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Hausdorff dimension of the scaling limit of loop-erased random walk in three dimensions

Daisuke Shiraishi

Department of Advanced Mathematical Sciences, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan.
E-mail: shiraishi@acs.i.kyoto-u.ac.jp

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Abstract. Let M_n be the length (number of steps) of the loop-erasure of a simple random walk up to the first exit from a ball of radius n centered at its starting point. It is shown in (*Ann. Probab.* **46** (2) (2018) 687–774) that there exists $\beta \in (1, \frac{5}{3}]$ such that $E(M_n)$ is of order n^β in 3 dimensions. In the present article, we show that the Hausdorff dimension of the scaling limit of the loop-erased random walk in 3 dimensions is equal to β almost surely.

Résumé. Soit M_n la longueur (nombre de pas) d'une marche aléatoire simple à boucles effacées considérée jusqu'à la première sortie d'une boule de rayon n centrée en son point de départ. Il est démontré dans (*Ann. Probab.* **46** (2) (2018) 687–774) qu'il existe $\beta \in (1, \frac{5}{3}]$ tel que $E(M_n)$ est d'ordre n^β en dimension 3. Dans le présent article, nous montrons que la dimension de Hausdorff de la limite d'échelle de la marche aléatoire effacée en dimension 3 est égale à β presque sûrement.

MSC: 82B41; 60G50

Keywords: Loop-erased random walk; Scaling limit; Hausdorff dimension

1. Introduction

1.1. *Introduction*

Loop-erased random walk (LERW) is a simple path obtained by erasing all loops from a random walk path chronologically (see Section 2.1 for the precise definition), which was originally introduced in [7]. In this article, we study the Hausdorff dimension of the scaling limit of LERW in three dimensions.

It is known that the scaling limit of LERW in $\mathbb{Z}^d$ exists for every d. Let S be the simple random walk in $\mathbb{Z}^d$ started at the origin and τ_n be the first exit time from a ball of radius n. We write $\mathrm{LEW}_n = \frac{\mathrm{LE}(S[0,\tau_n])}{n}$ for the rescaled loop-erased random walk obtained by multiplying LERW up to τ_n by $\frac{1}{n}$ (see Section 2.1 for the definition of LE). We think of LEW_n as a random element of the metric space of compact subsets in the closed unit ball in $\mathbb{R}^d$ endowed with the Hausdorff distance. Then LEW_n converges weakly to a d-dimensional Brownian motion for $d \geq 4$ (Theorem 7.7.6 of [8]), and converges weakly to SLE_2 ([13,17]) for $d = 2$ (actually, even in a stronger sense). For $d = 3$, the sequence LEW_{2^n} is Cauchy in the metric space and it converges weakly to a random compact subset in the closed unit ball in $\mathbb{R}^3$, see [6]. We denote the weak convergence limit by $\mathcal{K}$ in $d = 3$ and call it the scaling limit of LERW in 3 dimensions. It is also known that $\mathcal{K}$ is invariant under rotations and dilations, see [6].

While the scaling limit of LERW for $d \geq 4$ and $d = 2$ are well-studied, little is known about $\mathcal{K}$ when $d = 3$. Recently some topological properties of $\mathcal{K}$ were studied in [16]. In [16], it was proved that $\mathcal{K}$ is a simple path almost surely, and that the random set obtained by adding the loops of the independent Brownian loop soup of parameter 1 that meet $\mathcal{K}$ (see [14] for the Brownian loop soup) to $\mathcal{K}$, has the same distribution as the trace of Brownian motion (see Section 2.3 for details). Furthermore, bounds on the Hausdorff dimension of $\mathcal{K}$ were also derived in [16]. Namely, one has

$$2-\xi \le \dim_{\mathrm{H}}(\mathcal{K}) \le \beta, \quad \text{almost surely,} \tag{1.1}$$

where $\xi \in (\frac{1}{2}, 1)$ is the intersection exponent for three dimensional Brownian motion (see [9] for ξ) and $\beta \in (1, \frac{5}{3}]$ is the growth exponent for LERW in $d=3$, i.e., if we write M_n for the length (the number of steps) of $\mathrm{LE}(S[0,\tau_n])$, then [18] shows that the following limit exists in 3 dimensions:

$$\lim_{n\to\infty} \frac{\log E(M_n)}{\log n} = \beta. \tag{1.2}$$

In particular, we have $1 < \dim_{\mathrm{H}}(\mathcal{K}) \le \frac{5}{3}$ almost surely.

In the present article, we will show that $\dim_{\mathrm{H}}(\mathcal{K}) \ge \beta$, i.e., the main result of the article is the following.

Theorem 1.1. *Let $d=3$. We have*

$$\dim_{\mathrm{H}}(\mathcal{K}) = \beta, \quad \textit{almost surely}. \tag{1.3}$$

The exact value of β is not known or even conjectured. Numerical simulations suggest that $\beta = 1.62 \pm 0.01$, see [4,19]. The best rigorous bounds are $\beta \in (1, \frac{5}{3}]$, see [11].

The Hausdorff dimension of the scaling limit of LERW is equal to 2 for $d \ge 4$ (Theorem 7.7.6 of [8]), and is equal to $\frac{5}{4}$ for $d=2$ ([2,13]) almost surely. The exponent $\frac{5}{4}$ is called the growth exponent for LERW for $d=2$, that is, it is known that $E(M_n)$ is of order $n^{\frac{5}{4}}$ in 2 dimensions (see [5,15] and [12]).

1.2. *Some words about the proof*

In order to show that the Hausdorff dimension of $\mathcal{K}$ is bounded below by β, we will use a standard technique referred to as Frostman's lemma (see Lemma 5.1). We explain how to apply it to our situation here.

By Frostman's lemma, we need to construct a positive (random) measure μ supported on $\mathcal{K}$ such that its $(\beta-\delta)$-energy $I_{\beta-\delta}(\mu)$ (see Lemma 5.1 for the β-energy) is finite with high probability for any $\delta > 0$.

With this in mind, we partition the unit ball D into a collection of ϵ-cubes formed by $b_x = \epsilon \prod_{i=1}^{3}[x_i, x_i+1]$ for $x=(x_1,x_2,x_3) \in \mathbb{Z}^3$. We first want to construct a random measure μ_ϵ which approximates μ as follows. We introduce a (random) measure μ_ϵ whose density, with respect to Lebesgue measure, is comparable to $\frac{1}{P(\mathcal{K}\cap b_x \neq \varnothing)}$ on each ϵ-cube b_x with $\frac{1}{3} \le |\epsilon x| \le \frac{2}{3}$ such that $\mathcal{K}$ hits b_x and assigns measure zero elsewhere (see Section 5.1 for the precise definition of μ_ϵ). Then the limit of the support of μ_ϵ as $\epsilon \to 0$ is contained in $\mathcal{K}$ almost surely. Therefore we need to show that for every $\delta > 0$ and $r > 0$ there exist constants $c_r > 0$, $C_{\delta,r} < \infty$, which do not depend on ϵ, such that

$$P\big(I_{\beta-\delta}(\mu_\epsilon) \le C_{\delta,r}\big) \ge 1-r, \tag{1.4}$$

$$P\big(\mu_\epsilon(\overline{D}) \ge c_r\big) \ge 1-r. \tag{1.5}$$

for all $\epsilon > 0$. Once (1.4) and (1.5) are proved, it follows that there exists some subsequential weak limit measure μ of the μ_ϵ such that μ is a positive measure satisfying that its support is contained in $\mathcal{K}$ and the $(\beta-\delta)$-energy is finite with probability at least $1-2r$. Using Frostman's lemma, we get $\dim_{\mathrm{H}}(\mathcal{K}) \ge \beta-\delta$ with probability $\ge 1-2r$, and Theorem 1.1 is proved.

let μ be any weak limit of the μ_ϵ. Then the measure μ is a positive measure satisfying that its support is contained in $\mathcal{K}$ and the $(\beta-\delta)$-energy is finite with probability at least $1-r$. Using Frostman's lemma, we get $\dim_{\mathrm{H}}(\mathcal{K}) \ge \beta-\delta$ with probability $\ge 1-r$, and Theorem 1.1 is proved.

Next we explain how to prove (1.4) and (1.5). For (1.4), by Markov's inequality, it suffices to show that the first moment of $I_{\beta-\delta}(\mu_\epsilon)$ is bounded above by some constant C_δ uniformly in ϵ. In order to estimate the first moment, by definition of μ_ϵ, we need to give an upper bound of the probability that $\mathcal{K}$ hits two distinct ϵ-cubes b_x and b_y with $\frac{1}{3} \le |\epsilon x|, |\epsilon y| \le \frac{2}{3}$ (see the proof of Lemma 5.2). Such a bound will be given in the important Theorem 3.1. Theorem 3.1 roughly claims that $P^0(\mathcal{K} \cap b_x \neq \emptyset, \mathcal{K} \cap b_y \neq \emptyset)$ is bounded above by $CP^0(\mathcal{K} \cap b_x \neq \emptyset) P^{\epsilon x}(\mathcal{K} \cap b_y \neq \emptyset)$, where P^z denotes the probability measure for $\mathcal{K}$ started at z. Since the domain Markov property of $\mathcal{K}$ has not been established up to now, we will consider the corresponding probability for LERW as follows. As LEW_{2^k} converges to $\mathcal{K}$ by [6], we can couple them on the same probability space such that the Hausdorff distance between LEW_n and $\mathcal{K}$ is bounded above by ϵ^2 for large $n = 2^k$ with high probability. Then the problem boils down to estimates of $P(\mathrm{LE}(S[0, \tau_n]) \cap nb_x \neq \emptyset, \mathrm{LE}(S[0, \tau_n]) \cap nb_y \neq \emptyset)$. It is crucial to control the dependence of these two events with the help of the domain Markov property for LERW (see Lemma 2.3 for the domain Markov property). This key step will be done in Theorem 3.1. In Theorem 3.1, we will show that the probability is bounded above by $CP(\mathrm{LE}(S[0, \tau_n]) \cap nb_x \neq \emptyset) P^{\epsilon nx}(\mathrm{LE}(S[0, \tau_n]) \cap nb_y \neq \emptyset)$, and using some results derived in [18], we will derive a bound of this product in terms of escape probabilities defined as follows. Let $R_1 \le R_2$ and let S^1, S^2 be two independent simple random walks started at the origin. We write τ^i_R for the first time that S^i hits the boundary of the ball of radius R. We define the escape probability $\mathrm{Es}(R_1, R_2)$ by

$$\mathrm{Es}(R_1, R_2) = P_1 \otimes P_2\big(\mathrm{LE}\big(S^1\big[0, \tau^1_{R_2}\big]\big)[s, u] \cap S^2\big[0, \tau^2_{R_2}\big] = \emptyset\big),$$

where u is the length of $\mathrm{LE}(S^1[0, \tau^1_{R_2}])$, and s is its last visit to the ball of radius R_1 before time u (see Section 2.2 for escape probabilities). In order for z to be in $\mathrm{LE}(S[0, \tau_n])$, by definition of LE (see Definition 2.1), the following two conditions are required: (i) S hits z before τ_n. (ii) The loop-erasure of S from the origin to the last visit of z does not intersect the remaining part of S from z to $S(\tau_n)$. Reversing a path, the probability for z to be in the LERW is equal to the probability that with $S^1(0) = S^2(0) = z$,

- S^1 hits the origin before exiting the ball of radius n,
- The loop-erasure of S^1 from z to the last visit of the origin does not intersect S^2 up to exiting the ball.

It turns out that this probability is comparable to $n^{-1}\,\mathrm{Es}(0, n)$ if $\frac{n}{3} \le |z| \le \frac{2n}{3}$. Furthermore, a similar consideration gives that the probability of $\mathrm{LE}(S[0, \tau_n])$ hitting nb_x is comparable to $\epsilon\,\mathrm{Es}(\epsilon n, n)$, which leads that the probability of $\mathcal{K}$ hitting b_x is also comparable to $\epsilon\,\mathrm{Es}(\epsilon n, n)$. (In fact, we will set $\frac{1}{\epsilon\,\mathrm{Es}(\epsilon n, n)}$ on each b_x hit by $\mathcal{K}$ for the density of μ_ϵ, where we chose n as an arbitrary large integer so that the distance between LEW_n and $\mathcal{K}$ is small with high probability as explained above. We also point out that for all large n, $\mathrm{Es}(\epsilon n, n)$ is of order $\epsilon^{\alpha+o(1)}$ for some constant α, see Theorem 2.7.) Finally Theorem 3.1 concludes that

$$P(\mathcal{K} \cap b_x \neq \emptyset, \mathcal{K} \cap b_y \neq \emptyset) \le \frac{C\epsilon}{|x-y|}\,\mathrm{Es}(\epsilon n, n)\,\mathrm{Es}\big(\epsilon n, \epsilon n|x-y|\big), \tag{1.6}$$

which is a new result to our knowledge. Combining (1.6) with estimates for the escape probabilities obtained in [18] (see Section 2.2), we get (1.4).

Next we consider (1.5). The definition of μ_ϵ immediately gives that $\mu_\epsilon(\overline{D})$ is equal to $\frac{\epsilon^2 Y^\epsilon}{\mathrm{Es}(\epsilon n, n)}$, where Y^ϵ stands for the number of ϵ-cubes b_x with $\frac{1}{3} \le |\epsilon x| \le \frac{2}{3}$ such that $\mathcal{K}$ hits b_x. (Recall that we choose n large enough so that the Hausdorff distance between LEW_n and $\mathcal{K}$ is smaller than ϵ^2 with high probability in the coupling explained as above.) Therefore, in order to prove (1.5), it suffices to show that for all $r > 0$ there exists $c_r > 0$ such that

$$P\big(Y^\epsilon \ge c_r \epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)\big) \ge 1 - r, \tag{1.7}$$

for all $\epsilon > 0$.

Since the probability of $\mathcal{K}$ hitting b_x is comparable to $\epsilon\,\mathrm{Es}(\epsilon n, n)$, the first moment of Y^ϵ is of order $\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)$. Using (1.6), it turns out that the second moment of Y^ϵ is comparable to the square of its first moment. So the second moment method gives that Y^ϵ is bounded below by $c\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)$ with positive probability for some $c > 0$ (Corollary 3.11). However this is not enough to prove (1.7) and we need more careful considerations that we will explain below.

In order to prove (1.7), again we use the coupling of $\mathcal{K}$ and LEW_n explained as above. Then (1.7) boils down to the corresponding estimates for LERW as follows. Let Y^ϵ_n be the number of ϵn-cubes nb_x with $\frac{1}{3} \le |\epsilon x| \le \frac{2}{3}$ such that

LE($S[0,\tau_n]$) hits nb_x. Then (1.7) is reduced to proving that for all $r>0$ there exists $c_r>0$ such that

$$P\big(Y_n^{\epsilon} \geq c_r \epsilon^{-2}\operatorname{Es}(\epsilon n, n)\big) \geq 1-r. \tag{1.8}$$

We will prove (1.8) in Proposition 4.5 using Markovian-type "iteration arguments" that we will briefly explain here. In order to prove (1.8), we consider N cubes A_i ($i=1,\dots,N$) of side length $\frac{n}{3}+\frac{in}{3N}$. We are interested in a subpath γ_i of $\gamma := \mathrm{LE}(S[0,\tau_n])$ which consists of γ between its first visit to ∂A_i and that to ∂A_{i+1} (see the beginning of Section 4 for the precise definition of γ_i). We want to show that for all $r>0$, by choosing $N=N_r$ and c_r suitably, the probability that at least one of γ_i hits $c_r\epsilon^{-2}\operatorname{Es}(\epsilon n, n)$ ϵn-cubes is bigger than $1-r$. To achieve this, we prove in Lemma 4.4 that given $\gamma_1,\dots,\gamma_i$ the probability that γ_{i+1} hits $c_r\epsilon^{-2}\operatorname{Es}(\epsilon n, n)$ cubes is bigger than some universal constant $c>0$ for each i. This enables us to show that the probability in (1.8) is bigger than $1-(1-c)^N$ and finish the proof of (1.8) by taking N such that $(1-c)^N<r$. To establish Lemma 4.4, it is crucial to deal with some sort of independence of γ_i. The domain Markov property (see Lemma 2.3) tells that we need to study a random walk conditioned not to intersect $\gamma_1,\dots,\gamma_i$. We will study such a conditioned random walk in Section 4.1. Then we will prove Lemma 4.4 and (1.7) by using results derived there in Section 4.2. To our knowledge the tail estimate of Y^{ϵ} as in (1.7) is also new. This iteration argument is based on the same spirit of the proof of Theorem 6.7 of [1] and Theorem 8.2.6 of [18] where exponential lower tail bounds of M_n were established for $d=2$ ([1]) and $d=3$ ([18]).

Remark 1.2. As we discussed above, $E(Y^{\epsilon})$ is comparable to $\epsilon^{-2}\operatorname{Es}(\epsilon n, n)$, which is of order $\epsilon^{-(2-\alpha)+o(1)}$ for some exponent $\alpha\in[\frac{1}{3},1)$ (see Theorem 2.7). It turns out that β in Theorem 1.1 is equal to $2-\alpha$.

Remark 1.3. It is crucial that both the upper bound in the right hand side of (1.6) and the lower bound of Y^{ϵ} are given in terms of the escape probabilities. Since $\operatorname{Es}(\epsilon n, n)=\epsilon^{\alpha+o(1)}$ (see Theorem 2.7), one may suppose that in order to prove Theorem 1.1, it suffices to show that for every $\delta>0$, $Y^{\epsilon}\geq c\epsilon^{-(2-\alpha)+\delta}$ with high probability instead of proving (1.7). However this is not the case. Energy estimates as in Lemma 5.2 do not work if we rely on only such estimates without using the escape probabilities.

1.3. *Structure of the paper*

The organization of the paper is as follows. In the next subsection, we will give a list of notation used throughout the paper.

In Section 2, we will review known facts about LERW. We explain some basic properties of LERW in Section 2.1. In order to show Theorem 1.1, the probability that an LERW and an independent simple random walk do not intersect up to exiting a large ball, which is referred to as an escape probability, is a key tool. That probability will be considered in Section 2.2. The precise definition and some properties of $\mathcal{K}$ will be given in Section 2.3.

One of the key results in the paper is Theorem 3.1, which gives an upper bound of the probability that $\mathcal{K}$ hits two small boxes. The proof of Theorem 3.1 will be given in Section 3.1. Using Theorem 3.1, we study the number of small boxes hit by $\mathcal{K}$ in Section 3.2. By the second moment method, we give a lower bound of the number of those boxes hit by $\mathcal{K}$ in Corollary 3.11.

To establish (1.3) almost surely, we need to show (1.7) which is an improvement of Corollary 3.11 in Section 4. Following iteration arguments used in the proof of Theorem 6.7 [1] and Proposition 8.2.5 of [18], we study a random walk conditioned not to intersect a given simple path in Section 4.1. Using estimates derived there, we will prove (1.7) and (1.5) in Section 4.2.

We will prove (1.4) in Section 5.1. Finally, using Frostman's lemma (see Lemma 5.1), we will prove Theorem 1.1 in Section 5.2.

1.4. *Notation*

In this subsection, we will give some definitions which will be used throughout the paper.

Let $\lambda=[\lambda(0),\lambda(1),\dots,\lambda(m)]$ be a sequence of points in $\mathbb{Z}^d$. We call it a path if $|\lambda(j-1)-\lambda(j)|=1$ for all j. In that case we say λ has a length m and denote the length of λ by $\operatorname{len}\lambda$. We call λ a simple path if $\lambda(i)\neq\lambda(j)$ for all $i\neq j$.

We use $|\cdot|$ for the Euclid distance in $\mathbb{R}^d$. For $n \geq 0$ and $z \in \mathbb{Z}^d$, define $B_{z,n} = B(z,n) := \{x \in \mathbb{Z}^d \mid |x-z| < n\}$. If $z = 0$, we write $B_{0,n} = B(0,n) = B(n)$. We write $D = \{x \in \mathbb{R}^d \mid |x| < 1\}$ and $\overline{D}$ for its closure. For $r > 0$, let $D_r = \{x \in \mathbb{R}^d \mid |x| < r\}$ and $\overline{D_r}$ for its closure.

For a subset $A \subset \mathbb{Z}^d$, we let $\partial A = \{x \notin A \mid \text{there exists } y \in A \text{ such that } |x-y| = 1\}$ and $\partial_i A = \{x \in A \mid \text{there exists } y \notin A \text{ such that } |x-y| = 1\}$. We write $\overline{A} := A \cup \partial A$. Given a subset $A \subset \mathbb{Z}^d$ and $r > 0$, we write $rA := \{ry \mid y \in A\}$.

Throughout the paper, S, S^1, S^2, S^3 and S^4 denote independent simple random walks on $\mathbb{Z}^d$. For the probability law and the expectation of S started at z, we use P^z and E^z respectively. If $z = 0$, we write $P^0 = P$ and $E^0 = E$. For the probability law and the expectation of S^i started at z, we use P_i^z and E_i^z respectively. If $z = 0$, we write $P_i^0 = P_i$ and $E_i^0 = E_i$.

Given $n \geq 1$, let $\tau_n := \inf\{k \mid S(k) \notin B(n)\}$ and $\tau_n^i := \inf\{k \mid S^i(k) \notin B(n)\}$. For $z \in \mathbb{Z}^d$, we write $T_{z,n} := \inf\{k \mid S(k) \in \partial B(z,n)\}$ and $T_{z,n}^i := \inf\{k \mid S^i(k) \in \partial B(z,n)\}$.

For a subset $A \subset \mathbb{Z}^d$, define Green's function in A by $G(x,y,A) = G_A(x,y) = E^x(\sum_{j=0}^{\tau-1} \mathbf{1}\{S(j) = y\})$ for $x, y \in A$, where $\tau = \inf\{t \mid S(t) \in \partial A\}$.

We use $c, C, C_1, \cdots$ to denote arbitrary positive constants which may change from line to line. If a constant is to depend on some other quantity, this will be made explicit. For example, if C depends on δ, we write C_δ. To avoid complication of notation, we do not use $\lfloor r \rfloor$ (the largest integer $\leq r$) even though it is necessary to carry it.

2. Loop-erased random walk

In this section, we will review some known facts about loop-erased random walk (LERW). In Section 2.1, we begin with the definition of loop-erasure and LERW. Then we state the time reversibility and the domain Markov property of LERW. All results in Section 2.1 hold for LERW in $\mathbb{Z}^d$ (even in any graphs).

As we discussed in Section 1.2, the probability that an LERW and an independent simple random walk do not intersect up to exiting a large ball, which is referred to as escape probability, is a key tool in the paper. We will define and consider the escape probability for LERW in $\mathbb{Z}^3$ in Section 2.2. Most of estimates for escape probabilities stated there are results derived in [18], and those results will be repeatedly used throughout the paper.

We will explain some known results about the scaling limit of LERW in 3 dimensions in Section 2.3.

2.1. *Basic properties*

In this subsection, we first define the loop-erasure of a given path in Definition 2.1. LERW is a (random) simple path obtained by loop-erasing from a random walk. It satisfies the time reversibility (see Lemma 2.2). LERW is not a Markov process by definition, but it satisfies the domain Markov property (see Lemma 2.3). Lemma 2.2 and Lemma 2.3 hold for LERW in $\mathbb{Z}^d$ for all d.

We begin with the definition of loop-erasure of a path.

Definition 2.1. Given a path $\lambda = [\lambda(0), \lambda(1), \dots, \lambda(m)] \subset \mathbb{Z}^d$, define its loop-erasure $\mathrm{LE}(\lambda)$ as follows. Let

$$s_0 := \max\{t \mid \lambda(t) = \lambda(0)\}, \tag{2.1}$$

and for $i \geq 1$, let

$$s_i := \max\{t \mid \lambda(t) = \lambda(s_{i-1} + 1)\}. \tag{2.2}$$

We write $n = \min\{i \mid s_i = m\}$. Then define $\mathrm{LE}(\lambda)$ by

$$\mathrm{LE}(\lambda) = [\lambda(s_0), \lambda(s_1), \dots, \lambda(s_n)]. \tag{2.3}$$

If $\lambda = [\lambda(0), \lambda(1), \dots] \subset \mathbb{Z}^d$ is an infinite path satisfying that $\{k \geq 0 \mid \lambda(k) = \lambda(n)\}$ is finite for each n, then we define $\mathrm{LE}(\lambda)$ similarly.

Throughout the paper, we are interested in the loop-erasure of random walks running until some stopping time, the loop-erased random walk.

For two paths $\lambda_1 = [\lambda_1(0), \lambda_1(1), \dots, \lambda_1(m_1)]$ and $\lambda_2 = [\lambda_2(0), \lambda_2(1), \dots, \lambda_2(m_2)]$ in $\mathbb{Z}^d$ with $\lambda_1(m_1) = \lambda_2(0)$, we write

$$\lambda_1 + \lambda_2 := \bigl[\lambda_1(0), \lambda_1(1), \dots, \lambda_1(m_1), \lambda_2(1), \dots, \lambda_2(m_2)\bigr]. \tag{2.4}$$

We will use repeatedly the following notation for $\mathrm{LE}(\lambda_1 + \lambda_2)$. Let $u = \min\{t \mid \mathrm{LE}(\lambda_1)(t) \in \lambda_2\}$ and let $s = \max\{t \mid \lambda_2(t) = \mathrm{LE}(\lambda_1)(u)\}$. Define

$$\mathrm{LE}^{(1)} = \mathrm{LE}_1(\lambda_1, \lambda_2) := \mathrm{LE}(\lambda_1)[0, u], \qquad \mathrm{LE}^{(2)} = \mathrm{LE}_2(\lambda_1, \lambda_2) := \mathrm{LE}\bigl(\lambda_2[s, m_2]\bigr). \tag{2.5}$$

Then it is easy to check that $\mathrm{LE}(\lambda_1 + \lambda_2) = \mathrm{LE}^{(1)} + \mathrm{LE}^{(2)}$.

For a path $\lambda = [\lambda(0), \lambda(1), \dots, \lambda(m)] \subset \mathbb{Z}^d$, define its time reversal λ^R by $\lambda^R := [\lambda(m), \lambda(m-1), \dots, \lambda(0)]$. Note that in general, $\mathrm{LE}(\lambda) \neq (\mathrm{LE}(\lambda^R))^R$. However, as next lemma shows, the time reversal of LERW has same distribution to the original LERW. Let Λ_m be the set of paths of length m started at the origin.

Lemma 2.2 (Lemma 7.2.1 [8]). *For each $m \geq 0$, there exists a bijection $T^m : \Lambda_m \to \Lambda_m$ such that for each $\lambda \in \Lambda_m$,*

$$\mathrm{LE}(\lambda) = \bigl(\mathrm{LE}\bigl((T^m\lambda)^R\bigr)\bigr)^R. \tag{2.6}$$

Moreover, it follows that λ and $T^m\lambda$ visit the same edges in the same directions with the same multiplicities.

Note that LERW is not a Markov process. However it satisfies the domain Markov property in the following sense.

Lemma 2.3 (Proposition 7.3.1 [8]). *Let $D \subset \mathbb{Z}^d$ be a finite subset. Suppose that λ_i $(i = 1, 2)$ are simple paths of length m_i with $\lambda_1 \subset D$, $\lambda_1(m_1) = \lambda_2(0)$. Suppose also that $\lambda_1 + \lambda_2$ is a simple path from $\lambda_1(0)$ terminated at ∂D. Let Y be a random walk R started at $\lambda_2(0)$ conditioned on $R[1, \sigma_D^R] \cap \lambda_1 = \emptyset$. Here $\sigma_D^R = \inf\{t \mid R(t) \notin D\}$. Then we have*

$$P^{\lambda_1(0)}\bigl(\mathrm{LE}\bigl(S[0, \sigma_D]\bigr) = \lambda_1 + \lambda_2 \mid \mathrm{LE}\bigl(S[0, \sigma_D]\bigr)[0, m_1] = \lambda_1\bigr) = P\bigl(\mathrm{LE}\bigl(Y\bigl[0, \sigma_D^Y\bigr]\bigr) = \lambda_2\bigr), \tag{2.7}$$

where σ_D (resp. σ_D^Y) is the first exit time from D for S (resp. Y).

Suppose that S is the simple random walk started at the origin. Since S is transient for $d \geq 3$, we can define the *infinite* loop-erased random walk $\mathrm{LE}(S[0, \infty))$. Let $R > 0$. We write

$$\gamma = \mathrm{LE}\bigl(S[0, \tau_{4R}]\bigr), \qquad \gamma' = \mathrm{LE}\bigl(S[0, \infty)\bigr). \tag{2.8}$$

Let

$$\tau_R^\gamma = \inf\bigl\{k \mid \gamma(k) \notin B(R)\bigr\}, \qquad \tau_R^{\gamma'} = \inf\bigl\{k \mid \gamma'(k) \notin B(R)\bigr\}. \tag{2.9}$$

We write

$$\Lambda(R) = \bigl\{\lambda \mid P\bigl(\gamma\bigl[0, \tau_R^\gamma\bigr] = \lambda\bigr) > 0\bigr\}. \tag{2.10}$$

for a set of simple paths which can be $\gamma[0, \tau_R^\gamma]$ with positive probability.

It is clear that $\gamma[0, \tau_R^\gamma]$ does not coincide with $\gamma'[0, \tau_R^{\gamma'}]$. However, the next proposition says that their distributions are comparable.

Proposition 2.4 (Proposition 4.4 [15]). *Let $d \geq 3$. There exists $c > 0$ such that for all $R > 0$ and a simple path $\lambda \in \Lambda(R)$,*

$$cP\bigl(\gamma\bigl[0, \tau_R^\gamma\bigr] = \lambda\bigr) \leq P\bigl(\gamma'\bigl[0, \tau_R^{\gamma'}\bigr] = \lambda\bigr) \leq \frac{1}{c} P\bigl(\gamma\bigl[0, \tau_R^\gamma\bigr] = \lambda\bigr),$$

where $\gamma, \gamma', \tau_R^{\gamma}, \tau_R^{\gamma'}$ *and* $\Lambda(R)$ *are defined in* (2.8), (2.9) *and* (2.10), *respectively.*

2.2. *Escape probabilities*

As we discussed in Section 1.2, the probability that an LERW and an independent simple random walk do not intersect up to hitting the boundary of a large ball is a key ingredient in the present paper. Such a probability is called an escape probability. The escape probability was studied in order to estimate the length of LERW for $d = 2$ in [1,15] and for $d = 3$ in [18]. In this subsection, we will explain it. In this subsection we recall several results proved in [18]. Throughout this subsection, we will assume $d = 3$.

Definition 2.5. Let $m < n$. Suppose that S^1 and S^2 are independent simple random walks started at the origin on $\mathbb{Z}^3$. Define escape probabilities $\mathrm{Es}(n)$, $\mathrm{Es}^{\star}(n)$ and $\mathrm{Es}(m, n)$ as follows: Let

$$\mathrm{Es}(n) := P_1 \otimes P_2\big(S^1\big[1, \tau_n^1\big] \cap \mathrm{LE}\big(S^2\big[0, \tau_n^2\big]\big) = \varnothing\big), \tag{2.11}$$

i.e., $\mathrm{Es}(n)$ is the probability that a simple random walk up to exiting $B(n)$ does not intersect the loop erasure of an independent simple random walk up to exiting $B(n)$. Let

$$\mathrm{Es}^{\star}(n) := P_1 \otimes P_2\big(S^1\big[1, \tau_n^1\big] \cap \eta^1_{0,n}\big(\mathrm{LE}\big(S^2\big[0, \tau_{4n}^2\big]\big)\big) = \varnothing\big), \tag{2.12}$$

where $\eta^1_{z,n}(\lambda) = \lambda[0, u]$ with $u = \inf\{t \mid \lambda(t) \in \partial B(z, n)\}$. For $\mathrm{Es}^{\star}(n)$, we first consider the loop erasure of a random walk up to exiting $B(4n)$, then we only look at the loop erasure from the origin to the first visit to $\partial B(n)$. $\mathrm{Es}^{\star}(n)$ is the probability that this part of the loop erasure does not intersect an independent simple random walk up to exiting $B(n)$. Finally, let

$$\mathrm{Es}(m, n) := P_1 \otimes P_2\big(S^1\big[1, \tau_n^1\big] \cap \eta^2_{0,m,n}\big(\mathrm{LE}\big(S^2\big[0, \tau_n^2\big]\big)\big) = \varnothing\big), \tag{2.13}$$

where $\eta^2_{z,m,n}(\lambda) = \lambda[s, u]$ with $s = \sup\{t \le u \mid \lambda(t) \in \partial B(z, m)\}$ (u was defined as above). For $\mathrm{Es}(m, n)$, we first consider the loop erasure of a random walk up to exiting $B(n)$, then we only look at the loop erasure after the last visit to $B(m)$. $\mathrm{Es}(m, n)$ is the probability that this part of the loop erasure does not intersect an independent simple random walk up to exiting $B(n)$.

In the next proposition we collect various relations between the escape probabilities on various scales.

Proposition 2.6 (Propositions 6.2.1, 6.2.2, and 6.2.4 [18]). *Let $d = 3$. There exists a constant $C < \infty$ such that for all $l \le m \le n$,*

$$\begin{aligned} &\frac{1}{C}\,\mathrm{Es}^{\star}(n) \le \mathrm{Es}(n) \le C\,\mathrm{Es}^{\star}(n), \\ &\frac{1}{C}\,\mathrm{Es}\big(n'\big) \le \mathrm{Es}(n) \le C\,\mathrm{Es}\big(n'\big), \quad \text{for all } n \le n' \le 4n, \\ &\frac{1}{C}\,\mathrm{Es}(n) \le \mathrm{Es}(m)\,\mathrm{Es}(m, n) \le C\,\mathrm{Es}(n), \\ &\frac{1}{C}\,\mathrm{Es}(l, n) \le \mathrm{Es}(l, m)\,\mathrm{Es}(m, n) \le C\,\mathrm{Es}(l, n). \end{aligned} \tag{2.14}$$

The next theorem deals with the rate of growth for $\mathrm{Es}(n)$ and $\mathrm{Es}(m, n)$ in $d = 3$.

Theorem 2.7 (Theorem 7.2.1 and Lemma 7.2.2 [18]). *Let $d = 3$. There exists $\alpha \in [\frac{1}{3}, 1)$ such that*

$$\lim_{n \to \infty} \frac{\log \mathrm{Es}(n)}{\log n} = -\alpha. \tag{2.15}$$

Furthermore, for all $\kappa > 0$ *there exists* $c_\kappa > 0$ *and* $n_\kappa \in \mathbb{N}$ *such that*

$$c_\kappa \left(\frac{n}{m}\right)^{-\alpha-\kappa} \leq \mathrm{Es}(m, n) \leq \frac{1}{c_\kappa}\left(\frac{n}{m}\right)^{-\alpha+\kappa}, \tag{2.16}$$

for all $n_\kappa \leq m \leq n$.

The next lemma gives bounds of the ratio of escape probabilities, which will be used repeatedly in the paper.

Lemma 2.8 (Lemma 7.2.3 [18]). *Let* $d = 3$. *For all* $\kappa > 0$, *there exists* $C_\kappa < \infty$ *such that for all* $1 \leq m \leq n$,

$$m^{\alpha+\kappa}\,\mathrm{Es}(m) \leq C_\kappa n^{\alpha+\kappa}\,\mathrm{Es}(n). \tag{2.17}$$

Furthermore, for $l \leq m$, *by dividing both sides above by* $\mathrm{Es}(l)$ *and using Proposition* 2.6, *we see that for all* $1 \leq l \leq m \leq n$

$$m^{\alpha+\kappa}\,\mathrm{Es}(l, m) \leq C^2 C_\kappa n^{\alpha+\kappa}\,\mathrm{Es}(l, n), \tag{2.18}$$

where C *is a constant as in Proposition* 2.6.

Let $\tau_n = \inf\{t \mid S(t) \in \partial B(n)\}$ and let $M_n = \mathrm{len}\,\mathrm{LE}(S[0, \tau_n])$. The next theorem relates the length of LERW with the escape probability.

Theorem 2.9 (Theorem 8.1.4 and Proposition 8.1.5 [18]). *Let* $d = 3$. *There exists a constant* $C < \infty$ *such that for all* $n \geq 1$,

$$\frac{1}{C} n^2\,\mathrm{Es}(n) \leq E(M_n) \leq C n^2\,\mathrm{Es}(n). \tag{2.19}$$

In particular, we have

$$\lim_{n\to\infty} \frac{\log E(M_n)}{\log n} = 2 - \alpha. \tag{2.20}$$

In the rest of this subsection, we will give some extension of Theorem 6.1.5 [18] which is referred to as the "separation lemma". Let $R \geq 4$, $n \geq 1$ and $Rn \leq L \leq 4Rn$. We are interested in the following event.

$$F_{L,R,n} := \{\eta^2_{0,n,Rn}(\mathrm{LE}(S^1[0, \tau^1_L])) \cap S^2[0, \tau^2_{Rn}] = \varnothing\}, \tag{2.21}$$

where η^2 was defined right after (2.13) in Definition 2.5. Let

$$\begin{aligned} A^+_{R,n} &:= \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1 \geq \frac{2Rn}{3}\right\} \cup B\left(\frac{3Rn}{4}\right), \\ A^-_{R,n} &:= \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1 \leq -\frac{2Rn}{3}\right\} \cup B\left(\frac{3Rn}{4}\right). \end{aligned} \tag{2.22}$$

Define

$$\mathrm{Sep}_{L,R,n} := \{\eta^2_{0,n,Rn}(\mathrm{LE}(S^1[0, \tau^1_L])) \subset A^-_{R,n}, S^2[0, \tau^2_{Rn}] \subset A^+_{R,n}\}. \tag{2.23}$$

The next lemma shows that when a simple random walk does not intersect an independent LERW, they are "well-separated" with positive probability, i.e., the simple random walk lies in $A^+_{R,n}$ and the LERW lies in $A^-_{R,n}$ with positive conditional probability under the conditioning. The lemma will be used to compare escape probabilities on various scales by attaching paths to the separated paths (see Lemma 2.11, Proposition 3.10 and Lemma 4.3 for the applications of Lemma 2.10).

Lemma 2.10. *Let $d = 3$. There exists $c > 0$ such that for all $R \geq 4$, $n \geq 1$ and $Rn \leq L \leq 4Rn$, we have*

$$P_1 \otimes P_2(\mathrm{Sep}_{L,R,n} \mid F_{L,R,n}) \geq c. \tag{2.24}$$

Proof. Throughout the proof of this lemma, let $\gamma := \mathrm{LE}(S^1[0, \tau^1_L])$ and $\gamma' := \mathrm{LE}(S^1[0, \infty))$. Note that γ' is well-defined since S^1 is transient for $d = 3$. For $m < k$, let $\Lambda_{m,k}$ be the set of pairs of two paths (λ^1, λ^2) satisfying that

- λ^1 is a simple path started at the origin. λ^2 is a path started at the origin.
- $\lambda^i[0, \mathrm{len}\,\lambda^i] \subset B(k)$ and $\lambda^i(\mathrm{len}\,\lambda^i) \in \partial B(k)$ for each $i = 1, 2$.
- $\eta^2_{0,m,k}(\lambda^1) \cap \lambda^2 = \varnothing$.

We write $\tau^\gamma_m = \inf\{t \mid \gamma(t) \in \partial B(m)\}$ and define $\tau^{\gamma'}_m$ similarly. For $(\lambda^1, \lambda^2) \in \Lambda_{n,\frac{Rn}{8}}$, let

$$\begin{aligned} G' &:= \bigl\{\gamma'\bigl[\tau^{\gamma'}_{\frac{Rn}{8}}, \tau^{\gamma'}_{\frac{Rn}{4}}\bigr] \cap S^2\bigl[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\bigr] = \varnothing, \gamma'\bigl[\tau^{\gamma'}_{\frac{Rn}{8}}, \tau^{\gamma'}_{\frac{Rn}{4}}\bigr] \cap \lambda^2 = \varnothing, S^2\bigl[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\bigr] \cap \lambda^1 = \varnothing\bigr\}, \\ H' &:= \bigl\{\gamma'\bigl[\tau^{\gamma'}_{\frac{Rn}{8}}, \tau^{\gamma'}_{\frac{Rn}{4}}\bigr] \subset A^-_{\frac{R}{4},n}, S^2\bigl[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\bigr] \subset A^+_{\frac{R}{4},n}\bigr\}. \end{aligned}$$

(We define G and H by replacing γ' by γ above.) Then (6.15) of [18] shows that there exists an absolute constant $c > 0$ such that for all $(\lambda^1, \lambda^2) \in \Lambda_{n,\frac{Rn}{8}}$, we have

$$P_1 \otimes P_2\bigl(G', H' \mid \gamma'\bigl[0, \tau^{\gamma'}_{\frac{Rn}{8}}\bigr] = \lambda^1, S^2\bigl[0, \tau^2_{\frac{Rn}{8}}\bigr] = \lambda^2\bigr) \geq c P_1 \otimes P_2\bigl(G' \mid \gamma'\bigl[0, \tau^{\gamma'}_{\frac{Rn}{8}}\bigr] = \lambda^1, S^2\bigl[0, \tau^2_{\frac{Rn}{8}}\bigr] = \lambda^2\bigr). \tag{2.25}$$

Taking sum for $(\lambda^1, \lambda^2) \in \Lambda_{n,\frac{Rn}{8}}$, we have

$$\begin{aligned} &P_1 \otimes P_2\bigl(G'_1, H', \eta^2_{0,n,\frac{Rn}{8}}\bigl(\gamma'\bigl[0, \tau^{\gamma'}_{\frac{Rn}{8}}\bigr]\bigr) \cap S^2\bigl[0, \tau^2_{\frac{Rn}{8}}\bigr] = \varnothing\bigr) \\ &\quad \geq c P_1 \otimes P_2\bigl(G'_1, \eta^2_{0,n,\frac{Rn}{8}}\bigl(\gamma'\bigl[0, \tau^{\gamma'}_{\frac{Rn}{8}}\bigr]\bigr) \cap S^2\bigl[0, \tau^2_{\frac{Rn}{8}}\bigr] = \varnothing\bigr), \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} G'_1 := \bigl\{&\gamma'\bigl[\tau^{\gamma'}_{\frac{Rn}{8}}, \tau^{\gamma'}_{\frac{Rn}{4}}\bigr] \cap S^2\bigl[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\bigr] = \varnothing, \gamma'\bigl[\tau^{\gamma'}_{\frac{Rn}{8}}, \tau^{\gamma'}_{\frac{Rn}{4}}\bigr] \cap S^2\bigl[0, \tau^2_{\frac{Rn}{8}}\bigr] = \varnothing, \\ &S^2\bigl[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\bigr] \cap \gamma'\bigl[0, \tau^{\gamma'}_{\frac{Rn}{8}}\bigr] = \varnothing\bigr\}. \end{aligned}$$

(Again we define G_1 by replacing γ' by γ above.)

But by Proposition 2.4, the distribution of $\gamma'[0, \tau^{\gamma'}_{\frac{Rn}{4}}]$ is comparable to that of $\gamma[0, \tau^{\gamma}_{\frac{Rn}{4}}]$. Therefore,

$$\begin{aligned} &P_1 \otimes P_2\bigl(G_1, H, \eta^2_{0,n,\frac{Rn}{8}}\bigl(\gamma\bigl[0, \tau^{\gamma}_{\frac{Rn}{8}}\bigr]\bigr) \cap S^2\bigl[0, \tau^2_{\frac{Rn}{8}}\bigr] = \varnothing\bigr) \\ &\quad \geq c P_1 \otimes P_2\bigl(G_1, \eta^2_{0,n,\frac{Rn}{8}}\bigl(\gamma\bigl[0, \tau^{\gamma}_{\frac{Rn}{8}}\bigr]\bigr) \cap S^2\bigl[0, \tau^2_{\frac{Rn}{8}}\bigr] = \varnothing\bigr). \end{aligned} \tag{2.27}$$

Once $\gamma[0, \tau^{\gamma}_{\frac{Rn}{4}}]$ and $S^2[0, \tau^2_{\frac{Rn}{4}}]$ are separated as in H, by attaching paths from $\partial B(\frac{Rn}{4})$ to $\partial B(L)$, we see that

$$P_1 \otimes P_2(\mathrm{Sep}_{L,R,n}, F_{L,R,n}) \geq c P_1 \otimes P_2\bigl(G_1, H, \eta^2_{0,n,\frac{Rn}{8}}\bigl(\gamma\bigl[0, \tau^{\gamma}_{\frac{Rn}{8}}\bigr]\bigr) \cap S^2\bigl[0, \tau^2_{\frac{Rn}{8}}\bigr] = \varnothing\bigr), \tag{2.28}$$

for some $c > 0$.

Suppose that $\gamma[\tau^{\gamma}_{\frac{Rn}{8}}, \tau^{\gamma}_{Rn}] \cap \partial B(n) = \varnothing$. Then $\eta^2_{0,n,\frac{Rn}{8}}(\gamma[0, \tau^{\gamma}_{\frac{Rn}{8}}]) \cup \gamma[\tau^{\gamma}_{\frac{Rn}{8}}, \tau^{\gamma}_{\frac{Rn}{4}}] \subset \eta^2_{0,n,Rn}(\gamma)$. Therefore if we write $\sigma := \max\{t \leq \tau^{\gamma}_{Rn} \mid \gamma(t) \in B(n)\}$, then

$$\begin{aligned}
&P_1 \otimes P_2\big(G_1, \eta^2_{0,n,\frac{Rn}{8}}\big(\gamma\big[0, \tau^{\gamma}_{\frac{Rn}{8}}\big]\big) \cap S^2\big[0, \tau^2_{\frac{Rn}{8}}\big] = \varnothing\big) \\
&\quad \geq P_1 \otimes P_2\big(\gamma\big[\tau^{\gamma}_{\frac{Rn}{8}}, \tau^{\gamma}_{Rn}\big] \cap \partial B(n) = \varnothing, \eta^2_{0,n,Rn}(\gamma) \cap S^2\big[0, \tau^2_{Rn}\big] = \varnothing, \gamma[0,\sigma] \cap S^2\big[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\big] = \varnothing\big) \\
&\quad \geq P_1 \otimes P_2\big(\eta^2_{0,n,Rn}(\gamma) \cap S^2\big[0, \tau^2_{Rn}\big] = \varnothing, \gamma[0,\sigma] \cap S^2\big[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\big] = \varnothing\big) \\
&\qquad - P_1 \otimes P_2\big(\gamma\big[\tau^{\gamma}_{\frac{Rn}{8}}, \tau^{\gamma}_{Rn}\big] \cap \partial B(n) \neq \varnothing\big) \\
&\quad \geq P_1 \otimes P_2\big(\eta^2_{0,n,Rn}(\gamma) \cap S^2\big[0, \tau^2_{Rn}\big] = \varnothing, \gamma[0,\sigma] \cap S^2\big[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\big] = \varnothing\big) - \frac{C}{R},
\end{aligned}$$

for some $C < \infty$. Here we used Proposition 1.5.10 [8] in the last inequality. Let

$$q := \max\big\{k \mid \gamma[0,\sigma] \subset B\big(2^k n\big)\big\}.$$

Since $\gamma[0,\sigma] \subset B(Rn)$, we have $q \leq \log_2 R + 1$. Therefore,

$$\begin{aligned}
&P_1 \otimes P_2\big(\eta^2_{0,n,Rn}(\gamma) \cap S^2\big[0, \tau^2_{Rn}\big] = \varnothing, \gamma[0,\sigma] \cap S^2\big[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\big] = \varnothing\big) - \frac{C}{R} \\
&\quad \geq P_1 \otimes P_2(F_{L,R,n}) - P_1 \otimes P_2\big(\gamma[0,\sigma] \cap S^2\big[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\big] \neq \varnothing\big) - \frac{C}{R}.
\end{aligned} \tag{2.29}$$

But by Proposition 1.5.10 [8],

$$\begin{aligned}
&P_1 \otimes P_2\big(\gamma[0,\sigma] \cap S^2\big[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\big] \neq \varnothing\big) \\
&\quad \leq \sum_{k=1}^{\log_2 R+1} P_1 \otimes P_2\big(q = k, \gamma[0,\sigma] \cap S^2\big[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\big] \neq \varnothing\big) \\
&\quad \leq \sum_{k=1}^{\log_2 R+1} P_1 \otimes P_2\big(q = k, B\big(2^k n\big) \cap S^2\big[\tau^2_{\frac{Rn}{8}}, \tau^2_{\frac{Rn}{4}}\big] \neq \varnothing\big) \leq \sum_{k=1}^{\log_2 R+1} C 2^{-k} \frac{2^k}{R} \leq \frac{C \log R}{R}.
\end{aligned}$$

Combining this with (2.28) and (2.29), we have

$$P_1 \otimes P_2(\mathrm{Sep}_{L,R,n}, F_{L,R,n}) \geq c_1 P_1 \otimes P_2(F_{L,R,n}) - \frac{C_1 \log R}{R},$$

for some $c_1 > 0$, $C_1 < \infty$. However, by Corollary 4.2 [10], it follows that there exist $c_2 > 0$ and $\xi \in (\frac{1}{2}, 1)$ such that

$$P_1 \otimes P_2(F_{L,R,n}) \geq P_1 \otimes P_2\big(S^1\big[\tau^1_n, \tau^1_L\big] \cap S^2\big[0, \tau^2_{Rn}\big] = \varnothing\big) \geq c_2 R^{-\xi},$$

where ξ is referred to as the intersection exponent (see [10] for ξ). Since we know that $\xi < 1$ (see [10]), there exists $C < \infty$ such that $c_1 c_2 R^{-\xi} > \frac{2C_1 \log R}{R}$ for all $R \geq C$. Then for all $R \geq C$, we see that $P_1 \otimes P_2(\mathrm{Sep}_{L,R,n} \cap F_{L,R,n}) \geq \frac{c_1}{2} P_1 \otimes P_2(F_{L,R,n})$, which finishes the proof for $R \geq C$. It is easy to check that the lemma holds for $R \leq C$, so we finish the proof of lemma. $\square$

Once we show Lemma 2.10, using the same argument as in the proof of Proposition 6.2.1 [18], we get the following lemma immediately. We shall omit its proof and leave it to the reader.

Lemma 2.11. *Let $d = 3$. There exists $C < \infty$ such that for all for all $R \geq 4$, $n \geq 1$ and $Rn \leq L \leq 4Rn$, we have*

$$\frac{1}{C} P_1 \otimes P_2(F_{L,R,n}) \leq \mathrm{Es}(n, L) \leq C P_1 \otimes P_2(F_{L,R,n}), \tag{2.30}$$

where $F_{L,R,n}$ was defined as in (2.21).

2.3. *Scaling limit of LERW in three dimensions*

In this subsection, we will review some known facts about the scaling limit of LERW in three dimensions. As we explain in Section 1.1, the scaling limit of LERW for $d = 3$ exists [6], and some properties of it were studied in [16]. We will explain the details here.

Let $D = \{x \in \mathbb{R}^3 \mid |x| < 1\}$ and $\overline{D}$ be its closure. Let

$$\mathrm{LEW}_n = \frac{\mathrm{LE}(S[0, \tau_n])}{n}. \tag{2.31}$$

Here S is a simple random walk started at the origin on $\mathbb{Z}^3$ and $\tau_n = \inf\{t \mid S(t) \in \partial B(n)\}$.

We write $\mathcal{H}(\overline{D})$ for the metric space of the set of compact subsets in $\overline{D}$ with the Hausdorff distance d_{H}. Thinking of LEW_n as random elements of $\mathcal{H}(\overline{D})$, let $P^{(n)}$ be the probability measure on $\mathcal{H}(\overline{D})$ induced by LEW_n. Then [6] shows that $P^{(2^j)}$ is Cauchy with respect to the weak convergence topology, and therefore $P^{(2^j)}$ converges weakly. Let ν be its limit probability measure. We call ν the scaling limit measure of LERW in three dimensions. We write $\mathcal{K}$ for the random compact subset associated with ν. We call $\mathcal{K}$ the scaling limit of LERW in three dimensions. It is also shown in [6] that $\mathcal{K}$ is invariant under rotations and dilations.

Some properties of $\mathcal{K}$ were studied in [16]. In [16], it is shown that $\mathcal{K}$ is a simple path almost surely (Theorem 1.2 [16]). Furthermore, if we let Y be the union of $\mathcal{K}$ and loops from independent Brownian loop soup in D which intersect $\mathcal{K}$, more precisely,

$$Y := \mathcal{K} \cup \{\ell \in \mathrm{BS} \mid \ell \cap \mathcal{K} \neq \varnothing\}, \tag{2.32}$$

then Y has the same distribution in $\mathcal{H}(\overline{D})$ as the trace of three dimensional Brownian motion up to exiting from D (Theorem 1.1 [16]). Here BS is the Brownian loop soup in D which is independent of $\mathcal{K}$ (see [14] for the Brownian loop soup).

We denote the Hausdorff dimension by $\dim_{\mathrm{H}}(\cdot)$. Bounds of $\dim_{\mathrm{H}}(\mathcal{K})$ were given in Theorem 1.4 [16] as follows. Let ξ be the intersection exponent for three dimensional Brownian motion (see [9] for ξ). Let $\beta = 2 - \alpha$, where α is the exponent as in Theorem 2.7. Then Theorem 1.4 [16] shows that

$$2 - \xi \leq \dim_{\mathrm{H}}(\mathcal{K}) \leq \beta, \quad \text{almost surely.} \tag{2.33}$$

In particular, since $\xi \in (\frac{1}{2}, 1)$ (see [9]) and $\beta \in (1, \frac{5}{3}]$ (see [11]), we have

$$1 < \dim_{\mathrm{H}}(\mathcal{K}) \leq \frac{5}{3}, \quad \text{almost surely.} \tag{2.34}$$

The main purpose of the present paper is to show that

$$\dim_{\mathrm{H}}(\mathcal{K}) \geq \beta, \quad \text{almost surely,} \tag{2.35}$$

which concludes that $\dim_{\mathrm{H}}(\mathcal{K}) = \beta$ almost surely.

3. The number of small boxes hit by $\mathcal{K}$

From here to the end of the present paper, we will assume $d = 3$. In this section, we will give bounds of the number of small boxes hit by $\mathcal{K}$. To do it, we will first estimate the probability that $\mathcal{K}$ hits two distinct small boxes (see Theorem 3.1), which is one of the key result in the paper. We will show Theorem 3.1 in Section 3.1. Then using the second moment method, we will give some bounds of the number of boxes hit by $\mathcal{K}$ in Section 3.2.

3.1. *Probability of $\mathcal{K}$ hitting two small boxes*

Recall that $D = \{x \in \mathbb{R}^3 \mid |x| < 1\}$ and $\overline{D}$ is its closure. For $r > 0$, we write $D_r = \{x \in \mathbb{R}^3 \mid |x| < r\}$ and let $\overline{D_r}$ be its closure. For $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$, let

$$B_x = \prod_{i=1}^{3} [x_i, x_i + 1]. \tag{3.1}$$

In this subsection, we will establish an upper bound of the probability that $\mathcal{K}$ hits both ϵB_x and ϵB_y with $\frac{1}{3} \le |\epsilon x|, |\epsilon y| \le \frac{2}{3}$ and $x, y \in \mathbb{Z}^3$ (see Theorem 3.1). The upper bound will be given in terms of escape probabilities defined in Section 2.2. In the proof of Theorem 3.1, we will repeatedly use several properties of escape probabilities explained in Section 2.2 as well as Proposition 4.2, 4.4 and 4.6 in [15].

Let $\mathrm{LEW}_n = \frac{\mathrm{LE}(S[0,\tau_n])}{n}$. Here S is a simple random walk started at the origin on $\mathbb{Z}^3$ and $\tau_n = \inf\{t \mid S(t) \in \partial B(n)\}$. Since LEW_{2^j} converges weakly to $\mathcal{K}$ (see Section 2.3), we can define $\{\mathrm{LEW}_{2^j}\}_{j \ge 1}$ and $\mathcal{K}$ on the same probability space $(\Omega, \mathcal{F}, P)$ such that

$$\lim_{j \to \infty} d_{\mathrm{H}}(\mathrm{LEW}_{2^j}, \mathcal{K}) = 0, \quad P\text{-almost surely}, \tag{3.2}$$

where d_{H} is the Hausdorff metric on $\mathcal{H}(\overline{D})$ (see Section 2.3 for $\mathcal{H}(\overline{D})$).

Take $\epsilon > 0$. By (3.2), for P-a.s., ω, there exists $N_\epsilon(\omega) < \infty$ such that

$$d_{\mathrm{H}}(\mathrm{LEW}_{2^j}, \mathcal{K}) < \epsilon^2, \quad \text{for all } j \ge N_\epsilon.$$

Since $P(N_\epsilon < \infty) = 1$, there exists j_ϵ such that

$$P(N_\epsilon < j_\epsilon) \ge 1 - \epsilon^{100}. \tag{3.3}$$

On the event $\{N_\epsilon < j_\epsilon\}$, if we write $n_\epsilon := 2^{j_\epsilon}$, then

$$d_{\mathrm{H}}(\mathrm{LEW}_{n_\epsilon}, \mathcal{K}) < \epsilon^2. \tag{3.4}$$

From now on, we fix $n = n_\epsilon = 2^{j_\epsilon}$ for each $\epsilon > 0$ such that (3.3) holds.

One of the key results in this paper is the following theorem.

Theorem 3.1. *Fix $\epsilon > 0$ and take $n = n_\epsilon = 2^{j_\epsilon}$ such that (3.3) holds. Suppose that $x \neq y \in \mathbb{Z}^3$ satisfy*

$$\epsilon B_x \subset D_{\frac{2}{3}} \setminus D_{\frac{1}{3}} \quad \text{and} \quad \epsilon B_y \subset D_{\frac{2}{3}} \setminus D_{\frac{1}{3}}. \tag{3.5}$$

Let $l := |x - y|$. Then there exists an absolute constant $C < \infty$ such that

$$P(\mathcal{K} \cap \epsilon B_x \neq \varnothing \text{ and } \mathcal{K} \cap \epsilon B_y \neq \varnothing) \le C\, \mathrm{Es}(\epsilon n, l\epsilon n)\, \mathrm{Es}(\epsilon n, n) \frac{\epsilon}{l}. \tag{3.6}$$

Remark 3.2. Since the proof of Theorem 3.1 is quite long, we explain some of its ideas here. Take $n = n_\epsilon = 2^{j_\epsilon}$ such that (3.3) holds. Since $\epsilon^{100} \ll \mathrm{Es}(\epsilon n, l\epsilon n)\, \mathrm{Es}(\epsilon n, n)\frac{\epsilon}{l}$, we may suppose that $d_{\mathrm{H}}(\mathrm{LEW}_n, \mathcal{K}) < \epsilon^2$. In that case if $\mathcal{K}$ hits both ϵB_x and ϵB_y, then $\gamma := \mathrm{LE}(S[0, \tau_n])$ hits both $\epsilon n B'_x$ and $\epsilon n B'_y$, where $B'_z = \prod_{i=1}^{3} [z_i - 2, z_i + 2]$. So we need to estimate

$$P\big(\gamma \cap \epsilon n B'_x \neq \varnothing, \gamma \cap \epsilon n B'_y \neq \varnothing\big) \le P\big(\tau^{\gamma,x} < \tau^{\gamma,y} < \infty\big) + P\big(\tau^{\gamma,y} < \tau^{\gamma,x} < \infty\big). \tag{3.7}$$

Here $\tau^{\gamma,z} := \inf\{t \mid \gamma(t) \in \epsilon n B'_z\}$. We want to show that

$$P\big(\tau^{\gamma,x} < \tau^{\gamma,y} < \infty\big) \le C P\big(\tau^{\gamma,x} < \infty\big) P^{\epsilon n x}\big(\tau^{\gamma,y} < \infty\big). \tag{3.8}$$

Note that if γ were $S[0,\tau_n]$, (3.8) would hold because of the strong Markov property. However, since $\gamma = \mathrm{LE}(S[0,\tau_n])$ is not a Markov process, the distribution of $\gamma[\tau^{\gamma,x},\tau^{\gamma,y}]$ strongly depends on the shape of $\gamma[0,\tau^{\gamma,x}]$. We need to control such dependence and this will be done in Lemma 3.6, Lemma 3.7 and 3.8. Then we will prove (3.8). Once (3.8) is proved then Theorem 3.1 immediately follows because

$$P\big(\tau^{\gamma,x}<\infty\big)P^{\epsilon nx}\big(\tau^{\gamma,y}<\infty\big)\le C\,\mathrm{Es}(\epsilon n,l\epsilon n)\,\mathrm{Es}(\epsilon n,n)\frac{\epsilon}{l}, \tag{3.9}$$

and the second probability in RHS of (3.7) can be estimated similarly.

We will split the proof of Theorem 3.1 as follows. Since we want to estimate the probability in LHS of (3.8) in terms of escape probabilities, we first rewrite the probability in terms of independent random walks by reversing paths in Lemma 3.3. Such independent random walks consist of three walks $S^1,\dots,S^3$ with $S^1(0)=S^2(0)\in\partial(\epsilon nB'_x)$ and $S^3(0)\in\partial(\epsilon nB'_y)$ (see Figure 1). In order for $\gamma=\mathrm{LE}(S[0,\tau_n])$ to hit $\epsilon nB'_x$, the loop erasure of the time reverse of S^1, say $(S^1)^R$, does not intersect a composition of two walks S^2 and S^3. In addition, in order for γ to hit $\epsilon nB'_y$, the loop erasure of a composition of two walks $(S^1)^R$ and S^2 does not intersect S^3. To control the independence, we will replace the latter event by events that the loop erasure of $(S^2)^R$ up to some stopping time does not intersect S^3 in Lemma 3.6 (S^4 corresponds to $(S^2)^R$ up to that stopping time in Lemma 3.6). The distribution of the loop erasure of $(S^2)^R$ up to the stopping time will be studied in Lemma 3.7, which allows us to think that the latter event is independent from the former one, and to estimate the probability of the latter event in terms of escape probabilities. Finally in Lemma 3.8 we will estimate the probability of the former event using escape probabilities, and then prove Theorem 3.1.

Proof. It suffices to show (3.6) for $l\ge 10^6$. Indeed, $\mathrm{Es}(\epsilon n,l\epsilon n)\ge c$ for $l\le 10^6$ and we already showed that

$$P(\mathcal{K}\cap\epsilon B_x\ne\varnothing)\le C\,\mathrm{Es}(\epsilon n,n)\epsilon.$$

(See the proof of Lemma 7.1 [16] for this inequality.) Therefore,

$$P(\mathcal{K}\cap\epsilon B_x\ne\varnothing\text{ and }\mathcal{K}\cap\epsilon B_y\ne\varnothing)\le C\,\mathrm{Es}(\epsilon n,n)\epsilon\le C\,\mathrm{Es}(\epsilon n,l\epsilon n)\,\mathrm{Es}(\epsilon n,n)\frac{\epsilon}{l},$$

for $l\le 10^6$.

Thus we may assume that $10^6\le l\le\frac{2}{\epsilon}$. Note that by (2.16),

$$\epsilon^{100}\le C\,\mathrm{Es}(\epsilon n,l\epsilon n)\,\mathrm{Es}(\epsilon n,n)\frac{\epsilon}{l}.$$

So by (3.3),

$$\begin{aligned}P\big(\mathcal{K}\cap\epsilon B_x\ne\varnothing,\mathcal{K}\cap\epsilon B_y\ne\varnothing,d_{\mathrm{H}}(\mathrm{LEW}_n,\mathcal{K})\ge\epsilon^2\big)&\le P(\mathcal{K}\cap\epsilon B_x\ne\varnothing,\mathcal{K}\cap\epsilon B_y\ne\varnothing,N_\epsilon\ge j_\epsilon)\\&\le\epsilon^{100}\le C\,\mathrm{Es}(\epsilon n,l\epsilon n)\,\mathrm{Es}(\epsilon n,n)\frac{\epsilon}{l}.\end{aligned}$$

Therefore it suffices to show that

$$P\big(\mathcal{K}\cap\epsilon B_x\ne\varnothing,\mathcal{K}\cap\epsilon B_y\ne\varnothing,d_{\mathrm{H}}(\mathrm{LEW}_n,\mathcal{K})<\epsilon^2\big)\le C\,\mathrm{Es}(\epsilon n,l\epsilon n)\,\mathrm{Es}(\epsilon n,n)\frac{\epsilon}{l}. \tag{3.10}$$

Suppose that $\mathcal{K}\cap\epsilon B_x\ne\varnothing,\mathcal{K}\cap\epsilon B_y\ne\varnothing,d_{\mathrm{H}}(\mathrm{LEW}_n,\mathcal{K})<\epsilon^2$. Let $B'_x=\prod_{i=1}^3[x_i-2,x_i+2]$ and $B'_y=\prod_{i=1}^3[y_i-2,y_i+2]$. Then

$$\mathrm{LEW}_n\cap\epsilon B'_x\ne\varnothing,\qquad\mathrm{LEW}_n\cap\epsilon B'_y\ne\varnothing. \tag{3.11}$$

So we have to estimate

$$P\big(\mathrm{LEW}_n\cap\epsilon B'_x\ne\varnothing,\mathrm{LEW}_n\cap\epsilon B'_y\ne\varnothing\big)=P\big(\mathrm{LE}\big(S[0,\tau_n]\big)\cap\epsilon nB'_x\ne\varnothing,\mathrm{LE}\big(S[0,\tau_n]\big)\cap\epsilon nB'_y\ne\varnothing\big).$$

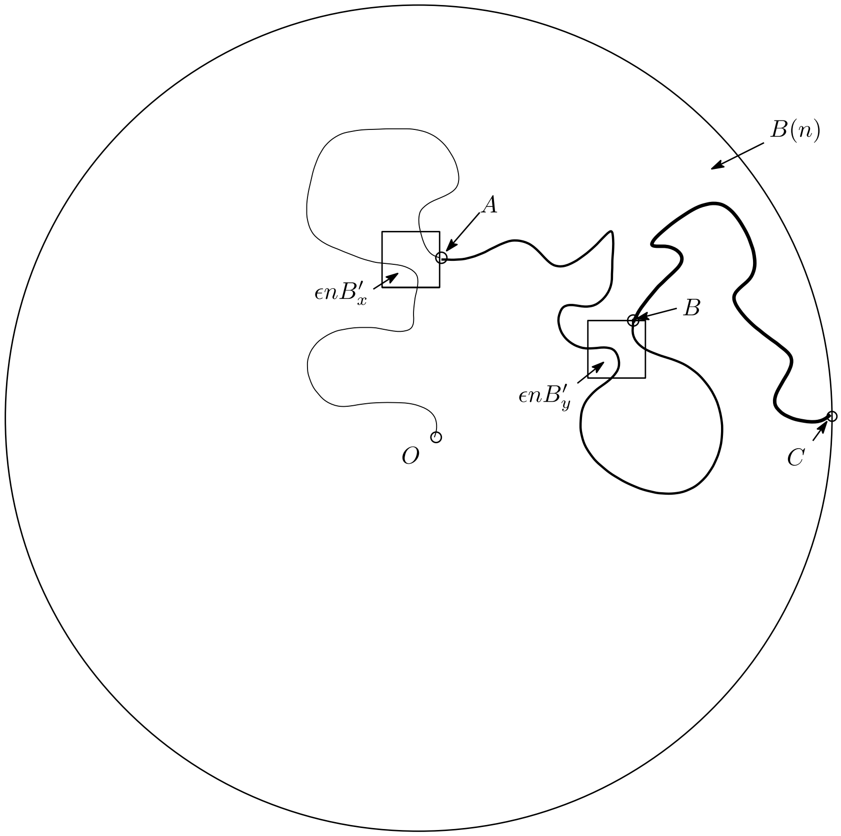

Fig. 1. A simple path from O to C stands for $\gamma = \mathrm{LE}(S[0,\tau_n])$. Points A and B stand for last visits of γ to $\epsilon n B'_x$ and $\epsilon n B'_y$. Then $A = S^1(0) = S^2(0)$ and $B = S^3(0)$. We let S^1 run until it hits O, let S^2 run until it hits B, and let S^3 run until it hits C. The simple path from O to A corresponds to the loop erasure of the time reverse of S^1, say $(S^1)^R$. The simple path from O to B corresponds to the loop erasure of $(S^1)^R + S^2$. Finally, γ corresponds to the loop erasure of $(S^1)^R + S^2 + S^3$.

Suppose that $\mathrm{LE}(S[0,\tau_n]) \cap \epsilon n B'_x \neq \varnothing, \mathrm{LE}(S[0,\tau_n]) \cap \epsilon n B'_y \neq \varnothing$. Then clearly $S[0,\tau_n] \cap \epsilon n B'_x \neq \varnothing, S[0,\tau_n] \cap \epsilon n B'_y \neq \varnothing$. So we may define

$$T^x = \max\bigl\{t \le \tau_n \mid S(t) \in \partial\bigl(\epsilon n B'_x\bigr)\bigr\}, \qquad T^y = \max\bigl\{t \le \tau_n \mid S(t) \in \partial\bigl(\epsilon n B'_y\bigr)\bigr\}.$$

Then

$$\begin{aligned} &P\bigl(\mathrm{LE}\bigl(S[0,\tau_n]\bigr) \cap \epsilon n B'_x \neq \varnothing, \mathrm{LE}\bigl(S[0,\tau_n]\bigr) \cap \epsilon n B'_y \neq \varnothing\bigr) \\ &\quad \le P\bigl(\mathrm{LE}\bigl(S[0,\tau_n]\bigr) \cap \epsilon n B'_x \neq \varnothing, \mathrm{LE}\bigl(S[0,\tau_n]\bigr) \cap \epsilon n B'_y \neq \varnothing, T^x < T^y\bigr) \\ &\qquad + P\bigl(\mathrm{LE}\bigl(S[0,\tau_n]\bigr) \cap \epsilon n B'_x \neq \varnothing, \mathrm{LE}\bigl(S[0,\tau_n]\bigr) \cap \epsilon n B'_y \neq \varnothing, T^x > T^y\bigr). \end{aligned} \tag{3.12}$$

We will deal with only the first probability in the right hand side of (3.12). The second probability can be estimated similarly.

Define

$$\sigma^i_z = \max\bigl\{t \le \tau^1_n \mid S^i(t) = z\bigr\}, \tag{3.13}$$

and

$$\begin{aligned} &\overline{\sigma}_1 := \inf\bigl\{t \mid \mathrm{LE}\bigl(S^1\bigl[0,\sigma^1_z\bigr]\bigr)(t) \in \partial\bigl(\epsilon n B'_x\bigr)\bigr\}, \\ &\overline{\sigma}_2 := \inf\bigl\{t \mid \mathrm{LE}\bigl(S^1\bigl[0,\sigma^1_z\bigr] + S^2\bigl[0,\sigma^2_w\bigr]\bigr)(t) \in \partial\bigl(\epsilon n B'_y\bigr)\bigr\}. \end{aligned} \tag{3.14}$$

The estimate of the first probability in the right hand side of (3.12) will be carried out below, but it is quite long. So we will split it into shorter claims (Lemma 3.3, 3.6, 3.7, and 3.8).

In order to estimate the first probability in the right hand side of (3.12) in terms of the escape probabilities, we need to decompose the simple random walk path into three parts; S from the origin to the ϵn cube around x, S from the ϵn cube around x to the cube around y, and S from the cube around y to the boundary of $B(n)$. By using a standard technique called "last exit decomposition" (see Proposition 2.4.1 [8] for details), Lemma 3.3 below deals with this decomposition. In the Lemma 3.3, these three parts in the decomposition correspond to S^1, S^2, and S^3 respectively.

Lemma 3.3. *There exists a $C < \infty$ such that*

$$\begin{aligned}
&P\big(\mathrm{LE}\big(S[0,\tau_n]\big)\cap \epsilon n B'_x \neq \varnothing, \mathrm{LE}\big(S[0,\tau_n]\big)\cap \epsilon n B'_y \neq \varnothing, T^x < T^y\big)\\
&\quad\le C \sum_{z\in\partial(\epsilon n B'_x)} \sum_{w\in\partial(\epsilon n B'_y)} P^0_1 \otimes P^z_2 \otimes P^w_3 \big(\sigma^1_z < \tau^1_n, \sigma^2_w < \tau^2_n,\\
&\qquad S^2\big[1,\sigma^2_w\big]\cap \overline{\big(\epsilon n B'_x\big)} = \varnothing, S^3\big[1,\tau^3_n\big]\cap \overline{\big(\epsilon n B'_y\big)} = \varnothing\\
&\qquad \mathrm{LE}\big(S^1\big[0,\sigma^1_z\big]\big)[0,\overline{\sigma}_1]\cap\big(S^2\big[0,\sigma^2_w\big]\cup S^3\big[0,\tau^3_n\big]\big) = \varnothing,\\
&\qquad \mathrm{LE}\big(S^1\big[0,\sigma^1_z\big]+S^2\big[0,\sigma^2_w\big]\big)[0,\overline{\sigma}_2]\cap S^3\big[0,T^3_{w,\frac{\epsilon l n}{4}}\big] = \varnothing\big).
\end{aligned} \tag{3.15}$$

Proof. Suppose that $\mathrm{LE}(S[0,\tau_n])\cap \epsilon n B'_x \neq \varnothing$. Let

$$\sigma'_1 = \inf\big\{t \mid \mathrm{LE}\big(S[0,\tau_n]\big)(t)\in\partial\big(\epsilon n B'_x\big)\big\}, \qquad \overline{\sigma}'_1 = \inf\big\{t \mid \mathrm{LE}\big(S\big[0,T^x\big]\big)(t)\in\partial\big(\epsilon n B'_x\big)\big\}. \tag{3.16}$$

Note that

$$\mathrm{LE}\big(S\big[0,T^x\big]\big)\big[0,\overline{\sigma}'_1\big]\cap S\big[T^x,\tau_n\big] = \varnothing. \tag{3.17}$$

To see this, we let $\mathrm{LE}^{(1)} = \mathrm{LE}_1(\lambda_1,\lambda_2)$ and $\mathrm{LE}^{(2)} = \mathrm{LE}_2(\lambda_1,\lambda_2)$ where $\lambda_1 = \mathrm{LE}(S[0,T^x])$ and $\lambda_2 = S[T^x,\tau_n]$ (see (2.5) for $\mathrm{LE}^{(i)}$). Then $\mathrm{LE}(S[0,\tau_n]) = \mathrm{LE}^{(1)} + \mathrm{LE}^{(2)}$. Let $u = \inf\{t \mid \lambda_1(t)\in\lambda_2\}$ and $s = \sup\{\lambda_2(t) = \lambda_1(u)\}$. Then $\mathrm{LE}^{(1)} = \lambda_1[0,u]$ and $\mathrm{LE}^{(2)} = \mathrm{LE}(\lambda_2[s,\mathrm{len}\,\lambda_2])$. If $\mathrm{LE}(S[0,T^x])[0,\overline{\sigma}'_1]\cap S[T^x,\tau_n]\neq\varnothing$, then $u \le \overline{\sigma}'_1$. By definition of $\overline{\sigma}'_1$, this implies that $\mathrm{LE}(S[0,T^x])[0,u]\cap \epsilon n B'_x = \varnothing$. Moreover, since $S[T^x,\tau_n]\cap \epsilon n B'_x = \varnothing$, we see that $\mathrm{LE}^{(2)}\cap\epsilon n B'_x = \varnothing$. This implies that $\mathrm{LE}(S[0,\tau_n])\cap \epsilon n B'_x = \varnothing$ and we get a contradiction. Therefore (3.17) holds and $\overline{\sigma}'_1 < u$. Thus $\mathrm{LE}(S[0,\tau_n])[0,\overline{\sigma}'_1] = \mathrm{LE}(S[0,T^x])[0,\overline{\sigma}'_1]$ and $\sigma'_1 = \overline{\sigma}'_1$.

Thus,

$$\begin{aligned}
&P\big(\mathrm{LE}\big(S[0,\tau_n]\big)\cap \epsilon n B'_x \neq \varnothing, \mathrm{LE}\big(S[0,\tau_n]\big)\cap \epsilon n B'_y \neq \varnothing, T^x < T^y\big)\\
&\quad\le P\big(\mathrm{LE}\big(S\big[0,T^x\big]\big)\big[0,\overline{\sigma}'_1\big]\cap S\big[T^x,\tau_n\big] = \varnothing,\\
&\qquad \mathrm{LE}\big(S\big[0,T^y\big]\big)\big[0,\overline{\sigma}'_2\big]\cap S\big[T^y,\tau_n\big] = \varnothing, T^x < T^y < \tau_n\big),
\end{aligned} \tag{3.18}$$

where $\overline{\sigma}'_2 = \inf\{t \mid \mathrm{LE}(S[0,T^y])(t)\in\partial(\epsilon n B'_y)\}$.

Next we will decompose $S[0,\tau_n]$ into three parts, $S[0,T^x]$, $S[T^x,T^y]$ and $S[T^y,\tau_n]$. Note that by the Markov property at time k_1 and k_1+k_2, we have

$$\begin{aligned}
&P\big(\mathrm{LE}\big(S\big[0,T^x\big]\big)\big[0,\overline{\sigma}'_1\big]\cap S\big[T^x,\tau_n\big] = \varnothing, \mathrm{LE}\big(S\big[0,T^y\big]\big)\big[0,\overline{\sigma}'_2\big]\cap S\big[T^y,\tau_n\big] = \varnothing, T^x < T^y < \tau_n\big)\\
&\quad= \sum_{k_1>0}\sum_{k_2>0}\sum_{z\in\partial(\epsilon n B'_x)}\sum_{w\in\partial(\epsilon n B'_y)} P\big(T^x = k_1, S(k_1) = z, T^y = k_1+k_2, S(k_1+k_2) = w, k_1+k_2 < \tau_n\\
&\qquad \mathrm{LE}\big(S\big[0,T^x\big]\big)\big[0,\overline{\sigma}'_1\big]\cap S\big[T^x,\tau_n\big] = \varnothing, \mathrm{LE}\big(S\big[0,T^y\big]\big)\big[0,\overline{\sigma}'_2\big]\cap S\big[T^y,\tau_n\big] = \varnothing\big)\\
&\quad= \sum_{k_1>0}\sum_{k_2>0}\sum_{z\in\partial(\epsilon n B'_x)}\sum_{w\in\partial(\epsilon n B'_y)} P^0_1\otimes P^z_2\big(S^1(k_1) = z, k_1 < \tau^1_n, S^2(k_2) = w, k_2 < \tau^2_n
\end{aligned}$$

$$S^2[1,\tau_n^2]\cap\overline{(\epsilon n B'_x)}=\varnothing,\ S^2[k_2+1,\tau_n^2]\cap\overline{(\epsilon n B'_y)}=\varnothing$$
$$\mathrm{LE}(S^1[0,k_1])[0,\overline{\sigma}_1]\cap S^2[0,\tau_n^2]=\varnothing,\ \mathrm{LE}(S^1[0,k_1]+S^2[0,k_2])[0,\overline{\sigma}_2]\cap S^2[k_2,\tau_n^2]=\varnothing\big)$$
$$=\sum_{z\in\partial(\epsilon n B'_x)}\sum_{w\in\partial(\epsilon n B'_y)}\frac{1}{p_z}\frac{1}{p_w}P_1^0\otimes P_2^z\otimes P_3^w\big(\sigma_z^1<\tau_n^1,\sigma_w^2<\tau_n^2,$$
$$S^2[1,\sigma_w^2]\cap\overline{(\epsilon n B'_x)}=\varnothing,\ S^3[1,\tau_n^3]\cap\big(\overline{(\epsilon n B'_x)}\cup\overline{(\epsilon n B'_y)}\big)=\varnothing$$
$$\mathrm{LE}(S^1[0,\sigma_z^1])[0,\overline{\sigma}_1]\cap\big(S^2[0,\sigma_w^2]\cup S^3[0,\tau_n^3]\big)=\varnothing,$$
$$\mathrm{LE}(S^1[0,\sigma_z^1]+S^2[0,\sigma_w^2])[0,\overline{\sigma}_2]\cap S^3[0,\tau_n^3]=\varnothing\big)$$
$$\big(\text{where } p_z=P^z\big(z\notin S[1,\tau_n]\big)\big) \tag{3.19}$$

which finishes the proof of Lemma 3.3. □

Remark 3.4. There are six events in the probability in the right hand side of (3.15). We want to say they are "independent up to constant". Namely, we will show that the probability in RHS of (3.15) is comparable to the product of six probabilities coming from each of six events. Then we need to estimate each of those probabilities. The first four events are easy to estimate. The fifth event corresponds to the probability that the loop erasure of a random walk from the ϵn cube around x to the origin does not intersect a random walk from the cube around x to the boundary of $B(n)$. This probability is comparable to $\mathrm{Es}(\epsilon n, n)$. Similarly we will see that the sixth event corresponds to $\mathrm{Es}(\epsilon n, \epsilon l n)$.

With the strategy in Remark 3.4 in mind, we introduce some notation before going to the next lemma.

We write

$$F^1:=\big\{\sigma_z^1<\tau_n^1,\sigma_w^2<\tau_n^2,S^2[1,\sigma_w^2]\cap\overline{(\epsilon n B'_x)}=\varnothing,\ S^3[1,\tau_n^3]\cap\overline{(\epsilon n B'_y)}=\varnothing$$
$$\mathrm{LE}(S^1[0,\sigma_z^1])[0,\overline{\sigma}_1]\cap\big(S^2[0,\sigma_w^2]\cup S^3[0,\tau_n^3]\big)=\varnothing,$$
$$\mathrm{LE}(S^1[0,\sigma_z^1]+S^2[0,\sigma_w^2])[0,\overline{\sigma}_2]\cap S^3[0,\tau_n^3]=\varnothing\big\}. \tag{3.20}$$

By Lemma 3.3, we have to estimate $P_1^0\otimes P_2^z\otimes P_3^w(F^1)$. To do so, define $A_z^r=B(z,2^r\epsilon n)\setminus B(z,2^{r-1}\epsilon n)$ for $r\geq 1$ and $A_r^0=B(z,\epsilon n)$. Let $u^{(1)}=\mathrm{len}\,\mathrm{LE}(S^1[0,\sigma_z^1])$ and

$$q^{(1)}=\max\big\{r\geq 0\mid \mathrm{LE}(S^1[0,\sigma_z^1])[\overline{\sigma}_1,u^{(1)}]\cap A_z^r\neq\varnothing\big\}. \tag{3.21}$$

($q^{(1)}$ is well-defined because $\mathrm{LE}(S^1[0,\sigma_z^1])[\overline{\sigma}_1,u^{(1)}]\cap A_r^0\neq\varnothing$.) We will first deal with the case of $q^{(1)}\leq\log_2 l-3$ so that $2^{q^{(1)}}\epsilon n\leq 2^{-3}l\epsilon n$. So suppose that $q^{(1)}=r\leq\log_2 l-3$. Let

$$T^2_{z,\frac{l\epsilon n}{2}}=\inf\bigg\{t\ \bigg|\ S^2(t)\in\partial B\bigg(z,\frac{l\epsilon n}{2}\bigg)\bigg\}. \tag{3.22}$$

Then by the strong Markov property for S^2 at $T^2_{z,\frac{l\epsilon n}{2}}$,

$$P_1^0\otimes P_2^z\otimes P_3^w\big(F^1,q^{(1)}=r\big)$$
$$=\sum_{z'\in\partial B(z,\frac{l\epsilon n}{2})}P_1^0\otimes P_2^z\otimes P_3^w\big(F^1,q^{(1)}=r,S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big)=z'\big)$$
$$=\sum_{z'\in\partial B(z,\frac{l\epsilon n}{2})}P_1^0\otimes P_2^z\otimes P_4^{z'}\otimes P_3^w\big(\sigma_z^1<\tau_n^1,\sigma_w^4<\tau_n^4,$$
$$\big(S^2\big[1,T^2_{z,\frac{l\epsilon n}{2}}\big]\cup S^4[0,\sigma_w^4]\big)\cap\overline{(\epsilon n B'_x)}=\varnothing,\ S^3[1,\tau_n^3]\cap\overline{(\epsilon n B'_y)}=\varnothing$$

$$\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap\big(S^2\big[0,T^2_{z,\frac{l\epsilon n}{2}}\big]\cup S^4\big[0,\sigma_w^4\big]\cup S^3\big[0,\tau_n^3\big]\big)=\varnothing$$

$$\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]+S^2\big[0,T^2_{z,\frac{l\epsilon n}{2}}\big]+S^4\big[0,\sigma_w^4\big]\big)[0,\overline{\sigma}_2]\cap S^3\big[0,\tau_n^3\big]=\varnothing, q^{(1)}=r, S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big)=z'\big). \tag{3.23}$$

We define an event F^2 by

$$F^2=\big\{\sigma_z^1<\tau_n^1, S^2\big[1,T^2_{z,\frac{l\epsilon n}{2}}\big]\cap\overline{\big(\epsilon n B_x'\big)}=\varnothing,$$
$$\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap S^2\big[0,T^2_{z,\frac{l\epsilon n}{2}}\big]=\varnothing, q^{(1)}=r, S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big)=z'\big\}. \tag{3.24}$$

Define a sequence of stopping times T_i by $T_0=0$ and

$$\begin{aligned}T_{2i+1}&=\inf\left\{t\geq T_{2i}\;\middle|\;S^4(t)\in\partial B\left(w,\frac{\epsilon l n}{4}\right)\right\},\\ T_{2i}&=\inf\left\{t\geq T_{2i-1}\;\middle|\;S^4(t)\in\partial B\left(w,\frac{\epsilon l n}{800}\right)\right\}.\end{aligned} \tag{3.25}$$

Let

$$\begin{aligned}u_4'=u_{4,i}'&:=\inf\left\{t\;\middle|\;\mathrm{LE}\big(S^4[0,T_{2i+1}]\big)(t)\in\partial B\left(w,\frac{\epsilon l n}{1600}\right)\right\},\\ \overline{\sigma}_4^\star=\overline{\sigma}_{4,i}^\star&:=\max\big\{t\leq u_4'\mid \mathrm{LE}\big(S^4[0,T_{2i+1}]\big)(t)\in\partial B(w,8\epsilon n)\big\}.\end{aligned} \tag{3.26}$$

Remark 3.5. Recall that z and w are points in the ϵn neighborhood of x and y. By reversing paths of S^1 and S^2 in the probability in RHS of (3.23), S^1 is a random walk from z to the origin, S^2 is a random walk from z to z', S^4 is a random walk from w to z', and S^3 is a random walk from w to $\partial B(n)$. We want to deal with eight events in the probability of (3.23) as if they were independent. Some technical issues arise when we deal with the fifth, sixth, and seventh events. We will first deal with the sixth event in the next lemma below, by using entrance and exit times defined as in (3.25).

We have to estimate the probability in RHS of (3.23). With the strategy in Remark 3.5 in mind, we first deal with the sixth event of the probability in (3.23). The sixth event is written in terms of the loop-erasure of three walks S^1, S^2 and S^4. We want to replace it by the loop-erasure of S^4 only. In the next lemma, we will do the replacement by using entrance and exit times defined in (3.25).

Lemma 3.6. *Suppose that $r\leq\log_2 l-3$. Then there exists $C<\infty$ such that*

$$\begin{aligned}&P_1^0\otimes P_2^z\otimes P_3^w\big(F^1,q^{(1)}=r,S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big)=z'\big)\\ &\quad\leq CE_1^0\otimes E_2^z\Bigg\{\mathbf{1}_{F^2}\\ &\qquad\times\left(\frac{C}{\epsilon n}\frac{1}{\epsilon l n}\sum_{i=0}^{\infty}P_4^w\otimes P_3^w\big(T_{2i+1}<\tau_n^4,\mathrm{LE}\big(S^4[0,T_{2i+1}]\big)\big[\overline{\sigma}_4^\star,u_4'\big]\cap S^3\big[0,T^3_{w,\frac{l\epsilon n}{4}}\big]=\varnothing\big)+\frac{C}{(\epsilon l n)^2}\right)\\ &\qquad\times\max_{w_1\in\partial B(w,\frac{\epsilon l n}{4})}P_3^{w_1}\big(\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap S^3\big[0,\tau_n^3\big]=\varnothing\big)\Bigg\}.\end{aligned} \tag{3.27}$$

(*See* (3.25) *and* (3.26) *for* T_i, $\overline{\sigma}_4^\star$, *and* u_4'.)

Proof. Condition on $S^1[0,\tau_n^1]$ and $S^2[0,T^2_{z,\frac{l\epsilon n}{2}}]$ on F^2, let

$$\begin{aligned}\gamma &= \mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]+S^2\big[0,T^2_{z,\frac{l\epsilon n}{2}}\big]\big),\\ \gamma_1 &= \mathrm{LE}_1\big(S^1\big[0,\sigma_z^1\big],S^2\big[0,T^2_{z,\frac{l\epsilon n}{2}}\big]\big),\\ \gamma_2 &= \mathrm{LE}_2\big(S^1\big[0,\sigma_z^1\big],S^2\big[0,T^2_{z,\frac{l\epsilon n}{2}}\big]\big),\end{aligned}\tag{3.28}$$

so that $\gamma=\gamma_1+\gamma_2$. Since $\mathrm{LE}(S^1[0,\sigma_z^1])[0,\overline{\sigma}_1]\cap S^2[0,T^2_{z,\frac{l\epsilon n}{2}}]=\varnothing$ on F^2, we see that $\mathrm{len}\,\gamma_1>\overline{\sigma}_1$, $\mathrm{LE}(S^1[0,\sigma_z^1])[0,\overline{\sigma}_1]\subset\gamma_1$, $\gamma_2(0)\in\mathrm{LE}(S^1[0,\sigma_z^1])[\overline{\sigma}_1,u^{(1)}]$ and $\gamma_2(\mathrm{len}\,\gamma_2)=z'$.

Conditioning $S^1[0,\tau_n^1]$ and $S^2[0,T^2_{z,\frac{l\epsilon n}{2}}]$ on F^2, we will deal with S^4 and S^3. Suppose that $\mathrm{LE}(S^1[0,\sigma_z^1])[0,\overline{\sigma}_1]\cap S^4[0,\sigma_w^4]=\varnothing$. Let

$$\begin{aligned}\lambda &= \mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]+S^2\big[0,T^2_{z,\frac{l\epsilon n}{2}}\big]+S^4\big[0,\sigma_w^4\big]\big),\\ \lambda_1 &= \mathrm{LE}_1\big(\gamma,S^4\big[0,\sigma_w^4\big]\big),\\ \lambda_2 &= \mathrm{LE}_2\big(\gamma,S^4\big[0,\sigma_w^4\big]\big),\end{aligned}\tag{3.29}$$

so that $\lambda=\lambda_1+\lambda_2$. Since $\mathrm{LE}(S^1[0,\sigma_z^1])[0,\overline{\sigma}_1]\cap S^4[0,\sigma_w^4]=\varnothing$, we see that $\mathrm{len}\,\lambda_1>\overline{\sigma}_1$, $\lambda_2(0)\in\gamma_1[\overline{\sigma}_1,\mathrm{len}\,\gamma_1]\cup\gamma_2$ and $\mathrm{LE}(S^1[0,\sigma_z^1])[0,\overline{\sigma}_1]\subset\lambda_1$.

Let $u_2=\mathrm{len}\,\lambda_1$ and let $T'=\max\{t\le\sigma_w^4\mid S^4(t)=\lambda_1(u_2)\}$. We see that $S^4(T')=\lambda_1(u_2)=\lambda_2(0)\in\gamma_1[\overline{\sigma}_1,\mathrm{len}\,\gamma_1]\cup\gamma_2$. Suppose that $q^{(1)}=r\le\log_2 l-3$. Then $\gamma_1[\overline{\sigma}_1,\mathrm{len}\,\gamma_1]\subset\mathrm{LE}(S^1[0,\sigma_z^1])[\overline{\sigma}_1,u^{(1)}]\subset B(z,2^r\epsilon n)\subset B(z,\frac{l\epsilon n}{4})$. Thus $S^4(T')\in B(z,\frac{l\epsilon n}{2})$.

Note that $u_2=\inf\{t\mid\gamma(t)\in S^4[0,\sigma_w^4]\}$ and $T'=\max\{t\le\sigma_w^4\mid S^4(t)=\gamma(u_2)\}$. Conditioning $S^1[0,\tau_n^1]$ and $S^2[0,T^2_{z,\frac{l\epsilon n}{2}}]$ on F^2, we are interested in

$$\begin{aligned}\tilde{p}_1 :=\;& P_4^{z'}\otimes P_3^w\big(T'\le\sigma_w^4<\tau_n^4,S^4\big(T'\big)\in\gamma_1[\overline{\sigma}_1,\mathrm{len}\,\gamma_1]\cup\gamma_2,S^3\big[1,\tau_n^3\big]\cap\overline{\big(\epsilon nB_y'\big)}=\varnothing,\\ &\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap S^3\big[T^3_{w,\frac{l\epsilon n}{4}},\tau_n^3\big]=\varnothing,\mathrm{LE}\big(S^4\big[T',\sigma_w^4\big]\big)[0,\overline{\sigma}_4]\cap S^3\big[0,T^3_{w,\frac{l\epsilon n}{4}}\big]=\varnothing\big)\\ &\big(\text{where }\overline{\sigma}_4:=\inf\big\{t\mid\mathrm{LE}\big(S^4\big[T',\sigma_w^4\big]\big)(t)\in\partial\big(\epsilon nB_y'\big)\big\}\big)\\ \le\;& P_4^{z'}\otimes P_3^w\big(T'\le\sigma_w^4<\tau_n^4,S^4\big(T'\big)\in\gamma_1[\overline{\sigma}_1,\mathrm{len}\,\gamma_1]\cup\gamma_2,\\ &S^3\big[1,T^3_{w,\frac{l\epsilon n}{4}}\big]\cap\overline{\big(\epsilon nB_y'\big)}=\varnothing,\mathrm{LE}\big(S^4\big[T',\sigma_w^4\big]\big)[0,\overline{\sigma}_4]\cap S^3\big[0,T^3_{w,\frac{l\epsilon n}{4}}\big]=\varnothing\big)\\ &\times\max_{w_1\in\partial B(w,\frac{\epsilon ln}{4})}P_3^{w_1}\big(\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap S^3\big[0,\tau_n^3\big]=\varnothing\big).\end{aligned}\tag{3.30}$$

We will consider the time reverse of $S^4[0,\sigma_w^4]$. Note that for each SRW path $\eta=[\eta(0),\dots,\eta(m)]$ with $\eta(0)=z'$ and $\eta(m)=w$, we have $P_4^{z'}(S^4[0,\sigma_w^4]=\eta)=\frac{p_w}{p_z}P_4^w(S^4[0,\sigma_z^4]=\eta^R)$. Suppose that $S^4(0)=w$ and $\sigma_{z'}^4<\tau_n^4$ (this is equivalent to $\tau_{z'}^4<\tau_n^4$). Define $u_2':=\inf\{t\mid\gamma(t)\in S^4[0,\sigma_w^4]\}$ and $T'':=\inf\{t\mid S^4(t)=\gamma(u_2')\}$. Let $\overline{\sigma}_4':=\max\{t\mid\mathrm{LE}(S^4[0,T''])(t)\in\partial(\epsilon nB_y')\}$ and $u_4:=\mathrm{len}\,\mathrm{LE}(S^4[0,T''])$. Then by the time reversibility of LERW (see Lemma 2.2), the distribution of $\mathrm{LE}(S^4[T',\sigma_w^4])[0,\overline{\sigma}_4]$ under $P_4^{z'}$ is same to that of $(\mathrm{LE}(S^4[0,T''])[\overline{\sigma}_4',u_4])^R$ under P_4^w. Therefore,

$$\begin{aligned}&P_4^{z'}\otimes P_3^w\big(T'\le\sigma_w^4<\tau_n^4,S^4\big(T'\big)\in\gamma_1[\overline{\sigma}_1,\mathrm{len}\,\gamma_1]\cup\gamma_2,\\ &\quad S^3\big[1,T^3_{w,\frac{l\epsilon n}{4}}\big]\cap\overline{\big(\epsilon nB_y'\big)}=\varnothing,\mathrm{LE}\big(S^4\big[T',\sigma_w^4\big]\big)[0,\overline{\sigma}_4]\cap S^3\big[0,T^3_{w,\frac{l\epsilon n}{4}}\big]=\varnothing\big)\end{aligned}$$

$$= \frac{p_w}{p_z} P_4^w \otimes P_3^w\big(T'' \le \sigma_{z'}^4 < \tau_n^4, S^4\big(T''\big) \in \gamma_1[\overline{\sigma}_1, \operatorname{len}\gamma_1] \cup \gamma_2,$$
$$S^3\big[1, T^3_{w,\frac{l\epsilon n}{4}}\big] \cap \overline{\big(\epsilon n B'_y\big)} = \varnothing, \mathrm{LE}\big(S^4\big[0, T''\big]\big)\big[\overline{\sigma}'_4, u_4\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \varnothing\big)$$
$$\le \frac{c}{\epsilon n} \times \max_{w_3 \in \partial B(w, 6\epsilon n)} P_4^w \otimes P_3^{w_3}\big(T'' \le \sigma_{z'}^4 < \tau_n^4, S^4\big(T''\big) \in \gamma_1[\overline{\sigma}_1, \operatorname{len}\gamma_1] \cup \gamma_2,$$
$$\mathrm{LE}\big(S^4\big[0, T''\big]\big)\big[\overline{\sigma}''_4, u_4\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \varnothing\big), \tag{3.31}$$

where $\overline{\sigma}''_4 := \max\{t \mid \mathrm{LE}(S^4[0, T''])(t) \in \partial B(w, 8\epsilon n)\}$ and we used $P_3^w(S^3[1, T^3_{w,6\epsilon n}] \cap \overline{(\epsilon n B'_y)} = \varnothing) \le \frac{c}{\epsilon n}$ in the last inequality (see Proposition 1.5.10 [8] for this). By the Harnack principle (see Theorem 1.7.6 [8]),

$$\text{RHS of (3.31)} \le \frac{C}{\epsilon n} P_4^w \otimes P_3^w\big(T'' \le \sigma_{z'}^4 < \tau_n^4, S^4\big(T''\big) \in \gamma_1[\overline{\sigma}_1, \operatorname{len}\gamma_1] \cup \gamma_2,$$
$$\mathrm{LE}\big(S^4\big[0, T''\big]\big)\big[\overline{\sigma}''_4, u_4\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \varnothing\big). \tag{3.32}$$

Let $\tau' := \inf\{t \ge T^4_{w,\frac{\epsilon l n}{1600}} \mid S^4(t) \in \partial B(w, 8\epsilon n)\}$. Then $P_4^w(\tau' < \sigma_{z'}^4 < \tau_n^4) \le \frac{C}{l^2 \epsilon n}$ (see Proposition 1.5.10 [8]). Therefore,

$$P_4^w \otimes P_3^w\big(T'' \le \sigma_{z'}^4 < \tau_n^4, S^4\big(T''\big) \in \gamma_1[\overline{\sigma}_1, \operatorname{len}\gamma_1] \cup \gamma_2, \mathrm{LE}\big(S^4\big[0, T''\big]\big)\big[\overline{\sigma}''_4, u_4\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \varnothing\big)$$
$$\le P_4^w \otimes P_3^w\big(T'' \le \sigma_{z'}^4 < \tau_n^4, \tau' > \sigma_{z'}^4, S^4\big(T''\big) \in \gamma_1[\overline{\sigma}_1, \operatorname{len}\gamma_1] \cup \gamma_2,$$
$$\mathrm{LE}\big(S^4\big[0, T''\big]\big)\big[\overline{\sigma}''_4, u_4\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \varnothing\big) + \frac{C}{l^2 \epsilon n}. \tag{3.33}$$

Suppose that $q^{(1)} = r \le \log_2 l - 3$, $T'' \le \sigma_{z'}^4 < \tau_n^4$ and $S^4(T'') \in \gamma_1[\overline{\sigma}_1, \operatorname{len}\gamma_1] \cup \gamma_2$. Then $S^4(T'') \in B(z, \frac{\epsilon l n}{2}) \subset B(w, \frac{\epsilon l n}{3})^c$. Let i_0 be the unique index i such that $T_{2i+1} < T'' \le \min\{T_{2i+2}, \sigma_{z'}^4\}$. Suppose that $\tau' > \sigma_{z'}^4$. Since $T_{2i_0+1} < T'' < T_{2i_0+2}$ and $S^4[T_{2i_0+1}, T''] \cap B(w, \frac{\epsilon l n}{800}) = \varnothing$, we have

$$\overline{\sigma}''_4 = \max\big\{t \mid \mathrm{LE}\big(S^4[0, T_{2i_0+1}]\big)(t) \in \partial B(w, 8\epsilon n)\big\}.$$

(Recall that $\overline{\sigma}''_4 := \max\{t \mid \mathrm{LE}(S^4[0, T''])(t) \in \partial B(w, 8\epsilon n)\}$.) Furthermore, if we let

$$u''_4 = \inf\left\{t \;\middle|\; \mathrm{LE}\big(S^4[0, T_{2i_0+1}]\big)(t) \in \partial B\left(w, \frac{\epsilon l n}{1600}\right)\right\},$$

then $\overline{\sigma}''_4 < u''_4$, $\overline{\sigma}''_4 = \max\{t \le u''_4 \mid \mathrm{LE}(S^4[0, T_{2i_0+1}])(t) \in \partial B(w, 8\epsilon n)\}$ and

$$u''_4 = \inf\left\{t \;\middle|\; \mathrm{LE}\big(S^4\big[0, T''\big]\big)(t) \in \partial B\left(w, \frac{\epsilon l n}{1600}\right)\right\}.$$

Therefore we see that $\mathrm{LE}(S^4[0, T''])[\overline{\sigma}''_4, u''_4] = \mathrm{LE}(S^4[0, T_{2i_0+1}])[\overline{\sigma}''_4, u''_4]$ and

The first term of RHS of (3.33)

$$\le \sum_{i=0}^{\infty} P_4^w \otimes P_3^w\big(T_{2i+1} < \sigma_{z'}^4 < \tau_n^4, \tau' > \sigma_{z'}^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)\big[\overline{\sigma}^\star_4, u'_4\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \varnothing\big)$$

$\big($Recall that $\overline{\sigma}^\star_4$ and u'_4 were defined as in (3.26)$\big)$

$$\le \sum_{i=0}^{\infty} \frac{C}{\epsilon l n} P_4^w \otimes P_3^w\big(T_{2i+1} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)\big[\overline{\sigma}^\star_4, u'_4\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \varnothing\big), \tag{3.34}$$

where we used the strong Markov property and the fact that $\max_{w' \in \partial B(w, \frac{\epsilon l n}{4})} P_4^{w'}(\tau_{z'}^4 < \infty) \le \frac{C}{\epsilon l n}$ in the last inequality (see Proposition 1.5.10 [8]), and we finish the proof of Lemma 3.6. $\square$

Recall the strategy in Remark 3.5. By Lemma 3.6, we replaced the sixth event in (3.23) by the event the loop-erasure of S^4 up to some stopping time does not intersect S^3. We want to show that the probability of that event is bounded above by an escape probability, i.e., we want to prove that

$$\sum_{i=0}^{\infty} P_4^w \otimes P_3^w \big(T_{2i+1} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)\big[\overline{\sigma}_4^{\star}, u_4'\big] \cap S^3\big[0, T^3_{w, \frac{l\epsilon n}{4}}\big] = \varnothing\big) \le C\, \mathrm{Es}(\epsilon n, \epsilon l n). \tag{3.35}$$

In order to show (3.35), we need to study the distribution of $\mathrm{LE}(S^4[0, T_{2i+1}])$. The next lemma compares the distribution of $\mathrm{LE}(S^4[0, T_{2i+1}])$ with that of $\mathrm{LE}(S^4[0, T_1])$. Note that the probability of $T_{2i+1} < \infty$ is bounded above by ci for some $c < 1$. The next lemma shows that conditioned on $T_{2i+1} < \infty$, the distribution of $\mathrm{LE}(S^4[0, T_{2i+1}])$ is comparable to that of $\mathrm{LE}(S^4[0, T_1])$.

Lemma 3.7. *There exists a $c \in (\frac{1}{2}, 1)$ such that for all $i \ge 0$ and for every simple path $\eta = [\eta(0), \dots, \eta(m)]$ with $\eta(0) = w$ and $\eta \subset \overline{B(w, \frac{\epsilon l n}{1600})}$, we have*

$$P_4^w\big(T_{2i+1} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta\big) \le c^i P_4^w\big(\mathrm{LE}\big(S^4[0, T_1]\big)[0, m] = \eta\big), \tag{3.36}$$

where T_i was defined as in (3.25).

Proof. We will show this sublemma by induction. Take a simple path $\eta = [\eta(0), \dots, \eta(m)]$ with $\eta(0) = w$ and $\eta \subset \overline{B(w, \frac{\epsilon l n}{1600})}$. The inequality (3.36) trivially holds when $i = 0$. So suppose that (3.36) holds for $c \in (\frac{1}{2}, 1)$ and $i - 1$. Note that

$$\begin{aligned} &P_4^w\big(T_{2i+1} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta\big) \\ &\quad = P_4^w\big(T_{2i} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta\big) \\ &\quad = P_4^w\big(T_{2i} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta, S^4[T_{2i}, T_{2i+1}] \cap \eta = \varnothing\big) \\ &\qquad + P_4^w\big(T_{2i} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta, S^4[T_{2i}, T_{2i+1}] \cap \eta \neq \varnothing\big). \end{aligned} \tag{3.37}$$

Suppose that $T_{2i} < \tau_n^4$, $\mathrm{LE}(S^4[0, T_{2i+1}])[0, m] = \eta$ and $S^4[T_{2i}, T_{2i+1}] \cap \eta = \varnothing$. Let $\mathrm{LE}^{(1)} = \mathrm{LE}_1(\lambda_1, \lambda_2)$ and $\mathrm{LE}^{(2)} = \mathrm{LE}_2(\lambda_1, \lambda_2)$ where $\lambda_1 = \mathrm{LE}(S^4[0, T_{2i-1}])$ and $\lambda_2 = S^4[T_{2i-1}, T_{2i+1}]$ (see (2.5) for $\mathrm{LE}^{(i)}$). Then $\mathrm{LE}(S^4[0, T_{2i+1}]) = \mathrm{LE}^{(1)} + \mathrm{LE}^{(2)}$. Let $u = \mathrm{len}\,\mathrm{LE}^{(1)} = \inf\{t \mid \lambda_1(t) \in \lambda_2\}$. Then $u > m$. Indeed, if $u \le m$, then $\mathrm{LE}^{(1)}(u) = \mathrm{LE}(S^4[0, T_{2i+1}])(u) = \eta(u)$. This implies $\eta(u) \in \lambda_2$. Since $S^4[T_{2i-1}, T_{2i}] \cap \overline{B(w, \frac{\epsilon l n}{1600})} = \varnothing$, we see that $\eta(u) \in S^4[T_{2i}, T_{2i+1}]$, and we get a contradiction. Thus $u > m$. Therefore $\eta = \mathrm{LE}(S^4[0, T_{2i+1}])[0, m] = \mathrm{LE}^{(1)}[0, m] = \mathrm{LE}(S^4[0, T_{2i-1}])[0, m]$. So

$$\begin{aligned} &P_4^w\big(T_{2i} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta, S^4[T_{2i}, T_{2i+1}] \cap \eta = \varnothing\big) \\ &\quad \le P_4^w\big(T_{2i-1} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i-1}]\big)[0, m] = \eta\big) \times \max_{w' \in \partial B(w, \frac{\epsilon l n}{4})} P_4^{w'}\big(t^1 < \tau_n^4, S^4\big[t^1, t^2\big] \cap \eta = \varnothing\big), \end{aligned} \tag{3.38}$$

where $t^1 = \inf\{t \mid S^4(t) \in \partial B(w, \frac{\epsilon l n}{800})\}$ and $t^2 = \inf\{t \ge t^1 \mid S^4(t) \in \partial B(w, \frac{\epsilon l n}{4})\}$.

Next we will deal with the second term in the RHS of (3.37). Suppose that $T_{2i} < \tau_n^4$, $S^4[T_{2i}, T_{2i+1}] \cap \eta \neq \varnothing$ and $\mathrm{LE}(S^4[0, T_{2i+1}])[0, m] = \eta$. On this event, we may define $u' := \inf\{t \mid \eta(t) \in S^4[T_{2i}, T_{2i+1}]\}$. Then $u' \le m$. Note that

$$\begin{aligned} &P_4^w\big(T_{2i} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta, S^4[T_{2i}, T_{2i+1}] \cap \eta \neq \varnothing\big) \\ &\quad = \sum_{j=0}^{m} P_4^w\big(T_{2i} < \tau_n^4, u' = j, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta\big). \end{aligned} \tag{3.39}$$

Suppose that $T_{2i} < \tau_n^4$, $u' = j$ and $\mathrm{LE}(S^4[0, T_{2i+1}])[0, m] = \eta$. Since $S^4[T_{2i-1}, T_{2i+1}]$ does not intersect $\eta[0, j-1]$, in order for $\mathrm{LE}(S^4[0, T_{2i+1}])[0, m]$ to be η, $\eta[0, j]$ must be contained in the loop-erasure of S^4 up to T_{2i-1}. The rest part of η, say $\eta[j+1, m]$, is constructed by the loop-erasure of $S^4[T_{2i}, T_{2i+1}]$. Therefore, we have

- $\mathrm{LE}(S^4[0, T_{2i-1}])[0, j] = \eta[0, j]$,
- If we let $\tau^{\star}_{\eta(j)} := \inf\{t \geq T_{2i-1} \mid S^4(t) = \eta(j)\}$, then $T_{2i} < \tau^{\star}_{\eta(j)} < T_{2i+1}$ and $S^4[T_{2i}, \tau^{\star}_{\eta(j)}] \cap \eta[0, j-1] = \emptyset$,
- $\mathrm{LE}(S^4[\tau^{\star}_{\eta(j)}, T_{2i+1}])[0, m-j] = \eta[j, m]$,
- $S^4[\tau^{\star}_{\eta(j)}, T_{2i+1}] \cap \eta[0, j-1] = \emptyset$.

So the probability in RHS of (3.39) is bounded above by the probability of four events above as follows.

$$\begin{aligned}
&P_4^w\big(T_{2i} < \tau_n^4, u' = j, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta\big) \\
&\quad \leq P_4^w\big(T_{2i} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i-1}]\big)[0, j] = \eta[0, j], T_{2i} < \tau^{\star}_{\eta(j)} < T_{2i+1}, S^4\big[T_{2i}, \tau^{\star}_{\eta(j)}\big] \cap \eta[0, j-1] = \emptyset, \\
&\qquad \mathrm{LE}\big(S^4\big[\tau^{\star}_{\eta(j)}, T_{2i+1}\big]\big)[0, m-j] = \eta[j, m], S^4\big[\tau^{\star}_{\eta(j)}, T_{2i+1}\big] \cap \eta[0, j-1] = \emptyset\big) \\
&\quad = \sum_{w' \in \partial B(w, \frac{\epsilon l n}{4})} P_4^w\big(T_{2i-1} < \tau_n^4, S^4(T_{2i-1}) = w', \mathrm{LE}\big(S^4[0, T_{2i-1}]\big)[0, j] = \eta[0, j]\big) \\
&\qquad \times P_4^{w'}\big(t^1 < \tau_n^4, \tau^4_{\eta(j)} < t^2, S^4\big[t^1, \tau^4_{\eta(j)}\big] \cap \eta[0, j-1] = \emptyset, \\
&\qquad \mathrm{LE}\big(S^4\big[\tau^4_{\eta(j)}, t^2\big]\big)[0, m-j] = \eta[j, m], S^4\big[\tau^4_{\eta(j)}, t^2\big] \cap \eta[0, j-1] = \emptyset\big),
\end{aligned} \tag{3.40}$$

where $\tau^4_{\eta(j)} = \inf\{t \mid S^4(t) = \eta(j)\}$. Since $\eta \subset \overline{B(w, \frac{\epsilon l n}{1600})}$, in order for S^4 to hit η, S^4 must intersect $\partial B(w, \frac{\epsilon l n}{800})$ before $\tau^4_{\eta(j)}$. So by using the strong Markov property at t^1 first, then using it again at $\tau^4_{\eta(j)}$, we have

$$\begin{aligned}
&P_4^{w'}\big(t^1 < \tau_n^4, \tau^4_{\eta(j)} < t^2, S^4\big[t^1, \tau^4_{\eta(j)}\big] \cap \eta[0, j-1] = \emptyset, \\
&\qquad \mathrm{LE}\big(S^4\big[\tau^4_{\eta(j)}, t^2\big]\big)[0, m-j] = \eta[j, m], S^4\big[\tau^4_{\eta(j)}, t^2\big] \cap \eta[0, j-1] = \emptyset\big) \\
&\quad = \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w'}\big(S^4\big(t^1\big) = w''\big) \\
&\qquad \times P_4^{w''}\big(\tau^4_{\eta(j)} < t^2, S^4\big[0, \tau^4_{\eta(j)}\big] \cap \eta[0, j-1] = \emptyset, \\
&\qquad \mathrm{LE}\big(S^4\big[\tau^4_{\eta(j)}, t^2\big]\big)[0, m-j] = \eta[j, m], S^4\big[\tau^4_{\eta(j)}, t^2\big] \cap \eta[0, j-1] = \emptyset\big) \\
&\quad = \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w'}\big(S^4\big(t^1\big) = w''\big) \times P_4^{w''}\big(\tau^4_{\eta(j)} < t^2, S^4\big[0, \tau^4_{\eta(j)}\big] \cap \eta[0, j-1] = \emptyset\big) \\
&\qquad \times P_4^{\eta(j)}\big(\mathrm{LE}\big(S^4\big[0, t^2\big]\big)[0, m-j] = \eta[j, m], S^4\big[0, t^2\big] \cap \eta[0, j-1] = \emptyset\big).
\end{aligned} \tag{3.41}$$

By the equation in line 10, page 199 of [11], we can write the distribution of LERW in terms of Green's functions and non-intersecting probabilities of η as follows.

$$\begin{aligned}
&P_4^{\eta(j)}\big(\mathrm{LE}\big(S^4\big[0, t^2\big]\big)[0, m-j] = \eta[j, m], S^4\big[0, t^2\big] \cap \eta[0, j-1] = \emptyset\big) \\
&\quad = \prod_{q=j}^{m-1} G\big(\eta(q), \eta(q), B \setminus \eta[0, q-1]\big) P_4^{\eta(q)}\big(S^4(1) = \eta(q+1)\big) G\big(\eta(m), \eta(m), B \setminus \eta[0, m-1]\big) \\
&\qquad \times P_4^{\eta(m)}\big(S^4\big[1, t^2\big] \cap \eta[0, m] = \emptyset\big),
\end{aligned} \tag{3.42}$$

where $B = B(w, \frac{\epsilon l n}{4})$ and $G(\cdot, \cdot, \cdot)$ is Green's function defined in Section 1.4.

Take $w_0' \in \partial B(w, \frac{\epsilon l n}{4})$ and $w_0'' \in \partial B(w, \frac{\epsilon l n}{800})$ such that

$$P_4^{w_0'}\big(S^4\big(t^1\big) = w_0''\big) = \max_{w' \in \partial B(w, \frac{\epsilon l n}{4}), w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w'}\big(S^4\big(t^1\big) = w''\big).$$

Then by using Proposition 1.5.10 [8], we see that

$$P_4^{w_0'}\big(S^4\big(t^1\big) = w_0''\big) \le \frac{6400}{(\epsilon l n)^2}. \tag{3.43}$$

By (3.40)–(3.42), and by definition of w_0' and w_0'',

$$\begin{aligned}
&P_4^w\big(T_{2i} < \tau_n^4, u' = j, \mathrm{LE}\big(S^4[0, T_{2i+1}]\big)[0, m] = \eta\big) \\
&\quad \le \sum_{w' \in \partial B(w, \frac{\epsilon l n}{4})} P_4^w\big(T_{2i-1} < \tau_n^4, S^4(T_{2i-1}) = w', \mathrm{LE}\big(S^4[0, T_{2i-1}]\big)[0, j] = \eta[0, j]\big) \\
&\qquad \times \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w'}\big(S^4\big(t^1\big) = w''\big) \times P_4^{w''}\big(\tau_{\eta(j)}^4 < t^2, S^4\big[0, \tau_{\eta(j)}^4\big] \cap \eta[0, j-1] = \varnothing\big) \\
&\qquad \times \prod_{q=j}^{m-1} G\big(\eta(q), \eta(q), B \setminus \eta[0, q-1]\big) P_4^{\eta(q)}\big(S^4(1) = \eta(q+1)\big) G\big(\eta(m), \eta(m), B \setminus \eta[0, m-1]\big) \\
&\qquad \times P_4^{\eta(m)}\big(S^4\big[1, t^2\big] \cap \eta[0, m] = \varnothing\big) \\
&\quad \le P_4^w\big(T_{2i-1} < \tau_n^4, \mathrm{LE}\big(S^4[0, T_{2i-1}]\big)[0, j] = \eta[0, j]\big) \\
&\qquad \times \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w_0'}\big(S^4\big(t^1\big) = w_0''\big) \times P_4^{w''}\big(\tau_{\eta(j)}^4 < t^2, S^4\big[0, \tau_{\eta(j)}^4\big] \cap \eta[0, j-1] = \varnothing\big) \\
&\qquad \times \prod_{q=j}^{m-1} G\big(\eta(q), \eta(q), B \setminus \eta[0, q-1]\big) P_4^{\eta(q)}\big(S^4(1) = \eta(q+1)\big) G\big(\eta(m), \eta(m), B \setminus \eta[0, m-1]\big) \\
&\qquad \times P_4^{\eta(m)}\big(S^4\big[1, t^2\big] \cap \eta[0, m] = \varnothing\big).
\end{aligned} \tag{3.44}$$

In order to estimate the RHS of (3.44), now we use the assumption of the induction for $\eta[0, j]$. By using it as well as the equation in line 10, page 199 of [11] for the distribution of $\mathrm{LE}(S^4[0, T_1])[0, m]$, we see that

$$\begin{aligned}
&\le c^{i-1} P_4^w\big(\mathrm{LE}\big(S^4[0, T_1]\big)[0, j] = \eta[0, j]\big) \\
&\quad \times \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w_0'}\big(S^4\big(t^1\big) = w_0''\big) \times P_4^{w''}\big(\tau_{\eta(j)}^4 < t^2, S^4\big[0, \tau_{\eta(j)}^4\big] \cap \eta[0, j-1] = \varnothing\big) \\
&\quad \times \prod_{q=j}^{m-1} G\big(\eta(q), \eta(q), B \setminus \eta[0, q-1]\big) P_4^{\eta(q)}\big(S^4(1) = \eta(q+1)\big) G\big(\eta(m), \eta(m), B \setminus \eta[0, m-1]\big) \\
&\quad \times P_4^{\eta(m)}\big(S^4\big[1, t^2\big] \cap \eta[0, m] = \varnothing\big) \\
&= c^{i-1} P_4^w\big(\mathrm{LE}\big(S^4[0, T_1]\big)[0, m] = \eta[0, m]\big) \\
&\quad \times \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w_0'}\big(S^4\big(t^1\big) = w_0''\big) \times P_4^{w''}\big(\tau_{\eta(j)}^4 < t^2, S^4\big[0, \tau_{\eta(j)}^4\big] \cap \eta[0, j-1] = \varnothing\big) \\
&\quad \times G\big(\eta(j), \eta(j), B \setminus \eta[0, j-1]\big) P_4^{\eta(j)}\big(S^4\big[1, t^2\big] \cap \eta[0, j] = \varnothing\big).
\end{aligned} \tag{3.45}$$

Let $u^\star := \inf\{t \mid \eta(t) \in S^4[0, t^2]\}$. In order for $u^\star$ to be j, first S^4 hits $\eta(j)$ before t^2 and intersecting $\eta[0, j-1]$, then S^4 does not hit $\eta[0, j-1]$ from $\tau^4_{\eta(j)}$ to the last visit of $\eta(j)$, and finally it exits $B(w, \frac{\epsilon l n}{4})$ without intersecting $\eta[0, j-1]$. Thus we have

$$
\begin{aligned}
&P_4^{w''}\bigl(u^\star = j\bigr)\\
&\quad = \sum_{k=0}^{\infty} P_4^{w''}\bigl(\tau^4_{\eta(j)} < t^2, t^\star = \tau^4_{\eta(j)} + 2k, S^4\bigl[0, t^2\bigr] \cap \eta[0, j-1] = \varnothing\bigr)\\
&\qquad \bigl(\text{where } t^\star := \max\bigl\{t \le t^2 \mid S^4(t) = \eta(j)\bigr\}\bigr)\\
&\quad = \sum_{k=0}^{\infty} P_4^{w''}\bigl(\tau^4_{\eta(j)} < t^2, S^4\bigl[0, \tau^4_{\eta(j)}\bigr] \cap \eta[0, j-1] = \varnothing\bigr)\\
&\qquad \times P_4^{\eta(j)}\bigl(S^4(2k) = \eta(j), S^4[0, 2k] \cap \eta[0, j-1] = \varnothing, S^4[0, 2k] \subset B\bigr) P_4^{\eta(j)}\bigl(S^4\bigl[1, t^2\bigr] \cap \eta[0, j] = \varnothing\bigr)\\
&\quad = P_4^{w''}\bigl(\tau^4_{\eta(j)} < t^2, S^4\bigl[0, \tau^4_{\eta(j)}\bigr] \cap \eta[0, j-1] = \varnothing\bigr) G\bigl(\eta(j), \eta(j), B \setminus \eta[0, j-1]\bigr)\\
&\qquad \times P_4^{\eta(j)}\bigl(S^4\bigl[1, t^2\bigr] \cap \eta[0, j] = \varnothing\bigr).
\end{aligned}
$$

Combining this with (3.45), we have

$$
\begin{aligned}
&P_4^{w}\bigl(T_{2i} < \tau^4_n, u' = j, \mathrm{LE}\bigl(S^4[0, T_{2i+1}]\bigr)[0, m] = \eta\bigr)\\
&\quad \le c^{i-1} P_4^{w}\bigl(\mathrm{LE}\bigl(S^4[0, T_1]\bigr)[0, m] = \eta[0, m]\bigr) \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w'_0}\bigl(S^4\bigl(t^1\bigr) = w''_0\bigr) P_4^{w''}\bigl(u^\star = j\bigr).
\end{aligned}
\tag{3.46}
$$

Clearly events $\{u' = j\}$ are disjoint, and the same thing holds for events $\{u^\star = j\}$. So taking sum for j in (3.46), we have

$$
\begin{aligned}
&P_4^{w}\bigl(T_{2i} < \tau^4_n, \mathrm{LE}\bigl(S^4[0, T_{2i+1}]\bigr)[0, m] = \eta, S^4[T_{2i}, T_{2i+1}] \cap \eta \ne \varnothing\bigr)\\
&\quad \le c^{i-1} P_4^{w}\bigl(\mathrm{LE}\bigl(S^4[0, T_1]\bigr)[0, m] = \eta[0, m]\bigr)\\
&\qquad \times \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w'_0}\bigl(S^4\bigl(t^1\bigr) = w''_0\bigr) P_4^{w''}\bigl(S^4\bigl[0, t^2\bigr] \cap \eta[0, m] \ne \varnothing\bigr).
\end{aligned}
\tag{3.47}
$$

The estimate of the case that $S^4[T_{2i}, T_{2i+1}] \cap \eta \ne \varnothing$ was given as in (3.47). For the case that $S^4[T_{2i}, T_{2i+1}] \cap \eta = \varnothing$, by (3.38) and the assumption of the induction,

$$
\begin{aligned}
&P_4^{w}\bigl(T_{2i} < \tau^4_n, \mathrm{LE}\bigl(S^4[0, T_{2i+1}]\bigr)[0, m] = \eta, S^4[T_{2i}, T_{2i+1}] \cap \eta = \varnothing\bigr)\\
&\quad \le c^{i-1} P_4^{w}\bigl(\mathrm{LE}\bigl(S^4[0, T_1]\bigr)[0, m] = \eta[0, m]\bigr)\\
&\qquad \times \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w'_0}\bigl(S^4\bigl(t^1\bigr) = w''_0\bigr) P_4^{w''}\bigl(S^4\bigl[0, t^2\bigr] \cap \eta[0, m] = \varnothing\bigr).
\end{aligned}
\tag{3.48}
$$

But (3.43) shows that $P_4^{w'_0}(S^4(t^1) = w''_0)$ is small enough compared with the number of lattice points in $\partial B(w, \frac{\epsilon l n}{800})$. Since we assume $c \in (\frac{1}{2}, 1)$ in the assumption of the induction, this leads to finish the proof of the induction as follows.

$$
\begin{aligned}
&P_4^{w}\bigl(T_{2i+1} < \tau^4_n, \mathrm{LE}\bigl(S^4[0, T_{2i+1}]\bigr)[0, m] = \eta\bigr)\\
&\quad \le c^{i-1} P_4^{w}\bigl(\mathrm{LE}\bigl(S^4[0, T_1]\bigr)[0, m] = \eta[0, m]\bigr)
\end{aligned}
$$

$$\times \sum_{w'' \in \partial B(w, \frac{\epsilon l n}{800})} P_4^{w_0'}\big(S^4(t^1) = w_0''\big)\big\{P_4^{w''}\big(S^4[0,t^2] \cap \eta[0,m] \neq \emptyset\big) + P_4^{w''}\big(S^4[0,t^2] \cap \eta[0,m] = \emptyset\big)\big\}$$

$$\leq c^{i-1} P_4^w\big(\mathrm{LE}\big(S^4[0,T_1]\big)[0,m] = \eta[0,m]\big)\frac{6400}{(\epsilon l n)^2}\frac{50(\epsilon l n)^2}{640{,}000}$$

$$\leq c^i P_4^w\big(\mathrm{LE}\big(S^4[0,T_1]\big)[0,m] = \eta[0,m]\big), \tag{3.49}$$

which finishes the proof of Lemma 3.7. □

Recall the strategy in Remark 3.5. Since $\mathrm{LE}(S^4[0,T_{2i+1}])[\overline{\sigma}_4^\star, u_4'] \subset \overline{B(w, \frac{\epsilon l n}{1600})}$, by Lemma 3.7, Lemma 2.11 and (2.14)

$$\sum_{i=0}^{\infty} P_4^w \otimes P_3^w\big(T_{2i+1} < \tau_n^4, \mathrm{LE}\big(S^4[0,T_{2i+1}]\big)\big[\overline{\sigma}_4^\star, u_4'\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \emptyset\big)$$

$$\leq \sum_{i=0}^{\infty} c^i P_4^w \otimes P_3^w\big(\mathrm{LE}\big(S^4[0,T_1]\big)\big[t_2^\star, t_1^\star\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \emptyset\big)$$

$$\left(\text{where } t_1^\star := \inf\left\{t \;\middle|\; \mathrm{LE}\big(S^4[0,T_1]\big)(t) \in \partial B\left(w, \frac{\epsilon l n}{1600}\right)\right\}\right.$$

$$\left.\text{and } t_2^\star := \max\big\{t \leq t_1^\star \mid \mathrm{LE}\big(S^4[0,T_1]\big)(t) \in \partial B(w, 8\epsilon n)\big\}\right)$$

$$\leq C\,\mathrm{Es}(\epsilon n, \epsilon l n). \tag{3.50}$$

Therefore, by (3.33),

$$P_4^w \otimes P_3^w\big(T'' \leq \sigma_{z'}^4 < \tau_n^4, S^4(T'') \in \gamma_1[\overline{\sigma}_1, \mathrm{len}\,\gamma_1] \cup \gamma_2, \mathrm{LE}\big(S^4[0,T'']\big)\big[\overline{\sigma}_4'', u_4\big] \cap S^3\big[0, T^3_{w,\frac{l\epsilon n}{4}}\big] = \emptyset\big)$$

$$\leq \frac{C}{\epsilon l n}\mathrm{Es}(\epsilon n, \epsilon l n), \tag{3.51}$$

where we used $l^{-1} \leq \mathrm{Es}(\epsilon n, \epsilon l n)$ in the last inequality (see (2.16)).

Thus by (3.30),

$$\tilde{p}_1 \leq \frac{C}{\epsilon n}\frac{1}{\epsilon l n}\mathrm{Es}(\epsilon n, \epsilon l n) \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1}\big(\mathrm{LE}\big(S^1\big[0, \sigma_z^1\big]\big)[0, \overline{\sigma}_1] \cap S^3\big[0, \tau_n^3\big] = \emptyset\big). \tag{3.52}$$

Combining (3.52) with (3.23), we have

$$P_1^0 \otimes P_2^z \otimes P_3^w\big(F^1, q^{(1)} = r, S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big) = z'\big) \leq E_1^0 \otimes E_2^z(\tilde{p}_1 \mathbf{1}_{F^2})$$

$$\big(\text{Recall that } F^2 \text{ was defined in (3.24)}\big)$$

$$\leq \frac{C}{\epsilon n}\frac{1}{\epsilon l n}\mathrm{Es}(\epsilon n, \epsilon l n) E_1^0 \otimes E_2^z\Big\{\mathbf{1}_{F^2} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1}\big(\mathrm{LE}\big(S^1\big[0, \sigma_z^1\big]\big)[0, \overline{\sigma}_1] \cap S^3\big[0, \tau_n^3\big] = \emptyset\big)\Big\}. \tag{3.53}$$

We need to estimate the expectation in RHS of (3.53). Using the time reversibility of LERW (see Lemma 2.2), we can replace the loop erasure of S^1 from the origin to z by the loop erasure of S^1 from z to the origin. Therefore we have

$$E_1^0 \otimes E_2^z \Big\{ \mathbf{1}_{F^2} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\mathrm{LE}\big(S^1\big[0, \sigma_z^1\big]\big)[0, \overline{\sigma}_1] \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\}$$

$$= E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^2} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\tilde{u}_1, \tilde{u}_2] \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\}, \tag{3.54}$$

where

$$\tilde{u}_1 := \max\big\{t \mid \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)(t) \in \partial\big(\epsilon n B_x'\big)\big\}, \qquad \tilde{u}_2 := \mathrm{len}\, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big), \tag{3.55}$$

and

$$\begin{aligned} \tilde{F}^2 := \big\{ & \tau_0^1 < \tau_n^1, S^2\big[1, T^2_{z, \frac{l\epsilon n}{2}}\big] \cap \overline{(\epsilon n B_x')} = \varnothing, S^2\big(T^2_{z, \frac{l\epsilon n}{2}}\big) = z', \\ & \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\tilde{u}_1, \tilde{u}_2] \cap S^2\big[0, T^2_{z, \frac{l\epsilon n}{2}}\big] = \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[0, \tilde{u}_1] \cap A_z^r \neq \varnothing, \\ & \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[0, \tilde{u}_1] \subset B\big(z, 2^r \epsilon n\big)\big\}. \end{aligned} \tag{3.56}$$

We have to estimate the expectation in RHS of (3.54). As we discussed, we want to deal with all events in $\tilde{F}^2$ as if they were independent. That will be done in the next lemma. In order to control the independence of LERW, we will use Proposition 4.6 [15], which states that $\eta^1_{0,R}(S[0, \tau_n])$ and $\eta^2_{0,4R,n}(S[0, \tau_n])$ are "independent up to constant".

Lemma 3.8. *Suppose that $r \leq \log_2 l - 3$. Then there exist universal constants $C < \infty$ and $\delta > 0$ such that*

$$E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^2} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\tilde{u}_1, \tilde{u}_2] \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\}$$

$$\leq \frac{C}{n} \frac{1}{\epsilon n} 2^{-\delta r} \, \mathrm{Es}(\epsilon n, n) P_2^z\big(S^2\big(T^2_{z, \frac{l\epsilon n}{2}}\big) = z'\big). \tag{3.57}$$

Proof. Throughout the proof, let

$$\eta_R^2(\lambda) := \lambda[s, u],$$

where $u = \inf\{t \mid \lambda(t) \in \partial B(z, R)\}$ and $s = \sup\{t \leq u \mid \lambda(t) \in \partial(\epsilon n B_x')\}$. Suppose that $\mathrm{LE}(S^1[0, \tau_0^1])[0, \tilde{u}_1] \cap A_z^r \neq \varnothing$ and $\mathrm{LE}(S^1[0, \tau_0^1])[0, \tilde{u}_1] \subset B(z, 2^r \epsilon n)$. Since $r \leq \log_2 l - 3$, we see that

$$\eta^2_{\frac{n}{16}}\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)\big) \subset \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\tilde{u}_1, \tilde{u}_2], \qquad \eta^2_{\epsilon l n}\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)\big) \subset \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\tilde{u}_1, \tilde{u}_2].$$

Therefore, if we write

$$\begin{aligned} \tilde{F}^3 := \big\{ & \tau_0^1 < \tau_n^1, S^2\big[1, T^2_{z, \frac{l\epsilon n}{2}}\big] \cap \overline{(\epsilon n B_x')} = \varnothing, S^2\big(T^2_{z, \frac{l\epsilon n}{2}}\big) = z', \\ & \eta^2_{\epsilon l n}\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)\big) \cap S^2\big[0, T^2_{z, \frac{l\epsilon n}{2}}\big] = \varnothing, \\ & \big(\eta^1_{z, 2^r \epsilon n}\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)\big) \setminus \eta^1_{z, 2^{r-1} \epsilon n}\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)\big)\big) \cap \overline{(\epsilon n B_x')} \neq \varnothing\big\}, \end{aligned} \tag{3.58}$$

(recall that η^1 was defined as in Definition 2.5) then we can replace $\mathrm{LE}(S^1[0, \tau_0^1])[\tilde{u}_1, \tilde{u}_2]$ as follows.

$$E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^2 \cap G} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\tilde{u}_1, \tilde{u}_2] \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\}$$

$$\leq E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^3 \cap G} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\eta^2_{\frac{n}{16}}\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)\big) \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\}. \tag{3.59}$$

By Proposition 4.2, 4.4 [15] and Proposition 1.5.10 [8], the distribution of the loop erasure of a random walk conditioned to hit the origin before exiting $B(n)$ is equal to (up to multiplicative constants) the distribution of the loop erasure of S^1 up to exiting $B(z, \frac{n}{4})$. So we have

$$\begin{aligned}
&E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^3 \cap G} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\eta^2_{\frac{n}{16}} \big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)\big) \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\} \\
&\quad \le \frac{C}{n} E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^4 \cap G} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\eta^2_{\frac{n}{16}} \big(\mathrm{LE}\big(S^1\big[0, T^1_{z, \frac{n}{4}}\big]\big)\big) \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\}, \qquad (3.60)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{F}^4 := \big\{ &S^2\big[1, T^2_{z, \frac{l\epsilon n}{2}}\big] \cap \overline{(\epsilon n B'_x)} = \varnothing, S^2\big(T^2_{z, \frac{l\epsilon n}{2}}\big) = z', \eta^2_{\epsilon l n}\big(\mathrm{LE}\big(S^1\big[0, T^1_{z, \frac{n}{4}}\big]\big)\big) \cap S^2\big[0, T^2_{z, \frac{l\epsilon n}{2}}\big] = \varnothing, \\
&\big(\eta^1_{z, 2^r \epsilon n}\big(\mathrm{LE}\big(S^1\big[0, T^1_{z, \frac{n}{4}}\big]\big)\big) \setminus \eta^1_{z, 2^{r-1}\epsilon n}\big(\mathrm{LE}\big(S^1\big[0, T^1_{z, \frac{n}{4}}\big]\big)\big)\big) \cap \overline{(\epsilon n B'_x)} \neq \varnothing \big\}. \qquad (3.61)
\end{aligned}$$

We will estimate the expectation in the RHS of (3.60). To do it, let $\gamma := \mathrm{LE}(S^1[0, T^1_{z, \frac{n}{4}}])$ and $\tau_R^\gamma := \inf\{t \mid \gamma(t) \in \partial B(z, R)\}$. Suppose that $\tilde{F}^4$ and $\gamma[\tau^\gamma_{\frac{n}{16}}, \tau^\gamma_{\frac{n}{4}}] \cap B(z, 2^{r+4}\epsilon n) \neq \varnothing$ occur. Then S^1 returns to $B(z, 8\epsilon n)$ after hitting $\partial B(z, 2^{r-1}\epsilon n)$. After S^1 returns to $B(z, 8\epsilon n)$ and goes to $\partial B(z, \frac{n}{16})$, S^1 must return to $B(z, 2^{r+4}\epsilon n)$. By Proposition 1.5.10 [8], that probability is bounded above by $C \frac{2^r \epsilon n}{n} \frac{\epsilon n}{2^r \epsilon n} = C\epsilon$. Thus by the strong Markov property, Proposition 1.5.10 [8], and (2.16),

$$\begin{aligned}
&\frac{C}{n} P_1^z \otimes P_2^z \big(\tilde{F}^4, \gamma\big[\tau^\gamma_{\frac{n}{16}}, \tau^\gamma_{\frac{n}{4}}\big] \cap B\big(z, 2^{r+4}\epsilon n\big) \neq \varnothing \big) \le \frac{C}{n} \frac{1}{\epsilon n} P_2^z\big(S^2\big(T^2_{z, \frac{l\epsilon n}{2}}\big) = z'\big) \times \epsilon \\
&\quad \le \frac{C}{n} \frac{1}{\epsilon n} 2^{-cr} \, \mathrm{Es}(\epsilon n, n) P_2^z\big(S^2\big(T^2_{z, \frac{l\epsilon n}{2}}\big) = z'\big), \qquad (3.62)
\end{aligned}$$

for some $c > 0$. So it suffices to consider the case that $\gamma[\tau^\gamma_{\frac{n}{16}}, \tau^\gamma_{\frac{n}{4}}] \cap B(z, 2^{r+4}\epsilon n) = \varnothing$. With this in mind, define $k_0 := \min\{k \mid \gamma[\tau^\gamma_{\frac{n}{16}}, \tau^\gamma_{\frac{n}{4}}] \cap B(z, 2^{-k} n) = \varnothing\}$. Then we may assume that $2^{-k_0} n \ge 2^{r+4}\epsilon n$. Now we consider two cases.

Case-1: $2^{-k_0} n \ge 4 l \epsilon n$.

In this case, we have $\eta^2_{z, 4\epsilon l n, \frac{n}{16}}(\gamma) = \eta^1_{z, \frac{n}{16}}(\eta^2_{z, 4\epsilon l n, \frac{n}{4}}(\gamma))$ (see Definition 2.5 for η^i). Thus by the Harnack principle (see Theorem 1.7.6 [8]),

$$\begin{aligned}
&\frac{C}{n} E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^4 \cap \text{Case-1}} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\eta^2_{\frac{n}{16}}(\gamma) \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\} \\
&\quad \le \frac{C}{n} E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^4 \cap \text{Case-1}} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1} \big(\eta^1_{z, \frac{n}{16}}\big(\eta^2_{z, 4\epsilon l n, \frac{n}{4}}(\gamma)\big) \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \Big\} \\
&\quad \le \frac{C}{n} E_1^z \otimes E_2^z \big\{ \mathbf{1}_{\tilde{F}^4 \cap \text{Case-1}} P_3^z \big(\eta^1_{z, \frac{n}{16}}\big(\eta^2_{z, 4\epsilon l n, \frac{n}{4}}(\gamma)\big) \cap S^3\big[0, \tau_n^3\big] = \varnothing \big) \big\}. \qquad (3.63)
\end{aligned}$$

But by Proposition 4.6 [15], the distribution of γ from z to $\partial B(z, \epsilon l n)$ and the distribution of γ after last visit to $B(z, 4\epsilon l n)$ are independent (up to multiplicative constants). Therefore the RHS of (3.63) is bounded above by

$$\frac{C}{n} P_1^z \otimes P_2^z \big(\tilde{F}^4\big) \times P_1^z \otimes P_3^z \big(\eta^1_{z, \frac{n}{16}}\big(\eta^2_{z, 4\epsilon l n, \frac{n}{4}}(\gamma)\big) \cap S^3\big[0, \tau_n^3\big] = \varnothing \big). \qquad (3.64)$$

By Lemma 2.11, Proposition 2.6, and the strong Markov property, we see that (3.64) is bounded above by

$$\frac{C}{n} \frac{1}{\epsilon n} 2^{-cr} \, \mathrm{Es}(\epsilon n, n) P_2^z\big(S^2\big(T^2_{z, \frac{l\epsilon n}{2}}\big) = z'\big). \qquad (3.65)$$

So we finish Case-1.

Case-2: $2^{r+4}\epsilon n \le 2^{-k_0} n \le 4l\epsilon n$.

Suppose that $k_0 = k$ with $2^{r+4}\epsilon n \le 2^{-k} n \le 4l\epsilon n$. Note that $\eta^2_{z,2^{-k}n,\frac{n}{16}}(\gamma) = \eta^1_{z,\frac{n}{16}}(\eta^2_{z,2^{-k}n,\frac{n}{4}}(\gamma))$, and $(\eta^2_{z,2^{-k}n,\frac{n}{4}}(\gamma) \setminus \eta^1_{z,\frac{n}{16}}(\eta^2_{z,2^{-k}n,\frac{n}{4}}(\gamma))) \cap B(z, 2^{-(k-1)}n) \neq \varnothing$. So by similar arguments using Proposition 4.6 [15] as above, we see that

$$\begin{aligned}
&\frac{C}{n} E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^4 \cap \{k_0 = k\}} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1}\big(\eta^2_{\frac{n}{16}}(\gamma) \cap S^3\big[0, \tau_n^3\big] = \varnothing\big) \Big\} \\
&\quad \le \frac{C}{n} P_1^z \otimes P_2^z \Big(\big(\eta^2_{z,2^{-k}n,\frac{n}{4}}(\gamma) \setminus \eta^1_{z,\frac{n}{16}}\big(\eta^2_{z,2^{-k}n,\frac{n}{4}}(\gamma)\big)\big) \cap B\big(z, 2^{-(k-1)}n\big) \neq \varnothing, \tilde{F}_k^4 \Big) \\
&\quad \le \frac{C}{n} P_1^z \otimes P_2^z \Big(\big(\eta^2_{z,2^{-k}n,\frac{n}{4}}(\gamma) \setminus \eta^1_{z,\frac{n}{16}}\big(\eta^2_{z,2^{-k}n,\frac{n}{4}}(\gamma)\big)\big) \cap B\big(z, 2^{-(k-1)}n\big) \neq \varnothing \Big) \times P_1^z \otimes P_2^z \big(\tilde{F}_k^4\big) \\
&\quad \le \frac{C}{n} \operatorname{Es}\big(2^{-k}n, n\big) 2^{-ck} P_1^z \otimes P_2^z \big(\tilde{F}_k^4\big),
\end{aligned} \tag{3.66}$$

for some $c > 0$. Here $\tilde{F}_k^4$ is defined by

$$\begin{aligned}
\tilde{F}_k^4 := \Big\{ & S^2\big[1, T^2_{z,\frac{l\epsilon n}{2}}\big] \cap \overline{(\epsilon n B'_x)} = \varnothing, S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big) = z', \eta^2_{2^{-(k+2)}n}(\gamma) \cap S^2\big[0, T^2_{z,\frac{l\epsilon n}{2}}\big] = \varnothing, \\
& \big(\eta^1_{z,2^r \epsilon n}(\gamma) \setminus \eta^1_{z,2^{r-1}\epsilon n}(\gamma)\big) \cap \overline{(\epsilon n B'_x)} \neq \varnothing \Big\}.
\end{aligned}$$

But by Lemma 2.11, Proposition 2.6, and the strong Markov property, RHS of (3.66) is bounded above by

$$\frac{C}{n} \frac{1}{\epsilon n} 2^{-ck} 2^{-cr} \operatorname{Es}(\epsilon n, n) P_2^z \big(S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big) = z'\big), \tag{3.67}$$

for some $c > 0$. Taking sum for k, we have

$$\begin{aligned}
&\frac{C}{n} E_1^z \otimes E_2^z \Big\{ \mathbf{1}_{\tilde{F}^4 \cap \text{Case-2}} \max_{w_1 \in \partial B(w, \frac{\epsilon l n}{4})} P_3^{w_1}\big(\eta^2_{\frac{n}{16}}(\gamma) \cap S^3\big[0, \tau_n^3\big] = \varnothing\big) \Big\} \\
&\quad \le \frac{C}{n} \frac{1}{\epsilon n} 2^{-cr} \operatorname{Es}(\epsilon n, n) P_2^z \big(S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big) = z'\big).
\end{aligned} \tag{3.68}$$

So we finish Case-2, and Lemma 3.8 is proved. $\square$

Now we return to the proof of Theorem 3.1. Using Lemma 3.3, 3.6, 3.7, and 3.8, by (3.53),

$$\begin{aligned}
&P_1^0 \otimes P_2^z \otimes P_3^w \big(F^1, q^{(1)} = r, S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big) = z'\big) \\
&\quad \le E_1^0 \otimes E_2^z (\tilde{p}_1 \mathbf{1}_{F^2}) \\
&\quad \le \frac{C}{\epsilon n} \frac{1}{\epsilon l n} \frac{1}{n} \frac{1}{\epsilon n} 2^{-\delta r} \operatorname{Es}(\epsilon n, n) \operatorname{Es}(\epsilon n, \epsilon l n) P_2^z \big(S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big) = z'\big).
\end{aligned} \tag{3.69}$$

Taking sum for $z' \in \partial B(z, \frac{\epsilon l n}{2})$ and $0 \le r \le \log_2 l - 3$, by (3.23), we see that

$$P_1^0 \otimes P_2^z \otimes P_3^w \big(F^1, q^{(1)} \le \log_2 l - 3\big) \le \frac{C}{\epsilon n} \frac{1}{\epsilon l n} \frac{1}{n} \frac{1}{\epsilon n} \operatorname{Es}(\epsilon n, n) \operatorname{Es}(\epsilon n, \epsilon l n). \tag{3.70}$$

(Recall that F^1 was defined in (3.20).) For the case that $q^{(1)} \ge \log_2 l - 3$, by the same argument as above, one can prove that

$$P_1^0 \otimes P_2^z \otimes P_3^w \big(F^1, q^{(1)} \ge \log_2 l - 3\big) \le \frac{C}{\epsilon n} \frac{1}{\epsilon l n} \frac{1}{n} \frac{1}{\epsilon n} \operatorname{Es}(\epsilon n, n) \operatorname{Es}(\epsilon n, \epsilon l n). \tag{3.71}$$

(We shall omit the proof of (3.71) and leave it to the reader.) Taking sum for $z \in \partial(\epsilon n B'_x)$ and $w \in \partial(\epsilon n B'_y)$, by (3.19), we have

$$P\big(\mathrm{LE}\big(S\big[0, T^x\big]\big)\big[0, \overline{\sigma}'_1\big] \cap S\big[T^x, \tau_n\big] = \varnothing, \mathrm{LE}\big(S\big[0, T^y\big]\big)\big[0, \overline{\sigma}'_2\big] \cap S\big[T^y, \tau_n\big] = \varnothing, T^x < T^y < \tau_n\big)$$
$$\leq \frac{C\epsilon}{l} \operatorname{Es}(\epsilon n, n) \operatorname{Es}(\epsilon n, \epsilon l n). \tag{3.72}$$

(Note that $\sharp\{z \in \partial(\epsilon n B'_x) \cap \mathbb{Z}^3\} \leq C(\epsilon n)^2$.) Combining (3.72) with (3.12) and (3.18), we finish the proof of Theorem 3.1. □

3.2. *Estimates of the number of boxes hit by $\mathcal{K}$*

Now we are ready to estimate the first and the second moment of the number of cubes hit by $\mathcal{K}$. For $\epsilon > 0$, let

$$Y^\epsilon := \sharp\big\{x \in \mathbb{Z}^3 \mid \epsilon B_x \subset D_{\frac{2}{3}} \setminus D_{\frac{1}{3}}, \mathcal{K} \cap \epsilon B_x \neq \varnothing\big\}. \tag{3.73}$$

(Recall that B_x was defined in (3.1).) In this subsection, we will give a lower bound of Y^ϵ in Corollary 3.11. In order to prove it, we first estimate the second moment of Y^ϵ (see Corollary 3.9 below) using Theorem 3.1. Then we also give a lower bound of $E(Y^\epsilon)$ in Proposition 3.10, and using the second moment method we get Corollary 3.11 in the end of this subsection.

Theorem 3.1 and estimates of escape probabilities introduced as in Section 2.2 immediately show the following corollary, which gives a second moment estimate of Y^ϵ.

Corollary 3.9. *Take $\epsilon > 0$ and fix $n = n_\epsilon = 2^{j_\epsilon}$ such that (3.3) holds. Then there exists an absolute constant $C < \infty$ such that*

$$E\big(\big(Y^\epsilon\big)^2\big) \leq C\big\{\epsilon^{-2} \operatorname{Es}(\epsilon n, n)\big\}^2. \tag{3.74}$$

Proof. By Theorem 3.1, we have

$$E\big(\big(Y^\epsilon\big)^2\big) \leq \sum_{|x|,|y| \in [\frac{1}{3\epsilon}, \frac{2}{3\epsilon}]} P(\mathcal{K} \cap \epsilon B_x \neq \varnothing, \mathcal{K} \cap \epsilon B_y \neq \varnothing)$$
$$\leq C \sum_{|x| \in [\frac{1}{3\epsilon}, \frac{2}{3\epsilon}]} \sum_{l=1}^{\frac{2}{\epsilon}} l^2 \operatorname{Es}(\epsilon n, l\epsilon n) \operatorname{Es}(\epsilon n, n) \frac{\epsilon}{l} + \sum_{|x| \in [\frac{1}{3\epsilon}, \frac{2}{3\epsilon}]} P(\mathcal{K} \cap \epsilon B_x \neq \varnothing)$$
$$\leq C\epsilon^{-2} \operatorname{Es}(\epsilon n, n) \sum_{l=1}^{\frac{2}{\epsilon}} l \operatorname{Es}(\epsilon n, l\epsilon n). \tag{3.75}$$

By Lemma 2.8, for any $\delta > 0$ there exists $C = C_\delta < \infty$ such that

$$(\epsilon l n)^{\alpha+\delta} \operatorname{Es}(\epsilon l n) \leq C_\delta n^{\alpha+\delta} \operatorname{Es}(n).$$

Dividing both sides by $(\epsilon l n)^{\alpha+\delta} \operatorname{Es}(\epsilon n)$ and using (2.14), we have

$$\operatorname{Es}(\epsilon n, \epsilon l n) \leq C_\delta \epsilon^{-\alpha-\delta} l^{-\alpha-\delta} \operatorname{Es}(\epsilon n, n).$$

Fix $\delta > 0$ so that $1 - \alpha - \delta > 0$. Combining this with (3.75), we have

$$\begin{aligned}
&\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)\sum_{l=1}^{\frac{2}{\epsilon}} l\,\mathrm{Es}(\epsilon n, l\epsilon n) \\
&\quad \le C_\delta \epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)\sum_{l=1}^{\frac{2}{\epsilon}} l\epsilon^{-\alpha-\delta} l^{-\alpha-\delta}\,\mathrm{Es}(\epsilon n, n) \\
&\quad = C_\delta \epsilon^{-2-\alpha-\delta}\,\mathrm{Es}(\epsilon n, n)^2 \sum_{l=1}^{\frac{2}{\epsilon}} l^{1-\alpha-\delta} \le C_\delta \epsilon^{-2-\alpha-\delta}\,\mathrm{Es}(\epsilon n, n)^2 \epsilon^{-2+\alpha+\delta} = C_\delta \epsilon^{-4}\,\mathrm{Es}(\epsilon n, n)^2,
\end{aligned} \tag{3.76}$$

which finishes the proof. □

Take $\epsilon > 0$. Fix $n = n_\epsilon = 2^{j_\epsilon}$ such that (3.3) holds. Recall that $B'_x := \prod_{i=1}^{3}[x_i - 2, x_i + 2]$ was defined just before (3.11). In the proof of Lemma 7.1 [16], it was shown that

$$P\big(\mathrm{LE}\big(S[0, \tau_n]\big) \cap \epsilon n B'_x \neq \varnothing\big) \le C\epsilon\,\mathrm{Es}(\epsilon n, n),$$

for $x \in \mathbb{Z}^3$ with $\frac{1}{3} \le |\epsilon x| \le \frac{2}{3}$. Using this and (3.3), we have

$$\begin{aligned}
&P(\mathcal{K} \cap \epsilon B_x \neq \varnothing) \\
&\quad \le P\big(\mathcal{K} \cap \epsilon B_x \neq \varnothing, d_{\mathrm{H}}(\mathrm{LEW}_n, \mathcal{K}) < \epsilon^2\big) + P\big(\mathcal{K} \cap \epsilon B_x \neq \varnothing, d_{\mathrm{H}}(\mathrm{LEW}_n, \mathcal{K}) \ge \epsilon^2\big) \\
&\quad \le P\big(\mathrm{LEW}_n \cap \epsilon B'_x \neq \varnothing\big) + \epsilon^{100} \le C\epsilon\,\mathrm{Es}(\epsilon n, n).
\end{aligned}$$

So we see that

$$E\big(Y^\epsilon\big) \le C\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n). \tag{3.77}$$

In the next proposition, we will give the lower bound of $E(Y^\epsilon)$. As Remark 7.2 [16] states, its proof is almost included in the proof of Lemma 7.1 [16]. However we will give the proof for completeness.

Proposition 3.10. *Take $\epsilon > 0$ and fix $n = n_\epsilon = 2^{j_\epsilon}$ such that (3.3) holds. Then there exists an absolute constant $c > 0$ such that*

$$E\big(Y^\epsilon\big) \ge c\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n). \tag{3.78}$$

Proof. Take $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ with $\frac{1}{3} \le |\epsilon x| \le \frac{2}{3}$. Let $x' = (x_1 + \frac{1}{2}, x_2 + \frac{1}{2}, x_3 + \frac{1}{2})$ be the center of B_x. Let $y = \epsilon n x'$. We write $B_1 = B(y, r\epsilon n)$, $B_2 = B(y, \frac{\epsilon n}{3})$ and $B_3 = B(y, \frac{\epsilon n}{2})$ throughout the proof, where $0 < r \ll 1$ is a small constant which will be fixed later.

Using (3.3), it suffices to show that

$$P\big(\mathrm{LE}\big(S[0, \tau_n]\big) \cap B_3 \neq \varnothing\big) \ge c\epsilon\,\mathrm{Es}(\epsilon n, n), \tag{3.79}$$

Let $T := \max\{t \mid S(t) \in B_1\}$ and $\tau := \min\{t \mid \mathrm{LE}(S[0, T])(t) \in B_3\}$. Then we have

$$P\big(\mathrm{LE}\big(S[0, \tau_n]\big) \cap B_3 \neq \varnothing\big) \ge P\big(T < \tau_n, \mathrm{LE}\big(S[0, T]\big)[0, \tau] \cap S[T+1, \tau_n] = \varnothing\big).$$

By the decomposition as in (3.19) and reversing a path, we have

$$\begin{aligned}
&P\big(T < \tau_n, \mathrm{LE}\big(S[0, T]\big)[0, \tau] \cap S[T+1, \tau_n] = \varnothing\big) \\
&\quad \ge \sum_{z \in \partial_i B_1} P_1^z \otimes P_2^z\big(\tau_0^1 < \tau_n^1, S^2\big[1, \tau_n^2\big] \cap B_1 = \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big),
\end{aligned}$$

where $\tau_0^1 = \inf\{t \mid S^1(t) = 0\}$, $\sigma_1 = \max\{t \mid \mathrm{LE}(S^1[0, \tau_0^1])(t) \in B_3\}$ and $\sigma_2 = \mathrm{len}\,\mathrm{LE}(S^1[0, \tau_0^1])$.

Let $\tau^2_{B_2} := \{t \mid S^2(t) \in \partial B_2\}$. Then for each $z \in \partial_i B_1$,

$$\begin{aligned}
&P_1^z \otimes P_2^z\big(\tau_0^1 < \tau_n^1, S^2\big[1, \tau_n^2\big] \cap B_1 = \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\quad \geq P_2^z\big(S^2\big[1, \tau_{B_2}^2\big] \cap B_1 = \varnothing\big) \\
&\qquad \times E_1^z\Big\{\mathbf{1}_{\{\tau_0^1 < \tau_n^1\}} \min_{w \in \partial B_2} P_2^w\big(S^2\big[0, \tau_n^2\big] \cap B_1 = \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big)\Big\}.
\end{aligned}$$

However, by the Harnack principle (see Theorem 1.7.6 [8]), there exists $c_1 > 0$ such that for any $w \in \partial B_2$,

$$\begin{aligned}
&P_2^w\big(S^2\big[0, \tau_n^2\big] \cap B_1 = \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\quad = P_2^w\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\qquad - P_2^w\big(S^2\big[0, \tau_n^2\big] \cap B_1 \neq \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\quad \geq c_1 P_2^z\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\qquad - P_2^w\big(S^2\big[0, \tau_n^2\big] \cap B_1 \neq \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big).
\end{aligned}$$

On the other hand, by Proposition 1.5.10 [8] and the Harnack principle again, there exists $C_2 < \infty$ such that

$$\begin{aligned}
&P_2^w\big(S^2\big[0, \tau_n^2\big] \cap B_1 \neq \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\quad \leq C_2 r \max_{z' \in \partial B_1} P_2^{z'}\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\quad \leq C_2 r \frac{1}{c_1} P_2^z\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big).
\end{aligned}$$

Now we take $r > 0$ sufficiently small so that $C_2 r \frac{1}{c_1} \leq \frac{c_1}{2}$. Then we have

$$\begin{aligned}
&P_2^w\big(S^2\big[0, \tau_n^2\big] \cap B_1 = \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\quad \geq \frac{c_1}{2} P_2^z\big(\mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&P_1^z \otimes P_2^z\big(\tau_0^1 < \tau_n^1, S^2\big[1, \tau_n^2\big] \cap B_1 = \varnothing, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\quad \geq \frac{c}{\epsilon n} P_1^z \otimes P_2^z\big(\tau_0^1 < \tau_n^1, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big).
\end{aligned}$$

Let $B = B(z, \frac{n}{4})$ and $\tau_i = \inf\{t \mid S^i(t) \in \partial B\}$ for $i = 1, 2$. We write $\gamma = \mathrm{LE}(S^1[0, \tau_1])$ and let $\sigma := \max\{t \mid \gamma(t) \in B_3\}$. We define events F and G by

$$\begin{aligned}
F &= \left\{\mathrm{dist}\big(\gamma(\mathrm{len}\,\gamma), S^2[0, \tau_2]\big) \geq \frac{n}{12}, \mathrm{dist}\big(\gamma[\sigma, \mathrm{len}\,\gamma), S^2(\tau_2)\big) \geq \frac{n}{12}\right\}, \\
G &= \left\{\gamma[0, \sigma] \cap B\left(\gamma(\mathrm{len}\,\gamma), \frac{n}{12}\right) = \varnothing\right\}.
\end{aligned}$$

By Proposition 1.5.10 [8], we have

$$P_1^z \otimes P_2^z\big(G^c\big) \leq \frac{C\epsilon}{n}.$$

By the strong Markov property,

$$\begin{aligned}
&P_1^z \otimes P_2^z\big(\tau_0^1 < \tau_n^1, \mathrm{LE}\big(S^1\big[0, \tau_0^1\big]\big)[\sigma_1, \sigma_2] \cap S^2\big[0, \tau_n^2\big] = \varnothing\big) \\
&\quad \geq P_1^z \otimes P_2^z\bigg(\tau_0^1 < \tau_n^1, F, G, \gamma[\sigma, \operatorname{len}\gamma) \cap S^2[0, \tau_2] = \varnothing, S^1\big[\tau_1, \tau_0^1\big] \cap S^2\big[\tau_2, \tau_n^2\big] = \varnothing, \\
&\qquad \big(S^1\big[\tau_1, \tau_0^1\big] \cap B\big) \subset B\bigg(S^1(\tau_1), \frac{n}{12}\bigg), \big(S^2\big[\tau_2, \tau_n^2\big] \cap B\big) \subset B\bigg(S^2(\tau_2), \frac{n}{12}\bigg)\bigg) \\
&\quad \geq \frac{c}{n} P_1^z \otimes P_2^z\big(F, G, \gamma[\sigma, \operatorname{len}\gamma) \cap S^2[0, \tau_2] = \varnothing\big) \\
&\quad \geq \frac{c}{n} P_1^z \otimes P_2^z\big(F, \gamma[\sigma, \operatorname{len}\gamma) \cap S^2[0, \tau_2] = \varnothing\big) - \frac{C\epsilon}{n}.
\end{aligned}$$

However, by Lemma 2.10 and 2.11, we have

$$P_1^z \otimes P_2^z\big(F, \gamma[\sigma, \operatorname{len}\gamma) \cap S^2[0, \tau_2] = \varnothing\big) \geq c P_1^z \otimes P_2^z\big(\gamma[\sigma, \operatorname{len}\gamma) \cap S^2[0, \tau_2] = \varnothing\big) \geq c \operatorname{Es}(\epsilon n, n).$$

Combining these estimates, we see that

$$P\big(\mathrm{LE}\big(S[0, \tau_n]\big) \cap B_3 \neq \varnothing\big) \geq \sum_{z \in \partial_i B_1} \frac{c}{\epsilon n} \frac{1}{n} \operatorname{Es}(\epsilon n, n) \geq c\epsilon \operatorname{Es}(\epsilon n, n), \tag{3.80}$$

which finishes the proof. □

By Corollary 3.9, Proposition 3.10 and the second moment method, we get the following lower bound of Y^ϵ.

Corollary 3.11. *Take $\epsilon > 0$ and fix $n = n_\epsilon = 2^{j_\epsilon}$ such that (3.3) holds. Then there exists an absolute constant $c > 0$ such that*

$$P\big(Y^\epsilon \geq c\epsilon^{-2} \operatorname{Es}(\epsilon n, n)\big) \geq c. \tag{3.81}$$

4. Tightness of $\frac{Y^\epsilon}{E(Y^\epsilon)}$

As we discussed in Section 1.2, in order to show that the Hausdorff dimension of $\mathcal{K}$ is equal to $2 - \alpha$ almost surely (α is the exponent as in Theorem 2.7), we need to improve Corollary 3.11, i.e., we have to prove that for all $r > 0$ there exists $c_r > 0$ such that

$$P\big(Y^\epsilon \geq c_r \epsilon^{-2} \operatorname{Es}(\epsilon n, n)\big) \geq 1 - r, \tag{4.1}$$

where $\epsilon > 0$ is an arbitrary positive number and $n = n_\epsilon = 2^{j_\epsilon}$ is an integer satisfying (3.3).

In order to prove (4.1), again we use the coupling of $\mathcal{K}$ and LEW_n explained as in Section 1.2. Then (4.1) boils down to the corresponding estimates for LERW as follows. Let Y_n^ϵ be the number of ϵn-cubes nb_x with $\frac{1}{3} \leq |\epsilon x| \leq \frac{2}{3}$ such that $\mathrm{LE}(S[0, \tau_n])$ hits nb_x. Then (4.1) is reduced to proving that for all $r > 0$ there exists $c_r > 0$ such that

$$P\big(Y_n^\epsilon \geq c_r \epsilon^{-2} \operatorname{Es}(\epsilon n, n)\big) \geq 1 - r. \tag{4.2}$$

To show (4.2), we will use "iteration arguments" as in the proof of Theorem 6.7 of [1] and Theorem 8.2.6 of [18] where exponential lower tail bounds of M_n were established for $d = 2$ ([1]) and $d = 3$ ([18]). We explain it here. Take integer N. Define a sequence of boxes A_i by $A_i = [-\frac{n}{3} - \frac{in}{N}, \frac{n}{3} + \frac{in}{N}]^3$ for $0 \leq i \leq \frac{N}{6}$. We write $\gamma = \mathrm{LE}(S[0, \tau_n])$ and let $\tau(i) = \tau^\gamma(i)$ be the first time that γ exits from A_i. It turns out that the expected number of ϵn-cubes hit by $\gamma_i := \gamma[\tau(i), \tau(i+1)]$ is of order $(\epsilon N)^{-2} \operatorname{Es}(\epsilon n, \frac{n}{N})$ for each i. Conditioned on $\gamma[0, \tau(i)] = \lambda$ for a given path λ, we are interested in the probability that the number of ϵn-cubes hit by γ_i is bigger than $c_1(\epsilon N)^{-2} \operatorname{Es}(\epsilon n, \frac{n}{N})$ (we denote this probability by $p(\lambda)$). The domain Markov property (see Lemma 2.3) tells that we need to study a random walk conditioned not to intersect λ. We will study such a conditioned random walk in Section 4.1 and show that there exists a universal constant $c_1 > 0$ which does not depend on λ such that the probability $p(\lambda)$ above is larger than c_1 for every i (see Lemma 4.4). Using this and the domain Markov property, we have

$$P\left(Y_n^\epsilon \leq c_1(\epsilon N)^{-2}\,\mathrm{Es}\left(\epsilon n, \frac{n}{N}\right)\right) \leq (1-c_1)^{\frac{N}{6}}. \tag{4.3}$$

Since $\mathrm{Es}(\epsilon n, \frac{n}{N}) \geq c_2\,\mathrm{Es}(\epsilon n, n)$ for some absolute constant $c_2 > 0$, taking $N = N_r$ such that $(1-c_1)^{\frac{N_r}{6}} < r$ first, then letting $c_r := c_1 c_2 N_r^{-2}$, we get (4.2) and (4.1) (see Proposition 4.5 and 4.6).

4.1. *Loop-erasure of conditioned random walks*

Given a box and a simple path γ contained in the inside of the box except the end point $\gamma(\operatorname{len}\gamma)$ which is lying on the boundary of the box. Following same spirits of Theorem 6.7 [1] and Theorem 8.2.6 [18], we are interested in a random walk X staring from $\gamma(\operatorname{len}\gamma)$ conditioned that $X[1,\tau] \cap \gamma = \emptyset$ for some stopping time τ. Estimates of such a conditioned random walk X are crucial to prove (4.1). In this subsection, we will study X.

We begin with some notation.

Definition 4.1. Let $M \geq 20$. Fix $\epsilon > 0$ and take $n = n_\epsilon = 2^{j_\epsilon}$ such that (3.3) holds. Define

$$\begin{aligned} &k_i = \frac{1}{3} + \frac{i}{M} \quad \text{for } i = 0, 1, \dots, \frac{M}{20}, \\ &D(i) = [-k_i, k_i]^3, \qquad A(i) = D(i+1) \setminus D(i), \qquad D_{i,n} = nD(i) \cap \mathbb{Z}^3, \qquad A_{i,n} = nA(i) \cap \mathbb{Z}^3. \end{aligned} \tag{4.4}$$

Take $i \in \{0, 1, \dots, \frac{M}{20}\}$. Suppose that $\gamma = \gamma_i$ is a simple path in $\mathbb{Z}^3$ with $\gamma(0) = 0$, $\gamma[0, \operatorname{len}\gamma - 1] \subset D_{i,n}$ and $\gamma(\operatorname{len}\gamma) \in \partial D_{i,n}$. Let $v = \gamma(\operatorname{len}\gamma)$. We denote a face of $\partial D_{i,n}$ containing v by π_1. Let ℓ_1 be the line segment starting at v and terminating at $\partial D_{i+1,n}$ which is perpendicular to $\partial D_{i,n}$. We denote the middle point of ℓ_1 by o_1. We define a set $F_{i,n}^\gamma$ by $F_{i,n}^\gamma := (o_1 + [-\frac{n}{8M}, \frac{n}{8M}]^3) \cap \mathbb{Z}^3$.

Let $X = X^\gamma$ be the random walk conditioned to hit ∂B_n before hitting γ, i.e., X is the simple random walk S started at v conditioned on $\{S[1,\tau_n] \cap \gamma = \emptyset\}$.

Suppose that $x, y \in \mathbb{Z}^3$ satisfy $\epsilon n B'_x \subset F_{i,n}^\gamma$ and $\epsilon n B'_y \subset F_{i,n}^\gamma$. ($B'_x$ was defined just before (3.11).) Let $l := |x-y|$. As in (3.12), we are interested in

$$P_X\big(\mathrm{LE}\big(X\big[0, \tau_n^X\big]\big) \cap \epsilon n B'_x \neq \emptyset, \mathrm{LE}\big(X\big[0, \tau_n^X\big]\big) \cap \epsilon n B'_y \neq \emptyset\big), \tag{4.5}$$

where we write P_X for the probability law of X and let $\tau_n^X := \inf\{t \mid X(t) \in \partial B_n\}$. For this probability, we have the following lemma, which is an analog of Theorem 3.1 for the probability that the loop erasure of X hits two distinct cubes.

Lemma 4.2. *Let X be the conditioned random walk defined in Definition 4.1 and suppose that $x, y \in \mathbb{Z}^3$ satisfy $\epsilon n B'_x \subset F_{i,n}^\gamma$ and $\epsilon n B'_y \subset F_{i,n}^\gamma$ (see Definition 4.1 for $F_{i,n}^\gamma$). Then there exists an absolute constant $C < \infty$ such that*

$$P_X\big(\mathrm{LE}\big(X\big[0, \tau_n^X\big]\big) \cap \epsilon n B'_x \neq \emptyset, \mathrm{LE}\big(X\big[0, \tau_n^X\big]\big) \cap \epsilon n B'_y \neq \emptyset\big) \leq \frac{C\epsilon M}{l}\,\mathrm{Es}(\epsilon n, \epsilon l n)\,\mathrm{Es}\left(\epsilon n, \frac{n}{M}\right), \tag{4.6}$$

where $l = |x-y|$.

Proof. Throughout the proof, we will use same notation defined in the proof of Theorem 3.1.

Define

$$T_X^x := \max\big\{t \leq \tau_n^X \mid X(t) \in \partial\big(\epsilon n B'_x\big)\big\}, \qquad T_X^y := \max\big\{t \leq \tau_n^X \mid X(t) \in \partial\big(\epsilon n B'_y\big)\big\}. \tag{4.7}$$

As in (3.12), it suffices to estimate

$$
\begin{aligned}
&P_X\big(\mathrm{LE}\big(X\big[0,\tau_n^X\big]\big)\cap \epsilon n B'_x\neq\varnothing, \mathrm{LE}\big(X\big[0,\tau_n^X\big]\big)\cap \epsilon n B'_y\neq\varnothing, T_X^x<T_X^y\big)\\
&\quad=\frac{P^v(\mathrm{LE}(S[0,\tau_n])\cap \epsilon n B'_x\neq\varnothing, \mathrm{LE}(S[0,\tau_n])\cap \epsilon n B'_y\neq\varnothing, T^x<T^y, S[1,\tau_n]\cap\gamma=\varnothing)}{P^v(S[1,\tau_n]\cap\gamma=\varnothing)}. \qquad (4.8)
\end{aligned}
$$

By the last exit decomposition as in (3.19), we have

$$
\begin{aligned}
&P^v\big(\mathrm{LE}\big(S[0,\tau_n]\big)\cap \epsilon n B'_x\neq\varnothing, \mathrm{LE}\big(S[0,\tau_n]\big)\cap \epsilon n B'_y\neq\varnothing, T^x<T^y, S[1,\tau_n]\cap\gamma=\varnothing\big)\\
&\quad\le C\sum_{z\in\partial(\epsilon n B'_x)}\sum_{w\in\partial(\epsilon n B'_y)} P_1^v\otimes P_2^z\otimes P_3^w\big(F^1, S^1\big[1,\sigma_z^1\big]\cap\gamma=\varnothing, S^3\big[0,\tau_n^3\big]\cap\gamma=\varnothing\big), \qquad (4.9)
\end{aligned}
$$

where σ_z^1 and F^1 were defined as in (3.13) and (3.20), respectively.

Let

$$
W:=\left(o_1+\left[-\frac{n}{4M},\frac{n}{4M}\right]^3\right)\cap\mathbb{Z}^3, \qquad (4.10)
$$

where o_1 was defined in Definition 4.1. Then

$$
\begin{aligned}
&P_1^v\otimes P_2^z\otimes P_3^w\big(F^1, S^1\big[1,\sigma_z^1\big]\cap\gamma=\varnothing, S^3\big[0,\tau_n^3\big]\cap\gamma=\varnothing\big)\\
&\quad\le P_1^v\otimes P_2^z\otimes P_3^w\big(F_\star^1, S^1\big[1,\sigma_z^1\big]\cap\gamma=\varnothing, S^3\big[\tau_{\partial W}^3,\tau_n^3\big]\cap\gamma=\varnothing\big), \qquad (4.11)
\end{aligned}
$$

where $\tau_{\partial W}^3=\{t\mid S^3(t)\in\partial W\}$ and

$$
\begin{aligned}
F_\star^1:=\big\{&\sigma_z^1<\tau_n^1, \sigma_w^2<\tau_n^2, S^2\big[1,T_{z,6\epsilon n}^2\big]\cap\overline{(\epsilon n B'_x)}=\varnothing, S^3\big[1,T_{w,6\epsilon n}^3\big]\cap\overline{(\epsilon n B'_y)}=\varnothing\\
&\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap\big(S^2\big[0,\sigma_w^2\big]\cup S^3\big[0,\tau_{\partial W}^3\big]\big)=\varnothing,\\
&\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]+S^2\big[0,\sigma_w^2\big]\big)[0,\overline{\sigma}_2]\cap S^3\big[0,\tau_{\partial W}^3\big]=\varnothing\big\}. \qquad (4.12)
\end{aligned}
$$

(See (3.14) for $\overline{\sigma}_1$ and $\overline{\sigma}_2$.) By the strong Markov property,

$$
\begin{aligned}
&P_1^v\otimes P_2^z\otimes P_3^w\big(F_\star^1, S^1\big[1,\sigma_z^1\big]\cap\gamma=\varnothing, S^3\big[\tau_{\partial W}^3,\tau_n^3\big]\cap\gamma=\varnothing\big)\\
&\quad\le P_1^v\otimes P_2^z\otimes P_3^w\big(F_\star^1, S^1\big[1,\sigma_z^1\big]\cap\gamma=\varnothing\big)\max_{w_\star\in\partial W} P_3^{w_\star}\big(S^3\big[0,\tau_n^3\big]\cap\gamma=\varnothing\big). \qquad (4.13)
\end{aligned}
$$

Recall that $q^{(1)}$ was defined as in (3.21) (we use the same notation here). Suppose that $0\le r\le\log_2 l-3$. We will first deal with $P_1^v\otimes P_2^z\otimes P_3^w(F_\star^1, S^1[1,\sigma_z^1]\cap\gamma=\varnothing, q^{(1)}=r)$. However, as in (3.53), we have

$$
\begin{aligned}
&P_1^v\otimes P_2^z\otimes P_3^w\big(F_\star^1, S^1\big[1,\sigma_z^1\big]\cap\gamma=\varnothing, q^{(1)}=r\big)\\
&\quad\le\frac{C}{\epsilon n}\frac{1}{\epsilon l n}\mathrm{Es}(\epsilon n,\epsilon l n)\\
&\qquad\times\sum_{z'\in\partial B(z,\frac{\epsilon l n}{2})} E_1^v\otimes E_2^z\Big\{\mathbf{1}_{F_\star^2\cap\{S^1[1,\sigma_z^1]\cap\gamma=\varnothing\}}\max_{w_1\in\partial B(w,\frac{\epsilon l n}{4})} P_3^{w_1}\big(\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap S^3\big[0,\tau_{\partial W}^3\big]=\varnothing\big)\Big\},\\
&\qquad\qquad (4.14)
\end{aligned}
$$

where

$$
\begin{aligned}
F_\star^2=\big\{&\sigma_z^1<\tau_n^1, S^2\big[1,T^2_{z,\frac{l\epsilon n}{2}}\big]\cap\overline{(\epsilon n B'_x)}=\varnothing,\\
&\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap S^2\big[0,T^2_{z,\frac{l\epsilon n}{2}}\big]=\varnothing, q^{(1)}=r, S^2\big(T^2_{z,\frac{l\epsilon n}{2}}\big)=z'\big\}. \qquad (4.15)
\end{aligned}
$$

(Recall that in order to show (4.14), we have to estimate $\tilde{p}_1$ defined as in (3.30). Note that we don't need to care about the "non-intersecting with γ" conditions as long as we deal with $\tilde{p}_1$.)

Using the time reversibility of LERW (see Lemma 2.2) as in (3.54), we see that

$$E_1^v \otimes E_2^z\Big\{\mathbf{1}_{F_\star^2\cap\{S^1[1,\sigma_z^1]\cap\gamma=\varnothing\}} \max_{w_1\in\partial B(w,\frac{\epsilon ln}{4})} P_3^{w_1}\big(\mathrm{LE}\big(S^1\big[0,\sigma_z^1\big]\big)[0,\overline{\sigma}_1]\cap S^3\big[0,\tau_{\partial W}^3\big]=\varnothing\big)\Big\}$$

$$= E_1^z \otimes E_2^z\Big\{\mathbf{1}_{\tilde{F}_\star^2\cap\{S^1[0,\tau_v^1]\cap(\gamma\setminus\{v\})=\varnothing\}} \max_{w_1\in\partial B(w,\frac{\epsilon ln}{4})} P_3^{w_1}\big(\mathrm{LE}\big(S^1\big[0,\tau_v^1\big]\big)[\tilde{u}_1,\tilde{u}_2]\cap S^3\big[0,\tau_{\partial W}^3\big]=\varnothing\big)\Big\}, \tag{4.16}$$

where

$$\tilde{u}_1 := \max\big\{t \mid \mathrm{LE}\big(S^1\big[0,\tau_v^1\big]\big)(t)\in\partial\big(\epsilon nB_x'\big)\big\}, \qquad \tilde{u}_2 := \mathrm{len}\,\mathrm{LE}\big(S^1\big[0,\tau_v^1\big]\big), \tag{4.17}$$

and

$$\begin{aligned}\tilde{F}_\star^2 := \big\{&\tau_v^1<\tau_n^1,\ S^2\big[1,T_{z,\frac{l\epsilon n}{2}}^2\big]\cap\overline{\big(\epsilon nB_x'\big)}=\varnothing,\ S^2\big(T_{z,\frac{l\epsilon n}{2}}^2\big)=z',\\ &\mathrm{LE}\big(S^1\big[0,\tau_v^1\big]\big)[\tilde{u}_1,\tilde{u}_2]\cap S^2\big[0,T_{z,\frac{l\epsilon n}{2}}^2\big]=\varnothing,\ \mathrm{LE}\big(S^1\big[0,\tau_v^1\big]\big)[0,\tilde{u}_1]\cap A_z^r\neq\varnothing,\\ &\mathrm{LE}\big(S^1\big[0,\tau_v^1\big]\big)[0,\tilde{u}_1]\subset B\big(z,2^r\epsilon n\big)\big\}.\end{aligned} \tag{4.18}$$

Let $\beta=\mathrm{LE}(S^1[0,\tau_v^1])$ and $\tau_{z,\frac{n}{12M}}^\beta := \inf\{t \mid \beta(t)\in\partial B(z,\frac{n}{12M})\}$. Since $r\le\log_2 l-3$, on $\tilde{F}_\star^2$, we have

$$\beta\big[\tau_{z,\frac{n}{12M}}^\beta,\tilde{u}_2\big]\cap\overline{\big(\epsilon nB_x'\big)}=\varnothing. \tag{4.19}$$

Indeed, if $\beta[\tau_{z,\frac{n}{12M}}^\beta,\tilde{u}_2]\cap\overline{(\epsilon nB_x')}\neq\varnothing$, then $\tau_{z,\frac{n}{12M}}^\beta<\tilde{u}_1$, which implies that $\beta[0,\tilde{u}_1]\cap\partial B(z,\frac{n}{12M})\neq\varnothing$. Since $\epsilon nB_x'\subset F_{i,n}^\gamma$ and $\epsilon nB_y'\subset F_{i,n}^\gamma$, we have $|\epsilon nx-\epsilon ny|\le\frac{\sqrt{3}n}{4M}$ which implies that $\epsilon l\le\frac{\sqrt{3}}{4M}$. Thus

$$2^r\epsilon n\le 2^{\log_2 -3}\epsilon n\le\frac{1}{8}\epsilon ln\le\frac{\sqrt{3}n}{32M}<\frac{n}{12M},$$

which contradicts $\beta[0,\tilde{u}_1]\subset B(z,2^r\epsilon n)$. Therefore, we get (4.19). Thus if we define $\tilde{u}_1^\star$ by

$$\tilde{u}_1^\star := \max\big\{t\le\tau_{z,\frac{n}{12M}}^\beta \mid \mathrm{LE}\big(S^1\big[0,\tau_v^1\big]\big)(t)\in\partial\big(\epsilon nB_x'\big)\big\},$$

then $\tilde{u}_1=\tilde{u}_1^\star$. Since $\beta[0,\tilde{u}_1^\star]$ is $\beta[0,\tau_{z,\frac{n}{12M}}^\beta]$-measurable, by Proposition 4.2 and 4.4 [15], we see that

$$\begin{aligned}&E_1^z \otimes E_2^z\Big\{\mathbf{1}_{\tilde{F}_\star^2\cap\{S^1[0,\tau_v^1]\cap(\gamma\setminus\{v\})=\varnothing\}} \max_{w_1\in\partial B(w,\frac{\epsilon ln}{4})} P_3^{w_1}\big(\mathrm{LE}\big(S^1\big[0,\tau_v^1\big]\big)[\tilde{u}_1,\tilde{u}_2]\cap S^3\big[0,\tau_{\partial W}^3\big]=\varnothing\big)\Big\}\\ &\quad\le CP_1^z\big(\tau_v^1<\tau_n^1,\ S^1\big[0,\tau_v^1\big]\cap\big(\gamma\setminus\{v\}\big)=\varnothing\big)\\ &\qquad\times E_1^z\otimes E_2^z\Big\{\mathbf{1}\big[S^2\big[1,T_{z,\frac{l\epsilon n}{2}}^2\big]\cap\overline{\big(\epsilon nB_x'\big)}=\varnothing,\ S^2\big(T_{z,\frac{l\epsilon n}{2}}^2\big)=z',\\ &\qquad\tilde{\beta}[\tilde{u}_3,\tilde{u}_4]\cap S^2\big[0,T_{z,\frac{l\epsilon n}{2}}^2\big]=\varnothing,\ \tilde{\beta}[0,\tilde{u}_3]\cap A_z^r\neq\varnothing,\ \tilde{\beta}[0,\tilde{u}_3]\subset B\big(z,2^r\epsilon n\big)\big]\\ &\qquad\times\max_{w_1\in\partial B(w,\frac{\epsilon ln}{4})} P_3^{w_1}\big(\tilde{\beta}[\tilde{u}_3,\tilde{u}_4]\cap S^3\big[0,\tau_{\partial W}^3\big]=\varnothing\big)\Big\},\end{aligned} \tag{4.20}$$

where $\tilde{\beta}=\mathrm{LE}(S^1[0,T_{z,\frac{n}{3M}}^1])$, $\tilde{u}_4=\inf\{t \mid \tilde{\beta}(t)\in\partial B(z,\frac{n}{12M})\}$ and $\tilde{u}_3=\max\{t\le\tilde{u}_4 \mid \tilde{\beta}(t)\in\partial(\epsilon nB_x')$.

However, as in (3.69), we have

$$\begin{aligned}
&(\text{RHS of (4.20)}) \\
&\quad \le C P_1^z\big(\tau_v^1 < \tau_n^1, S^1\big[0, \tau_v^1\big] \cap \big(\gamma \setminus \{v\}\big) = \varnothing\big) \\
&\qquad \times \frac{1}{\epsilon n} P_2^z\big(S^2\big(T^2_{z, \frac{l\epsilon n}{2}}\big) = z'\big) ES(\epsilon n, \epsilon l n) 2^{-\delta r} \operatorname{Es}\bigg(\epsilon l n, \frac{n}{M}\bigg).
\end{aligned} \tag{4.21}$$

Therefore, by (4.14),

$$\begin{aligned}
&P_1^v \otimes P_2^z \otimes P_3^w\big(F_\star^1, S^1\big[1, \sigma_z^1\big] \cap \gamma = \varnothing, q^{(1)} = r\big) \\
&\quad \le \frac{C}{\epsilon n} \frac{1}{\epsilon l n} \frac{1}{\epsilon n} 2^{-\delta r} ES(\epsilon n, \epsilon l n) \operatorname{Es}\bigg(\epsilon n, \frac{n}{M}\bigg) P_1^z\big(\tau_v^1 < \tau_n^1, S^1\big[0, \tau_v^1\big] \cap \big(\gamma \setminus \{v\}\big) = \varnothing\big).
\end{aligned} \tag{4.22}$$

Taking sum for $r \le \log_2 l - 3$, by (4.13),

$$\begin{aligned}
&P_1^v \otimes P_2^z \otimes P_3^w\big(F_\star^1, S^1\big[1, \sigma_z^1\big] \cap \gamma = \varnothing, S^3\big[\tau_{\partial W}^3, \tau_n^3\big] \cap \gamma = \varnothing, q^{(1)} \le \log_2 l - 3\big) \\
&\quad \le \frac{C}{\epsilon n} \frac{1}{\epsilon l n} \frac{1}{\epsilon n} ES(\epsilon n, \epsilon l n) \operatorname{Es}\bigg(\epsilon n, \frac{n}{M}\bigg) P_1^z\big(\tau_v^1 < \tau_n^1, S^1\big[0, \tau_v^1\big] \cap \big(\gamma \setminus \{v\}\big) = \varnothing\big) \\
&\qquad \times \max_{w_\star \in \partial W} P_3^{w_\star}\big(S^3\big[0, \tau_n^3\big] \cap \gamma = \varnothing\big).
\end{aligned} \tag{4.23}$$

Similar argument gives that

$$\begin{aligned}
&P_1^v \otimes P_2^z \otimes P_3^w\big(F_\star^1, S^1\big[1, \sigma_z^1\big] \cap \gamma = \varnothing, S^3\big[\tau_{\partial W}^3, \tau_n^3\big] \cap \gamma = \varnothing\big) \\
&\quad \le \frac{C}{\epsilon n} \frac{1}{\epsilon l n} \frac{1}{\epsilon n} ES(\epsilon n, \epsilon l n) \operatorname{Es}\bigg(\epsilon n, \frac{n}{M}\bigg) P_1^z\big(\tau_v^1 < \tau_n^1, S^1\big[0, \tau_v^1\big] \cap \big(\gamma \setminus \{v\}\big) = \varnothing\big) \\
&\qquad \times \max_{w_\star \in \partial W} P_3^{w_\star}\big(S^3\big[0, \tau_n^3\big] \cap \gamma = \varnothing\big).
\end{aligned} \tag{4.24}$$

By reversing the path, we see that

$$P_1^z\big(\tau_v^1 < \tau_n^1, S^1\big[0, \tau_v^1\big] \cap \big(\gamma \setminus \{v\}\big) = \varnothing\big) \le \frac{CM}{n} P^v\big(S[1, T_{v, \frac{n}{8M}}] \cap \gamma = \varnothing\big).$$

Recall that ℓ_1 was defined in Definition 4.1. Note that ℓ_1 intersects with $\partial B(v, \frac{n}{8M})$ at only one point. We call the point v'. Let $A := \partial B(v, \frac{n}{8M}) \cap B(v', \frac{n}{16M})$. By Proposition 6.1.1 [18] and by the Harnack principle (see Theorem 1.7.6 [8]),

$$\begin{aligned}
&P^v\big(S[1, \tau_n] \cap \gamma = \varnothing\big) \\
&\quad \ge P^v\big(S[1, T_{v, \frac{n}{8M}}] \cap \gamma = \varnothing, S(T_{v, \frac{n}{8M}}) \in A, \tau_{\partial W} < \tau_n, S[T_{v, \frac{n}{8M}}, \tau_{\partial W}] \cap \gamma = \varnothing, S[\tau_{\partial W}, \tau_n] \cap \gamma = \varnothing\big) \\
&\quad \ge c P^v\big(S[1, T_{v, \frac{n}{8M}}] \cap \gamma = \varnothing\big) \max_{w_\star \in \partial W} P_3^{w_\star}\big(S^3\big[0, \tau_n^3\big] \cap \gamma = \varnothing\big).
\end{aligned} \tag{4.25}$$

Thus by (4.24),

$$\begin{aligned}
&P_1^v \otimes P_2^z \otimes P_3^w\big(F_\star^1, S^1\big[1, \sigma_z^1\big] \cap \gamma = \varnothing, S^3\big[\tau_{\partial W}^3, \tau_n^3\big] \cap \gamma = \varnothing\big) \\
&\quad \le \frac{C}{\epsilon n} \frac{1}{\epsilon l n} \frac{1}{\epsilon n} \frac{M}{n} ES(\epsilon n, \epsilon l n) \operatorname{Es}\bigg(\epsilon n, \frac{n}{M}\bigg) P^v\big(S[1, T_{v, \frac{n}{8M}}] \cap \gamma = \varnothing\big) \\
&\qquad \times \max_{w_\star \in \partial W} P_3^{w_\star}\big(S^3\big[0, \tau_n^3\big] \cap \gamma = \varnothing\big) \\
&\quad \le \frac{C}{\epsilon n} \frac{1}{\epsilon l n} \frac{1}{\epsilon n} \frac{M}{n} ES(\epsilon n, \epsilon l n) \operatorname{Es}\bigg(\epsilon n, \frac{n}{M}\bigg) P^v\big(S[1, \tau_n] \cap \gamma = \varnothing\big).
\end{aligned} \tag{4.26}$$

Taking sum for $z \in \partial(\epsilon n B'_x)$ and $w \in \partial(\epsilon n B'_y)$, by (4.9),

$$P^v\big(\mathrm{LE}\big(S[0,\tau_n]\big) \cap \epsilon n B'_x \neq \emptyset, \mathrm{LE}\big(S[0,\tau_n]\big) \cap \epsilon n B'_y \neq \emptyset, T^x < T^y, S[1,\tau_n] \cap \gamma = \emptyset\big)$$
$$\leq \frac{C\epsilon M}{l} \mathrm{Es}(\epsilon n, \epsilon l n) \mathrm{Es}\left(\epsilon n, \frac{n}{M}\right) P^v\big(S[1,\tau_n] \cap \gamma = \emptyset\big).$$

Combining this with (4.8), we finish the proof. □

Next we will consider the lower bound of the probability that $\mathrm{LE}(X[0,\tau_n^X]) \cap \epsilon n B'_x \neq \emptyset$. Assume that γ is a simple path and X is a conditioned random walk defined as in Definition 4.1. Let

$$t_{i,n}^{\gamma} := \inf\big\{t \mid \mathrm{LE}\big(X\big[0,\tau_n^X\big]\big)(t) \in \partial D_{i+1,n}\big\}, \tag{4.27}$$

where $D_{i+1,n}$ was defined in Definition 4.1. Then we have the following lemma, which is an analog of (3.79) for the probability that the loop erasure of X hits a cube.

Lemma 4.3. *Suppose that $x \in \mathbb{Z}^3$ satisfies $\epsilon n B'_x \subset F_{i,n}^{\gamma}$ (see Definition 4.1 for $F_{i,n}^{\gamma}$). Then there exists an absolute constant $c > 0$ such that*

$$P_X\big(\mathrm{LE}\big(X\big[0,\tau_n^X\big]\big)\big[0,t_{i,n}^{\gamma}\big] \cap \epsilon n B'_x \neq \emptyset\big) \geq c\epsilon M \,\mathrm{Es}\left(\epsilon n, \frac{n}{M}\right). \tag{4.28}$$

Proof. Take $w \in \partial_i B(\epsilon n x, \frac{\epsilon n}{1000})$. Throughout the proof, we write

$$B_1 := B\left(\epsilon n x, \frac{\epsilon n}{1000}\right), \qquad B_2 := B\left(w, \frac{\epsilon n}{8}\right), \qquad B_3 := B\left(w, \frac{n}{4M}\right). \tag{4.29}$$

Suppose that S^1 and S^2 are independent simple random walks started at w. Let

$$t^i := \inf\big\{t \mid S^i(t) \in \partial B_3\big\}, \tag{4.30}$$

for each $i = 1, 2$. Recall that the line segment ℓ_1 was defined in Definition 4.1. We define random sets A_i as follows. Let ℓ^1 be the line segment started at $y_1 := S^1(t^1)$ and terminated at v. Define A_1 by $A_1 := \{y \mid \mathrm{dist}(y, \ell^1) \leq \frac{n}{20M}\}$. Let w^1 be the intersection point of the line segment connecting v with w and ∂B_3, and let $w^2 \in \partial B_3$ be the point such that $\frac{w^1+w^2}{2} = w$. Let ℓ^2 be the line segment starting from w^2 terminated at $\partial B(v, \frac{L_0 n}{M})$ which is parallel to ℓ_1. Here L_0 is a (large) constant which will be defined later. Define A_2 by $A_2 := \{y \mid \mathrm{dist}(y, \ell^2) \leq \frac{n}{20M}\}$. Let $\partial_2 := \partial A_2 \cap \{y \mid \mathrm{dist}(y, \partial B(v, \frac{L_0 n}{M})) \leq \frac{n}{20M}\}$. For each $i = 1, 2$, we write $u^i := \{t \geq t^i \mid S^i(t) \in \partial A_i\}$. Finally, let $H^i := B(w, \frac{n}{6M}) \cup B(w^i, \frac{n}{8M})$ for each $i = 1, 2$ and $\partial_1 := \partial B_3 \cap B(w^1, \frac{n}{8M})$.

We write

$$\sigma^1 := \max\big\{t \mid \mathrm{LE}\big(S^1\big[0,t^1\big]\big)(t) \in B_2\big\} \quad \text{and} \quad \sigma^2 := \mathrm{len}\,\mathrm{LE}\big(S^1\big[0,t^1\big]\big).$$

Let

$$\tau_{i,n} := \inf\big\{t \mid \mathrm{LE}\big(S[0,\tau_n]\big)(t) \in \partial D_{i+1,n}\big\}.$$

Then we have

$$P_X\big(\mathrm{LE}\big(X\big[0,\tau_n^X\big]\big)\big[0,t_{i,n}^{\gamma}\big] \cap \epsilon n B'_x \neq \emptyset\big) = \frac{P^v(\mathrm{LE}(S[0,\tau_n])[0,\tau_{i,n}] \cap \epsilon n B'_x \neq \emptyset, S[1,\tau_n] \cap \gamma = \emptyset)}{P^v(S[1,\tau_n] \cap \gamma = \emptyset)}. \tag{4.31}$$

By considering the last exit from B_1 and by reversing a path, we have

$$\begin{aligned}
&P^v\big(\mathrm{LE}\big(S[0,\tau_n]\big)[0,\tau_{i,n}]\cap \epsilon n B'_x\neq\varnothing,\ S[1,\tau_n]\cap\gamma=\varnothing\big)\\
&\geq \sum_{w\in\partial_i B_1} P_1^w\otimes P_2^w\bigg(S^2\big[1,t^2\big]\cap B_1=\varnothing,\mathrm{LE}\big(S^1\big[0,t^1\big]\big)\big[\sigma^1,\sigma^2\big]\cap S^2\big[0,t^2\big]=\varnothing,\\
&\quad \mathrm{LE}\big(S^1\big[0,t^1\big]\big)\big[\sigma^1,\sigma^2\big]\subset H^1,\mathrm{LE}\big(S^1\big[0,t^1\big]\big)\big[0,\sigma^1\big]\cap B\bigg(w^1,\frac{n}{8M}\bigg)=\varnothing, S^2\big[0,t^2\big]\subset H^2,\\
&\quad S^1\big[t^1,\tau_v^1\big]\subset A_1\cap\big(\gamma\setminus\{v\}\big)^c, S^2\big(u^2\big)\in\partial_2, S^2\big[u^2,\tau_n^2\big]\cap\gamma=\varnothing, S^2\big[u^2,\tau_n^2\big]\cap B\bigg(v,\frac{n}{M}\bigg)=\varnothing\bigg).
\end{aligned}\tag{4.32}$$

By using Lemma 2.10 and 2.11 as in (3.80) and by the strong Markov property, we have

$$\begin{aligned}
&\big(\text{The probability in RHS of (4.32)}\big)\\
&\geq \frac{c_{L_0}}{\epsilon n}\mathrm{Es}\bigg(\epsilon n,\frac{n}{M}\bigg)\\
&\quad\times\min_{y_1\in\partial_1,y_2\in\partial_2} P_1^{y_1}\big(S^1\big[0,\tau_v^1\big]\subset A_1\cap\big(\gamma\setminus\{v\}\big)^c\big)\\
&\quad\times P_2^{y_2}\bigg(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing, S^2\big[0,\tau_n^2\big]\cap B\bigg(v,\frac{n}{M}\bigg)=\varnothing\bigg).
\end{aligned}\tag{4.33}$$

But by reversing a path and by Proposition 6.1.1 [18], we see that for each $y_1\in\partial_1$,

$$P_1^{y_1}\big(S^1\big[0,\tau_v^1\big]\subset A_1\cap\big(\gamma\setminus\{v\}\big)^c\big)\geq\frac{cM}{n}P^v\big(S[1,T_{v,\frac{n}{30M}}]\cap\gamma=\varnothing\big).\tag{4.34}$$

On the other hand, by Lemma 6.1.2 [18] and the Harnack principle (see Theorem 1.7.6 [8]), there exists an absolute constant $C_0<\infty$ such that

$$\max_{y\in B(v,\frac{n}{M})}P^y\big(S[0,\tau_n]\cap\gamma=\varnothing\big)\leq C_0\min_{y\in\partial_2}P^y\big(S[0,\tau_n]\cap\gamma=\varnothing\big).\tag{4.35}$$

Now take L_0 such that $\frac{2C_0}{L_0}<\frac{1}{2}$. Then by the strong Markov property and Proposition 1.5.10 [8], we see that for each $y_2\in\partial_2$,

$$\begin{aligned}
&P_2^{y_2}\bigg(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing, S^2\big[0,\tau_n^2\big]\cap B\bigg(v,\frac{n}{M}\bigg)=\varnothing\bigg)\\
&\geq P_2^{y_2}\big(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing\big)-P_2^{y_2}\bigg(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing, S^2\big[0,\tau_n^2\big]\cap B\bigg(v,\frac{n}{M}\bigg)\neq\varnothing\bigg)\\
&\geq P_2^{y_2}\big(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing\big)-\frac{2}{L_0}\max_{y'\in B(v,\frac{n}{M})}P^{y'}\big(S[0,\tau_n]\cap\gamma=\varnothing\big)\\
&\geq P_2^{y_2}\big(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing\big)-\frac{2C_0}{L_0}\min_{y'\in\partial_2}P^{y'}\big(S[0,\tau_n]\cap\gamma=\varnothing\big)\\
&\geq P_2^{y_2}\big(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing\big)-\frac{2C_0}{L_0}P^{y}\big(S[0,\tau_n]\cap\gamma=\varnothing\big)\\
&\geq\frac{1}{2}\min_{y_2\in\partial_2}P_2^{y_2}\big(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing\big).
\end{aligned}\tag{4.36}$$

Again by Lemma 6.1.2 [18] and the Harnack principle (see Theorem 1.7.6 [8]), we have

$$\max_{y\in B(v,\frac{n}{30M})} P^y\big(S[0,\tau_n]\cap\gamma=\varnothing\big)\le C\min_{y_2\in\partial_2} P_2^{y_2}\big(S^2\big[0,\tau_n^2\big]\cap\gamma=\varnothing\big). \tag{4.37}$$

Combining these estimates, we have

$$\begin{aligned}&P^v\big(\mathrm{LE}\big(S[0,\tau_n]\big)[0,\tau_{i,n}]\cap\epsilon nB'_x\ne\varnothing, S[1,\tau_n]\cap\gamma=\varnothing\big)\\ &\quad c\epsilon M\,\mathrm{Es}\Big(\epsilon n,\frac{n}{M}\Big)P^v\big(S[1,T_{v,\frac{n}{30M}}]\cap\gamma=\varnothing\big)\max_{y\in B(v,\frac{n}{30M})}P^y\big(S[0,\tau_n]\cap\gamma=\varnothing\big)\\ &\quad c\epsilon M\,\mathrm{Es}\Big(\epsilon n,\frac{n}{M}\Big)P^v\big(S[1,\tau_n]\cap\gamma=\varnothing\big),\end{aligned} \tag{4.38}$$

which finishes the proof. □

4.2. *Proof of* (4.1)

Suppose that γ is a simple path and X is a conditioned random walk not to hit γ as in Definition 4.1. Let

$$J_{i,n}^{\gamma}:=\sharp\big\{x\in\mathbb{Z}^3\mid\epsilon nB'_x\subset F_{i,n}^{\gamma},\mathrm{LE}\big(X\big[0,\tau_n^X\big]\big)\big[0,t_{i,n}^{\gamma}\big]\cap\epsilon nB'_x\ne\varnothing\big\}, \tag{4.39}$$

where $F_{i,n}^{\gamma}$ and $t_{i,n}^{\gamma}$ were defined in Definition 4.1 and (4.27), respectively. We are interested in the lower bound of $J_{i,n}^{\gamma}$. Using Lemma 4.2, 4.3, and the second moment method as in Corollary 3.11, we will prove Lemma 4.4 below. Lemma 4.4 is an analog of Corollary 3.11 for X. Then using iteration arguments as in Theorem 6.7 [1] and Proposition 8.2.5 [18], we will prove Proposition 4.5 which immediately concludes (4.1).

We begin with the following lemma, which shows that the number of cubes hit by the loop erasure of the conditioned random walk is bigger than the expected number of such cubes with positive probability. We may think of the next lemma as an analog of Corollary 3.11 for the number of cubes hit by the loop erasure of X.

Lemma 4.4. *There exists an absolute constant $c_1>0$ such that*

$$P_X\bigg(J_{i,n}^{\gamma}\ge c_1(\epsilon M)^{-2}\,\mathrm{Es}\Big(\epsilon n,\frac{n}{M}\Big)\bigg)\ge c_1. \tag{4.40}$$

Proof. By Lemma 4.2 and (2.16)

$$\begin{aligned}&E_X\big(\big(J_{i,n}^{\gamma}\big)^2\big)\\ &\quad\le\sum_{x,y\in\mathbb{Z}^3,\epsilon nB'_x,\epsilon nB'_y\subset F_{i,n}^{\gamma}}P_X\big(\mathrm{LE}\big(X\big[0,\tau_n^X\big]\big)\cap\epsilon nB'_x\ne\varnothing,\mathrm{LE}\big(X\big[0,\tau_n^X\big]\big)\cap\epsilon nB'_y\ne\varnothing\big)\\ &\quad\le C\sum_{x\in\mathbb{Z}^3,\epsilon nB'_x\subset F_{i,n}^{\gamma}}\sum_{l=1}^{\frac{1}{\epsilon M}}l^2\frac{\epsilon M}{l}\,\mathrm{Es}(\epsilon n,\epsilon ln)\,\mathrm{Es}\Big(\epsilon n,\frac{n}{M}\Big)\\ &\quad=C\sum_{x\in\mathbb{Z}^3,\epsilon nB'_x\subset F_{i,n}^{\gamma}}\epsilon M\,\mathrm{Es}\Big(\epsilon n,\frac{n}{M}\Big)\sum_{l=1}^{\frac{1}{\epsilon M}}l\,\mathrm{Es}(\epsilon n,\epsilon ln)\\ &\quad\le C\sum_{x\in\mathbb{Z}^3,\epsilon nB'_x\subset F_{i,n}^{\gamma}}\epsilon M\,\mathrm{Es}\Big(\epsilon n,\frac{n}{M}\Big)(\epsilon M)^{-\alpha-\delta}\,\mathrm{Es}\Big(\epsilon n,\frac{n}{M}\Big)\sum_{l=1}^{\frac{1}{\epsilon M}}l^{1-\alpha-\delta}\end{aligned}$$

$$\left(\text{Here we used (2.16) to say that } \mathrm{Es}(\epsilon n, \epsilon l n) \leq C(\epsilon M)^{-\alpha-\delta} l^{-\alpha-\delta}\, \mathrm{Es}\left(\epsilon n, \frac{n}{M}\right)\right.$$

$$\left.\text{for some } \delta > 0\right)$$

$$\leq C\left\{(\epsilon M)^{-2}\, \mathrm{Es}\left(\epsilon n, \frac{n}{M}\right)\right\}^2. \tag{4.41}$$

On the other hand, by Lemma 4.3, we have

$$\begin{aligned} & E_X\left(J^{\gamma}_{i,n}\right) \\ & \geq \sum_{x\in\mathbb{Z}^3, \epsilon n B'_x \subset F^{\gamma}_{i,n}} P_X\left(\mathrm{LE}\left(X\left[0, \tau^X_n\right]\right)\left[0, t^{\gamma}_{i,n}\right] \cap \epsilon n B'_x \neq \varnothing\right) \\ & \geq c(\epsilon M)^{-2}\, \mathrm{Es}\left(\epsilon n, \frac{n}{M}\right). \end{aligned} \tag{4.42}$$

Therefore, we see that $E_X((J^{\gamma}_{i,n})^2) \leq C(E_X(J^{\gamma}_{i,n}))^2$. By the second moment method, we finish the proof. □

Let

$$J_{\epsilon,n} := \sharp\left\{x \in \mathbb{Z}^3 \,\middle|\, \epsilon n B'_x \subset B\left(\frac{2n}{3}\right) \setminus B\left(\frac{n}{3}\right), \mathrm{LE}\left(S[0, \tau_n]\right) \cap \epsilon n B'_x \neq \varnothing\right\}. \tag{4.43}$$

Now we use iteration arguments explained in the beginning of Section 4. Using the iteration argument, we prove next proposition which gives (4.2).

Proposition 4.5. *For every $r > 0$, there exists $c_r > 0$ such that*

$$P\left(J_{\epsilon,n} \geq c_r \epsilon^{-2}\, \mathrm{Es}(\epsilon n, n)\right) \geq 1 - r, \tag{4.44}$$

for all $\epsilon > 0$ and $n = n_\epsilon = 2^{j_\epsilon}$ satisfying (3.3).

Proof. Take $r > 0$. Let M_r be an integer satisfying $(1 - c_1)^{\lfloor \frac{M_r}{20} \rfloor} < r$ where $c_1 > 0$ is a constant as in Lemma 4.4. We write $N_r := \lfloor \frac{M_r}{20} \rfloor$.

Let $\beta := \mathrm{LE}(S[0, \tau_n])$ and recall that $\tau_{i,n} = \inf\{t \mid \beta(t) \in \partial D_{i+1,n}\}$ where $D_{i,n}$ was defined as in Definition 4.1. For each $i = 1, \dots, N_r$, define

$$J_i := \sharp\left\{x \in \mathbb{Z}^3 \mid \epsilon n B'_x \subset F^{\beta[0,\tau_{i-1,n}]}_{i,n}, \beta[\tau_{i-1,n}, \tau_{i,n}] \cap \epsilon n B'_x \neq \varnothing\right\}. \tag{4.45}$$

(See Definition 4.1 for $F^{\gamma}_{i,n}$.)

Then by the domain Markov property of LERW (see Lemma 2.3) and by Lemma 4.4,

$$\begin{aligned} & P\left(J_{\epsilon,n} < c_1(\epsilon M_r)^{-2}\, \mathrm{Es}\left(\epsilon n, \frac{n}{M_r}\right)\right) \\ & \leq P\left(J_i < c_1(\epsilon M_r)^{-2}\, \mathrm{Es}\left(\epsilon n, \frac{n}{M_r}\right) \text{ for all } i = 1, \dots, N_r\right) \\ & \leq E\left\{\bigcap_{i=1}^{N_r-1}\left\{J_i < c_1(\epsilon M_r)^{-2}\, \mathrm{Es}\left(\epsilon n, \frac{n}{M_r}\right)\right\} P\left(J_{N_r} < c_1(\epsilon M_r)^{-2}\, \mathrm{Es}\left(\epsilon n, \frac{n}{M_r}\right) \,\middle|\, \beta[0, \tau_{N_r-1,n}]\right)\right\} \end{aligned}$$

$$\le (1-c_1)P\left(\bigcap_{i=1}^{N_r-1}\left\{J_i < c_1(\epsilon M_r)^{-2}\,\mathrm{Es}\left(\epsilon n, \frac{n}{M_r}\right)\right\}\right)$$
$$\le (1-c_1)^{N_r} < r. \tag{4.46}$$

Since $\mathrm{Es}(\epsilon n, \frac{n}{M_r}) \ge c\,\mathrm{Es}(\epsilon n, n)$ for some $c > 0$, if we let $c_r := cc_1 M_r^{-2}$, we finish the proof. □

By Proposition 4.5, we get the following proposition immediately. Recall that Y^ϵ was defined in (3.73).

Proposition 4.6. *For all* $r > 0$, *there exists* $c_r > 0$ *such that*

$$P\big(Y^\epsilon \ge c_r \epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)\big) \ge 1 - r - \epsilon^{100}, \tag{4.47}$$

for every $\epsilon > 0$ *and* $n = n_\epsilon = 2^{j_\epsilon}$ *satisfying* (3.3).

Proof. Note that $cY^\epsilon \le J_{\epsilon,n} \le \frac{1}{c} Y^\epsilon$ for some absolute constant $c > 0$ on $\{d_{\mathrm{H}}(\mathcal{K}, \mathrm{LEW}_n) < \epsilon^2\}$. Thus if $n = n_\epsilon = 2^{j_\epsilon}$ satisfies (3.3), by (3.4) and Proposition 4.5,

$$P\big(Y^\epsilon \ge cc_r\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)\big)$$
$$\ge P\big(Y^\epsilon \ge cc_r\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n), d_{\mathrm{H}}(\mathrm{LEW}_{n_\epsilon}, \mathcal{K}) < \epsilon^2\big)$$
$$\ge P\big(J_{\epsilon,n} \ge c_r\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n), d_{\mathrm{H}}(\mathrm{LEW}_{n_\epsilon}, \mathcal{K}) < \epsilon^2\big)$$
$$\ge P\big(J_{\epsilon,n} \ge c_r\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)\big) - P\big(d_{\mathrm{H}}(\mathrm{LEW}_{n_\epsilon}, \mathcal{K}) \ge \epsilon^2\big)$$
$$\ge 1 - r - \epsilon^{100}, \tag{4.48}$$

which finishes the proof. □

Remark 4.7. By (3.77) and Markov's inequality, we see that for all $r > 0$,

$$P\left(Y^\epsilon \ge \frac{C}{r}\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)\right) \le \frac{E(Y^\epsilon)}{\frac{C}{r}\epsilon^{-2}\,\mathrm{Es}(\epsilon n, n)} \le r, \tag{4.49}$$

where C is a constant as in (3.77).

5. Lower bound of $\dim_{\mathrm{H}}(\mathcal{K})$

In this section, we will prove that

$$\dim_{\mathrm{H}}(\mathcal{K}) \ge 2 - \alpha, \quad \text{almost surely.} \tag{5.1}$$

Combining this with Theorem 1.4 [16], we have

$$\dim_{\mathrm{H}}(\mathcal{K}) = 2 - \alpha, \quad \text{almost surely.} \tag{5.2}$$

In order to prove (5.1), we will use a standard technique so called Frostman's lemma (see Lemma 5.1). We will review that lemma in Section 5.1. We then give some energy estimates for suitable sequence of measures whose supports converge to $\mathcal{K}$ (see Lemma 5.2). Using Lemma 5.2, we will prove (5.1) in Section 5.2.

5.1. *Preliminaries*

In order to give a lower bound of the Hausdorff dimension of a set in $\mathbb{R}^d$, the Lemma 5.1 below is a standard criterion referred to as Frostman's lemma. In this subsection, we first state it. Then in Lemma 5.2, we will estimate β-energy for suitable measures μ_k defined below.

Lemma 5.1 (Theorem 4.13 [3]). *Suppose that $K \subset \overline{D}$ is a closed set and let μ be a positive measure supported on K with $\mu(K) > 0$. Define β-energy $I_\beta(\mu)$ by*

$$I_\beta(\mu) = \int_{\overline{D}} \int_{\overline{D}} |x-y|^{-\beta} \, d\mu(x) \, d\mu(y). \tag{5.3}$$

If $I_\beta(\mu) < \infty$, then $\dim_{\mathrm{H}}(K) \geq \beta$.

According to Lemma 5.1, we need to construct a positive (random) measure μ supported on $\mathcal{K}$ such that its $(\beta-\delta)$-energy $I_{\beta-\delta}(\mu)$ is finite with high probability for any $\delta > 0$, where $\beta := 2 - \alpha$ (see Theorem 2.7 for α). With this in mind, let $\epsilon = \epsilon_k = 2^{-k}$ for $k \geq 1$ and let $n = n_\epsilon = 2^{j_\epsilon}$ be an integer satisfying (3.3). Now we define a sequence of measures μ_k which approximates μ as follows. Let μ_k be the (random) measure whose density, with respect to Lebesgue measure, is $\frac{1}{\epsilon \operatorname{Es}(\epsilon n, n)}$ on each box ϵB_x with $x \in \mathbb{Z}^3$ and $\epsilon B_x \subset D_{\frac{2}{3}} \setminus D_{\frac{1}{3}}$ or $\epsilon B_x \cap \partial D_{\frac{i}{3}} \neq \emptyset$ for $i = 1, 2$ such that $\mathcal{K} \cap \epsilon B_x \neq \emptyset$, and assigns measure zero elsewhere. Then it is easy to check that $\operatorname{supp}(\mu_{k+1}) \subset \operatorname{supp}(\mu_k)$ and with probability one $\bigcap_{k=1}^{\infty} \operatorname{supp}(\mu_k) \subset \mathcal{K}$.

Therefore, as we discussed as in Section 1.2, we need to show that for every $\delta > 0$ and $r > 0$ there exist constants $c_r > 0$, $C_{\delta,r} < \infty$ which do not depend on ϵ such that

$$P\big(I_{\beta-\delta}(\mu_k) \leq C_{\delta,r}\big) \geq 1 - r, \tag{5.4}$$

$$P\big(\mu_k(\overline{D}) \geq c_r\big) \geq 1 - r. \tag{5.5}$$

for all $k > 0$. Once (5.4) and (5.5) are proved, it follows that there exists some subsequential weak limit measure μ of the μ_k such that the measure μ is a positive measure satisfying that its support is contained in $\mathcal{K}$ and the $(\beta-\delta)$-energy is finite with probability at least $1 - r$. Using Lemma 5.1, we get $\dim_{\mathrm{H}}(\mathcal{K}) \geq \beta - \delta$ with probability $\geq 1 - r$, and Theorem 1.1 is proved.

For (5.5), we have the following. Take an arbitrary $r > 0$. By Proposition 4.6, with probability at least $1 - r - \epsilon^{100}$, we have

$$\mu_k(\overline{D}) \geq \sum_{x \in \mathbb{Z}^3, \epsilon B_x \subset D_{\frac{2}{3}} \setminus D_{\frac{1}{3}}} \mathbf{1}\{\mathcal{K} \cap \epsilon B_x \neq \emptyset\} \frac{\epsilon^2}{\operatorname{Es}(\epsilon n, n)} = Y^\epsilon \frac{\epsilon^2}{\operatorname{Es}(\epsilon n, n)} \geq c_r, \tag{5.6}$$

for all k, which proves (5.5).

For (5.4), we start with the following lemma which gives a first moment estimate of $I_{\beta-\delta}(\mu_k)$ for an arbitrary positive number δ.

Lemma 5.2. *For every $\delta > 0$, there exists $C_\delta < \infty$ such that*

$$E\big(I_{\beta-\delta}(\mu_k)\big) \leq C_\delta, \tag{5.7}$$

for all k. Here $\beta := 2 - \alpha$.

Proof. Recall that we write $\epsilon = \epsilon_k = 2^{-k}$ for $k \geq 1$ and let $n = n_\epsilon = 2^{j_\epsilon}$ be an integer satisfying (3.3). Then by Theorem 3.1,

$$E\big(I_{\beta-\delta}(\mu_k)\big)$$
$$\leq \sum_{x, y \in \mathbb{Z}^3, \epsilon B_x, \epsilon B_y \subset D_{\frac{2}{3}} \setminus D_{\frac{1}{3}}} E\left\{ \int_{\epsilon B_x} \int_{\epsilon B_y} |z-w|^{-(\beta-\delta)} \, d\mu_k(z) \, d\mu_k(w) \right\}$$

$$= \sum_{x,y\in\mathbb{Z}^3,\epsilon B_x,\epsilon B_y\subset D_{\frac{2}{3}}\setminus D_{\frac{1}{3}}} \big(\epsilon\,\mathrm{Es}(\epsilon n,n)\big)^{-2}\int_{\epsilon B_x}\int_{\epsilon B_y}|z-w|^{-(\beta-\delta)}\,dz\,dw\,P(\mathcal{K}\cap\epsilon B_x\neq\varnothing \text{ and } \mathcal{K}\cap\epsilon B_y\neq\varnothing)$$

$$\le C\sum_{x\in\mathbb{Z}^3,\epsilon B_x\subset D_{\frac{2}{3}}\setminus D_{\frac{1}{3}}}\sum_{l=1}^{\frac{2}{\epsilon}} l^2\big(\epsilon\,\mathrm{Es}(\epsilon n,n)\big)^{-2}(\epsilon l)^{-(\beta-\delta)}\epsilon^6\,\mathrm{Es}(\epsilon n,l\epsilon n)\,\mathrm{Es}(\epsilon n,n)\frac{\epsilon}{l}$$

$$+C\sum_{x\in\mathbb{Z}^3,\epsilon B_x\subset D_{\frac{2}{3}}\setminus D_{\frac{1}{3}}}\big(\epsilon\,\mathrm{Es}(\epsilon n,n)\big)^{-2}\int_{\epsilon B_x}\int_{\epsilon B_x}|z-w|^{-(\beta-\delta)}\,dz\,dw\,\mathrm{Es}(\epsilon n,n)\epsilon. \tag{5.8}$$

But RHS of (5.8) is bounded above by

$$\le C\epsilon^{2-(\beta-\delta)}\frac{1}{\mathrm{Es}(\epsilon n,n)}\sum_{l=1}^{\frac{2}{\epsilon}} l^{1-(\beta-\delta)}\,\mathrm{Es}(\epsilon n,l\epsilon n)+\frac{C\epsilon^{\alpha+\delta}}{\mathrm{Es}(\epsilon n,n)}$$

$$\le C\epsilon^{2-(\beta-\delta)}\frac{1}{\mathrm{Es}(\epsilon n,n)}\sum_{l=1}^{\frac{2}{\epsilon}} l^{1-(\beta-\delta)}(\epsilon l)^{-\alpha-\eta}\,\mathrm{Es}(\epsilon n,n)+\frac{C\epsilon^{\alpha+\delta}}{\mathrm{Es}(\epsilon n,n)}$$

$$\left(\text{Here } \eta:=\frac{\delta}{2} \text{ and we used (2.18)}\right)$$

$$\le C\epsilon^{2-(\beta-\delta)-\alpha-\eta}\sum_{l=1}^{\frac{2}{\epsilon}} l^{1-(\beta-\delta)-\alpha-\eta}+\frac{C\epsilon^{\alpha+\delta}}{\mathrm{Es}(\epsilon n,n)}\le C_\delta$$

$$\big(\text{Here we used (2.16)}\big), \tag{5.9}$$

which finishes the proof. □

5.2. *Proof of* (5.1)

Now we are ready to prove the following theorem.

Theorem 5.3. *Let $d=3$. Then*

$$\dim_{\mathrm{H}}(\mathcal{K})\ge 2-\alpha, \quad \textit{almost surely}. \tag{5.10}$$

Proof. Recall that $r>0$ is an arbitrary positive number. Let $\delta>0$ be an arbitrary positive number also. Define $C_{\delta,r}:=\frac{C_\delta}{r}$ where C_δ is a constant as in Lemma 5.2. Take a constant $c_r>0$ as in Proposition 4.6. Let $\beta:=2-\alpha$. By Lemma 5.2 and Markov's inequality,

$$P\big(I_{\beta-\delta}(\mu_k)\ge C_{\delta,r}\big)\le\frac{C_\delta}{C_{\delta,r}}=r, \tag{5.11}$$

for all k. Combining this with (5.6), we have

$$P\big(\mu_k(\overline{D})\ge c_r, I_{\beta-\delta}(\mu_k)\le C_{\delta,r}\big)\ge 1-2r-2^{-100k},$$

for all k. By Fatou's lemma, this implies

$$P\big(\mu_k(\overline{D})\ge c_r, I_{\beta-\delta}(\mu_k)\le C_{\delta,r} \text{ i.o.}\big)\ge 1-2r.$$

On the event above, there exists some subsequential weak limit measure μ of the μ_k such that μ is supported on $\mathcal{K}$, $\mu(\mathcal{K}) \geq c_r$, and $I_{\beta-\delta}(\mu) \leq C_{\delta,r}$. By Lemma 5.1, we have

$$P\big(\dim_{\mathrm{H}}(\mathcal{K}) \geq 2-\alpha-\delta\big) \geq 1-2r.$$

Since this holds for every $r > 0$ which is independent of $\delta > 0$,

$$P\big(\dim_{\mathrm{H}}(\mathcal{K}) \geq 2-\alpha-\delta\big) = 1.$$

Since this holds for every $\delta > 0$, we see that

$$P\big(\dim_{\mathrm{H}}(\mathcal{K}) \geq 2-\alpha\big) = 1,$$

which finishes the proof. $\square$

Remark 5.4. We expect that

$$\mathrm{Es}(n) \asymp n^{-\alpha}, \tag{5.12}$$

in 3 dimensions. Here we write $a_n \asymp b_n$ if there exists $c > 0$ such that $cb_n \leq a_n \leq \frac{1}{c} b_n$ for all n.

This is proved for $d = 2$ [12]. The main steps in [12] are

- Write $\mathrm{Es}(n)$ in terms of simple random walk quantities.
- Estimate the simple random walk quantities.

The simple random walk quantities as above come from the random walk loop measure which is related to the winding number of loops (see [12]). In [12], by estimating such simple random walk quantities carefully, not only the relation as in (5.12) but the exact value of α were also obtained in two dimensions ($\alpha = \frac{3}{4}$ in two dimensions).

Is it possible to find suitable simple random walk quantities to calculate $\mathrm{Es}(n)$ and to compute the exact value of α for $d = 3$?

Remark 5.5. Recall that we write Y for the union of $\mathcal{K}$ and loops from independent Brownian loop soup in D which intersect $\mathcal{K}$, see (2.32). Theorem 1.1 of [16] shows that Y has the same distribution as the trace of three-dimensional Brownian motion. In Conjecture 1.3 of [16], we conjectured that the law of $\mathcal{K}$ would be characterized uniquely by this decomposition. Namely, if the union of a random simple path $\tilde{\mathcal{K}}$ and loops from independent Brownian loop soup in D which intersect $\tilde{\mathcal{K}}$ has the same distribution as the trace of three-dimensional Brownian motion, then we expect that $\tilde{\mathcal{K}}$ has the same distribution as $\mathcal{K}$. Thanks to Theorem 1.3, in this characterization, we may add one additional assumption for $\tilde{\mathcal{K}}$, i.e., $\dim_{\mathrm{H}}(\tilde{\mathcal{K}}) = \beta$ almost surely. We believe that this might be useful to prove the conjecture.

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