \mathrm{d}s \cdot \int_{-A}^A z^2 \mu^{\varepsilon}(\mathrm{d}z).$$

This yields

$$\mathbf{E} \sup_{t \in [0,T]} \big| V_{t \wedge \tau_m^{\varepsilon}}^{\varepsilon} \big|^2 \le v_0^2 + T \int_{-A}^A z^2 \mu^{\varepsilon}(\mathrm{d}z) + 4 \bigg(T \int_{-A}^A z^2 \mu^{\varepsilon}(\mathrm{d}z) \bigg)^{1/2} \bigg(\mathbf{E} \sup_{t \in [0,T]} \big| V_{t \wedge \tau_m^{\varepsilon}}^{\varepsilon} \big|^2 \bigg)^{1/2}.$$

Thus these exists a constant $C > 0$, independent on ε, such that

$$\sup_m \mathbf{E} \sup_{t \in [0,T]} \big| V_{t \wedge \tau_m^{\varepsilon}}^{\varepsilon} \big|^2 \le C.$$

Since $\tau_m^\varepsilon \to \infty, m \to \infty$, a.s., by the Fatou lemma we get

$$\mathbf{E} \sup_{t \in [0,T]} |V_t^\varepsilon|^2 \leq C. \tag{29}$$

This yields (16) by the Chebyshev inequality.

3. To prove (17), we note that M^ε defined in (28) is a square integrable martingale by (29). Then by (27) we have

$$\mathbf{E}|V_T^\varepsilon|^2 = v_0^2 - \frac{2}{\varepsilon}\mathbf{E}\int_0^T |V_s^\varepsilon|^{\beta+1}\mathbb{I}(|V_s^\varepsilon| \neq 0)\,\mathrm{d}s + T\int_{-A}^A z^2 \mu^\varepsilon(\mathrm{d}z). \tag{30}$$

Hence

$$\mathbf{E}\int_0^T |V_s^\varepsilon|^{\beta+1}\mathbb{I}(|V_s^\varepsilon| \neq 0)\,\mathrm{d}s \leq \frac{\varepsilon}{2}\left(v_0^2 + T\int_{-A}^A z^2 \mu^\varepsilon(\mathrm{d}z)\right) \to 0, \quad \varepsilon \to 0.$$

For $\beta > -1$ this yields that, for any $\delta > 0$,

$$\int_0^T \mathbb{I}(|V_s^\varepsilon| > \delta)\,\mathrm{d}s \to 0, \quad \varepsilon \to 0, \tag{31}$$

in probability. For $\beta \leq -1$, we have for any $R > 0$

$$\mathbf{E}\int_0^T |V_s^\varepsilon|^{\beta+1}\mathbb{I}(|V_s^\varepsilon| \neq 0)\,\mathrm{d}s \geq R^{\beta+1}\mathbf{E}\left[\mathbb{I}\left(\sup_{t \in [0,T]}|V_t^\varepsilon| \leq R\right) \cdot \int_0^T \mathbb{I}(|V_s^\varepsilon| \neq 0)\,\mathrm{d}s\right].$$

Combined with (16), this gives

$$\int_0^T \mathbb{I}(|V_s^\varepsilon| \neq 0)\,\mathrm{d}s \to 0, \quad \varepsilon \to 0,$$

in probability. In each of these cases, we have that, for any given $\zeta > 0$, $t_0 \geq 0$, the stopping times

$$\theta_\zeta^\varepsilon(t_0) = \inf\{t \geq t_0 \colon |V_t^\varepsilon| \leq \zeta\}$$

satisfy

$$\theta_\zeta^\varepsilon(t_0) \to t_0, \quad \varepsilon \to 0 \tag{32}$$

in probability.

Now we can finalize the proof of (17). For a given $t > 0$, fix $t_0 \in [0, t)$ and $\zeta > 0$, and consider the set

$$C_{\zeta,t_0,t}^\varepsilon = \{\theta_\zeta^\varepsilon(t_0) \leq t\} \in \mathcal{F}_{\theta_\zeta^\varepsilon(t_0)}.$$

Then by (27) and Doob's optional sampling theorem, we have

$$\mathbf{E}|V_t^\varepsilon|^2 \mathbb{I}_{C_{\zeta,t_0,t}^\varepsilon} \leq \mathbf{E}|V_{\theta_\zeta^\varepsilon(t_0)}^\varepsilon|^2 \mathbb{I}_{C_{\zeta,t_0,t}^\varepsilon} + \mathbf{E}(t - \theta_\zeta^\varepsilon(t_0))\mathbb{I}_{C_{\zeta,t_0,t}^\varepsilon}\left(\int_{-A}^A z^2 \mu^\varepsilon(\mathrm{d}z)\right)$$

$$\leq \zeta^2 + (t - t_0)\left(\int_{-A}^A z^2 \mu^\varepsilon(\mathrm{d}z)\right).$$

This implies that

$$\mathbf{P}(|V_t^\varepsilon| > \delta) \leq \mathbf{P}(\Omega \setminus C_{\zeta,t_0,t}^\varepsilon) + \frac{\zeta^2}{\delta^2} + \frac{t - t_0}{\delta^2}\left(\int_{-A}^A z^2 \mu^\varepsilon(\mathrm{d}z)\right).$$

By (32) we have

$$\mathbf{P}\big(\Omega \setminus C^{\varepsilon}_{\zeta,t_0,t}\big) \to 0, \quad \varepsilon \to 0.$$

Hence by (12)

$$\limsup_{\varepsilon \searrow 0} \mathbf{P}\big(\big|V^{\varepsilon}_t\big| > \delta\big) \le \frac{\zeta^2}{\delta^2} + C\frac{t - t_0}{\delta^2}.$$

Since $\zeta > 0$ and $t_0 < t$ are arbitrary, this proves (17).

4. Proof of Theorem 2.3: Regular case

To simplify the notation, in what follows we fix $\alpha \in [0, 2]$ such that $\alpha + 2\beta < 4$ and for some $C > 0$

$$\sup_{\varepsilon \in (0,1]} \mu^{\varepsilon}\big(|z| \ge r\big) \le Cr^{-\alpha}, \quad r \in (0, 1]. \tag{33}$$

Such α exists by the assumption (8) and the definition of the Blumenthal–Getoor index for the family $\{Z^{\varepsilon}\}$. If $\alpha_{BG}(\{Z^{\varepsilon}\}) < 2$, we can take $\alpha < 2$. Otherwise $\alpha = 2$; recall that in this case (18) is additionally assumed.

We will prove Theorem 2.3 in two different cases. First, we consider the *regular* case

$$(\alpha, \beta) \in \Xi_{\text{regular}} = \big\{\alpha \in [0, 2], \alpha + \beta < 2\big\} \cup \big\{(2, 0)\big\}, \tag{34}$$

see Figure 1. This name and the main idea of the proof are explained in Section 4.1 below.

4.1. *Outline*

Let us apply, yet just formally, the Itô formula to the function $F(v) = \frac{1}{2-\beta}|v|^{2-\beta} \operatorname{sgn} v$ and the process V^{ε} given by (7):

$$F\big(V^{\varepsilon}_t\big) = F(v_0) - \frac{1}{\varepsilon}\int_0^t V^{\varepsilon}_{s-}\,\mathrm{d}s + M^{\varepsilon}_t + \int_0^t H^{\varepsilon}\big(V^{\varepsilon}_s\big)\,\mathrm{d}s, \tag{35}$$

where

$$M^{\varepsilon}_t = \int_0^t \int_{\mathbb{R}} \big(F\big(V^{\varepsilon}_{s-} + z\big) - F\big(V^{\varepsilon}_{s-}\big)\big)\widetilde{N}^{\varepsilon}(\mathrm{d}z\,\mathrm{d}s), \tag{36}$$

$$H^{\varepsilon}(v) = \int_0^{\infty} \big(F(v + z) + F(v - z) - 2F(v)\big)\mu^{\varepsilon}(\mathrm{d}z). \tag{37}$$
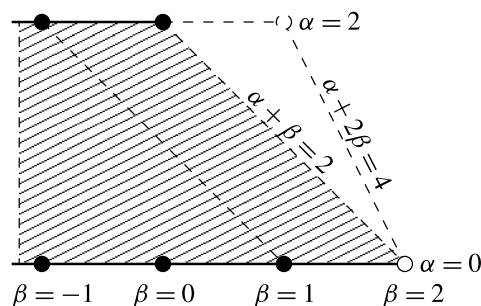

Fig. 1. The set of parameters $(\alpha, \beta) \in \Xi_{\text{regular}}$ corresponding to the regular case, see (34).

Then

$$X_t^\varepsilon + F(V_t^\varepsilon) = x_0 + \frac{1}{\varepsilon} \int_0^t V_{s-}^\varepsilon \, ds + F(V_t^\varepsilon) = x_0 + F(v_0) + M_t^\varepsilon + \int_0^t H^\varepsilon(V_s^\varepsilon) \, ds. \tag{38}$$

By Theorem 2.2 we have

$$F(V_t^\varepsilon) \to 0, \quad \varepsilon \to 0$$

in probability, and by (31) one can expect to have

$$M_t^\varepsilon - \mathbf{M}_t^\varepsilon \to 0, \quad \varepsilon \to 0 \tag{39}$$

in probability; here and below we denote

$$\mathbf{M}_t^\varepsilon := \int_0^t \int_{\mathbb{R}} \big(F(0+z) - F(0)\big) \widetilde{N}^\varepsilon(dz \, ds) = \int_0^t \int_{\mathbb{R}} F(z) \widetilde{N}^\varepsilon(dz \, ds).$$

It is easy to show that

$$\mathbf{M}^\varepsilon \overset{\text{f.d.d.}}{\to} \frac{1}{2 - \beta} \int_0^\cdot \int |z|^{2-\beta} \operatorname{sgn} z \, \widetilde{N}(dz \, ds), \quad \varepsilon \to 0. \tag{40}$$

Hence, to prove the required statement, it will be enough to show that

$$\int_0^t H^\varepsilon(V_s^\varepsilon) \, ds \to 0, \quad \varepsilon \to 0. \tag{41}$$

We note that, up to a certain point, this argument follows the strategy, frequently used in limit theorems, based on the use of a *correction term*. In one of its standard forms, which dates back to [19] (see also [20]), the correction term approach assumes that one adds to the process an asymptotically negligible term, which transforms it into a martingale. In our framework, the classical correction term would have the form $F^\varepsilon(V_t^\varepsilon)$, where F^ε is the solution to the *Poisson equation*

$$L^\varepsilon F^\varepsilon(v) = -v,$$

where

$$L^\varepsilon f(v) = -|v|^\beta \operatorname{sgn} v \cdot f'(v) + \varepsilon \int_{\mathbb{R}} \big(f(v+z) - f(v) - f'(v)z\big) \mu^\varepsilon(dz)$$

is the generator of the velocity process v^ε at the "microscopic time scale". Since we are not able to specify the solution F^ε to the Poisson equation, we use instead the function F, which in this context is just the solution to equation

$$L^0 F(v) = -v, \qquad L^0 f(v) = -|v|^\beta \operatorname{sgn} v \cdot f'(v).$$

Hence F can be understood as an approximate solution to the Poisson equation, and thus we call the entire argument the *approximate correction term* approach. Note that the non-martingale term $\int_0^t H^\varepsilon(V_s^\varepsilon) \, ds$ appears in (38) exactly because the exact solution to the Poisson equation is replaced by an approximate one. In what follows we will show that such an approximation is precise enough, and this integral term is negligible.

Of course, this is just an outline of the argument, and we have to take care about numerous technicalities. For $\beta < 0$ or $\beta = 1$, the function F belongs to $C^2(\mathbb{R}, \mathbb{R})$ and thus (35) follows by the usual Itô formula. Otherwise, we yet have to justify this relation, e.g. by an approximation procedure. We are actually able to do that when $(\alpha, \beta) \in \Xi_{\text{regular}}$; see Lemma A.2 in the Appendix. Note that this is exactly the case, where the functions H^ε can be proved to be equicontinuous at the point $v = 0$, see Lemma A.1. Otherwise, the functions H^ε are typically discontinuous, or even unbounded near the origin (see Figure 2) which makes the entire approach hardly applicable.

To summarize: when $(\alpha, \beta) \in \Xi_{\text{regular}}$, the function F is regular enough to allow the Itô formula to be applied, and the family $\{H^\varepsilon\}$ is equicontinuous at $v = 0$, which makes it possible to derive (41) from the convergence $V_t^\varepsilon \to 0$, $\varepsilon \to 0$. This is why we call this case *regular*.
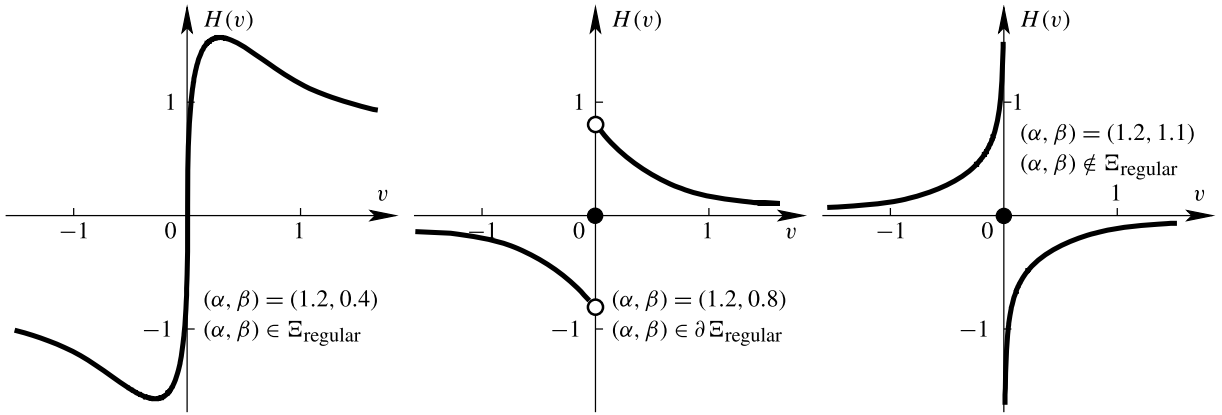
Fig. 2. The functions $H^\varepsilon = H$ defined in (37) corresponding to the damped symmetric α-stable process with the Lévy measure $\mu^\varepsilon(\mathrm{d}z) = \mu(\mathrm{d}z) = |z|^{-\alpha-1}\mathbb{I}_{(0,1]}(|z|)\,\mathrm{d}z$ for $\alpha = 1.2$ and $\beta = 0.4$, $\beta = 0.8$ and $\beta = 1.1$.

4.2. Detailed proof

We will use the same "truncation of large jumps" argument which now has the following form: if we can prove (21) under the additional assumption (26), then we actually have (21) in the general setting. Hence, in what follows we assume (26) to hold true for some $A > 0$.

To clarify the exposition, we postpone the proof of some technicalities to Appendix A. Namely, in Lemma A.2 we show that the Itô formula (35) holds true indeed for the function F. In Lemma A.1, we show that the family $\{H^\varepsilon\}_{\varepsilon \in (0,1]}$ is uniformly bounded on bounded sets, and that $\lim_{v \to 0} \sup_{\varepsilon \in (0,1]} H^\varepsilon(v) = 0$. By (16) and (31), this means that (41) holds true in probability and hence the integral term in (38) is negligible. Here, we focus on the convergence of martingales (39).

First, we observe that, because of the principal assumption $\alpha + 2\beta < 4$ and the truncation assumption (26) with the help of (83) we estimate

$$\int_{\mathbb{R}} (F(z))^2 \mu^\varepsilon(\mathrm{d}z) = \frac{2}{(2-\beta)^2} \int_0^A z^{4-2\beta} \mu^\varepsilon(\mathrm{d}z) = \frac{2(4-2\beta)}{(2-\beta)^2} \int_0^A z^{3-2\beta} \mu^\varepsilon([z, A])\,\mathrm{d}z \leq C A^{4-\alpha-2\beta},$$

that is, $\mathbf{M}^\varepsilon$ is a square integrable martingale. Denote for $\delta > 0$ and $R > 0$

$$\tau_R^\varepsilon = \inf\{t : |V_t^\varepsilon| > R\},$$

$$M_t^{\varepsilon,\delta} = \int_0^t \int_{|z|>\delta} (F(V_{s-}^\varepsilon + z) - F(V_{s-}^\varepsilon))\widetilde{N}^\varepsilon(\mathrm{d}z\,\mathrm{d}s) \quad \text{and} \quad \mathbf{M}_t^{\varepsilon,\delta} = \int_0^t \int_{|z|>\delta} F(z)\widetilde{N}^\varepsilon(\mathrm{d}z\,\mathrm{d}s).$$

Since F is continuous, we have by (31) and the dominated convergence theorem,

$$\mathbf{E}(M_{t \wedge \tau_R^\varepsilon}^{\varepsilon,\delta} - \mathbf{M}_{t \wedge \tau_R^\varepsilon}^{\varepsilon,\delta})^2 \leq \mathbf{E}\int_0^t \int_{|z|>\delta} (F(V_{s-}^\varepsilon + z) - F(V_{s-}^\varepsilon) - F(z))^2\mathbb{I}_{|V_{s-}^\varepsilon| \leq R}\mu^\varepsilon(\mathrm{d}z)\,\mathrm{d}s \to 0, \quad \varepsilon \to 0.$$

By (16),

$$\sup_{\varepsilon \in (0,1]} \mathbf{P}(\tau_R^\varepsilon < t) \to 0, \quad R \to \infty. \tag{42}$$

Hence the above estimate provides that for each $\delta > 0$

$$M_t^{\varepsilon,\delta} - \mathbf{M}_t^{\varepsilon,\delta} \to 0, \quad \varepsilon \to 0 \tag{43}$$

in probability.

Next, we have

$$\sup_{\varepsilon \in (0,1]} \mathbf{E}(\mathbf{M}_t^\varepsilon - \mathbf{M}_t^{\varepsilon,\delta})^2 = t \sup_{\varepsilon \in (0,1]} \int_{|z| \le \delta} (F(z))^2 \mu^\varepsilon(\mathrm{d}z) \le C\delta^{4-\alpha-2\beta} \to 0, \quad \delta \to 0. \tag{44}$$

If $\beta \in [1, 2)$, the function F is Hölder continuous with the index $2 - \beta$, and for M^ε we have essentially the same estimate:

$$\sup_{\varepsilon \in (0,1]} \mathbf{E}(M_t^\varepsilon - M_t^{\varepsilon,\delta})^2 \le Ct \sup_{\varepsilon \in (0,1]} \int_{|z| \le \delta} |z|^{4-2\beta} \mu^\varepsilon(\mathrm{d}z) \le C\delta^{4-\alpha-2\beta} \to 0, \quad \delta \to 0.$$

If $\beta < 1$, the function F has a locally bounded derivative, which gives for arbitrary R

$$\sup_{\varepsilon \in (0,1]} \mathbf{E}(M_{t \wedge \tau_R^\varepsilon}^\varepsilon - M_{t \wedge \tau_R^\varepsilon}^{\varepsilon,\delta})^2 \le tC_R \sup_{\varepsilon \in (0,1]} \int_{|z| \le \delta} z^2 \mu^\varepsilon(\mathrm{d}z) \to 0, \quad \delta \to 0.$$

In both these cases, we have for arbitrary $c > 0$

$$\sup_{\varepsilon \in (0,1]} \mathbf{P}(|M_t^\varepsilon - M_t^{\varepsilon,\delta}| > c) \to 0, \quad \delta \to 0. \tag{45}$$

Combining (43), (44), and (45), we complete the proof of (39).

Each $\mathbf{M}^\varepsilon$ is a Lévy process. Since (25) and (39) hold true and F is continuous, we have for any $t \ge 0$ and $\lambda \in \mathbb{R}$

$$\mathbf{E}\mathrm{e}^{\mathrm{i}\lambda \mathbf{M}_t^\varepsilon} = \exp\left(t \int (\mathrm{e}^{\mathrm{i}\lambda F(z)} - 1)\mu^\varepsilon(\mathrm{d}z)\right) \to \exp\left(t \int (\mathrm{e}^{\mathrm{i}\lambda F(z)} - 1)\mu(\mathrm{d}z)\right), \quad \varepsilon \to 0,$$

which gives (40). This completes the proof of the Theorem.

Remark 4.1. In the proof of (39) and (40), we have not used the regularity assumption $(\alpha, \beta) \in \Xi_{\text{regular}}$ and proved these relations under the principal assumption $\alpha + 2\beta < 4$ combined with the auxiliary truncation assumption (26).

5. Proof of Theorem 2.3: Non-regular/quasi-ergodic case

5.1. *Outline*

In this section, we prove Theorem 2.3, assuming

$$(\alpha, \beta) \notin \Xi_{\text{regular}}.$$

Combined with the principal assumption $\alpha + 2\beta < 4$, this yields

$$\alpha > 0, \qquad \beta > 0,$$

see Figure 3.

We call this case *non-regular* and *quasi-ergodic*. Let us explain the latter name and outline the proof. We make the change of variables

$$Y_t^\varepsilon = \varepsilon^{-\gamma} V_{t\varepsilon^{\alpha\gamma}}^\varepsilon, \qquad \gamma = \frac{1}{\alpha + \beta - 1} > 0,$$

so that the new process Y^ε satisfies the SDE

$$Y_t^\varepsilon = Y_0^\varepsilon - \int_0^t |Y_s^\varepsilon|^\beta \operatorname{sgn} Y_s^\varepsilon \, \mathrm{d}s + U_t^\varepsilon \tag{46}$$
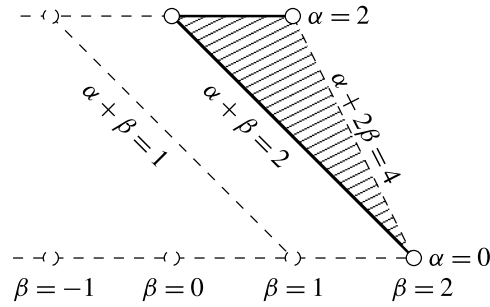
Fig. 3. The domain of parameters (α, β) corresponding to the non-regular/quasi-ergodic case.

with a Lévy process

$$U_t^\varepsilon = \varepsilon^{-\gamma} Z_{t\varepsilon^{\alpha\gamma}}^\varepsilon, \tag{47}$$

with a symmetric jump measure ν^ε. Such a space-time rescaling transforms the equation for the velocity in the original system (7) to a similar one, but without the term $1/\varepsilon$. In terms of Y^ε, the expression for X^ε takes the form

$$X_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t V_s^\varepsilon \, \mathrm{d}s = \varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} Y_s^\varepsilon \, \mathrm{d}s. \tag{48}$$

In the particularly important case where $Z^\varepsilon = Z$ and Z is symmetric α-stable, each process U^ε has the same law as Z, and thus the law of the solution to (46) does not depend on ε. The corresponding Markov processes Y^ε are also equal in law and ergodic for $\alpha + \beta > 1$, see [26, Section 3.4]. Hence one can expect the limit behaviour of the re-scaled integral functional (48) to be well controllable. We confirm this conjecture in the general (not necessarily α-stable) case, which we call *quasi-ergodic* because, instead of *one* ergodic process Y we have to consider *a family* of processes $\{Y^\varepsilon\}$, which, however, possesses a certain uniform stabilization property as $t \to \infty$ thanks to dissipativity of the drift coefficient in (46).

To study the limit behaviour of X^ε, we will follow the *approximate corrector term* approach, similar to the one used in Section 4. On this way, we meet two new difficulties. The first one is minor and technical: since we assume $(\alpha, \beta) \notin \Xi_{\text{regular}}$, we are not able to apply the Itô formula to the function F, see Figure 2. Consequently we consider a mollified function

$$\widehat{F} = F + \bar{F},$$

where $\bar{F}$ is an odd continuous function, vanishing outside of $[-1, 1]$, and such that $\widehat{F} \in C^3(\mathbb{R}, \mathbb{R})$. Now the Itô formula is applicable:

$$\widehat{F}(Y_t^\varepsilon) = \widehat{F}(Y_0^\varepsilon) - \int_0^t \widehat{F}(Y_s^\varepsilon)|Y_s^\varepsilon|^\beta \operatorname{sgn} Y_s^\varepsilon \, \mathrm{d}s + m_t^\varepsilon + \int_0^t J^\varepsilon(Y_s^\varepsilon) \, \mathrm{d}s,$$

where

$$m_t^\varepsilon = \int_0^t \int_{\mathbb{R}} \left(\widehat{F}(Y_{s-}^\varepsilon + u) - \widehat{F}(Y_{s-}^\varepsilon)\right) \widetilde{n}^\varepsilon(\mathrm{d}u \, \mathrm{d}s),$$

$$J^\varepsilon(y) = \int_0^\infty \left(\widehat{F}(y + u) + \widehat{F}(y - u) - 2\widehat{F}(y)\right) \nu^\varepsilon(\mathrm{d}u),$$

see the notation in Section 5.2 below. This gives

$$X_t^\varepsilon + \varepsilon^{(2-\beta)\gamma} \widehat{F}(Y_{t\varepsilon^{-\alpha\gamma}}^\varepsilon) = x_0 + \varepsilon^{(2-\beta)\gamma} \widehat{F}(Y_0^\varepsilon) + \varepsilon^{(2-\beta)\gamma} m_{t\varepsilon^{-\alpha\gamma}}^\varepsilon + \varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} R^\varepsilon(Y_s^\varepsilon) \, \mathrm{d}s, \tag{49}$$

where

$$R^\varepsilon(y) = -\widehat{F}'(y)|y|^\beta \operatorname{sgn} y + y + J^\varepsilon(y) = -\bar{F}'(y)|y|^\beta \operatorname{sgn} y + J^\varepsilon(y). \tag{50}$$

This representation is close to (38). This relation becomes even more visible, when one observes that

$$\varepsilon^{(2-\beta)\gamma} F\big(Y^\varepsilon_{t\varepsilon^{-\alpha\gamma}}\big) = F\big(V^\varepsilon_t\big).$$

Then (49) can be written as

$$X^\varepsilon_t + F\big(V^\varepsilon_t\big) = x_0 + F(v_0) + M^\varepsilon_t + \varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} R^\varepsilon\big(Y^\varepsilon_s\big) \, ds$$
$$- \varepsilon^{(2-\beta)\gamma} \bar{F}\big(Y^\varepsilon_{t\varepsilon^{-\alpha\gamma}}\big) + \varepsilon^{(2-\beta)\gamma} \bar{F}\big(Y^\varepsilon_0\big) + \varepsilon^{(2-\beta)\gamma} \bar{m}^\varepsilon_{t\varepsilon^{-\alpha\gamma}} \tag{51}$$

with

$$\bar{m}^\varepsilon_t = \int_0^t \int_{\mathbb{R}} \big(\bar{F}\big(Y^\varepsilon_{s-} + u\big) - \bar{F}\big(Y^\varepsilon_{s-}\big)\big) \widetilde{n}^\varepsilon(du \, ds).$$

Since $\bar{F}$ is bounded and $\beta < 2$, the terms $\varepsilon^{(2-\beta)\gamma} \bar{F}(Y^\varepsilon_{t\varepsilon^{-\alpha\gamma}})$ and $\varepsilon^{(2-\beta)\gamma} \bar{F}(Y^\varepsilon_0)$ are obviously negligible. Also, it will be not difficult to show that the last term in (51) is negligible, as well:

$$\varepsilon^{(2-\beta)\gamma} \bar{m}^\varepsilon_{t\varepsilon^{-\alpha\gamma}} \to 0, \quad \varepsilon \to 0, \tag{52}$$

in probability. Recall that we have (39) and (40), see Remark 4.1. Eventually, to establish (21), it is enough to show that

$$\varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} R^\varepsilon\big(Y^\varepsilon_s\big) \, ds \to 0, \quad \varepsilon \to 0, \tag{53}$$

in probability. The second, more significant, difficulty which we encounter now is that this relation cannot be obtained in the same way we did that in Section 4. We can transform it, in order to make visible that it is similar to (41):

$$\varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} R^\varepsilon\big(Y^\varepsilon_s\big) \, ds = \varepsilon^{(2-\alpha-\beta)\gamma} \int_0^t R^\varepsilon\big(Y^\varepsilon_{\varepsilon^{-\alpha\gamma}s}\big) \, ds = \int_0^t \widehat{H}^\varepsilon\big(V^\varepsilon_s\big) \, ds,$$
$$\widehat{H}^\varepsilon(v) = \varepsilon^{(2-\alpha-\beta)\gamma} R^\varepsilon\big(\varepsilon^{-\gamma}v\big). \tag{54}$$

We are now not in the regular case, $(\alpha, \beta) \notin \Xi_{\text{regular}}$, and thus the family $\{H^\varepsilon\}_{\varepsilon \in (0,1]}$ is typically unbounded in the neighbourhood of the point $v = 0$, see Figure 2. We have for each $\delta > 0$

$$\sup_{|v| > \delta} \big|\widehat{H}^\varepsilon(v) - H^\varepsilon(v)\big| \to 0, \quad \varepsilon \to 0, \tag{55}$$

the proof is postponed to Appendix C. Thus the family $\{\widehat{H}^\varepsilon\}_{\varepsilon \in (0,1]}$ is unbounded, and one can hardly derive (53) from (31), like we did that in Section 4. Instead, we will prove (53) using the stabilization properties of the family $\{Y^\varepsilon\}$.

5.2. *Preliminaries to the proof*

In what follows we assume (26) to hold true, i.e. the jumps of the processes Z^ε are bounded by some $A > 0$. Using the "truncation of large jumps" trick from the previous section, we guarantee that this assumption does not restrict the generality. We denote by ν^ε the Lévy measure of the Lévy process U^ε introduced in (47), and by n^ε and $\widetilde{n}^\varepsilon$ the corresponding Poisson and compensated Poisson random measures. More precisely, for $B \in \mathcal{B}(\mathbb{R})$ and $s \geq 0$

$$\nu^\varepsilon(B) := \varepsilon^{\alpha\gamma} \mu\big(z \colon \varepsilon^{-\gamma}z \in B\big),$$
$$n^\varepsilon\big(B \times [0,s]\big) := N^\varepsilon\big((z,t) \colon \big(\varepsilon^{-\gamma}z, \varepsilon^{\alpha\gamma}t\big) \in B \times [0,s]\big),$$
$$\widetilde{n}^\varepsilon(du \, ds) := n^\varepsilon(du \, ds) - \nu^\varepsilon(du) \, ds.$$

Each of the measures ν^ε is symmetric, and

$$\nu^\varepsilon\big(u\colon |u| > r\big) = \varepsilon^{\alpha\gamma}\mu^\varepsilon\big(z\colon |z| > \varepsilon^\gamma r\big), \quad r > 0.$$

Hence we have the following analogue of (33):

$$\sup_{\varepsilon \in (0,1]} \nu^\varepsilon\big(u\colon |u| \geq r\big) \leq Cr^{-\alpha}, \quad r > 0, \tag{56}$$

see also (83). In addition, we have

$$\operatorname{supp}\nu^\varepsilon \subseteq \big[-A\varepsilon^{-\gamma}, A\varepsilon^{-\gamma}\big] \tag{57}$$

by the assumption (26), and

$$\sup_{\varepsilon \in (0,1]} \int_{\mathbb{R}} \big(u^2 \wedge 1\big)\nu^\varepsilon(\mathrm{d}u) < \infty. \tag{58}$$

The latter inequality follows directly from (56) for $\alpha < 2$. For $\alpha = 2$, one should also use (18), which gives

$$\int_{|u| \leq 1} u^2 \nu^\varepsilon(\mathrm{d}u) = \int_{|z| \leq \varepsilon^\gamma} u^2 \mu^\varepsilon(\mathrm{d}z) \to 0, \quad \varepsilon \to 0.$$

Using these relations, it is easy to derive (52). Since $\widehat{F} \in C^3(\mathbb{R}, \mathbb{R})$ and $\bar{F} = \widehat{F} - F$ is compactly supported, $\bar{F}$ is $(2 - \beta)$-Hölder continuous for $\beta \geq 1$ and is Lipschitz continuous if $\beta < 1$. In addition, $\bar{F}$ is bounded, which gives

$$\mathbf{E}\big(\varepsilon^{(2-\beta)\gamma}\bar{m}^\varepsilon_{t\varepsilon^{-\alpha\gamma}}\big)^2 \leq C\varepsilon^{4-2\beta-\alpha} \int_{\mathbb{R}} \big(|u|^{4-2\beta} \wedge 1\big)\nu^\varepsilon(\mathrm{d}u)$$

if $\beta \geq 1$, and

$$\mathbf{E}\big(\varepsilon^{(2-\beta)\gamma}\bar{m}^\varepsilon_{t\varepsilon^{-\alpha\gamma}}\big)^2 \leq C\varepsilon^{4-2\beta-\alpha} \int_{\mathbb{R}} \big(u^2 \wedge 1\big)\nu^\varepsilon(\mathrm{d}u)$$

if $\beta < 1$. In the latter case, (52) follows by (58) and the basic assumption $\alpha + 2\beta < 4$. For $\beta \geq 1$, we have (52) by

$$\sup_{\varepsilon \in (0,1]} \int_{\mathbb{R}} \big(|u|^{4-2\beta} \wedge 1\big)\nu^\varepsilon(\mathrm{d}u) < \infty,$$

which follows from (56).

Let us explain the strategy of the proof of (53). The process Y^ε being a solution to (46) is a Markov process. Let us denote by $\mathbf{P}^{Y,\varepsilon}_y$ its law of this process with $Y^\varepsilon_0 = y$, and by $\mathbf{E}^{Y,\varepsilon}_y$ the corresponding expectation. Then

$$\mathbf{E}\bigg(\varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} R^\varepsilon\big(Y^\varepsilon_s\big)\,\mathrm{d}s\bigg)^2 = 2\varepsilon^{(4-2\beta)\gamma}\mathbf{E} \int_0^{t\varepsilon^{-\alpha\gamma}} \bigg(R^\varepsilon\big(Y^\varepsilon_s\big) \cdot \mathbf{E}^{Y,\varepsilon}_{Y^\varepsilon_s} \int_0^{t\varepsilon^{-\alpha\gamma}-s} R^\varepsilon\big(Y^\varepsilon_r\big)\,\mathrm{d}r\bigg)\,\mathrm{d}s.$$

Our aim will be to construct a non-negative function Q such that, for some $c, C > 0$,

- for all $y \in \mathbb{R}$, $t > 0$, and $\varepsilon > 0$

$$R^\varepsilon(y)\mathbf{E}^{Y,\varepsilon}_y \int_0^t R^\varepsilon\big(Y^\varepsilon_s\big)\,\mathrm{d}s \leq cQ(y); \tag{59}$$

- for all $t > 0$ and $\varepsilon > 0$

$$\mathbf{E} \int_0^t Q\big(Y^\varepsilon_s\big)\,\mathrm{d}s \leq c \cdot C\big(1 + t + \big|Y^\varepsilon_0\big|^\alpha\big). \tag{60}$$

Since $Y_0^\varepsilon = \varepsilon^{-\gamma} v_0$, this will provide (53) since

$$\mathbf{E}\left(\varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} R^\varepsilon(Y_s^\varepsilon) \, \mathrm{d}s\right)^2 \le C\varepsilon^{(4-2\beta)\gamma}\left(1 + t\varepsilon^{-\alpha\gamma} + |v_0|^\alpha \varepsilon^{-\alpha\gamma}\right) \to 0$$

by the principal assumption $\alpha + 2\beta < 4$.

The inequality (60) can be obtained in quite a standard way, based on a proper Lyapunov-type condition, see e.g. Section 2.8.2 and Section 3.2 in [26]. For the reader's convenience, we explain how this simple, but important argument can be applied in the current setting. Denote for $G \in C^2(\mathbb{R}, \mathbb{R})$

$$\mathcal{A}^\varepsilon G(y) = -|y|^\beta \operatorname{sgn} v \cdot G'(y) + \int_0^\infty \big(G(y+u) + G(y-u) - 2G(y)\big)\nu^\varepsilon(\mathrm{d}u). \tag{61}$$

Lemma 5.1. *Let a non-negative $G \in C^2(\mathbb{R}, \mathbb{R})$ be such that for some $c_1, c_2 > 0$*

$$\mathcal{A}^\varepsilon G(y) \le -c_1 Q(y) + c_2, \quad \varepsilon > 0. \tag{62}$$

Then for all $t \ge 0$ and $\varepsilon > 0$

$$\mathbf{E} \int_0^t Q(Y_s^\varepsilon) \, \mathrm{d}s \le \frac{1}{c_1} G(Y_0^\varepsilon) + \frac{c_2}{c_1} t.$$

Proof. By the Itô formula,

$$G(Y_t^\varepsilon) = \int_0^t \mathcal{A}^\varepsilon G(Y_s^\varepsilon) \, \mathrm{d}s + \mathcal{M}_t^\varepsilon,$$

where $\mathcal{M}^\varepsilon$ is a local martingale. Let $\tau_n^\varepsilon \nearrow \infty$ be a localizing sequence for $\mathcal{M}^\varepsilon$, then

$$\mathbf{E} \int_0^{t\wedge\tau_n^\varepsilon} Q(Y_s^\varepsilon) \, \mathrm{d}s \le \frac{c_2}{c_1} t - \frac{1}{c_1}\mathbf{E} \int_0^{t\wedge\tau_n^\varepsilon} \mathcal{A}^\varepsilon G(Y_s^\varepsilon) \, \mathrm{d}s$$

$$= \frac{c_2}{c_1} t + \frac{1}{c_1}\mathbf{E}G(Y_0^\varepsilon) - \frac{1}{c_1}\mathbf{E}G(Y_{t\wedge\tau_n^\varepsilon}^\varepsilon) \le \frac{c_2}{c_1} t + \frac{1}{c_1}G(Y_0^\varepsilon).$$

We complete the proof passing to the limit $n \to \infty$ and applying the Fatou lemma. $\qquad \square$

Now we specify the functions G and Q which we plug into this general statement. Fix

$$p \in (\beta - 1, \alpha + \beta - 1), \tag{63}$$

recall that $\alpha > 0$ and therefore the above interval is non-empty. Let a non-negative $G \in C^2(\mathbb{R}, \mathbb{R})$ be such that

$$G(y) \equiv 0 \quad \text{in some neighbourhood of } 0,$$
$$G(y) \le |y|^{p+1-\beta}, \quad |y| \le 1, \tag{64}$$
$$G(y) = |y|^{p+1-\beta}, \quad |y| > 1.$$

Then for $|y| \ge 1$

$$\mathcal{A}^\varepsilon G(y) = -(p + 1 - \beta)|y|^p + K^\varepsilon(y), \tag{65}$$

where

$$K^\varepsilon(y) = \int_0^\infty \big(G(y+u) + G(y-u) - 2G(y)\big)\nu^\varepsilon(\mathrm{d}u).$$

The function G satisfies the assumptions of Lemma B.1 with $\sigma = p + 1 - \beta$; note that assumption (63) means that $\sigma \in (0, \alpha)$. Since

$$\sigma - \alpha = p + 1 - \alpha - \beta < p,$$

we have by Lemma B.1

$$\sup_{\varepsilon \in (0,1]} |y|^{-p} K^{\varepsilon}(y) \to 0, \quad y \to \infty. \tag{66}$$

In addition, by the same Lemma the family $\{K^{\varepsilon}\}_{\varepsilon \in (0,1]}$ is uniformly bounded on each bounded set, hence the same property holds true for the family $\{\mathcal{A}^{\varepsilon} G\}_{\varepsilon \in (0,1]}$. This provides (62) with G specified above,

$$Q(y) = 1 + |y|^p,$$

and properly chosen c_1, c_2. Eventually by construction we have

$$G(y) \le |y|^{p+1-\beta} \le C(1 + |y|^{\alpha}),$$

therefore (60) holds true by Lemma 5.1.

By Lemma B.2, the family $\{R^{\varepsilon}\}_{\varepsilon \in (0,1]}$ satisfies

$$\left| R^{\varepsilon}(y) \right| \le C(1 + |y|)^{2-\alpha-\beta} \ln(2 + |y|).$$

Hence, to prove the bound (59) with Q specified above, it is enough to show that, for some $p' < p$

$$\left| \mathbf{E}_y^{Y,\varepsilon} \int_0^t R^{\varepsilon}(Y_s^{\varepsilon}) \, ds \right| \le C(1 + |y|)^{p'+\alpha+\beta-2}, \quad t > 0, \varepsilon > 0. \tag{67}$$

In the rest of the proof, we verify this relation for properly chosen p'. We fix y, and (with a slight abuse of notation) denote by Y^{ε}, $Y^{\varepsilon,0}$ the strong solutions to (46) with the same process U^{ε} and initial conditions $Y_0^{\varepsilon} = y$, $Y_0^{\varepsilon,0} = 0$. Recall that the Lévy process U^{ε} is symmetric. Since the drift coefficient $-|y|^{\beta} \operatorname{sgn} y$ in (46) is odd, the law of $Y^{\varepsilon,0}$ is symmetric as well. By Lemma B.2, the family of functions $\{R^{\varepsilon}\}_{\varepsilon \in (0,1]}$ is bounded: if $\alpha + \beta > 2$ this is straightforward, for $\alpha + \beta = 2$ one should recall that in the non-regular case this identity excludes the case $\alpha = 2$, see Figure 3. It is also easy to verify that functions R^{ε} are odd, which gives

$$\mathbf{E} R^{\varepsilon}(Y_t^{\varepsilon,0}) = 0, \quad t \ge 0, \varepsilon > 0.$$

Then

$$\left| \mathbf{E}_y^{Y,\varepsilon} \int_0^t R^{\varepsilon}(Y_s^{\varepsilon}) \, ds \right| = \left| \mathbf{E} \int_0^t R^{\varepsilon}(Y_s^{\varepsilon}) \, ds - \mathbf{E} \int_0^t R^{\varepsilon}(Y_s^{\varepsilon,0}) \, ds \right| \le \int_0^t \mathbf{E} \left| R^{\varepsilon}(Y_s^{\varepsilon}) - R^{\varepsilon}(Y_s^{\varepsilon,0}) \right| ds.$$

This bound will allow us to prove (67) using the dissipation, brought to the system by the drift coefficient $-|y|^{\beta} \operatorname{sgn} y$. In what follows, we consider separately two cases: $\beta \in [1, 2)$ ("strong dissipation") and $\beta \in (0, 1)$ ("Hölder dissipation").

5.3. *Strong dissipation*: $\beta \in [1, 2)$

Since the noise in the SDE (46) is additive, the difference $t \mapsto \Delta_t^{\varepsilon} = Y_t^{\varepsilon} - Y_t^{\varepsilon,0}$ is an absolutely continuous function and

$$d\Delta_t^{\varepsilon} = \left(|Y_t^{\varepsilon}|^{\beta} \operatorname{sgn} Y_t^{\varepsilon} - |Y_t^{\varepsilon,0}|^{\beta} \operatorname{sgn} Y_t^{\varepsilon,0} \right) dt.$$

Since $\Delta \mapsto |\Delta|$ is Lipschitz continuous, $t \mapsto |\Delta_t^\varepsilon|$ is an absolutely continuous function as well with

$$\mathrm{d}|\Delta_t^\varepsilon| = -\big(|Y_t^\varepsilon|^\beta \operatorname{sgn} Y_t^\varepsilon - |Y_t^{\varepsilon,0}|^\beta \operatorname{sgn} Y_t^{\varepsilon,0}\big) \operatorname{sgn}\big(Y_t^\varepsilon - Y_t^{\varepsilon,0}\big)\, \mathrm{d}t.$$

For $\beta \in (1, 2)$ we have the inequality

$$-\big(|y_1|^\beta \operatorname{sgn} y_1 - |y_2|^\beta \operatorname{sgn} y_2\big) \operatorname{sgn}(y_1 - y_2) \leq -2^{-\beta}|y_1 - y_2|^\beta, \quad y_1, y_2 \in \mathbb{R}. \tag{68}$$

To prove (68), it suffice to consider two cases. For $0 \leq y_1 \leq y_2$ we recall the elementary inequality $(t - 1)^\beta \leq t^\beta - 1$ for $t \geq 1$ to get

$$2^{-\beta}(y_2 - y_1)^\beta \leq (y_2 - y_1)^\beta = y_1^\beta \left(\frac{y_2}{y_1} - 1\right)^\beta \leq y_1^\beta \left(\frac{y_2^\beta}{y_1^\beta} - 1\right) = y_2^\beta - y_1^\beta.$$

For $y_1 \leq 0 \leq y_2$, we have

$$2^{-\beta}\big(|y_1| + y_2\big)^\beta = \left(\frac{|y_1| + y_2}{2}\right)^\beta \leq \max\{|y_1|^\beta, y_2^\beta\} \leq |y_1|^\beta + y_2^\beta.$$

Eventually, with the help of (68) we get a differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta_t^\varepsilon| \leq -2^{-\beta}|\Delta_t^\varepsilon|^\beta.$$

Denote by Υ the solution to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_t = -2^{-\beta}\Upsilon_t^\beta, \qquad \Upsilon_0 = |\Delta_0^\varepsilon| = |y|.$$

Then by the comparison theorem [27, Theorem 1.4.1], $|\Delta_t^\varepsilon| \leq \Upsilon_t$, $t \geq 0$. This solution is explicit:

$$\Upsilon_t = \begin{cases} |y|\mathrm{e}^{-2^{-\beta}t}, & \beta = 1, \\ (|y|^{1-\beta} + 2^{-\beta}(\beta - 1)t)^{\frac{1}{1-\beta}}, & \beta > 1, \end{cases}$$

and we have

$$\int_0^\infty \Upsilon_t\, \mathrm{d}t = \frac{2^\beta}{2 - \beta}|y|^{2-\beta}.$$

By Lemma B.3, derivatives of the functions R^ε are uniformly bounded, which gives for some $C > 0$

$$\left|\mathbf{E}_y^{Y,\varepsilon} \int_0^t R^\varepsilon\big(Y_s^\varepsilon\big)\, \mathrm{d}s\right| \leq C \int_0^t \mathbf{E}|\Delta_s^\varepsilon|\, \mathrm{d}s \leq C \int_0^t \Upsilon_s\, \mathrm{d}s \leq C|y|^{2-\beta}.$$

Eventually we obtain (67) with

$$p' = 4 - \alpha - 2\beta > 0.$$

If $\beta \in (1, 2)$, we have

$$p' - \alpha = 2(2 - \alpha - \beta) \leq 0 < \beta - 1,$$

that is,

$$p' < \alpha + \beta - 1.$$

Then we can take $p \in (p', \alpha + \beta - 1)$ and get that, for $Q(y) = 1 + |y|^p$, both (59) and (60) hold true which provides (53) and completes the entire proof.

For $\beta = 1$, the same argument applies with just a minor modification. Namely, since the functions $\{R^\varepsilon\}_{\varepsilon \in (0,1]}$ are uniformly bounded and have uniformly bounded derivatives, for each $\kappa \in (0, 1)$ these functions are uniformly κ-Hölder equicontinuous:

$$\left| R^\varepsilon(y_1) - R^\varepsilon(y_2) \right| \leq C|y_1 - y_2|^\kappa, \quad y_1, y_2 \in \mathbb{R}, \varepsilon > 0. \tag{69}$$

Hence

$$\left| \mathbf{E}_y^{Y,\varepsilon} \int_0^t R^\varepsilon\left(Y_s^\varepsilon\right) \mathrm{d}s \right| \leq C \int_0^t \mathbf{E}\left| \Delta_s^\varepsilon \right|^\kappa \mathrm{d}s \leq C \int_0^t \Upsilon_s^\kappa \, \mathrm{d}s \leq C|y|^\kappa.$$

That is, we have (67) with

$$p' = \kappa + 1 - \alpha.$$

Note that for $\beta = 1$ the principal assumption $\alpha + 2\beta < 4$ yields $\alpha < 2$, hence p' can be made positive by taking $\kappa < 1$ close enough to 1. On the other hand, we are considering the non-regular case now, hence $\alpha \geq 2 - \beta = 1$. That is,

$$p' \leq \kappa < 1 \leq \alpha + \beta - 1.$$

Again, we can take $p \in (p', \alpha + \beta - 1)$ and get that, for $Q(y) = 1 + |y|^p$, both (59) and (60) hold true, which provides (53) and completes the entire proof.

5.4. *Hölder dissipation*: $\beta \in (0, 1)$

Now the situation is more subtle because, instead of (68), which holds true on the entire $\mathbb{R}$, we have only a family of local inequalities: for each $D > 0$ there exists $c_D > 0$ such that

$$-\left(|y_1|^\beta \operatorname{sgn} y_1 - |y_2|^\beta \operatorname{sgn} y_2\right) \operatorname{sgn}(y_1 - y_2) \leq -c_D|y_1 - y_2|^\beta \wedge |y_1 - y_2| \cdot \mathbb{I}_{|y_2| \leq D}, \quad y_1, y_2 \in \mathbb{R}. \tag{70}$$

Indeed, first we observe that the inequality holds for $y_2 = 0$ and any $0 < c_D \leq 1$, and for $|y_2| > D$ for any $c_D > 0$. It is sufficient to consider $0 < y_2 \leq D$. We have

$$|y_2 - y_1|^\beta \leq \left(|y_1| + y_2\right)^\beta \leq |y_1|^\beta + y_2^\beta, \quad y_1 \in (-\infty, 0],$$

$$y_2^\beta - y_1^\beta \geq \beta y_2^{\beta-1}(y_2 - y_1) \geq \beta D^{\beta-1}(y_2 - y_1), \quad y_1 \in [0, y_2],$$

$$y_1^\beta - y_2^\beta \geq \left(2^\beta - 1\right) y_2^{\beta-1}(y_1 - y_2) \geq \left(2^\beta - 1\right) D^{\beta-1}(y_1 - y_2), \quad y_1 \in [y_2, 2y_2],$$

$$(y_1 - y_2)^\beta = y_2^\beta \left(\frac{y_1}{y_2} - 1\right)^\beta \leq \frac{1}{2^\beta - 1}\left(y_1^\beta - y_2^\beta\right), \quad y_1 \in [2y_2, \infty),$$

so that the inequality (70) holds true for $c_D = \beta D^{\beta-1} \wedge 1$.

We will prove (67) in two steps, considering separately the cases $|y| \leq D$ and $|y| > D$ for some fixed D. In both these cases, we will require the following recurrence bound. Denote

$$\theta_D^\varepsilon = \inf\left\{t \geq 0 : \left|Y_t^\varepsilon\right| \leq D\right\}.$$

Lemma 5.2. *Let $p \in (0, \alpha + \beta - 1)$ be fixed. Then there exist $D > 1$ large enough and a constant $C > 0$ such that*

$$\mathbf{E}_y^{Y,\varepsilon}\left(\theta_D^\varepsilon\right)^{\frac{p+1-\beta}{1-\beta}} \leq C|y|^{p+1-\beta}, \quad y \in \mathbb{R}. \tag{71}$$

Proof. Since $\beta \in (0, 1)$, we have $p > 0 > \beta - 1$. That is, p satisfies (63), and for the function G given by (64), the equality (65) holds true. By (66), we can fix D large enough and some $c > 0$ such that

$$\mathcal{A}^{\varepsilon} G(y) \leq -c\big(G(y)\big)^{p/(p+1-\beta)}, \quad |y| > D. \tag{72}$$

Note that, by the Itô formula,

$$G\big(Y_t^{\varepsilon}\big) = G(y) + \int_0^t \mathcal{A}^{\varepsilon} G\big(Y_s^{\varepsilon}\big) \, ds + M_t^{G,\varepsilon},$$

where the local martingale $M^{G,\varepsilon}$ is given by

$$M_t^{G,\varepsilon} = \int_0^t \int_{\mathbb{R}} \big(G\big(Y_{s-}^{\varepsilon} + u\big) - G\big(Y_{s-}^{\varepsilon}\big)\big) \widetilde{n}^{\varepsilon} (du \, ds).$$

The rest of the proof is based on the general argument explained in [21, Section 4.1.2]; see also [26, Lemma 3.2.4] and [15, Lemma 2]. Denote $q = (p + 1 - \beta)/(1 - \beta) > 1$ and let

$$H(t, g) = \left(\frac{c}{q} t + g^{1/q}\right)^q.$$

Then

$$H_t'(t, g) = c\left(\frac{c}{q} t + g^{1/q}\right)^{q-1} = c\big(H(t, g)\big)^{p/(p+1-\beta)},$$

$$H_g'(t, g) = g^{-(q-1)/q}\left(\frac{c}{q} t + g^{1/q}\right)^{q-1} = g^{-p/(p+1-\beta)}\big(H(t, g)\big)^{p/(p+1-\beta)},$$

and the function $g \mapsto H(t, g)$ is concave for each $t \geq 0$. Then by the Itô formula

$$H\big(t, G\big(Y_t^{\varepsilon}\big)\big) = G(y) + \int_0^t \big[c + \big(G\big(Y_s^{\varepsilon}\big)\big)^{-p/(p+1-\beta)} \mathcal{A}^{\varepsilon} G\big(Y_s^{\varepsilon}\big)\big] H\big(s, G\big(Y_s^{\varepsilon}\big)\big)^{p/(p+1-\beta)} \, ds$$

$$+ \int_0^t \Psi_s^{\varepsilon} \, ds + M_t^{H,\varepsilon},$$

where $M^{H,\varepsilon}$ is a local martingale and

$$\Psi_s^{\varepsilon} = \int_{\mathbb{R}} \big[H\big(s, G\big(Y_{s-}^{\varepsilon} + u\big)\big) - H\big(s, G\big(Y_{s-}^{\varepsilon}\big)\big) - H_g'\big(s, G\big(Y_{s-}^{\varepsilon}\big)\big)\big(G\big(Y_s^{\varepsilon} + u\big) - G\big(Y_{s-}^{\varepsilon}\big)\big)\big] \nu^{\varepsilon}(du) \leq 0$$

since $H(t, \cdot)$ is concave. Combined with (72), this provides that

$$H\big(t \wedge \theta_D^{\varepsilon}, G\big(Y_{t \wedge \theta_D^{\varepsilon}}^{\varepsilon}\big)\big), \quad t \geq 0$$

is a local super-martingale. Then, by the Fatou lemma,

$$\mathbf{E}_y^{Y,\varepsilon} H\big(t \wedge \theta_D^{\varepsilon}, G\big(Y_{t \wedge \theta_D^{\varepsilon}}^{\varepsilon}\big)\big) \leq G(y), \quad t \geq 0.$$

Note that $G(y) = |y|^{p+1-\beta}$ for $|y| > D$, and

$$H(t, g) \geq \left(\frac{c}{q} t\right)^q.$$

This gives (71) for $|y| > D$. For $|y| \leq D$ we have $\theta_D^{\varepsilon} = 0$ $\mathbf{P}_y^{Y,\varepsilon}$-a.s., and (71) holds true trivially. $\qquad \square$

By Jensen's inequality, (71) yields

$$\mathbf{E}_y^{Y,\varepsilon}\theta_D^\varepsilon \le C|y|^{1-\beta}, \quad y \in \mathbb{R}. \tag{73}$$

Since functions R^ε are uniformly bounded, by the strong Markov property this leads to the bound

$$\left|\mathbf{E}_y^{Y,\varepsilon}\int_0^t R^\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s\right| \le \left|\mathbf{E}_y^{Y,\varepsilon}\int_0^{t\wedge\theta_D^\varepsilon} R^\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s\right| + \left|\mathbf{E}_y^{Y,\varepsilon}\int_{t\wedge\theta_D^\varepsilon}^{t-t\wedge\theta_D^\varepsilon} R^\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s\right|$$

$$\le C\mathbf{E}_y^{Y,\varepsilon}\theta_D^\varepsilon + \left|\mathbf{E}_y^{Y,\varepsilon}\left[\mathbf{E}_{y'}^{Y,\varepsilon}\int_0^t R^\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s\right]_{y'=Y_{t\wedge\theta_D^\varepsilon}^\varepsilon}\right|$$

$$\le C|y|^{1-\beta} + \sup_{|y'|\le D, t'\le t}\left|\mathbf{E}_{y'}^{Y,\varepsilon}\int_0^{t'} R^\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s\right|.$$

That is, if we manage to show that

$$\sup_{|y|\le D, t\ge 0, \varepsilon\in(0,1]}\left|\mathbf{E}_y^{Y,\varepsilon}\int_0^t R^\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s\right| < \infty, \tag{74}$$

then we have (67) with

$$p' = (1-\beta) - (\alpha+\beta-2) = 3-\alpha-2\beta.$$

Since $\beta > 0$, we have

$$4 - 2\alpha - 3\beta = 2(2-\alpha-\beta) - \beta < 0 \quad \Rightarrow \quad p' < \alpha+\beta-1.$$

Taking $p \in (p'\vee 0, \alpha+\beta-1)$, we will get that, for $Q(y) = 1 + |y|^p$, both (59) and (60) hold true, which will provide (53) and complete the entire proof.

To prove (74), we modify the dissipativity-based argument from the previous section. We would like to use (70) with $y_1 = Y_t^\varepsilon$, $y_2 = Y_t^{\varepsilon,0}$, and for a given D we denote

$$\Lambda_D^\varepsilon(t) = c_D \int_0^t \mathbb{I}_{|Y_s^{\varepsilon,0}|\le D}\,\mathrm{d}s, \tag{75}$$

the total time spent by $|Y^{\varepsilon,0}|$ under the level D up to the time t multiplied by the corresponding local dissipativity index c_D. By (70),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\Delta_t^\varepsilon\right| \le -c_D\mathbb{I}_{|Y_t^{\varepsilon,0}|}\left(\left|\Delta_t^\varepsilon\right|^\beta \wedge \left|\Delta_t^\varepsilon\right|\right),$$

hence again by the comparison theorem

$$\left|\Delta_t^\varepsilon\right| \le \Upsilon\left(\Lambda_D^\varepsilon(t), |y|\right),$$

where we denote by $\Upsilon(t, r)$ the solution to the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(t, r) = -\left(\Upsilon(t, r)^\beta \wedge \Upsilon(t, r)\right), \quad \Upsilon(0, r) = r.$$

We have $\Upsilon(t, r) \le r$ for $t \ge 0$, and

$$\Upsilon(t, r) = \mathrm{e}^{t_r} \cdot \mathrm{e}^{-t}, \quad t \ge t_r,$$

where

$$t_r = \frac{(r^{1-\beta} - 1)_+}{1 - \beta}.$$

Since the derivatives of R^ε, $\varepsilon > 0$ are uniformly bounded, this provides for $|y| \leq D$

$$\left| \mathbf{E}_y^{Y,\varepsilon} \int_0^t R^\varepsilon (Y_s^\varepsilon) \, \mathrm{d}s \right| \leq CDt_D + Ce^{tD} \mathbf{E} \int_0^\infty e^{-\Lambda_D^\varepsilon(t)} \, \mathrm{d}t. \tag{76}$$

The rest of the proof is contained in the following

Lemma 5.3. *For any $q < \alpha/(2 - 2\beta)$, there exist $D > 0$, $a > 0$, and C such that*

$$\mathbf{P}\big(\Lambda_D^\varepsilon(t) \leq at\big) \leq C(1+t)^{-q}, \quad t \geq 0, \varepsilon > 0. \tag{77}$$

Once Lemma 5.3 is proved, we easily complete the entire proof. Namely, because $\alpha + \beta \geq 2$ and $\beta > 0$, we have

$$\frac{\alpha}{2 - 2\beta} \geq \frac{2 - \beta}{2 - 2\beta} > 1.$$

That is, (77) holds true for some $D > 1$, $a > 0$, and $q > 1$. Using the estimate

$$\mathbf{E}e^{-\Lambda_D^\varepsilon(t)} \leq \mathbf{P}\big(\Lambda_D^\varepsilon(t) \leq at\big) + e^{-at}$$

and (76), we guarantee (74) and complete the proof of Theorem 2.3.

Proof of Lemma 5.3:. Without loss of generality, we can assume that $q > 1/2$. Let p be such that

$$q = \frac{p + 1 - \beta}{2 - 2\beta},$$

then $p \in (0, \alpha + \beta - 1)$, and Lemma 5.2 is applicable. Let $D_0 > 1$ be such that (71) holds true with p specified above and $D = D_0$.

There exists $D > D_0$ large enough, such that

$$\sup_{|y| \leq D_0} \mathbf{P}_y^{Y,\varepsilon} \Big(\sup_{t \in [0,1]} |Y_t^\varepsilon| \geq D \Big) \leq \frac{1}{2}; \tag{78}$$

the calculation here is the same as in Section 3.2, and we omit the details. We fix these two levels D_0, D and define iteratively the sequence of stopping times

$$\theta_0^\varepsilon = 0,$$
$$\chi_k^\varepsilon = \inf\big\{t \geq \theta_{k-1}^\varepsilon \colon |Y_t^{\varepsilon,0}| > D\big\} \wedge \big(\theta_{k-1}^\varepsilon + 1\big),$$
$$\theta_k^\varepsilon = \inf\big\{t \geq \chi_k^\varepsilon \colon |Y_t^{\varepsilon,0}| \leq D_0\big\}, \quad k \geq 1.$$

We denote

$$S_k^{\varepsilon,\uparrow} = \sum_{j=1}^k \big(\chi_j^\varepsilon - \theta_{j-1}^\varepsilon\big), \qquad S_k^{\varepsilon,\downarrow} = \sum_{j=1}^k \big(\theta_j^\varepsilon - \chi_j^\varepsilon\big), \quad \theta_k^\varepsilon = S_k^{\varepsilon,\uparrow} + S_k^{\varepsilon,\downarrow}, k \geq 1,$$

and

$$N_t^\varepsilon = \min\big\{k \geq 1 \colon \theta_k^\varepsilon \geq t\big\}, \quad t > 0.$$

On each of the time intervals $\mathcal{I}_k = [\theta_{k-1}, \chi_k), k \geq 1$ we have $|Y_s^{\varepsilon,0}| \leq D, s \in \mathcal{I}_k$. In addition, $\mathcal{I}_k \subset [0, t]$ for $k < N_t^\varepsilon$. Thus

$$\Lambda_D^\varepsilon(t) \geq c_D S_{N_t^\varepsilon - 1}^{\varepsilon,\uparrow},$$

see (75) for the definition of $\Lambda_D^\varepsilon(t)$. Then for arbitrary $b > 0$ we have

$$\mathbf{P}\big(\Lambda_D^\varepsilon(t) \leq at\big) \leq \mathbf{P}\bigg(S_{[bt]}^{\varepsilon,\uparrow} \leq \frac{a}{c_D}t\bigg) + \mathbf{P}\big(N_t^\varepsilon \leq [bt] + 1\big).$$

On the other hand,

$$\theta_k^\varepsilon \leq S_k^{\varepsilon,\downarrow} + k,$$

which gives

$$\mathbf{P}\big(N_t^\varepsilon \leq [bt] + 1\big) = \mathbf{P}\big(\theta_{[bt]+1}^\varepsilon \geq t\big) \leq \mathbf{P}\big(S_{[bt]+1}^{\varepsilon,\downarrow} \geq t - [bt] - 1\big).$$

In what follows, we show that there exists $c > 0$ small enough, such that

$$\mathbf{P}\big(S_k^{\varepsilon,\downarrow} \geq c^{-1}k\big) \leq Ck^{-q} \quad \text{and} \quad \mathbf{P}\big(S_k^{\varepsilon,\uparrow} \leq ck\big) \leq Ck^{-q}, \quad k \geq 1. \tag{79}$$

Once we do that, the rest of the proof is easy. Namely, we take

$$b < \big(1 + c^{-1}\big)^{-1},$$

then $(1 - b)/b > c^{-1}$, and by the first inequality in (79) we get

$$\mathbf{P}\big(N_t^\varepsilon \leq [bt] + 1\big) \leq \mathbf{P}\big(S_{[bt]+1}^{\varepsilon,\downarrow} \geq t - [bt] - 1\big) \leq C\big([bt] + 1\big)^{-q}, \quad t \geq t_0 = \frac{3}{1-b}.$$

Then taking

$$a < \frac{c_D bc}{2}$$

we will have by the second inequality in (79)

$$\mathbf{P}\bigg(S_{[bt]}^{\varepsilon,\uparrow} \leq \frac{a}{c_D}t\bigg) \leq C\big([bt] + 1\big)^{-q}, \quad t \geq t_0 = \frac{3}{1-b}.$$

This will give (77) for $t \geq t_0$. One can easily extend (77) to the entire axis $[0, \infty)$ simply by increasing C.

Let us proceed with the proof of the first inequality in (79). Denote

$$s_k^{\varepsilon,\downarrow} = S_k^{\varepsilon,\downarrow} - S_{k-1}^{\varepsilon,\downarrow} = \theta_k^\varepsilon - \chi_k^\varepsilon, \qquad \mathcal{F}_k^{\varepsilon,\downarrow} = \mathcal{F}_{\theta_k^\varepsilon}, \quad k \geq 0,$$

then

$$M_k^{\varepsilon,\downarrow} = \sum_{j=1}^k \big(s_j^{\varepsilon,\downarrow} - \mathbf{E}\big[s_j^{\varepsilon,\downarrow}|\mathcal{F}_{j-1}^{\varepsilon,\downarrow}\big]\big), \quad k \geq 0$$

is an $\{\mathcal{F}_k^{\varepsilon,\downarrow}\}$-martingale. By Lemma 5.2, applied to $D = D_0$, and the strong Markov property, we have

$$\mathbf{E}\big[(s_k^{\varepsilon,\downarrow})^{2q}|\mathcal{F}_{k-1}^{\varepsilon,\downarrow}\big] = \mathbf{E}\big[(\theta_k^\varepsilon - \chi_k^\varepsilon)^{(p+1-\beta)/(1-\beta)}|\mathcal{F}_{\theta_{k-1}^\varepsilon}\big] \leq C\mathbf{E}_y^{Y,\varepsilon}|Y_{\chi_1^\varepsilon}^\varepsilon|^{p+1-\beta}|_{y=Y_{\theta_{k-1}^\varepsilon}^\varepsilon}.$$

Note that

$$\left|Y^{\varepsilon}_{\theta^{\varepsilon}_k}\right| \le D_0, \quad k \ge 0$$

by construction. Next, it is easy to show that

$$\sup_{|y| \le D_0, \varepsilon \in (0,1]} \mathbf{E}^{Y,\varepsilon}_y \left|Y^{\varepsilon}_{\chi^{\varepsilon}_1}\right|^{p+1-\beta} < \infty. \tag{80}$$

Indeed, let $G \in C^2(\mathbb{R}, \mathbb{R})$ be function specified in (64). Since $p < \alpha + \beta - 1$, we have $p + 1 - \beta < \alpha$ and thus by (61) the family of functions $\{\mathcal{A}^{\varepsilon}G\}_{\varepsilon \in (0,1]}$ is well defined and is uniformly bounded on the set $\{|y| \le D_0\}$. We have

$$\mathbf{E}^{Y,\varepsilon}_y \left|Y^{\varepsilon}_{\chi^{\varepsilon}_1}\right|^{p+1-\beta} \le G(y) + \mathbf{E}^{Y,\varepsilon}_y \int_0^{\chi^{\varepsilon}_1} \mathcal{A}^{\varepsilon}G\left(Y^{\varepsilon}_s\right) \mathrm{d}s \le G(y) + \sup_{|y'| \le D_0} \left|\mathcal{A}^{\varepsilon}G\left(y'\right)\right|$$

since $\chi^{\varepsilon}_1 \le 1$ by construction. This yields (80). Summarizing the above calculation, we conclude that

$$\mathbf{E}\left[\left(s^{\varepsilon,\downarrow}_k\right)^{2q}|\mathcal{F}^{\varepsilon,\downarrow}_{k-1}\right] \le C, \quad k \ge 1. \tag{81}$$

Consequently, for some $c^{\downarrow} > 0$ we have

$$\mathbf{E}\left[s^{\varepsilon,\downarrow}_k|\mathcal{F}^{\varepsilon,\downarrow}_{k-1}\right] \le c^{\downarrow}, \quad k \ge 1, \tag{82}$$

and

$$\mathbf{E}\left[\left|M^{\varepsilon,\downarrow}_k - M^{\varepsilon,\downarrow}_{k-1}\right|^{2q}|\mathcal{F}^{\varepsilon,\downarrow}_{k-1}\right] \le C, \quad k \ge 1.$$

By the Burkholder–Davis–Gundy inequality [24, Theorem 23.12], and Jensen's inequality, we have

$$\mathbf{E}\left|M^{\varepsilon,\downarrow}_k\right|^{2q} \le C(p)\mathbf{E}\left(\sum_{j=1}^k \left(M^{\varepsilon,\downarrow}_j - M^{\varepsilon,\downarrow}_{j-1}\right)^2\right)^q \le C(q)k^{q-1}\sum_{j=1}^k \mathbf{E}\left(M^{\varepsilon,\downarrow}_j - M^{\varepsilon,\downarrow}_{j-1}\right)^{2q} \le Ck^q.$$

Now we obtain the first inequality in (79): if $c > 0$ is such that $c^{-1} > c^{\downarrow}$, then

$$\mathbf{P}\left(S^{\varepsilon,\downarrow}_k \ge c^{-1}k\right) \le \mathbf{P}\left(M^{\varepsilon,\downarrow}_k \ge \left(c^{-1} - c^{\downarrow}\right)k\right) \le \left(c^{-1} - c^{\downarrow}\right)^{-2q}k^{-2q}\mathbf{E}\left|M^{\varepsilon,\downarrow}_k\right|^{2q} \le Ck^{-q}.$$

The proof of the second inequality in (79) is similar and simpler. We denote

$$s^{\varepsilon,\uparrow}_k = S^{\varepsilon,\uparrow}_k - S^{\varepsilon,\uparrow}_{k-1} = \chi^{\varepsilon}_k - \theta^{\varepsilon}_{k-1}, \qquad \mathcal{F}^{\varepsilon,\uparrow}_k = \mathcal{F}_{\chi^{\varepsilon}_k}, \quad k \ge 1, \qquad \mathcal{F}^{\varepsilon,\uparrow}_0 = \mathcal{F}_0,$$

and put

$$M^{\varepsilon,\uparrow}_k = \sum_{j=1}^k \left(s^{\varepsilon,\uparrow}_j - \mathbf{E}\left[s^{\varepsilon,\uparrow}_j|\mathcal{F}^{\varepsilon,\uparrow}_{j-1}\right]\right), \quad k \ge 0,$$

Now $s^{\varepsilon,\uparrow}_k \le 1$ by construction, hence analogues of (81) and (82) trivially hold true, which gives

$$\mathbf{E}\left|M^{\varepsilon,\uparrow}_k\right|^{2q} \le Ck^q.$$

On the other hand, by (78) and the strong Markov property,

$$\mathbf{E}\left[s^{\varepsilon,\uparrow}_k|\mathcal{F}^{\varepsilon,\uparrow}_{k-1}\right] \ge \frac{1}{2}, \quad k \ge 1.$$

Then for $c < 1/2$ we have

$$\mathbf{P}\big(S_k^{\varepsilon,\uparrow} \le ck\big) \le \mathbf{P}\bigg(\big|M_k^{\varepsilon,\uparrow}\big| \ge \bigg(\frac{1}{2} - c\bigg)k\bigg) \le \bigg(\frac{1}{2} - c\bigg)^{-2q} k^{-2q}\mathbf{E}\big|M_k^{\varepsilon,\uparrow}\big|^{2q} \le Ck^{-q}. \qquad \square$$

Appendix A: Auxiliaries to the proof of Theorem 2.3: Regular case

In this section, we assume conditions of Theorem 2.3 to hold true, and (26) to hold true for some A. First, we give some basic integral estimates. Denote

$$T^\varepsilon(r) = \mu^\varepsilon\big([r, \infty)\big), \quad r > 0,$$

the tail function for μ^ε. By (26), $T^\varepsilon(r) = 0$, $r > A$, and by (25), for each $r_0 > 0$

$$\sup_{r > r_0} \sup_{\varepsilon > 0} T^\varepsilon(r) < \infty.$$

Hence by (33) we have

$$\sup_{r > 0} \sup_{\varepsilon \in (0,1]} r^\alpha T^\varepsilon(r) < \infty. \tag{83}$$

Next, for each $c > 0$

$$\limsup_{\delta \searrow 0} \sup_{\varepsilon \in (0,1]} \delta^2 T^\varepsilon(\delta) \le \limsup_{\delta \searrow 0} \sup_{\varepsilon \in (0,1]} \delta^2 \mu^\varepsilon\big([c, \infty)\big) + \limsup_{\delta \searrow 0} \sup_{\varepsilon \in (0,1]} \int_{[\delta,c)} z^2 \mu^\varepsilon(\mathrm{d}z) = \sup_{\varepsilon > 0} \int_{(0,c)} z^2 \mu^\varepsilon(\mathrm{d}z).$$

Since $c > 0$ is arbitrary, the above inequality and (18) yield

$$\lim_{\delta \to 0} \sup_{\varepsilon \in (0,1]} \delta^2 T^\varepsilon(\delta) = 0 \tag{84}$$

for $\alpha = 2$. The same assertion holds true for $\alpha < 2$ by (83).

Using (84), we can perform integration by parts:

$$2\int_0^r z T^\varepsilon(z)\,\mathrm{d}z = z^2 T^\varepsilon(z)\big|_{0+}^r - \int_0^r z^2\,\mathrm{d}T^\varepsilon(z) = r^2 T^\varepsilon(r) + \int_0^r z^2 \mu^\varepsilon(\mathrm{d}z). \tag{85}$$

For $\alpha < 2$, by (83) and (26) this immediately gives

$$\sup_{\varepsilon \in (0,1]} \int_0^\infty z^2 \mu^\varepsilon(\mathrm{d}z) \le 2 \sup_{\varepsilon \in (0,1]} \int_0^\infty z T^\varepsilon(z)\,\mathrm{d}z < \infty \tag{86}$$

and

$$\sup_{\varepsilon \in (0,1]} \int_0^\delta z^2 \mu^\varepsilon(\mathrm{d}z) \le 2 \sup_{\varepsilon \in (0,1]} \int_0^\delta z T^\varepsilon(z)\,\mathrm{d}z \to 0, \quad \delta \to 0. \tag{87}$$

For $\alpha = 2$, the same relations hold true by (18).

From now on, we assume that $(\alpha, \beta) \in \Xi_{\text{regular}}$. The following lemma describes the local ($v \to 0$) and the asymptotic ($v \to \infty$) behaviour of the functions H^ε defined in (37).

Lemma A.1. *For each $\varepsilon \in (0, 1]$, the function*

$$H^\varepsilon(v) = \int_0^A \big(F(v + z) + F(v - z) - 2F(v)\big)\mu^\varepsilon(\mathrm{d}z)$$

is well defined, continuous, and odd. In addition,

$$\lim_{v \to 0} \sup_{\varepsilon \in (0,1]} \left| H^\varepsilon(v) \right| = 0, \tag{88}$$

and, for every $\delta > 0$,

$$\sup_{|v| \geq \delta} \sup_{\varepsilon \in (0,1]} |v|^\beta \left| H^\varepsilon(v) \right| < \infty. \tag{89}$$

Proof. First, let us consider the case $\alpha < 2$, note that in this case we have $\alpha + \beta < 2$. By the Fubini theorem,

$$H^\varepsilon(v) = \int_0^A \big(F(v+z) + F(v-z) - 2F(v) \big) \mu^\varepsilon(dz)$$

$$= -\int_0^A \int_0^z \big(F'(v+w) - F'(v-w) \big) \, dw \, T^\varepsilon(dz) = \int_0^A \big(F'(v+w) - F'(v-w) \big) T^\varepsilon(w) \, dw. \tag{90}$$

The r.h.s. in (90) is well defined because, by (83), for $v > 0$

$$\left| H^\varepsilon(v) \right| \leq C \int_0^A \left| |v+w|^{1-\beta} - |v-w|^{1-\beta} \right| \frac{dw}{w^\alpha} = C|v|^{2-\alpha-\beta} \int_0^{A/|v|} \frac{\psi(\rho)}{\rho^\alpha} \, d\rho, \tag{91}$$

where we denote

$$\psi(\rho) = \left| (1+\rho)^{1-\beta} - |1-\rho|^{1-\beta} \right|.$$

The latter integral in (91) is finite because

$$\frac{\psi(\rho)}{\rho^\alpha} \sim 2(1-\beta)\rho^{1-\alpha}, \quad \rho \to 0,$$

and the function ψ either is continuous for $\beta \leq 1$, or satisfies

$$\psi(\rho) \sim \frac{1}{|1-\rho|^{\beta-1}}, \quad \rho \to 1$$

for $\beta \in (1, 2)$. Since

$$\psi(\rho) \sim 2|1-\beta|\rho^{-\beta}, \quad \rho \to +\infty$$

one can easily derive for the function

$$I(\sigma) = \int_0^\sigma \frac{\psi(\rho)}{\rho^\alpha} \, d\rho$$

the following:

$$I(\sigma) \sim \begin{cases} \frac{2|1-\beta|}{1-\alpha-\beta} \sigma^{1-\alpha-\beta}, & \alpha + \beta < 1, \\ 2|1-\beta| \ln \sigma, & \alpha + \beta = 1, \\ c_I, & 1 < \alpha + \beta < 2, \end{cases} \quad \sigma \to \infty, \tag{92}$$

where

$$c_I = \int_0^\infty \frac{\psi(\rho)}{\rho^\alpha} \, d\rho \in (0, \infty), \quad \alpha + \beta > 1.$$

Thus there exist $v_0 > 0$ and $C > 0$ such that

$$\left| H^\varepsilon(v) \right| \leq C \cdot \begin{cases} |v|, & \alpha + \beta < 1, \\ |v| \ln \frac{1}{|v|}, & \alpha + \beta = 1, \\ |v|^{2-\alpha-\beta}, & 1 < \alpha + \beta < 2, \end{cases} \qquad |v| \leq v_0, \tag{93}$$

which gives (88). The proof of (89) is similar and is based on the relation

$$I(\sigma) \sim \frac{2|1 - \beta|}{2 - \alpha} \sigma^{2-\alpha}, \quad \sigma \to 0,$$

we omit the details.

Next, let $\alpha = 2$; note that, in this case $\beta \leq 0$. Then

$$\left| F'(v + w) - F'(v - w) \right| \leq C\left(1 + |v|^{-\beta}\right)w, \quad w \in (0, A), \tag{94}$$

and the integral in the right hand side of (90) is well defined by (86). The same inequality yields (89).

To prove (88), we restrict ourselves to the case $0 < |v| \leq 2A$, and decompose

$$H^\varepsilon(v) = \left(\int_0^{|v|/2} + \int_{|v|/2}^2 \right) \left(F'(v + w) - F'(v - w)\right) T^\varepsilon(w) \, \mathrm{d}w =: H_1^\varepsilon(v) + H_2^\varepsilon(v).$$

The term H_2^ε admits estimates similar to those we had above. Namely, we have

$$\left| H_2^\varepsilon(v) \right| \leq C|v|^{2-\alpha-\beta} I_2\left(\frac{A}{|v|}\right), \qquad I_2(\sigma) = \int_{1/2}^\sigma \left((1 + \rho)^{1-\beta} - |1 - \rho|^{1-\beta}\right) \frac{\mathrm{d}\rho}{\rho^2}.$$

For I_2 analogue of (92) holds true, and thus H_2^ε satisfies (93). To estimate H_1^ε we use the Lipschitz condition (94) and assumption $v \leq 2A$:

$$\left| H_1^\varepsilon(v) \right| \leq C \sup_{\varepsilon \in (0,1]} \int_0^{|v|/2} w T^\varepsilon(w) \, \mathrm{d}w < \infty.$$

Now (88) follows by (87).   □

In the following lemma, we justify the formal relation (35).

Lemma A.2. *Identity* (35) *holds true with the local martingale* M^ε *defined by* (36).

Proof. For $\beta < 0$, $F \in C^2(\mathbb{R}, \mathbb{R})$, and the standard Itô formula holds. For $\beta \in [0, 2)$, we consider an approximating family $F_m \in C^2(\mathbb{R}, \mathbb{R})$, $m \geq 1$, for F, which satisfies the following:

$$\sup_{v \in \mathbb{R}} \left| F(v) - F_\delta(v) \right| \leq \frac{C}{m^{2-\beta}}, \tag{95}$$

$$F'(v) \equiv F_m'(v) \quad \text{for } |v| \geq \frac{1}{m},$$

$$\sup_{|v| \leq \frac{1}{m}} \left| F_m'(v)|v|^\beta \operatorname{sgn} v - v \right| \leq \frac{c}{m} \quad \text{and} \quad F_m'(v)|v|^\beta \operatorname{sgn} v \equiv v \quad \text{for } |v| \geq \frac{1}{m}, \tag{96}$$

$$\lim_{m \to \infty} F_m''(v) = F''(v) \quad \text{for any } v \neq 0.$$

One particular example of such a family is given by

$$F_m(v) = \begin{cases} (\frac{1-\beta^2}{3(2-\beta)} \frac{1}{m^{2-\beta}} + \frac{1}{2-\beta}|v|^{2-\beta}) \operatorname{sgn} v, & |v| \geq \frac{1}{m}, \\ \frac{1+\beta}{2} \frac{1}{m^{1-\beta}} v + \frac{1-\beta}{6} m^{1+\beta} v^3, & |v| < \frac{1}{m}. \end{cases}$$

The Itô formula applied to F_m yields

$$F_m(V_t^\varepsilon) = F_m(V_0^\varepsilon) - \frac{1}{\varepsilon} \int_0^t F_m'(V_s^\varepsilon) |V_s^\varepsilon|^\beta \operatorname{sgn} V_s^\varepsilon \, ds + M_m^\varepsilon(t) + \int_0^t H^{\varepsilon,m}(V_s^\varepsilon) \, ds,$$

$$M_m^\varepsilon(t) = \int_0^t \int_{|z| \leq A} (F_m(V_{s-}^\varepsilon + z) - F_m(V_{s-}^\varepsilon)) \widetilde{N}^\varepsilon(ds, \, dz), \tag{97}$$

$$H^{\varepsilon,m}(v) = \int_0^\infty (F_m(v+z) + F_m(v-z) - 2F_m(v)) \mu^\varepsilon(dz).$$

By construction, we have

$$F_m(V_t^\varepsilon) \to F(V_t^\varepsilon), \qquad F_m(v_0^\varepsilon) \to F(v_0^\varepsilon), \quad m \to \infty,$$

$$\int_0^t (F_m'(V_s^\varepsilon) |V_s^\varepsilon|^\beta \operatorname{sgn} V_s^\varepsilon - V_s^\varepsilon) \, ds \to 0, \quad m \to \infty,$$

in probability. To analyse the behaviour of the martingale part $M_m^\varepsilon(\cdot)$, we repeat, with proper changes, the argument used to prove (39). Namely, truncating the small jumps, stopping the processes at the time moments

$$\tau_R^\varepsilon = \inf\{t : |V_t^\varepsilon| > R\}, \quad R > 0,$$

and using Theorem 2.2, we can show that

$$M_m^\varepsilon(t) \to M_t^\varepsilon, \quad m \to \infty,$$

in probability. Finally, one has

$$H^{\varepsilon,m} \to H^\varepsilon, \quad m \to \infty$$

uniformly of any bounded set. We omit the detailed proof of this statement, which is very similar to Lemma A.1 but is substantially simpler for it requires no uniform estimates in ε. Taking $m \to \infty$ in (97), we obtain the required Itô formula. $\qquad \square$

Appendix B: Auxiliaries to the proof of Theorem 2.3: Non-regular case

Lemma B.1. *Let $G \in C^2(\mathbb{R}, \mathbb{R})$ be such that for some $\sigma \in (0, \alpha)$ and all $|y| \geq 1$*

$$|G'(y)| \leq C|y|^{\sigma-1} \quad and \quad |G''(y)| \leq C|y|^{\sigma-2}. \tag{98}$$

Then the family

$$K^\varepsilon(y) = \int_0^\infty (G(y+u) + G(y-u) - 2G(y)) \nu^\varepsilon(du), \quad \varepsilon \in (0, 1], \tag{99}$$

satisfies

$$\sup_{\varepsilon \in (0,1]} |K^\varepsilon(y)| \leq C(1 + |y|)^{\sigma-\alpha}$$

for $\alpha \in (0, 2)$ and

$$\sup_{\varepsilon \in (0,1]} \big|K^\varepsilon(y)\big| \le C\big(1 + |y|\big)^{\sigma - 2} \ln\big(2 + |y|\big)$$

for $\alpha = 2$.

Proof. To simplify the notation, we assume $y \ge 0$; clearly, this does not restrict the generality. For $y \le 2$, we decompose

$$K^\varepsilon(y) = \int_0^3 \big(G(y + u) + G(y - u) - 2G(y)\big)v^\varepsilon(\mathrm{d}u)$$

$$+ \int_3^\infty \big(G(y + u) + G(y - u) - 2G(y)\big)v^\varepsilon(\mathrm{d}u) =: K_1^\varepsilon(y) + K_2^\varepsilon(y).$$

We have

$$\big|K_1^\varepsilon(y)\big| \le \max_{|v| \le 5}\big|G''(v)\big| \sup_{\varepsilon \in (0,1]} \int_{|u| < 3} u^2 v^\varepsilon(\mathrm{d}u) < \infty,$$

see (58). Next, we transform $K_2^\varepsilon(y)$ using the Newton–Leibniz formula and the Fubini theorem:

$$K_2^\varepsilon(y) = \int_3^\infty \int_0^u \big(G'(y + v) - G'(y - v)\big)\,\mathrm{d}v\,v^\varepsilon(\mathrm{d}u)$$

$$= \int_0^\infty \big(G'(y + v) - G'(y - v)\big)v^\varepsilon\big([3 \vee v, \infty)\big)\,\mathrm{d}v.$$

Since G' is locally bounded, by (56) this gives for some $C > 0$

$$\big|K_2^\varepsilon(y)\big| \le C + C \int_3^\infty \big|G'(y + v) - G'(y - v)\big|v^{-\alpha}\,\mathrm{d}v.$$

For $y \in [0, 2]$ and $v \ge 3$ we have $|y \pm v| \ge 1$, hence we can continue the above estimate:

$$\big|K_2^\varepsilon(y)\big| \le C\left(1 + \int_3^\infty \big(|y + v|^{\sigma - 1} + |y - v|^{\sigma - 1}\big)v^{-\alpha}\,\mathrm{d}v\right)$$

$$\le C\left(1 + \int_3^\infty \big(v^{\sigma - 1} + (v - 2)^{\sigma - 1}\big)v^{-\alpha}\,\mathrm{d}v\right) < \infty,$$

where the integral is finite because $\sigma < \alpha$. That is,

$$\sup_{y \le 2} \sup_{\varepsilon \in (0,1]} \big|K^\varepsilon(y)\big| < \infty. \tag{100}$$

For $y > 2$, we use another decomposition:

$$K^\varepsilon(y) = \int_0^{y/2} \big(G(y + u) + G(y - u) - 2G(y)\big)v^\varepsilon(\mathrm{d}u)$$

$$+ \int_{y/2}^\infty \big(G(y + u) + G(y - u) - 2G(y)\big)v^\varepsilon(\mathrm{d}u) =: K_3^\varepsilon(y) + K_4^\varepsilon(y).$$

We have $\sigma < \alpha \le 2$. Hence, for $u \le y/2$,

$$\big|G(y + u) + G(y - u) - 2G(y)\big| \le u^2 \sup_{v > y/2}\big|G''(v)\big| \le Cu^2 y^{\sigma - 2},$$

and

$$\left|K_3^\varepsilon(y)\right| \le Cy^{\sigma-2} \int_0^{y/2} u^2 v^\varepsilon(\mathrm{d}u).$$

We have by (56) and (58)

$$\int_0^r u^2 v^\varepsilon(\mathrm{d}u) = \int_0^1 u^2 v^\varepsilon(\mathrm{d}u) + \int_1^r u^2 v^\varepsilon(\mathrm{d}u) \le Cr^{2-\alpha} + C \int_1^r u^{1-\alpha}\,\mathrm{d}u.$$

That is, we have for $y > 2$

$$\left|K_3^\varepsilon(y)\right| \le Cy^{\sigma-\alpha}$$

if $\alpha \in (0,2)$, and

$$\left|K_3^\varepsilon(y)\right| \le Cy^{\sigma-2}\ln(2+y)$$

if $\alpha = 2$.

For $K_4^\varepsilon(y)$, we again use the Fubini theorem:

$$\begin{aligned}
K_4^\varepsilon(y) &= \int_{y/2}^\infty \int_0^u \left(G'(y+v) - G'(y-v)\right)\mathrm{d}v\, v^\varepsilon(\mathrm{d}u) \\
&= \int_0^\infty \left(G'(y+v) - G'(y-v)\right) v^\varepsilon\left([(y/2)\vee v, \infty)\right)\mathrm{d}v \\
&= \left[\int_0^{y/2} + \int_{y/2}^{y-1} + \int_{y-1}^{y+1} + \int_{y+1}^\infty\right]\left(G'(y+v) - G'(y-v)\right) v^\varepsilon\left([(y/2)\vee v, \infty)\right)\mathrm{d}v \\
&=: \sum_{j=1}^4 K_{4,j}^\varepsilon(y).
\end{aligned}$$

Since $y > 2$ we have $y + v > 1$ for any $v > 0$, and thus

$$\left|G'(y+v)\right| \le C(y+v)^{\sigma-1}.$$

In addition, for $v \in [0, y-1]$ we have

$$\left|G'(y+v)\right| \le C(y-v)^{\sigma-1}.$$

Thus by (56)

$$\left|K_{4,1}^\varepsilon(y)\right| \le Cy^{-\alpha}\int_0^{y/2}\left((y+v)^{\sigma-1} + (y-v)^{\sigma-1}\right)\mathrm{d}v \le Cy^{\sigma-\alpha},$$

$$\begin{aligned}
\left|K_{4,2}^\varepsilon(y)\right| &\le C\int_{y/2}^{y-1}\left((y+v)^{\sigma-1} + (y-v)^{\sigma-1}\right)v^{-\alpha}\,\mathrm{d}v \\
&\le Cy^{\sigma-\alpha}\int_{1/2}^1\left((1+\rho)^{\sigma-1} + (1-\rho)^{\sigma-1}\right)\rho^{-\alpha}\,\mathrm{d}\rho,
\end{aligned}$$

note that the latter integral is finite because $\sigma > 0$. Similarly,

$$\begin{aligned}
\left|K_{4,4}^\varepsilon(y)\right| &\le C\int_{y+1}^\infty\left((y+v)^{\sigma-1} + (v-y)^{\sigma-1}\right)v^{-\alpha}\,\mathrm{d}v \\
&\le Cy^{\sigma-\alpha}\int_1^\infty\left((1+\rho)^{\sigma-1} + (\rho-1)^{\sigma-1}\right)\rho^{-\alpha}\,\mathrm{d}\rho,
\end{aligned}$$

and the latter integral is finite because $\sigma > 0$ and $\sigma < \alpha$. Finally, since G' is locally bounded,

$$\left|K_{4,3}^{\varepsilon}(y)\right| \leq C \int_{y-1}^{y+1} \left((y+v)^{\sigma-1} + 1\right)v^{-\alpha}\,\mathrm{d}v \leq C\left(y^{\sigma-\alpha-1} + y^{-\alpha}\right).$$

Combining the estimates for K_3^{ε} and for $K_{4,j}^{\varepsilon}$, $j = 1, \ldots, 4$, we complete the proof. $\qquad\square$

Lemma B.2. *The functions R^{ε}, $\varepsilon \in (0, 1]$ satisfy*

$$\left|R^{\varepsilon}(y)\right| \leq C\left(1 + |y|\right)^{2-\alpha-\beta}$$

if $\alpha \in (0, 2)$, and

$$\left|R^{\varepsilon}(y)\right| \leq C\left(1 + |y|\right)^{2-\alpha-\beta} \ln\left(2 + |y|\right)$$

if $\alpha = 2$.

Proof. The family R^{ε}, $\varepsilon \in (0, 1]$ has the form (99) with $G = \widehat{F}$, and this function satisfies (98) with $\sigma = 2 - \beta > 0$. Hence, for $\alpha + \beta > 2$, the required statement follows directly from Lemma B.1. Let us prove this statement in the boundary case $\alpha + \beta = 2$. One can see that the estimates for K_1^{ε}, K_3^{ε} and $K_{4,j}^{\varepsilon}$, $j = 1, 2, 3$, from the previous proof remain true under the assumption $\sigma = 0$ as well. Next, for $G = \widehat{F}$ we have

$$G'(y) = \begin{cases} y^{1-\beta}, & y \geq 1, \\ (-y)^{1-\beta}, & y \leq -1. \end{cases}$$

Then for $y \leq 2$

$$K_2^{\varepsilon}(y) = \int_0^{\infty} \left((y+v)^{1-\beta} - (v-y)^{1-\beta}\right)v^{\varepsilon}\left([3 \vee v, \infty)\right)\,\mathrm{d}v,$$

and therefore

$$\left|K_2^{\varepsilon}(y)\right| \leq C\left(1 + \int_3^{\infty} \left|v^{\sigma-1} - (v-2)^{\sigma-1}\right|v^{-\alpha}\,\mathrm{d}v\right).$$

The latter integral is finite for $\sigma > \alpha - 1$ because

$$\left|v^{\sigma-1} - (v-2)^{\sigma-1}\right| \sim cv^{\sigma-2}, \quad v \to \infty.$$

Similarly,

$$\left|K_{4,4}^{\varepsilon}(y)\right| \leq Cy^{\sigma-\alpha} \int_1^{\infty} \left|(1+\rho)^{\sigma-1} - (\rho-1)^{\sigma-1}\right|\rho^{-\alpha}\,\mathrm{d}\rho,$$

and the latter integral is finite for $\sigma > \alpha - 1$. $\qquad\square$

Lemma B.3. *The derivatives of functions R^{ε}, $\varepsilon \in (0, 1]$, are uniformly bounded, namely there exists $C > 0$ such that*

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}R^{\varepsilon}(y)\right| \leq C, \quad y \in \mathbb{R}, \varepsilon \in (0, 1].$$

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}y}R^{\varepsilon}(y) = \int_0^{\infty} \left(\widehat{F}'(y+u) + \widehat{F}'(y-u) - 2\widehat{F}'(y)\right)v^{\varepsilon}(\mathrm{d}u),$$

and the integral is well defined because $\widehat{F}' \in C^2(\mathbb{R}, \mathbb{R})$. We have that the second derivative $(\widehat{F}')'' = \widehat{F}'''$ of $\widehat{F}'$ is bounded, and $\widehat{F}'$ is either bounded for $\beta \geq 1$, or $(1 - \beta)$-Hölder continuous for $\beta \in (0, 1)$. In the first case, we just have

$$\sup_{y \in \mathbb{R}, \varepsilon \in (0,1]} \left| \frac{\mathrm{d}}{\mathrm{d}y} R^\varepsilon(y) \right| \leq C \sup_{\varepsilon \in (0,1]} \int_{\mathbb{R}} (u^2 \wedge 1) v^\varepsilon(\mathrm{d}u) < \infty,$$

see (58). In the second case we have

$$\sup_{y \in \mathbb{R}, \varepsilon \in (0,1]} \left| \frac{\mathrm{d}}{\mathrm{d}y} R^\varepsilon(y) \right| \leq C \sup_{\varepsilon \in (0,1]} \int_{\mathbb{R}} (u^2 \wedge |u|^{1-\beta}) v^\varepsilon(\mathrm{d}u).$$

By (58),

$$\int_{|u|>1} |u|^{1-\beta} v^\varepsilon(\mathrm{d}u) = 2v^\varepsilon([1, \infty)) + 2(1 - \beta) \int_1^\infty \int_1^u v^{-\beta} \, \mathrm{d}v v^\varepsilon(\mathrm{d}u)$$

$$\leq C + C \int_1^\infty v^{-\alpha-\beta} \, \mathrm{d}v < \infty,$$

where we have used (56) and the assumption $\alpha + \beta \geq 2 > 1$. This provides the required statement for $\beta \in (0, 1)$. $\quad\square$

Appendix C: Proof of (55)

First, we observe that

$$\varepsilon^{(2-\alpha-\beta)\gamma} \int_0^\infty \left(F(\varepsilon^{-\gamma} v + u) + F(\varepsilon^{-\gamma} v - u) - 2F(\varepsilon^{-\gamma} v) \right) v^\varepsilon(\mathrm{d}u)$$

$$= \varepsilon^{(2-\beta)\gamma} \int_0^\infty \left(F(\varepsilon^{-\gamma} v + \varepsilon^{-\gamma} z) + F(\varepsilon^{-\gamma} v - \varepsilon^{-\gamma} z) - 2F(\varepsilon^{-\gamma} v) \right) \mu^\varepsilon(\mathrm{d}z)$$

$$= \int_0^\infty \left(F(v + z) + F(v - z) - 2F(y) \right) \mu^\varepsilon(\mathrm{d}z) = H^\varepsilon(v).$$

Hence

$$\widehat{H}^\varepsilon(v) - H^\varepsilon(v) = \varepsilon^{(2-\alpha-\beta)\gamma} R^{0,\varepsilon}(\varepsilon^{-\gamma} v) + \varepsilon^{(2-\alpha-\beta)\gamma} R^{1,\varepsilon}(\varepsilon^{-\gamma} v),$$

where

$$R^{0,\varepsilon}(y) = -\bar{F}'(y)|y|^\beta \operatorname{sgn} y, \quad R^{1,\varepsilon}(y) = \int_0^\infty \left(\bar{F}(y + u) + \bar{F}(y - u) - 2\bar{F}(y) \right) v^\varepsilon(\mathrm{d}u).$$

Since $\bar{F}$ vanishes outside of $[-1, 1]$, so does $R^{0,\varepsilon}(y)$, and for $y > 1$ we have

$$\left| R^{1,\varepsilon}(y) \right| = \left| \int_{[y-1, y+1]} F(y - u) v^\varepsilon(\mathrm{d}u) \right| \leq C v^\varepsilon([y - 1, \infty)) \leq C(y - 1)^{-\alpha};$$

here we have used that $\bar{F}$ is bounded and (56). Hence

$$\sup_{|v|>2\varepsilon^\gamma} \left| \widehat{H}^\varepsilon(v) - H^\varepsilon(v) \right| \leq C\varepsilon^{(2-\beta)\gamma} \to 0, \quad \varepsilon \to 0,$$

which yields (55).

Acknowledgements

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