

Finite rank perturbations in products of coupled random matrices: From one correlated to two Wishart ensembles

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Abstract. We compare finite rank perturbations of the following three ensembles of complex rectangular random matrices: First, a generalised Wishart ensemble with one random and two fixed correlation matrices introduced by Borodin and Pécché, second, the product of two independent random matrices where one has correlated entries, and third, the case when the two random matrices become also coupled through a fixed matrix. The singular value statistics of all three ensembles is shown to be determinantal and we derive double contour integral representations for their respective kernels. Three different kernels are found in the limit of infinite matrix dimension at the origin of the spectrum. They depend on finite rank perturbations of the correlation and coupling matrices and are shown to be integrable. The first kernel (I) is found for two independent matrices from the second, and two weakly coupled matrices from the third ensemble. It generalises the Meijer G -kernel for two independent and uncorrelated matrices. The third kernel (III) is obtained for the generalised Wishart ensemble and for two strongly coupled matrices. It further generalises the perturbed Bessel kernel of Desrosiers and Forrester. Finally, kernel (II), found for the ensemble of two coupled matrices, provides an interpolation between the kernels (I) and (III), generalising previous findings of part of the authors.

Résumé. Les perturbations de rang fini des trois ensembles de matrices aléatoires complexes rectangulaires suivants sont comparés: d'abord un ensemble de Wishart généralisé, avec une matrice aléatoire et deux matrices de corrélation fixées, introduit par Borodin et Pécché; ensuite le produit de deux matrices aléatoires indépendantes, dont une a des éléments corrélés; enfin le cas où deux matrices aléatoires sont couplées par une matrice fixée. Nous prouvons que la statistique des valeurs singulières des trois ensembles est déterminantale et nous dérivons des représentations en termes d'intégrales de contour doubles pour leurs noyaux respectifs. Dans la limite de dimension de matrice infinie à l'origine du spectre, on trouve trois noyaux différents, qui dépendent de la perturbation du rang fini des matrices de corrélation et du couplage et s'avèrent être intégrables. Le premier noyau (I) est trouvé pour le cas de deux matrices indépendantes du second ensemble, et pour celui de deux matrices faiblement couplées du troisième ensemble. Ce noyau généralise celui du type Meijer- G , valable pour deux matrices indépendantes et non corrélées. Le troisième noyau (III) est obtenu pour l'ensemble de Wishart généralisé et pour deux matrices couplées de façon forte. Celui-là généralise le noyau de Bessel perturbé de Desrosiers et Forrester. Finalement, le noyau (II), qui est trouvé pour l'ensemble de deux matrices couplées, représente une interpolation entre les noyaux (I) et (III), ce qui généralise des résultats précédemment obtenus par certains des auteurs.

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1. Introduction and main results

The topic of products of random matrices has seen a very rapid development in the past few years, in particular for a finite number of factors at finite matrix size. For example, it has been understood for a fixed product of r independent complex Gaussian random matrices that its singular value statistics is determinantal, and the same holds for the complex eigenvalues, cf. [2] for a recent review and a list of references. This has opened up the possibility for a detailed analysis of the local statistics. The global density of singular values of such a product, generalising the Marchenko–Pastur distribution with $r = 1$, develops a singularity at the origin depending on r (see e.g. [21]). It is thus not surprising that for the local statistics at the origin, representing a hard edge, a new family of Meijer G -kernels labelled by r has been found [34]. It generalises the Bessel kernel at $r = 1$. On the other hand, in the bulk and at the soft edge of the spectrum the respective Sine and Airy kernel have been recovered [36], agreeing with $r = 1$. The kernels at $r = 1$ are well known to be universal, see [30] and references therein.

What is known about the universality of the Meijer G -kernel? While it is known to appear in ensembles with Cauchy interaction [11] for $r \leq 3$ (and conjectured $\forall r$), or when multiplying other types of e.g. truncated unitary matrices [29], the question is open for products of matrices from unitary bi-invariant ensembles with general distributions. The difficulty is that the unitary group integrals needed after singular value decomposition are not available in general (see however [35]).

Another direction to address its universality – apart from Wigner matrices that drop invariance entirely – is to introduce correlations among the elements of each random matrix. This is the route we will choose here, for the product of $r = 2$ matrices as a starting point. We will combine this with a coupling among the two random matrices of scalar [5,6] or matrix-valued type [35], investigating the most general distribution that is quadratic in the two random matrices and remains determinantal. A further direction was taken in [20] by adding an external field to the product of r Gaussian matrices. All these deformations allow to study finite rank perturbations of the known Bessel and Meijer G -kernel, extending the results of [15] for a single Wishart matrix with external field for the former, and of [20,35] for the latter. Similar findings were made earlier for deformations of the Airy kernel at the soft edge [8,13,15], where a relation to directed percolation was pointed out. We can only speculate if the deformed Bessel, and Meijer G -kernel which we will find here, enjoy such a relation.

It is an open question if the analysis of [8], deforming the Tracy–Widom distribution by finite rank perturbations at the soft edge, could be extended to the smallest singular value distribution $p(s)$ in our setup. Apart from the Wishart–Laguerre ensemble, where several equivalent representations for $p(s)$ are available for the undeformed case, see e.g. [16], Table 3, for a list including references, for the product of $r \geq 2$ independent matrices the corresponding Painlevé type systems of equations [40] become very rapidly cumbersome, cf. [43]. This is the reason why we will focus on the kernel instead.

Let us introduce our most general ensemble of two correlated coupled matrices first. We consider two rectangular complex random matrices, G of size $L \times M$, and X of size $M \times N$. Throughout this work we will keep the following differences fixed:

$$\kappa = L - N \geq 0, \quad \nu = M - N \geq 0. \quad (1.1)$$

These two random matrices have the probability distribution with density

$$\mathcal{P}(G, X) = c \exp[-\text{Tr}(WGG^*) + \text{Tr}(\Omega GX + X^*G^*\Omega^*) - \text{Tr}(QXX^*)], \quad (1.2)$$

where c is a normalisation constant. Here, we have introduced three constant matrices. First, Ω is a fixed complex matrix of size $N \times L$ with squared singular values $\delta_1, \dots, \delta_N \geq 0$. It parametrises the coupling between the matrices G and X . Second, Q is a fixed Hermitian $M \times M$ matrix with positive eigenvalues $q_1, \dots, q_M > 0$. It introduces correlations among the matrix elements of X – typically Q^{-1} denotes a given empirical covariance matrix. Third, W is a fixed Hermitian matrix of size $L \times L$, that we will take to be proportional to the identity, $W = \alpha \mathbb{1}_L$, with $\alpha > 0$ constant. We will show later, why the choice of a more general fixed matrix W with positive eigenvalues, introducing correlations also among the matrix elements of G , leads out of the class of determinantal point processes. The joint

distribution (1.2) of the matrices G and X is convergent if the following conditions are satisfied:

$$\alpha q_i - \delta_j > 0 \quad \forall i = 1, \dots, M, \forall j = 1, \dots, N, \tag{1.3}$$

see the discussion around (2.41) below. In the sequel we will determine the correlation functions of the squared singular values of the product matrix $Y = GX$ of the two random matrices that are coupled and correlated. Before stating our main results for this ensemble let us state, in which special cases the distribution (1.2) has been investigated, and compare to two related ensembles.

In [3,4] it was shown that the distribution of squared singular values of the product Y of two independent rectangular matrices with $\Omega \equiv 0$ and $W, Q \sim \mathbb{1}$ forms a determinantal point process. There, its kernel was constructed in terms of biorthogonal functions for finite matrix size. In [34] a double contour representation was found, leading to a limiting kernel expressed in terms of Meijer G -functions, whence its name. Both, the results from [3] and [34], were derived for arbitrary but fixed products of $r \geq 1$ independent matrices. In [5] their independence was dropped and the product Y of $r = 2$ coupled matrices was considered for Y square, i.e. $L = N$ with $\nu \geq 0$, where

$$\Omega = \frac{1 - \mu}{2\mu} \mathbb{1}_N, \quad W = \frac{1 + \mu}{2\mu} \mathbb{1}_N \quad \text{and} \quad Q = \frac{1 + \mu}{2\mu} \mathbb{1}_M, \tag{1.4}$$

depending on the parameter $\mu \in (0, 1]$. It allowed to interpolate between the ensemble of the product of two independent matrices ($\mu = 1$), and a single random matrix ($\mu \rightarrow 0$), due to $G = X^*$ in this limit. Once again the interpolating ensemble was shown to be determinantal and was solved in terms of biorthogonal functions [5]. In [6] three different scaling limits were identified at the origin of the spectrum representing a hard edge, with the following limiting kernels: (I) the Meijer G -kernel for two independent matrices, proving its universality for a one-parameter family, (II) a parameter dependent kernel that interpolates between the limit (I) and the limit (III), where the well-known universal Bessel kernel (III) was obtained. In a subsequent paper [35] a full coupling matrix Ω was introduced as in (1.2), while keeping the conditions on $W = \frac{1+\mu}{2\mu} \mathbb{1}_N$ and $Q = \frac{1+\mu}{2\mu} \mathbb{1}_M$ as in [5]. There, the kernels in the limits (I), (II) and (III) were extended and finite rank perturbations in the limits (II) and (III) of [6] were found. In the present work we will study the most general case of a coupling matrix and correlations amongst the matrix elements of the two random matrices, that is compatible with a determinantal structure. Our findings generalise the kernels found in [6,35], to include also finite rank perturbations in limit (I).

Next, we introduce the following two ensembles related to (1.2). First, consider the product of two independent matrices $Y = GX$, where the second random matrix X has correlated entries

$$\mathcal{P}_2(G, X) = c_2 \exp[-\alpha \text{Tr}(GG^*) - \text{Tr}(QXX^*)]. \tag{1.5}$$

All conditions on the dimensions of G, X , and Q are as in (1.2), $\alpha > 0$ and c_2 is a normalisation constant. Related ensembles of products of r correlated random matrices have been considered by Forrester in the limit of infinite product size $r \rightarrow \infty$ [19], studying the Lyapunov exponents.

Second, following [13] we consider the generalised Wishart ensemble correlated from two sides:

$$\mathcal{P}_1(X) = c_1 \exp[-\text{Tr}(X\Sigma X^*) - \text{Tr}(QXX^*)], \tag{1.6}$$

with Q as before, Σ a fixed Hermitian $N \times N$ matrix with positive eigenvalues, and normalisation c_1 . For both Q and Σ having positive eigenvalues this ensemble is clearly convergent. However, we could also allow Σ (or Q) to have several or all eigenvalues to be negative, as long as they are bounded in absolute value by the smallest eigenvalue of Q (or Σ). In fact, when integrating out the random matrix G in ensemble (1.2), we arrive at $-\Sigma = \Omega\Omega^*/\alpha$, with the corresponding bounds (1.3). The ensemble (1.6) has been introduced in [13] for $M = N$ and was called generalised Wishart ensemble. The authors solved it for finite matrix size N using Schur processes, and then focused on the kernel at the soft edge, without considering the hard edge.

Let $\Delta_n(x_1, \dots, x_n) = \prod_{1 \leq j < k \leq n} (x_k - x_j) = \det[x_j^{i-1}]_{i,j=1}^n$ denote the Vandermonde determinant. We are now ready to state our main results.

Theorem 1.1. *Denote by x_1, \dots, x_N the squared singular values of X and by y_1, \dots, y_N the squared singular values of $Y = GX$, where X and G are distributed according to (1.2), all parameters $\delta_1, \dots, \delta_N$ and q_1, \dots, q_M are mutually*

distinct and satisfy (1.3). Their joint probability density function is given by

$$P(x_1, \dots, x_N; y_1, \dots, y_N) = \frac{1}{Z} \det[y_j^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\delta_l y_j})]_{j,l=1}^N \det\left[x_j^{-\kappa-1} \exp\left[-\alpha \frac{y_l}{x_j}\right]\right]_{j,l=1}^N \times \det[1, q_i, \dots, q_i^{\nu-1}, \exp[-q_i x_1], \dots, \exp[-q_i x_N]]_{i=1}^M, \tag{1.7}$$

where we recall (1.1). Here, $I_\kappa(z)$ denotes the modified Bessel function of the first kind and Z is a normalising constant given by

$$Z = (N!)^2 (-\alpha)^{N\nu + \frac{N(N-1)}{2}} \alpha^{-N\kappa} \Delta_M(q_1, \dots, q_M) \Delta_N(\delta_1, \dots, \delta_N) \prod_{j=1}^N \delta_j^{\frac{\kappa}{2}} \prod_{i=1}^M \prod_{j=1}^N (\alpha q_i - \delta_j)^{-1}. \tag{1.8}$$

In (1.7), in the determinant in the second line, the notation is such that the first ν columns are only present for $M > N$ and absent for $M = N$ ($\nu = 0$). When two or more parameters become degenerate the corresponding density follows from l'Hôpital's rule, see e.g. Appendix A.2. Integrating out the variables x_j leads to the joint probability density of the y_j alone, given by the following

Corollary 1.2. *The joint probability density of the squared singular values y_1, \dots, y_N of the product matrix $Y = GX$ from the ensemble (1.2) is equal to*

$$P(y_1, \dots, y_N) = \frac{2^N N!}{Z} \det[\psi_i(y_j)]_{i,j=1}^N \det[1, q_i, \dots, q_i^{\nu-1}, \varphi_i(y_1), \dots, \varphi_i(y_N)]_{i=1}^M, \tag{1.9}$$

where Z is given by (1.8) and we have introduced the following notation

$$\psi_j(y) = y^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\delta_j y}), \quad j = 1, \dots, N, \quad \varphi_i(y) = \left(\frac{q_i}{\alpha y}\right)^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\alpha q_i y}), \quad i = 1, \dots, M. \tag{1.10}$$

Here, $K_\kappa(z) = K_{-\kappa}(z)$ denotes the modified Bessel function of the second kind.

The joint probability density (1.9) lies outside the class of polynomial ensembles [31,32], that have many invariance properties [14]. Although the two determinants have different sizes, (1.9) can be mapped to a bona fide biorthogonal ensemble in the sense of [12] using the Schur complement formula, see Section 3. When we take the limit $\delta_l \rightarrow 0$ for all $l = 1, \dots, N$ in Theorem 1.1 (and Corollary 1.2), which corresponds to setting $\Omega = 0$, we arrive at the joint probability density of the ensemble (1.5) given by Theorem 2.2 (and Corollary 2.3). If in Theorem 1.1 we integrate out the squared singular values y_1, \dots, y_N , which corresponds to integrating out the matrix G in our ensemble (1.2), we arrive at the joint probability density for the squared singular values of the ensemble (1.6), as stated in Proposition 2.1. For details of these short-cuts we refer to the Appendix A.

In special cases the joint densities from (1.7) and (1.9) were known. Setting all parameters $q_j = \frac{1+\mu}{2\mu}$ equal for all $j = 1, \dots, M$, they reduce to the joint probability densities in [35], and setting furthermore all parameters $\delta_l = \frac{(1-\mu)^2}{4\mu^2}$ equal for all $l = 1, \dots, N$, see (1.4), we reobtain the joint probability densities in [5].

Our next result is an example for biorthogonal ensembles [12]. In Proposition 3.1 we show how to construct the kernel $K_N(x, y)$ for the determinantal point processes (1.9), given by

$$P(y_1, \dots, y_N) = \frac{1}{N!} \det[K_N(y_i, y_j)]_{i,j=1}^N, \tag{1.11}$$

and the resulting k -point correlation functions defined in (3.4). We quote here the final answer obtained for this kernel.

Theorem 1.3. *The correlation kernel for the ensemble (1.9) can be represented as a double contour integral:*

$$K_N(y_1, y_2) = 2 \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} \oint_{\gamma_q} \frac{d\zeta}{2\pi i} \left(\frac{\zeta}{\eta}\right)^{\frac{\kappa}{2}} \frac{I_\kappa(2\sqrt{\eta y_1}) K_\kappa(2\sqrt{\zeta y_2})}{\eta - \zeta} \prod_{l=1}^N \frac{\zeta - \delta_l}{\eta - \delta_l} \prod_{l=1}^M \frac{\eta - \alpha q_l}{\zeta - \alpha q_l}. \tag{1.12}$$

Here, γ_δ is a closed contour encircling $\delta_1, \dots, \delta_N \geq 0$ in a counter-clockwise way, and γ_q is a closed contour encircling $\alpha q_1, \dots, \alpha q_M > 0$ in counter-clockwise direction, excluding the origin and not intersecting γ_δ .

Note that the same formula for the kernel remains valid when two or more of the parameters become degenerate. In (1.12) we have suppressed a factor $(y_1/y_2)^{\kappa/2}$ on the right hand side. More generally speaking, due to (1.11) the following modification $K_N(y_1, y_2) \rightarrow K_N(y_1, y_2) f(y_1)/f(y_2)$ leads to an *equivalent* kernel, with the same joint probability density and k -point correlation functions. We will frequently use such a transformation. Corollary 1.2 together with the representation of the kernel (1.12) constitutes the solution of the ensemble (1.2) for finite matrix sizes. When setting all coupling parameters to zero, $\delta_{l=1, \dots, N} = 0$, we obtain the kernel of the ensemble (1.5), see Theorem 3.8.

Next, we turn to the main results taking three different large- N limits at the origin of the spectrum, with matrices Ω and Q having finite rank perturbations from (1.4). In order to prepare these limits let us introduce the following partial degeneracies among the sets of parameters $\delta_1, \dots, \delta_N$ and q_1, \dots, q_M , as parametrised by a single parameter $\mu \in (0, 1]$:

$$\delta_{n+1} = \dots = \delta_N = \frac{(1 - \mu)^2}{4\mu^2}, \quad q_{m+1} = \dots = q_M = \frac{1 + \mu}{2\mu}. \tag{1.13}$$

In addition we set

$$\alpha = \frac{1 + \mu}{2\mu}. \tag{1.14}$$

Let us first see what the degeneracies (1.13) imply. Clearly, because all parameters δ_l, q_j and α are positive, the condition for the convergence of the model (1.3) is equivalent to

$$1 - \frac{q_j}{\alpha} < 1 - \frac{\delta_l}{\alpha^2}, \quad j = 1, \dots, M, l = 1, \dots, N, \tag{1.15}$$

even before degeneracies (1.13) and (1.14) are imposed. Because we can insert both, on the left hand side, or on the right hand side of (1.15) the set of degenerate or non-degenerate parameters, together with (1.14) this leads to the following four different inequalities:

$$1 - \frac{2\mu q_j}{1 + \mu} < 1 - \frac{4\mu^2 \delta_l}{(1 + \mu)^2}, \quad j = 1, \dots, m, l = 1, \dots, n, \tag{1.16}$$

for the non-degenerate values,

$$1 - \frac{2\mu(1 + \mu)}{(1 + \mu)2\mu} = 0 < 1 - \frac{4\mu^2 \delta_l}{(1 + \mu)^2}, \quad l = 1, \dots, n, \tag{1.17}$$

$$1 - \frac{2\mu q_j}{1 + \mu} < 1 - \frac{(1 - \mu)^2}{(1 + \mu)^2} = \frac{4\mu}{(1 + \mu)^2}, \quad j = 1, \dots, m, \tag{1.18}$$

for one of each sets being degenerate, and

$$0 < \frac{4\mu}{(1 + \mu)^2}, \tag{1.19}$$

for both sets being degenerate, which is trivially satisfied due to $\mu \in (0, 1]$. This brings us to the sets of parameters that will be relevant for our limiting kernels. In view of (1.16) we define the quantities

$$\pi_l = \frac{(1 + \mu)^2}{4\mu} \left(1 - \frac{4\mu^2 \delta_l}{(1 + \mu)^2} \right), \quad l = 1, \dots, n, \tag{1.20}$$

$$\theta_j = \frac{(1 + \mu)^2}{4\mu} \left(1 - \frac{2\mu q_j}{1 + \mu} \right), \quad j = 1, \dots, m, \tag{1.21}$$

which satisfy

$$\theta_j < \pi_l, \quad j = 1, \dots, m, l = 1, \dots, n, \tag{1.22}$$

due to (1.16), normalised by the right hand side of (1.19). The bounds for these quantities resulting from (1.17) and (1.18), respectively, are

$$0 < \pi_l, \quad l = 1, \dots, n \quad \text{and} \quad \theta_j < 1, \quad j = 1, \dots, m, \tag{1.23}$$

These will be useful when taking the three different large- N limits next.

At large- N limit the parameter μ is now considered as a function of N , $\mu = \mu(N)$, taking values in $(0, 1]$. We will assume that also the remaining, non-degenerate parameters may become functions of N , that is, $\pi_l = \pi_l(N)$, $l = 1, \dots, n$ and $\theta_j = \theta_j(N)$, $j = 1, \dots, m$. Let us recall here that the parameters κ and ν in (1.1) as well as n, m will be kept fixed in these limits.

In the first limiting regime (I) the function $\mu(N)$ is such that $\lim_{N \rightarrow \infty} \mu(N)N = \infty$. This regime includes the situation when μ is constant. In this case the bounds in (1.23) together with (1.20) ensure that the limits exist

$$\lim_{N \rightarrow \infty} \frac{1}{N} \pi_l(N) = 0, \quad l = 1, \dots, n. \tag{1.24}$$

Noting the restriction (1.22), in order to obtain some nontrivial results we assume that $\theta_j(N)$ grows linearly with N as $N \rightarrow \infty$:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \theta_j(N) = \hat{\theta}_j \in (-\infty, 0], \quad j = 1, \dots, m. \tag{1.25}$$

The corresponding limiting kernel that we will encounter in Theorem 1.4 below is defined as

$$\begin{aligned} \mathbb{K}_I^{(m)}(y_1, y_2) &= \int_0^\infty dt \oint_{\Gamma_0} \frac{ds}{2\pi i} t^{\kappa-1} s^{-\kappa-1} \exp[s-t] \\ &\times \oint_{\Gamma_{\text{out}}} \frac{dv}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{du}{2\pi i} \frac{\exp[-\frac{v}{s}y_1 + \frac{u}{t}y_2]}{u-v} e^{-\frac{1}{u} + \frac{1}{v}} \left(\frac{v}{u}\right)^{\nu-m} \prod_{l=1}^m \frac{v - \hat{\theta}_l}{u - \hat{\theta}_l}. \end{aligned} \tag{1.26}$$

Here, Γ_0 is a closed contour encircling the origin counter-clockwise, Γ_{in} is a closed contour encircling $\{0, \hat{\theta}_1, \dots, \hat{\theta}_m\}$ in counter-clockwise direction, and Γ_{out} is a closed contour enclosing the contour Γ_{in} counter-clockwise. This kernel generalises the Meijer G -kernel for the product of two independent matrices [34] by a set of finite rank perturbations, and it holds $\mathbb{K}_I^{(m=0)}(x, y) = \mathbb{K}_{\text{Meijer}}(x, y)$, see (4.27) and (4.24) for integral representations of the latter. It was shown in [14] for this Meijer G -kernel that it can also be written as a double integral of the Bessel kernel. The same relation extends to the kernels with finite rank perturbations, comparing the second line of (1.26) with (1.30) below at $n = 0$, the generalised Bessel kernel. We will show that a representation of the kernel (1.26) with only two integrals exists, cf. (4.29).

In the second limiting regime (II) the function $\mu(N)$ behaves as $\lim_{N \rightarrow \infty} \mu(N)N = \tau/4$, with $\tau > 0$. In this limit, considering the definitions (1.20) and (1.21) together with (1.22) and (1.23), we may assume that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \pi_l(N) &= \hat{\pi}_l \in \left[0, \frac{1}{\tau}\right), \quad l = 1, \dots, n \quad \text{and} \\ \lim_{N \rightarrow \infty} \frac{1}{N} \theta_j(N) &= \hat{\theta}_j \in \bigcap_{k=1}^n (-\infty, \hat{\pi}_k], \quad j = 1, \dots, m. \end{aligned} \tag{1.27}$$

The limiting kernel in this regime (II) is a parameter dependent family of interpolating limiting kernels. It is defined as

$$\mathbb{K}_{\text{II}}^{(n,m)}(y_1, y_2; \tau) = \frac{2}{\tau} \oint_{\Gamma_{\text{out}}} \frac{dv}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{du}{2\pi i} I_\kappa(2\sqrt{(1-\tau v)y_1/\tau^2}) K_\kappa(2\sqrt{(1-\tau u)y_2/\tau^2})$$

$$\times e^{-\frac{1}{u} + \frac{1}{v}} \frac{1}{u-v} \left(\frac{1-\tau u}{1-\tau v} \right)^{\frac{\kappa}{2}} \left(\frac{v}{u} \right)^{v+n-m} \prod_{k=1}^n \frac{u-\hat{\pi}_k}{v-\hat{\pi}_k} \prod_{l=1}^m \frac{v-\hat{\theta}_l}{u-\hat{\theta}_l}. \quad (1.28)$$

Here, Γ_{in} is defined as in the first limit (I), such that $\Re(u) < 1/\tau$ for $u \in \Gamma_{\text{in}}$, and Γ_{out} is a closed contour encircling the parameters $\{\hat{\pi}_1, \dots, \hat{\pi}_m\}$ and contour Γ_{in} in counter-clockwise direction. It generalises the interpolating kernel of [6], where no such parameters $\hat{\pi}_k$ and $\hat{\theta}_l$ were present (although the representation of the kernel in [6] is different), and the kernel of [35], where the parameters $\hat{\pi}_k$ are present and $\hat{\theta}_k$ are absent. The kernel (1.28) can also be written as a double integral of the generalised Bessel kernel (1.30) below, cf. (4.52).

The third limiting regime (III) is given by $\lim_{N \rightarrow \infty} \mu(N)N = 0$, with limiting parameter assumptions

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \pi_l(N) &= \hat{\pi}_l \in [0, \infty), \quad l = 1, \dots, n \quad \text{and} \\ \lim_{N \rightarrow \infty} \frac{1}{N} \theta_j(N) &= \hat{\theta}_j \in \bigcap_{k=1}^n (-\infty, \hat{\pi}_k], \quad j = 1, \dots, m. \end{aligned} \quad (1.29)$$

The corresponding limiting kernel is a generalisation of the Bessel kernel with finite rank perturbations,

$$\mathbb{K}_{\text{III}}^{(n,m)}(y_1, y_2) = \oint_{\Gamma_{\text{out}}} \frac{dv}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{du}{2\pi i} \frac{\exp[-y_1 v + y_2 u]}{u-v} e^{-\frac{1}{u} + \frac{1}{v}} \left(\frac{v}{u} \right)^{v+n-m} \prod_{k=1}^n \frac{u-\hat{\pi}_k}{v-\hat{\pi}_k} \prod_{l=1}^m \frac{v-\hat{\theta}_l}{u-\hat{\theta}_l}, \quad (1.30)$$

where Γ_{in} is a closed contour encircling $\{0, \hat{\theta}_1, \dots, \hat{\theta}_m\}$ counter-clockwise, and Γ_{out} a closed contour enclosing $\{\hat{\pi}_1, \dots, \hat{\pi}_m\}$ and the contour Γ_{in} counter-clockwise. Without finite rank perturbations it coincides with the Bessel kernel, $\mathbb{K}_{\text{III}}^{(0,0)}(x, y) = \mathbb{K}_{\text{Bessel}}(x, y)$, as shown in [15], see (4.16) and (4.10) for representations of the latter. At $m = 0$ the kernel (1.30) was found earlier in [35] and agrees with the one from [15], Theorem 15, for a different ensemble. The kernel (1.30) enjoys a formal duality relation, $\mathbb{K}_{\text{III}}^{(n,m)}(x, y) \rightarrow \mathbb{K}_{\text{III}}^{(m,n)}(y, x)|_{v \rightarrow -v, \hat{\pi}_k \leftrightarrow -\hat{\theta}_l}$, as can be seen already in the corresponding ensemble (1.6), by interchanging $\Sigma \leftrightarrow Q$ and $N \leftrightarrow M$. It thus holds already for the kernel at finite- N , cf. (3.20), and in particular also for the extended Airy kernel of [13] at the soft edge.

In all three kernels the contours can be chosen differently, for example when showing that they are integrable in the sense of [25], see Corollaries 4.4, 4.5 and 4.2, respectively. Theorems 3.5, 4.1 and Corollary 4.2 solve an open problem stated in [15], Section 7.2. The integrability found here implies that the asymptotic analysis of all three kernels can be formulated as a Riemann–Hilbert problem. These three kernels can all be obtained from the kernel (1.12) of ensemble (1.2).

Theorem 1.4 (Hard edge scaling limits). *Consider the correlation kernel (1.12) from Theorem 1.3, with fixed non-negative integers $\nu = M - N$ and $\kappa = L - N$, together with the definitions (1.20) and (1.21). With the three kernels defined above, the following limits hold uniformly for any x, y in a compact subset of $(0, \infty)$ as $N \rightarrow \infty$.*

(I) *Suppose that $\mu(N)N \rightarrow \infty$ and (1.25) hold true, then we have*

$$\lim_{N \rightarrow \infty} \frac{\mu(N)}{N} K_N \left(\frac{\mu(N)}{N} x, \frac{\mu(N)}{N} y \right) \left(\frac{y}{x} \right)^{\frac{\kappa}{2}} = \mathbb{K}_{\text{I}}^{(m)}(x, y),$$

with the limiting parameters $\hat{\theta}_{k=1, \dots, m} \in (-\infty, 0]$.

(II) *Suppose that $\mu(N)N \rightarrow \tau/4$ with $\tau > 0$ and (1.27) hold true, then we have*

$$\lim_{N \rightarrow \infty} \frac{1}{4N^2} K_N \left(\frac{x}{4N^2}, \frac{y}{4N^2} \right) = \mathbb{K}_{\text{II}}^{(n,m)}(x, y; \tau),$$

with $\hat{\pi}_{l=1, \dots, n} \in [0, 1/\tau)$ and the limiting parameters $\hat{\theta}_{k=1, \dots, m} \in \bigcap_{l=1}^n (-\infty, \hat{\pi}_l]$.

(III) Suppose that $\mu(N)N \rightarrow 0$ and (1.29) hold true, then we have

$$\lim_{N \rightarrow \infty} \frac{1}{4N^2} K_N \left(\frac{x}{4N^2}, \frac{y}{4N^2} \right) e^{\frac{1}{2\mu(N)N}(\sqrt{y}-\sqrt{x})} = \frac{1}{2} (xy)^{-\frac{1}{4}} \mathbb{K}_{\text{III}}^{(n,m)}(\sqrt{x}, \sqrt{y}),$$

with $\hat{\pi}_{l=1,\dots,n} \in [0, \infty)$ and the limiting parameters $\hat{\theta}_{k=1,\dots,m} \in \bigcap_{l=1}^n (-\infty, \hat{\pi}_l]$.

The kernel in limit (I) is also obtained from ensemble (1.5) of the product of two independent random matrices $Y = GX$, where the matrix elements of X are correlated, see Theorem 4.3. The kernel in limit (III) is also obtained from ensemble (1.6) of a single random matrix, that has matrix elements correlated from both sides, see Theorem 4.1. The fact that in Theorem 1.4 (III) we obtain this generalised Bessel kernel, rescaled and with square root arguments, was already observed and explained in [5]. It is due to the fact that in this strongly coupled limit the squared singular values of $Y = X^*X$ are obtained. The kernel in the limit (II) is called interpolating in the following sense.

Theorem 1.5 (Interpolating kernel). *The parameter dependent family of kernels (1.28) is interpolating between the Meijer G-kernel and the Bessel kernel, both with finite rank perturbations. Namely, it holds for x, y in any compact subset of \mathbb{R}_+ :*

- (a) $\lim_{\tau \rightarrow \infty} \tau \mathbb{K}_{\text{II}}^{(n,m)}(\tau x, \tau y; \tau) \left(\frac{y}{x}\right)^{\frac{\kappa}{2}} = \mathbb{K}_{\text{I}}^{(m)}(x, y),$
- (b) $\lim_{\tau \rightarrow 0^+} \mathbb{K}_{\text{II}}^{(n,m)}(x, y; \tau) e^{2(\sqrt{y}-\sqrt{x})/\tau} = \frac{1}{2} (xy)^{-\frac{1}{4}} \mathbb{K}_{\text{III}}^{(n,m)}(\sqrt{x}, \sqrt{y}).$

This generalises the interpolating kernel derived in [35] for $m = 0$, and in [6] for $m = n = 0$, where an alternative integral representation was given. It is an open problem to map these two representations.

In [8] a modification of the Tracy–Widom distribution for the largest eigenvalue was observed from finite rank perturbations. One could ask if a similar phenomenon occurs here for the distribution of the smallest eigenvalue, applying a Fredholm determinant representation in terms of the three different limiting kernels from our ensemble (1.2). Here, we stress that the difference in scaling and thus of the fluctuations in limits (I) and (II) indicates such a transition. On the other hand, the same scaling in limits (II) and (III) indicates a smooth interpolation, consistent with the findings of [6] without finite rank perturbations.

The remainder of this article is organised as follows. In the next Section 2 we derive the joint densities of squared singular values for all three ensembles. In Section 3 we show that all three ensembles represent determinantal point processes, and derive their kernels at finite matrix size as double contour integrals. The last Section 4 is devoted to the asymptotic analysis of the three kernels at the origin, their integrability and the interpolating property of the kernel in limit (II).

2. Joint probability densities

For pedagogical reasons we will start with the derivation of the joint probability density of the generalised Wishart ensemble (1.6), where we extend the results of [13] to rectangular matrices. Then, we move to the product of two independent matrices (1.5) where one matrix has correlated entries, before coupling these two matrices in our most general ensemble (1.2).

2.1. Joint probability density of the generalised Wishart ensemble

This ensemble that is correlated from two sides is defined following [13]. Let X be a complex random matrix of size $M \times N$ with $M - N = \nu \geq 0$, and its matrix entries distributed as

$$\mathcal{P}_1(X) = c_1 \exp[-\text{Tr}(X \Sigma X^*) - \text{Tr}(Q X X^*)]. \quad (2.1)$$

Here, Σ is a fixed Hermitian matrix of size $N \times N$ with eigenvalues $\sigma_1, \dots, \sigma_N$, and Q is a fixed Hermitian matrix of size $M \times M$ with eigenvalues q_1, \dots, q_M . For simplicity we first assume that these are all pairwise distinct. The normalisation constant reads $c_1 = \pi^{-NM} \prod_{i=1}^M \prod_{j=1}^N (q_i + \sigma_j)$. When thinking of Σ and Q as originating from given empirical covariance matrices we would choose them to have only positive eigenvalues, ensuring convergence.

For what follows below we will choose Q to have positive eigenvalues, and allow Σ to have also negative eigenvalues, which leads to the constraint

$$q_i + \sigma_j > 0, \quad \forall i = 1, \dots, M, \forall j = 1, \dots, N. \tag{2.2}$$

Note that the ensemble (2.1) is different from the doubly correlated Wishart ensemble, with distribution $\tilde{\mathcal{P}}(X) \sim \exp[-\text{Tr}(QX\Sigma X^*)]$, that has been considered in [42]. Our first result is the following

Proposition 2.1. *The joint probability density of the squared singular values x_1, \dots, x_N of X distributed according to (2.1) equals*

$$P_1(x_1, \dots, x_N) = \frac{1}{Z_1} \det[e^{-\sigma_i x_j}]_{i,j=1}^N \det[1, q_i, \dots, q_i^{v-1}, e^{-q_i x_1}, \dots, e^{-q_i x_N}]_{i=1}^M, \tag{2.3}$$

where the normalising constant is given by

$$Z_1 = N!(-1)^{Nv} \Delta_M(q_1, \dots, q_M) \Delta_N(\sigma_1, \dots, \sigma_N) \prod_{i=1}^M \prod_{j=1}^N (q_i + \sigma_j)^{-1}. \tag{2.4}$$

In the case of quadratic matrices with $v = 0$, where the x -independent columns in the second determinant of the distribution (2.3) are absent, this result was derived in [13].

Proof. As the first step we will reduce the rectangular matrix X to a quadratic matrix, following [17]. In the sequel we will denote by O matrices with zero entries. Set

$$X = U \begin{pmatrix} X_0 \\ O \end{pmatrix}, \tag{2.5}$$

where $U \in U(M)/U(N) \times U(v)$ is an $M \times M$ unitary matrix, and X_0 is an $N \times N$ complex matrix. We note that $X^*X = X_0^*X_0$ is of size $N \times N$ and thus the matrices X and X_0 have the same squared singular values. Inserting the Jacobian [17] the joint probability distribution of X_0 and U following from (2.1) is thus proportional to

$$P_1(X)[dX] \sim e^{-\text{Tr}(\Sigma X_0^* X_0)} e^{-\text{Tr}(QU \begin{pmatrix} X_0 X_0^* & O \\ O & O \end{pmatrix} U^*)} \det[X_0^* X_0]^v [dX_0] d\mu(U). \tag{2.6}$$

Here, $d\mu(U)$ denotes the corresponding Haar measure, $[dX] = \prod_{i=1}^M \prod_{j=1}^N dX_{i,j}^R dX_{i,j}^I$ denotes the flat Lebesgue measure over all independent matrix elements X_{ij} , their real and imaginary parts $X_{i,j}^R$ and $X_{i,j}^I$. In the following we will suppress all proportionality constants that can be determined, and fix the normalisation Z_1 of the joint probability density of squared singular values (2.3) only at the end. The singular value decomposition for X_0 can be written as

$$X_0 = \mathfrak{U} \Lambda_x^{\frac{1}{2}} P^*, \quad \Lambda_x^{\frac{1}{2}} = \begin{pmatrix} \sqrt{x_1} & 0 & \dots & 0 \\ 0 & \sqrt{x_2} & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \sqrt{x_N} \end{pmatrix}, \tag{2.7}$$

where \mathfrak{U} and P are unitary matrices with $P \in U(N)$ and $\mathfrak{U} \in U(N)/U(1)^N$. Here and further on we will use the notion $\Lambda_x = \text{diag}(x_1, \dots, x_N)$ for this and other sets of variables. The measure $[dX_0]$ decomposes as

$$[dX_0] \sim \Delta_N(x_1, \dots, x_N)^2 dx_1 \dots dx_N d\mu(\mathfrak{U}) d\mu(P), \tag{2.8}$$

and we arrive at

$$P_1(X)[dX] \sim e^{-\text{Tr}(\Sigma P \Lambda_x P^*)} e^{-\text{Tr}[QU \begin{pmatrix} \mathfrak{U} \Lambda_x \mathfrak{U}^* & O \\ O & O \end{pmatrix}^*]} \prod_{k=1}^N x_k^v \Delta_N(x_1, \dots, x_N)^2$$

$$\times dx_1 \cdots dx_N d\mu(U) d\mu(\mathfrak{U}) d\mu(P). \quad (2.9)$$

To obtain (2.3) we need to compute the group integrals over P , and over U and \mathfrak{U} that have already decoupled. Furthermore, we have to diagonalise the fixed matrices, $\Sigma = V \Lambda_\sigma V^*$ and $Q = \tilde{V} \Lambda_q \tilde{V}^*$, and absorb these extra factors V and \tilde{V} of fixed unitary matrices through the invariance of the Haar measures of these group integrals. For the integral over P this is straightforward and we can readily apply the standard Harish-Chandra–Itzykson–Zuber (HCIZ) integral formula, reading [24,26]:

$$\int_{U(N)} d\mu(P) e^{-\text{Tr}(\Lambda_\sigma P \Lambda_x P^*)} = \text{const.} \frac{\det[e^{-\sigma_i x_j}]_{i,j=1}^N}{\Delta_N(\sigma_1, \dots, \sigma_N) \Delta_N(x_1, \dots, x_N)}. \quad (2.10)$$

The constant that is independent of the σ_j and x_j can be determined and depends only on the convention in normalising the Haar measure of the unitary group.

Next we can turn to the integrals over U and \mathfrak{U} . Here, an additional integral over \mathfrak{U}_1 can be introduced that we choose to be over $U(\nu)$,

$$e^{-\text{Tr}[Q U \begin{pmatrix} \mathfrak{U} \Lambda_x \mathfrak{U}^* & O \\ O & O \end{pmatrix} U^*]} = \text{const.} \int d\mu(\mathfrak{U}_1) e^{-\text{Tr}[\tilde{V} \Lambda_q \tilde{V}^* U \begin{pmatrix} \mathfrak{U} & O \\ O & \mathfrak{U}_1 \end{pmatrix} \tilde{\Lambda}_x \begin{pmatrix} \mathfrak{U}^* & O \\ O & \mathfrak{U}_1^* \end{pmatrix} U^*]},$$

where the extra zeros are denoted by the diagonal $M \times M$ matrix

$$\tilde{\Lambda}_x = \text{diag}(x_1, \dots, x_N, 0, \dots, 0).$$

The three integrations over U , \mathfrak{U} , and \mathfrak{U}_1 together parametrise the coset space $U(M)/U(1)^N$, which can be used to absorb \tilde{V} by invariance of the corresponding Haar measure. In order to apply the HCIZ formula (2.10) we need to take into account that ν eigenvalues of $\tilde{\Lambda}_x$ are equal to zero, which can be obtained by l'Hôpital's rule. We thus arrive at

$$\int d\mu(U) d\mu(\mathfrak{U}) e^{-\text{Tr}[Q U \begin{pmatrix} \mathfrak{U} \Lambda_x \mathfrak{U}^* & O \\ O & O \end{pmatrix} U^*]} = \text{const.} \frac{\det[1, q_i, \dots, q_i^{\nu-1}, e^{-q_i x_1}, \dots, e^{-q_i x_N}]_{i=1}^M}{\Delta_M(q_1, \dots, q_M) \Delta_N(x_1, \dots, x_N) \prod_{k=1}^N x_k^\nu}, \quad (2.11)$$

where U is integrated over $U(M)/U(N) \times U(\nu)$, \mathfrak{U} over $U(N)/U(1)^N$ (and \mathfrak{U}_1 over $U(\nu)$). Integrating (2.9) over the corresponding coset spaces, from (2.11) together with (2.10) we arrive at (2.3), up to the normalisation constant Z_1 . For its calculation we apply the generalisation [28], Appendix C, of the Andréief formula [7] that follows from simple linear algebra. We quote the following form for later use:

$$\begin{aligned} & \prod_{j=1}^N \int_0^\infty dx_j \det \begin{bmatrix} R_{a,b} |_{\substack{1 \leq b \leq N+k \\ 1 \leq a \leq k}} \\ \psi_b(x_a) |_{\substack{1 \leq b \leq N+k \\ 1 \leq a \leq N}} \end{bmatrix} \det \begin{bmatrix} S_{b,a} |_{\substack{1 \leq a \leq l \\ 1 \leq b \leq N+l}} & \varphi_b(x_a) |_{\substack{1 \leq a \leq N \\ 1 \leq b \leq N+l}} \end{bmatrix} \\ & = (-1)^{kl} N! \det \begin{bmatrix} O_{k \times l} & R_{a,b} |_{\substack{1 \leq b \leq N+k \\ 1 \leq a \leq k}} \\ S_{b,a} |_{\substack{1 \leq a \leq l \\ 1 \leq b \leq N+l}} & \int_0^\infty dx \varphi_b(x) \psi_a(x) |_{\substack{1 \leq a \leq N+k \\ 1 \leq b \leq N+l}} \end{bmatrix}. \end{aligned} \quad (2.12)$$

Here, we have explicitly spelled out the dimension of the matrix block with zero elements O . This identity is valid for two sets of functions $\psi_j(x)$ and $\varphi_k(x)$ that are suitably integrable, and two constant matrices R and S . The integration domains can also be chosen differently, cf. [28]. When specifying to $k = 0$, that is in the absence of matrix R , and to $l = \nu$ with $S_{b,a} = q_b^{a-1}$, we obtain

$$\begin{aligned} & \prod_{k=1}^N \int_0^\infty dx_k \det[\psi_i(x_j)]_{i,j=1}^N \det[1, q_j, \dots, q_j^{\nu-1}, \varphi_j(x_1), \dots, \varphi_j(x_N)]_{j=1}^M \\ & = N! \det \left[1, q_j, \dots, q_j^{\nu-1}, \int_0^\infty dx \psi_1(x) \varphi_j(x), \dots, \int_0^\infty dx \psi_N(x) \varphi_j(x) \right]_{j=1}^M. \end{aligned} \quad (2.13)$$

The standard Andréief formula is obtained when also setting $\nu = 0$, when the first ν columns on left and right hand side (and thus matrix S in (2.12)) are absent. Inserting

$$\psi_i^{(1)}(x) = e^{-\sigma_i x} \quad \text{and} \quad \varphi_j^{(1)}(x) = e^{-q_j x} \quad (2.14)$$

into (2.13) we have the resulting simple integral that we define for later purpose:

$$I_{i,j}^{(1)} = \int_0^\infty dx e^{-q_i x} e^{-\sigma_j x} = \frac{1}{q_i + \sigma_j} \quad \text{for } i = 1, \dots, M, j = 1, \dots, N. \quad (2.15)$$

We can now apply the generalised Cauchy determinant derived by Basor and Forrester [9], Lemma 2, to (2.13) and (2.15)

$$\det \left[1, q_j, \dots, q_j^{\nu-1}, \frac{1}{q_j + \sigma_1}, \dots, \frac{1}{q_j + \sigma_N} \right]_{j=1}^M = (-1)^{N\nu} \frac{\Delta_N(\sigma_1, \dots, \sigma_N) \Delta_M(q_1, \dots, q_M)}{\prod_{i=1}^N \prod_{j=1}^M (q_j + \sigma_i)}, \quad (2.16)$$

with $\nu = M - N$. Equation (2.16) yields the normalisation in (2.4). \square

2.2. Joint probability density of the product of two independent correlated matrices

Let us consider two independent complex random matrices G of size $L \times M$, and X of size $M \times N$. The matrix elements of each G and X are correlated, given by the probability distribution

$$\mathcal{P}_2(G, X) = c_2 \exp[-\text{Tr}(WGG^*) - \text{Tr}(QXX^*)]. \quad (2.17)$$

Here, W is a fixed Hermitian matrix of size $L \times L$ with positive eigenvalues, and Q is a fixed Hermitian matrix of size $M \times M$ with pairwise non-degenerate eigenvalues $q_1, \dots, q_M > 0$. In what follows we will restrict ourselves to the case of $W = \alpha \mathbb{1}_L$ being proportional to the identity, with $\alpha > 0$. The reason is that for generic W the joint probability density of squared singular values of the product $Y = GX$ is no longer determinantal, as we will show. The normalising constant c_2 in (2.17) can be computed by performing the Gaussian integrals over G and X , which leads to $c_2 = \pi^{-M(L+N)} \alpha^{ML} \prod_{j=1}^M q_j^N$. Our first result is the following

Theorem 2.2. Denote by x_1, \dots, x_N the squared singular values of X and by y_1, \dots, y_N the squared singular values of $Y = GX$, with G and X distributed according to (2.17) with $W = \alpha \mathbb{1}_L$. Their joint probability density is given by

$$\begin{aligned} \mathcal{P}_2(x_1, \dots, x_N; y_1, \dots, y_N) &= \frac{1}{Z_2} \prod_{j=1}^N y_j^\kappa \Delta_N(y_1, \dots, y_N) \det[x_j^{-\kappa-1} e^{-\alpha \frac{y_k}{x_j}}]_{j,k=1}^N \\ &\quad \times \det[1, q_i, \dots, q_i^{\nu-1}, e^{-q_i x_1}, \dots, e^{-q_i x_N}]_{i=1}^M, \end{aligned} \quad (2.18)$$

where Z_2 is a normalising constant given by

$$Z_2 = (N!)^2 (-\alpha)^{N\nu + \frac{N(N-1)}{2}} \alpha^{-N\kappa - NM} \left(\prod_{l=1}^N \Gamma(\kappa + l) \Gamma(l) \right) \Delta_M(q_1, \dots, q_M) \prod_{k=1}^M q_k^{-N}. \quad (2.19)$$

Proof. We start with a general fixed Hermitian matrix W up to the point where it becomes clear, why only $W = \alpha \mathbb{1}_L$ leads to a determinantal point process. We begin by decomposing the random matrix X as in (2.5), leading immediately to

$$\mathcal{P}_2(G, X)[dG][dX] \sim e^{-\text{Tr}(WGG^*)} e^{-\text{Tr}(QU \begin{pmatrix} X_0 X_0^* & O \\ O & O \end{pmatrix} U^*)} \det[X_0^* X_0]^\nu [dG][dX_0] d\mu(U), \quad (2.20)$$

in analogy to (2.6), where the corresponding measures are defined. We are interested in the singular values of the product matrix $Y = GX$, and in view of (2.5) we set $\widehat{G} = GU$ which is again a matrix of size $L \times M$. We split this

matrix \widehat{G} into its first N columns and its remaining ν columns by introducing the matrices \widehat{G}_0 of size $L \times N$ and \widehat{G}_1 of size $L \times \nu$ as

$$GU = \widehat{G} = (\widehat{G}_0, \widehat{G}_1), \quad (2.21)$$

with $[dG] = [d\widehat{G}_0][d\widehat{G}_1]$. It immediately follows that the matrices $\widehat{G}_0 X_0$ and GX have the same singular values. Furthermore, the product of GG^* can be written as

$$GG^* = \widehat{G}\widehat{G}^* = (\widehat{G}_0, \widehat{G}_1) \begin{pmatrix} \widehat{G}_0^* \\ \widehat{G}_1^* \end{pmatrix} = \widehat{G}_0\widehat{G}_0^* + \widehat{G}_1\widehat{G}_1^*.$$

For that reason the matrix \widehat{G}_1 completely decouples in the exponent in (2.20) and can be integrated out, being part of the normalisation. Now consider the change of variables for invertible X_0 :

$$\widehat{G}_0 \longrightarrow Y = \widehat{G}_0 X_0, \quad X_0 \longrightarrow X_0. \quad (2.22)$$

Note that Y is a matrix of size $L \times N$ and that the Jacobian of this transformation is given by $\det[X_0^* X_0]^{-L}$. Thus we obtain that the joint distribution of Y , X_0 and U is proportional to

$$e^{-\text{Tr}(WY(X_0^* X_0)^{-1} Y^*)} e^{-\text{Tr}(QU \begin{pmatrix} X_0 X_0^* & O \\ O & O \end{pmatrix} U^*)} \det[X_0^* X_0]^{\nu-L} [dY][dX_0] d\mu(U). \quad (2.23)$$

The singular value decomposition for Y is in analogy to that of X_0 in (2.7), using the same notation:

$$Y = \widetilde{U} \Lambda_y^{\frac{1}{2}} V. \quad (2.24)$$

Here, \widetilde{U} is an $L \times N$ matrix with $\widetilde{U}^* \widetilde{U} = \mathbb{1}_N$, whereas $V \in U(N)$ is a unitary matrix, cf. [17]. The measure $[dY]$ can be expressed through the singular values y_1, \dots, y_N in analogy to (2.6) and (2.8), leading to

$$[dY] \sim \prod_{l=1}^N y_l^k \Delta_N(y_1, \dots, y_N)^2 dy_1 \cdots dy_N d\mu(\widetilde{U}) d\mu(V).$$

The joint probability density of squared singular values of X_0 and Y is obtained from the following relation between probability measures:

$$\begin{aligned} \mathcal{P}_2(G, X)[dG][dX] \\ \sim e^{-\text{Tr}[(VP)^* \Lambda_y^{\frac{1}{2}} \widetilde{U}^* W \widetilde{U} \Lambda_y^{\frac{1}{2}} (VP) \Lambda_x^{-1}]} e^{-\text{Tr}[QU \begin{pmatrix} \mathfrak{U} \Lambda_x \mathfrak{U}^* & O \\ O & O \end{pmatrix} U^*]} \prod_{l=1}^N x_l^{\nu-L} y_l^k \Delta_N(y_1, \dots, y_N)^2 \\ \times \Delta_N(x_1, \dots, x_N)^2 dy_1 \cdots dy_N dx_1 \cdots dx_N d\mu(U) d\mu(\widetilde{U}) d\mu(V) d\mu(\mathfrak{U}) d\mu(P). \end{aligned} \quad (2.25)$$

It remains to integrate over all remaining Haar measures, after diagonalising $W = V_1 \Lambda_w V_1^*$ and $Q = V_2 \Lambda_q V_2^*$ by unitary transformations. Clearly the integrals over U and \mathfrak{U} decouple and lead to the same results as in (2.11). The remaining integrals are over P , \widetilde{U} and V , and after using the invariance of the Haar measure to absorb $VP \rightarrow P$ and $V_1^* \widetilde{U} \rightarrow \widetilde{U}$ we face the following group integral:

$$\mathcal{I} = \int d\mu(P) d\mu(\widetilde{U}) e^{-\text{Tr}[P^* \Lambda_y^{\frac{1}{2}} \widetilde{U}^* \Lambda_w \widetilde{U} \Lambda_y^{\frac{1}{2}} P \Lambda_x^{-1}]} \quad (2.26)$$

This integral was computed by Simon, Moustakas and Marinelli [39] using character expansion techniques. However, the final answer is given by a sum over representations that cannot be simplified to a determinantal expression, see

[39], Eq. (59), and details therein. For this reason from now on we will simplify to $W = \alpha \mathbb{1}_L$ as for the result stated in (2.18). In that case the integral (2.26) simplifies to the standard HCIZ integral (2.10), and we obtain

$$\int_{U(N)} d\mu(P) e^{-\alpha \text{Tr}(P^* \Lambda_y P \Lambda_x^{-1})} = \text{const.} \frac{\det[e^{-\alpha \frac{y_j}{x_i}}]_{i,j=1}^N}{\Delta_N(y_1, \dots, y_N) \Delta_N(x_1, \dots, x_N)} \prod_{j=1}^N x_j^{N-1}, \tag{2.27}$$

after using $\Delta_N(x_1^{-1}, \dots, x_N^{-1}) = \text{const.} \Delta_N(x_1, \dots, x_N) / \prod_{j=1}^N x_j^{N-1}$. Integrating (2.25) over the corresponding coset spaces, using (2.11) and (2.27) we arrive at the statement in (2.18). It remains to compute the normalisation constant Z_2 which we postpone to the proof of the next corollary. \square

From (2.18) we can easily deduce the joint probability density of the variables y_j alone, together with the corresponding normalisation constant, as summarised in the following

Corollary 2.3. *The joint probability density of the squared singular values y_1, \dots, y_N of the product matrix $Y = GX$, where G and X are distributed according to (2.17) with $W = \alpha \mathbb{1}_L$, is reading*

$$P_2(y_1, \dots, y_N) = \frac{2^N N!}{Z_2} \det[y_i^{\kappa+j-1}]_{i,j=1}^N \det[1, q_i, \dots, q_i^{\nu-1}, \varphi_i^{(2)}(y_1), \dots, \varphi_i^{(2)}(y_N)]_{i=1}^M. \tag{2.28}$$

Here, we have introduced the following notation

$$\varphi_i^{(2)}(y) = \left(\frac{q_i}{\alpha y}\right)^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\alpha q_i y}), \tag{2.29}$$

and the normalising constant Z_2 is defined as in (2.19).

Proof. We make use of the generalised Andréief formula (2.13) by integrating (2.18) over the x_j :

$$\begin{aligned} & \prod_{j=1}^N \int_0^\infty dx_j P_2(x_1, \dots, x_N; y_1, \dots, y_N) \\ &= \frac{N!}{Z_2} \prod_{j=1}^N y_j^\kappa \Delta_N(y_1, \dots, y_N) \det\left[1, q_i, \dots, q_i^{\nu-1}, \int_0^\infty dx x^{-\kappa-1} e^{-\frac{\alpha y_1}{x} - q_i x}, \dots, \int_0^\infty dx x^{-\kappa-1} e^{-\frac{\alpha y_N}{x} - q_i x}\right]_{i=1}^M. \end{aligned}$$

The remaining integrals are obtained using [22], Eq. 3.471.9, and the identity $K_{-\kappa}(x) = K_\kappa(x)$ for the modified Bessel function of the second kind,

$$\int_0^\infty dx x^{-\kappa-1} e^{-\frac{a}{x} - bx} = 2\left(\frac{a}{b}\right)^{-\frac{\kappa}{2}} K_\kappa(2\sqrt{ab}) \quad \text{for } \Re(a) > 0, \Re(b) > 0. \tag{2.30}$$

This yields (2.28) together with (2.29), after taking out factors of 2 of the determinant and the factors y_i^κ into the Vandermonde determinant. To finally compute the normalisation constant Z_2 we have to apply once again the generalised Andréief formula (2.13) to (2.28). In there we identify

$$\psi_j^{(2)}(y) = y^{\kappa+j-1}. \tag{2.31}$$

The integral that remains to be evaluated is thus

$$\begin{aligned} I_{i,j}^{(2)} &= \int_0^\infty dy y^{\kappa+j-1} \left(\frac{q_i}{\alpha y}\right)^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\alpha q_i y}) \\ &= \frac{\Gamma(\kappa + j)\Gamma(j)}{2\alpha^{\kappa+j}} q_i^{-j} \quad \text{for } i = 1, \dots, M, j = 1, \dots, N, \end{aligned} \tag{2.32}$$

which we obtain by using the formula [22], Eq. 6.561.16. We thus arrive at

$$Z_2 = (N!)^2 \left(\prod_{l=1}^N \Gamma(\kappa + l) \alpha^{-(\kappa+l)} \Gamma(l) \right) \det[1, q_i, \dots, q_i^{\nu-1}, q_i^{-1}, \dots, q_i^{-N}]_{i=1}^M, \quad (2.33)$$

which is equivalent to (2.19), after taking out factors of q_i^{-N} and rearranging columns. \square

2.3. Joint probability density of the product of two correlated coupled matrices

Following the derivations from the previous two subsections we are now in the position to prove Theorem 1.1, the joint probability density of the squared singular values of the random matrix X and of the product matrix $Y = GX$, distributed according to (1.2):

$$\mathcal{P}(G, X) = c \exp[-\alpha \operatorname{Tr}(GG^*) + \operatorname{Tr}(\Omega GX + X^* G^* \Omega^*) - \operatorname{Tr}(QXX^*)]. \quad (2.34)$$

Here, X and G are random, Q is fixed, as given before in Section 2.2, the second correlation matrix is $W = \alpha \mathbb{1}_L$, with $\alpha > 0$, and the fixed matrix Ω that provides the coupling is of size $N \times L$, with squared singular values $\delta_1, \dots, \delta_N \geq 0$. The normalisation is $c = \pi^{-M(L+N)} \prod_{i=1}^M \prod_{j=1}^N (\alpha q_i - \delta_j)$.

Proof of Theorem 1.1 and Corollary 1.2. Following the parametrisations (2.5), (2.21) and the change of variables (2.22) of the previous subsections, as well as the singular value decompositions (2.7) and (2.24), we immediately obtain that the probability measure from above is proportional to

$$\begin{aligned} & \mathcal{P}(G, X)[dG][dX] \\ & \sim e^{-\alpha \operatorname{Tr}[(VP)^* \Lambda_y (VP) \Lambda_x^{-1}]_1} e^{\operatorname{Tr}[\Omega \tilde{U} \Lambda_y^{\frac{1}{2}} V + V^* \Lambda_y^{\frac{1}{2}} \tilde{U}^* \Omega^*]} e^{-\operatorname{Tr}[QU \begin{pmatrix} \mathbb{1} \Lambda_x \mathbb{1}^* & 0 \\ 0 & 0 \end{pmatrix} U^*]} \prod_{j=1}^N x_j^{\nu-L} y_j^\kappa \Delta_N(y_1, \dots, y_N)^2 \\ & \times \Delta_N(x_1, \dots, x_N)^2 dx_1 \cdots dx_N dy_1 \cdots dy_N d\mu(U) d\mu(\tilde{U}) d\mu(V) d\mu(\mathbb{1}) d\mu(P). \end{aligned} \quad (2.35)$$

The unitary integrals over the third exponential factor in the second line obviously decouple. We can decouple also the first and second exponential factor by exploiting the invariance of the Haar measure $d\mu(P)$ under $VP \rightarrow P$. Furthermore, in the same way we can absorb the fixed unitary matrices from the diagonalisation of $Q = \tilde{V} \Lambda_q \tilde{V}^*$ and the singular value decomposition $\Omega = V_1 \Lambda_\delta^{\frac{1}{2}} V_2^*$, respectively. The new group integral over \tilde{U} and V that we encounter compared to the previous two subsections, and that is due to the coupling matrix Ω , is called Berezin–Karpelevich integral [10]

$$\int_{\tilde{U}^* \tilde{U} = \mathbb{1}_N} d\mu(\tilde{U}) \int_{U(N)} d\mu(V) e^{\operatorname{Tr}(\Lambda_\delta^{\frac{1}{2}} \tilde{U} \Lambda_y^{\frac{1}{2}} V + V^* \Lambda_y^{\frac{1}{2}} \tilde{U} \Lambda_\delta^{\frac{1}{2}})} = \text{const.} \frac{\det[I_\kappa(2\sqrt{\delta_k y_j})]_{j,k=1}^N \prod_{k=1}^N y_k^{-\frac{\kappa}{2}}}{\Delta_N(y_1, \dots, y_N) \Delta_N(\delta_1, \dots, \delta_N)}, \quad (2.36)$$

where the constant does not depend on y_1, \dots, y_N . This integral is an analogue of the Harish-Chandra–Itzykson–Zuber integral. Such integrals were studied in Guhr and Wettig [23], and Jackson, Şener and Verbaarschot [27]. In Liu [35] the same integral appears in the context of coupling uncorrelated Gaussian random matrices, see [35], Eq. (2.16). For a similar integral we refer to [18], Proposition 11.6.2. Integrating over the coset spaces in (2.35), we obtain from the HCIZ integrals (2.11) and (2.27) together with this integral (2.36) the following result for the joint probability density

$$\begin{aligned} & P(x_1, \dots, x_N; y_1, \dots, y_N) \\ & = \text{const.} \prod_{j=1}^N y_j^{\frac{\kappa}{2}} \det[I_\kappa(2\sqrt{\delta_l y_j})]_{j,l=1}^N \prod_{j=1}^N x_j^{-\kappa-1} \det[e^{-\alpha \frac{y_j}{x_i}}]_{i,j=1}^N \\ & \times \det[1, q_i, \dots, q_i^{\nu-1}, e^{-q_i x_1}, \dots, e^{-q_i x_N}]_{i=1}^M. \end{aligned} \quad (2.37)$$

This is equivalent to Theorem 1.1, recalling (1.1), up to the normalisation constant Z that remains to be determined. We will combine its calculation with the proof of Corollary 1.2. For this purpose we apply the generalised Andréief formula (2.13) twice to the joint probability distribution (2.37): First, integrating over the x_j we obtain

$$\begin{aligned} & \prod_{j=1}^N \int_0^\infty dx_j P(x_1, \dots, x_N; y_1, \dots, y_N) \\ &= \frac{N!2^N}{Z} \det[y_j^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\delta_l y_j})]_{j,l=1}^N \\ & \quad \times \det\left[1, q_i, \dots, q_i^{v-1}, \left(\frac{q_i}{\alpha y_1}\right)^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\alpha q_i y_1}), \dots, \left(\frac{q_i}{\alpha y_N}\right)^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\alpha q_i y_N})\right]_{i=1}^M, \end{aligned} \quad (2.38)$$

after using (2.30) and properties of the determinant. This is the statement (1.9) in Corollary 1.2, together with the identification (1.10) that we repeat here:

$$\psi_i(y) = y^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\delta_i y}), \quad \varphi_i(y) = \left(\frac{q_i}{\alpha y}\right)^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\alpha q_i y}). \quad (2.39)$$

For the determination of Z given by the second intergration of (2.37), this time over the y_j , we have

$$Z = (N!)^2 2^N \det[1, q_i, \dots, q_i^{v-1}, I_{i,1}, \dots, I_{i,N}]_{i=1}^M. \quad (2.40)$$

We are left with the following integral

$$I_{i,j} = \int_0^\infty dy \left(\frac{q_i}{\alpha y}\right)^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\alpha q_i y}) y^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\delta_j y}) = \frac{\delta_j^{\frac{\kappa}{2}}}{2\alpha^\kappa (\alpha q_i - \delta_j)}, \quad (2.41)$$

for $i = 1, \dots, M$, and $j = 1, \dots, N$, which is obtained using [22], Eq. 6.576.7. Its convergence follows from (1.3). Moreover, the determinant resulting from (2.40) can be identified with the degenerate Cauchy determinant from [9], cf. (2.16) where it was applied before. It reads

$$\begin{aligned} & \det\left[1, q_i, \dots, q_i^{v-1}, \frac{1}{\alpha q_i - \delta_1}, \dots, \frac{1}{\alpha q_i - \delta_N}\right]_{i=1}^M \\ &= (-\alpha)^{MN - \frac{1}{2}N(N+1)} \frac{\Delta_M(q_1, \dots, q_M) \Delta_N(\delta_1, \dots, \delta_N)}{\prod_{i=1}^M \prod_{j=1}^N (\alpha q_i - \delta_j)}. \end{aligned} \quad (2.42)$$

The last three equations together yield the normalisation constant Z in (1.8). \square

3. Determinantal point process, correlation kernel and its contour integral representation

In this section we will proceed in two steps. First, we will show that all our three ensembles are indeed representing determinantal point processes. Second, we use the inverse Gram matrix to explicitly construct complex contour integral representations for all three kernels in separate subsections.

We begin by recalling that the joint probability densities of all our ensembles (1.9), (2.3) and (2.28) are of the form

$$P(y_1, \dots, y_N) = \frac{1}{N! \det[A_{i,j}]_{i,j=1}^M} \det[\psi_i(y_j)]_{i,j=1}^N \det[1, q_i, \dots, q_i^{v-1}, \varphi_i(y_1), \dots, \varphi_i(y_N)]_{i=1}^M. \quad (3.1)$$

Here, the Gram type matrix A of size $M \times M$ is defined as

$$A = \begin{pmatrix} 1 & q_1 & \dots & q_1^{v-1} & I_{1,1} & \dots & I_{1,N} \\ 1 & q_2 & \dots & q_2^{v-1} & I_{2,1} & \dots & I_{2,N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & q_M & \dots & q_M^{v-1} & I_{M,1} & \dots & I_{M,N} \end{pmatrix}, \tag{3.2}$$

with

$$I_{i,j} = \int_0^\infty dy \varphi_i(y) \psi_j(y) \quad \text{for } i = 1, \dots, M, j = 1, \dots, N. \tag{3.3}$$

Part of showing that the class of joint densities (3.1) is determinantal includes to determine the k -point correlation functions, defined as

$$\rho_k(y_1, \dots, y_k) = \frac{N!}{(N-k)!} \int_0^\infty dy_{k+1} \dots dy_N P(y_1, \dots, y_N), \tag{3.4}$$

in terms of the kernel of the point process. Note that for $k = N$ there is no integral and the N -point function is just $N!$ times the joint probability density (3.1) itself, cf. (1.11). Our strategy is to first map the joint probability density (3.1) to the standard form of a biorthogonal ensemble of Borodin [12], having two determinants of equal size instead of (3.1), which shows that this density is indeed determinantal. In a second step we rewrite the resulting kernel in an alternative form, involving directly (part of) the inverse of the Gram type matrix (3.2), that will be more convenient for later use. This is stated by the following

Proposition 3.1. *Provided that the integrals in (3.3) of the two sets of functions φ_i and ψ_j from the joint probability density (3.1) exist, the k -point correlation functions are given by*

$$\rho_k(y_1, \dots, y_k) = \det [K_N(y_i, y_j)]_{i,j=1}^k. \tag{3.5}$$

The corresponding correlation kernel can be written as

$$K_N(x, y) = \sum_{i=1}^N \sum_{j=1}^M \psi_i(x) C_{i+v,j} \varphi_j(y), \tag{3.6}$$

where we denote the inverse Gram type matrix with $C = A^{-1}$. In particular the joint probability density (3.1) itself is determinantal, with $k = N$ in (3.5).

Proof. It is well known that for a block matrix $D = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with square blocks a and d the determinant of D can be reduced to determinants of smaller size as follows,

$$\det[D] = \det[a] \det[d - ba^{-1}c], \tag{3.7}$$

provided that matrix a is invertible. A similar formula exists for matrix d being invertible. The matrix $d - ba^{-1}c$ is called the Schur complement of matrix a in D . Choosing $(a)_{i,j} = q_i^{j-1}$ as the $v \times v$ matrix from the upper left block of the last determinant in (3.1) we obtain

$$\det[1, q_i, \dots, q_i^{v-1}, \varphi_i(y_1), \dots, \varphi_i(y_N)]_{i=1}^M = \det[a] \det[\tilde{\varphi}_i(y_j)]_{i,j=1}^N. \tag{3.8}$$

For the Schur complement we obtain

$$\tilde{\varphi}_i(y) = \varphi_{i+v}(y) - \sum_{k,l=1}^v b_{i,k} (a^{-1})_{k,l} \varphi_l(y) \quad \text{for } i = 1, \dots, N, \tag{3.9}$$

with $b_{i,k} = q_{i+v}^{k-1}$. Clearly, for all $q_{i=1,\dots,v}$ being mutually distinct, matrix a is invertible and in fact $\det[a] = \Delta_v(q_1, \dots, q_v)$. We can thus apply the result of Borodin [12] for the biorthogonal ensemble obtained from (3.1),

$$P(y_1, \dots, y_N) = \frac{\Delta_v(q_1, \dots, q_v)}{N! \det[A_{i,j}]_{i,j=1}^M} \det[\psi_i(y_j)]_{i,j=1}^N \det[\tilde{\varphi}_i(y_j)]_{i,j=1}^N, \quad (3.10)$$

and conclude that it is indeed a determinantal point process, with its correlation kernel given by

$$K_N(x, y) = \sum_{i,j=1}^N \psi_i(x) (g^{-1})_{i,j} \tilde{\varphi}_j(y) \quad \text{with } g_{i,j} = \int_0^\infty dy \tilde{\varphi}_i(y) \psi_j(y). \quad (3.11)$$

It remains to show (3.6). For that we insert (3.9) into our kernel (3.11) to obtain

$$K_N(x, y) = \sum_{i=1}^N \psi_i(x) \left[\sum_{j=1}^N (g^{-1})_{i,j} \varphi_{j+v}(y) - \sum_{l=1}^v \left(\sum_{j=1}^N \sum_{k=1}^v (g^{-1})_{i,j} b_{j,k} (a^{-1})_{k,l} \right) \varphi_l(y) \right]. \quad (3.12)$$

Considering now matrix A from (3.2) as a block matrix, $A = \begin{pmatrix} a & J \\ b & I \end{pmatrix}$, with matrices a and b as defined before, we immediately realise that

$$\det[A] = \det[a] \det[I_{i+v,j} - (b(a^{-1})J)_{i,j}] = \det[a] \det[g_{i,j}]_{i,j=1}^N \quad (3.13)$$

holds for the corresponding Schur complement. By making use of this block decomposition of A , it is well known that its inverse, $C = A^{-1}$, can be written in the following block form, cf. [37], Section 3.1:

$$C = \begin{pmatrix} a^{-1} + a^{-1} J g^{-1} b a^{-1} & -a^{-1} J g^{-1} \\ -g^{-1} b a^{-1} & g^{-1} \end{pmatrix}. \quad (3.14)$$

In particular its two lower blocks are given by

$$C_{i+v,j} = \begin{cases} -\sum_{l=1}^N \sum_{k=1}^v (g^{-1})_{i,l} b_{l,k} (a^{-1})_{k,j} & \text{for } j = 1, \dots, v, \\ (g^{-1})_{i,j-v} & \text{for } j = 1 + v, \dots, N + v, \end{cases} \quad (3.15)$$

where $i = 1, \dots, N$. Together with (3.12) this yields (3.6). \square

Remark 3.2. As an alternative to the formulation of the kernel (3.11) in terms of the inverse Gram matrix, in [12] the two sets of functions constituting the joint density (3.10) can also be orthogonalised. For the example from the last subsection, (2.39), this seems to be challenging, as (for $v = 0$) in each determinant these functions differ only by the parameters in the arguments. A biorthogonalisation can still be performed, see [1] for a similar example.

Remark 3.3. There exist alternative proofs of Proposition 3.1 without applying [12]. While in Appendix B we use simple ideas from functional analysis, we present here a short calculation applying the extended Andréief formula (2.12) [28]. Choosing $l = k + v$ and $N \rightarrow (N - k)$ in (2.12), we can directly perform the integration of (3.1) over $(N - k)$ variables as prescribed in (3.4):

$$\begin{aligned} \rho_k(y_1, \dots, y_k) &= \frac{(-1)^{k(k+v)}}{\det[A]} \det \begin{bmatrix} O_{k \times v} & O_{k \times k} & \psi_i(y_j) \Big|_{j=1, \dots, k}^{i=1, \dots, N} \\ q_j^{i-1} \Big|_{j=1, \dots, M}^{i=1, \dots, v} & \varphi_j(y_i) \Big|_{j=1, \dots, M}^{i=1, \dots, k} & \int_0^\infty dy \varphi_j(y) \psi_i(y) \Big|_{j=1, \dots, M}^{i=1, \dots, N} \end{bmatrix} \\ &= \frac{(-1)^{k^2}}{\det[A]} \det \begin{bmatrix} O_{k \times k} & O_{k \times v} & \psi_i(y_j) \Big|_{j=1, \dots, k}^{i=1, \dots, N} \\ \varphi_j(y_i) \Big|_{j=1, \dots, M}^{i=1, \dots, k} & q_j^{i-1} \Big|_{j=1, \dots, M}^{i=1, \dots, v} & I_{j,i} \Big|_{j=1, \dots, M}^{i=1, \dots, N} \end{bmatrix}. \end{aligned} \quad (3.16)$$

Here, the functions $\psi_i(y)$ and $\varphi_j(y)$ of unintegrated variables y_1, \dots, y_k are corresponding to the matrices R and S in (2.12), respectively. In the second step we have simply interchanged columns, such that the matrix A from

(3.2) is formed by the two lower right blocks. Using the equivalent formula to (3.7) for invertible $d = A$ this time, $\det[D] = \det[d] \det[a - cd^{-1}b]$, we can choose $a = O_{k \times k}$ here. After taking out all minus signs of the determinant and using that part of matrix c is $O_{k \times \nu}$, we arrive at the statement of Proposition 3.1:

$$\rho_k(y_1, \dots, y_k) = \det \left[\sum_{i=1+\nu}^M \sum_{j=1}^M \psi_{i-\nu}(y_n) C_{i,j} \varphi_j(y_m) \right]_{n,m=1}^k. \tag{3.17}$$

3.1. Kernel of the generalised Wishart ensemble

We begin by deriving an explicit form of the kernel of the generalised Wishart ensemble with joint probability density (2.3). In the simplest case, when $M = N$ ($\nu = 0$), its Gram type matrix (3.2) reads $A_{i,j} = I_{i,j}^{(1)} = (q_i + \sigma_j)^{-1}$, from (2.15). For its inversion we use the following result of [12], Lemma 3.1 (cf. [38] for an earlier work)

Lemma 3.4 (Borodin). *The inverse $C_{i,j}$ of matrix $A_{i,j} = (q_i + \sigma_j)^{-1}$ is given by*

$$C_{i,j} = \frac{1}{(q_j + \sigma_i)} \frac{\prod_{l=1}^N (q_l + \sigma_i)(q_j + \sigma_l)}{\prod_{k=1; k \neq i}^N (\sigma_i - \sigma_k) \prod_{l=1; l \neq j}^N (q_j - q_l)}. \tag{3.18}$$

From (3.6) together with (2.14) this explicitly determines the kernel of the generalised Wishart ensemble for $\nu = 0$:

$$\begin{aligned} K_N^{(1)}(x, y) &= \sum_{i,j=1}^N \frac{e^{-\sigma_i x - q_j y}}{(q_j + \sigma_i)} \frac{\prod_{l=1}^N (q_l + \sigma_i)(q_j + \sigma_l)}{\prod_{k=1; k \neq i}^N (\sigma_i - \sigma_k) \prod_{l=1; l \neq j}^N (q_j - q_l)} \\ &= \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} \oint_{\gamma_q} \frac{d\zeta}{2\pi i} \frac{e^{x\eta - y\zeta}}{\eta - \zeta} \prod_{l=1}^N \frac{\zeta + \sigma_l}{\eta + \sigma_l} \prod_{l=1}^N \frac{\eta - q_l}{\zeta - q_l}. \end{aligned} \tag{3.19}$$

In the second step we have used the Residue Theorem to express the double sum as a double contour integral. The contours are defined such that the closed contour γ_σ includes the poles at $-\sigma_l$, $l = 1, \dots, N$, running in counter-clockwise direction, and likewise γ_q includes the poles at q_l , $l = 1, \dots, N$ in counter-clockwise direction, such that the two contours do not intersect. Because of $q_i + \sigma_j > 0$, $\forall i, j$, this is always possible. For different choices of integration contours see Figure 1. Note that the form of the kernel (3.19) valid for $M = N$ can be found already in [13], see also [15] for the Multiple Laguerre kernel. It is very suggestive to expect that a similar form holds also for $M > N$, which is our main result of this subsection as stated below.

Theorem 3.5. *The correlation kernel $K_N^{(1)}(x, y)$ of the generalised Wishart ensemble (1.6) permits the following double contour integral representation*

$$K_N^{(1)}(x, y) = \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} \oint_{\gamma_q} \frac{d\zeta}{2\pi i} \frac{e^{x\eta - y\zeta}}{\eta - \zeta} \prod_{l=1}^N \frac{\zeta + \sigma_l}{\eta + \sigma_l} \prod_{k=1}^M \frac{\eta - q_k}{\zeta - q_k}, \tag{3.20}$$

where γ_σ is a closed contour encircling $-\sigma_1, \dots, -\sigma_N$ counter-clockwise, and γ_q is a closed contour encircling q_1, \dots, q_M counter-clockwise, without intersecting γ_σ , see Figure 1.

Proof. The idea of the proof is to obtain the double contour integral representation (3.20) without explicitly computing the inverse matrix C as we did for $M = N$. For that purpose we restate the orthogonality relation $AC = \mathbb{1}_M$ for the Gram type matrix (3.2), with $I_{i,j}^{(1)} = (q_j + \sigma_i)^{-1}$ for our ensemble from (2.15):

$$\sum_{k=1}^{\nu} q_j^{k-1} C_{k,l} + \sum_{k=1}^N \frac{1}{q_j + \sigma_k} C_{k+\nu,l} = \delta_{j,l} \quad \text{for } 1 \leq j, l \leq M. \tag{3.21}$$

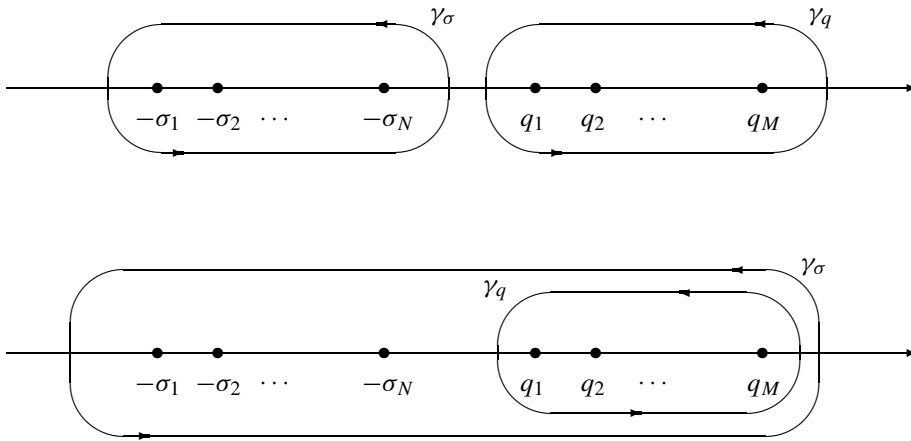


Fig. 1. Possible choices for the integration contours in Theorem 3.5: Independent non-intersecting contours (top) and nested non-intersecting contours (bottom). For simplicity we have ordered the parameters σ_i and q_j according to their index. We do not display a third possible choice that is also nested, where the inner contour encircles the $-\sigma_j$ and not the q_j .

This leads us to define the following set of $l = 1, \dots, M$ meromorphic functions

$$f_l(z) = \sum_{k=1}^v z^{k-1} C_{k,l} + \sum_{k=1}^N \frac{1}{z + \sigma_k} C_{k+v,l}. \tag{3.22}$$

They are uniquely determined in the complex plane by specifying all their zeros, poles, and by providing the value of the function at one further point. Namely, without specifying the constant matrix C on the right hand side, the functions $f_l(z)$ satisfy:

- (1) due to (3.21) each function f_l has $M - 1$ zeros, $f_l(q_i) = 0$ for $i = 1, \dots, l - 1, l + 1, \dots, M$,
- (2) because of definition (3.22), each function f_l has N poles at $z = -\sigma_k$ for $k = 1, \dots, N$,
- (3) the condition $f_l(q_l) = 1$ from (3.21) uniquely fixes the remaining constant coefficient,

leading to

$$f_l(z) = \prod_{i=1, i \neq l}^M \frac{z - q_i}{q_l - q_i} \prod_{k=1}^N \frac{q_l + \sigma_k}{z + \sigma_k}. \tag{3.23}$$

The fact that these are all poles and zeros follows from the behaviour at infinity, $\lim_{|z| \rightarrow \infty} f_l(z) = \mathcal{O}(z^{v-1})$, as required from the definition (3.22). The next step is to bring the kernel (3.6) to a form containing (3.22), such that we can apply (3.23), without determining C explicitly. From (2.14) we can rewrite for ensemble (1.6)

$$\psi_j^{(1)}(x) = e^{-\sigma_j x} = \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} \frac{e^{x\eta}}{\eta + \sigma_j}, \quad j = 1, \dots, N. \tag{3.24}$$

Here, γ_σ denotes a closed contour encircling $-\sigma_j$ in counter-clockwise direction. For later we choose γ_σ to contain already all σ_l , $l = 1, \dots, N$. Likewise, we may write zero in the form

$$0 = \oint_{\gamma_q} \frac{d\eta}{2\pi i} \eta^{j-1} e^{x\eta}, \quad j = 1, 2, \dots, \tag{3.25}$$

which holds trivially for any closed contour, due to the analyticity of the integrand. With these preparations, using the definitions (2.14) we can rewrite the kernel (3.6) for our ensemble as

$$\begin{aligned}
 K_N^{(1)}(x, y) &= \sum_{i=1}^N \sum_{j=1}^M \psi_i^{(1)}(x) C_{i+v,j} \varphi_j^{(1)}(y) \\
 &= \sum_{j=1}^M \left[\sum_{i=1}^N C_{i+v,j} \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} \frac{e^{x\eta}}{\eta + \sigma_i} + \sum_{i=1}^v C_{i,j} \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} \eta^{i-1} e^{x\eta} \right] \varphi_j^{(1)}(y) \\
 &= \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} \sum_{j=1}^M e^{x\eta} \left[\sum_{i=1}^v \eta^{i-1} C_{i,j} + \sum_{i=1}^N \frac{1}{\eta + \sigma_i} C_{i+v,j} \right] \varphi_j^{(1)}(y) \\
 &= \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} \sum_{j=1}^M e^{x\eta} e^{-q_j y} \left[\prod_{m=1, m \neq j}^M \frac{\eta - q_m}{q_j - q_m} \prod_{k=1}^N \frac{q_j + \sigma_k}{\eta + \sigma_k} \right]. \tag{3.26}
 \end{aligned}$$

In the second step we have taken out the contour integrals, and in the third step we have inserted (3.23) and the explicit representation $\varphi_j^{(1)}(y) = e^{-q_j y}$ from (2.14). Finally a simple application of the Residue Theorem leads to (3.20), when choosing γ_q as a closed contour that encircles all poles at q_l , $l = 1, \dots, M$ in counter-clockwise direction, and that does not intersect γ_σ . In view of the condition (2.2), $q_j + \sigma_i > 0 \forall i, j$, this is always possible. Two possible choices of such contours are depicted in Figure 1. \square

At first sight the second, nested choice in Figure 1 bottom may not seem to be useful: It forces us to do the integral over the inner contour γ_q first, before performing the second integral over γ_σ . However, when taking the large- N limit in Section 4 later, we will encounter the situation that two or more parameters $-\sigma_i$ and q_j coalesce. In the case of non-nested contours the contours would touch then, which is not allowed. In the nested case there is no such problem, as all $-\sigma_i$ and q_j remain enclosed by γ_σ , and none of the $-\sigma_i$ is a pole of the integral over γ_q . Of course there is a third choice, by letting γ_q enclose the contour γ_σ . Then the integral over the inner contour γ_σ has to be done first.

The double contour integral in (3.20) can be factorised, at the expense of a further real integral. For this to be possible we have to choose the contours to be non-nested as in Figure 1 top, and thus the order of integration to be independent. In that case the non-intersection condition of the contours γ_q and γ_σ and the fact that $q_j + \sigma_i > 0, \forall i, j$, implies that $\Re(\zeta - \eta) > 0$. Therefore, we can rewrite the term coupling the two contour integrals as

$$\frac{1}{\eta - \zeta} = - \int_0^1 du u^{\zeta - \eta - 1} \quad \text{for } \Re(\zeta - \eta) > 0. \tag{3.27}$$

This immediately leads to the following

Corollary 3.6. *The kernel $K_N^{(1)}(x, y)$ given by (3.20), with integration contours chosen as in Figure 1 top, can be written as*

$$K_N^{(1)}(x, y) = - \int_0^1 \frac{du}{u} F_1^{(1)}(x; u) F_2^{(1)}(y; u), \tag{3.28}$$

where the functions $F_1^{(1)}(x; u)$ and $F_2^{(1)}(y; u)$ are defined by the formulae

$$F_1^{(1)}(x; u) = \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} u^{-\eta} e^{x\eta} \frac{\prod_{l=1}^M (\eta - q_l)}{\prod_{l=1}^N (\eta + \sigma_l)}, \quad F_2^{(1)}(y; u) = \oint_{\gamma_q} \frac{d\zeta}{2\pi i} u^\zeta e^{-y\zeta} \frac{\prod_{l=1}^N (\zeta + \sigma_l)}{\prod_{l=1}^M (\zeta - q_l)}. \tag{3.29}$$

3.2. Kernel of the product of two correlated coupled matrices

Next we immediately turn to the ensemble (1.2) of two correlated coupled random matrices. The reason is that the Gram type matrix is very similar to the previous subsection, $I_{i,j} = \delta_j^{\frac{\kappa}{2}} (2\alpha^\kappa (\alpha q_i - \delta_j))^{-1}$ from (2.41), making it

straightforward to generalise the results from the previous subsection. For $N = M$ we can apply Lemma 3.4, replacing $q_j \rightarrow \alpha q_j$ and $\sigma_i \rightarrow -\delta_i$, and, apart from a trivial factor, directly read off the inverse matrix $C_{i,j}$ of $A_{i,j} = I_{i,j}$:

$$C_{i,j} = \frac{2\alpha^\kappa}{\delta_j^{\frac{\kappa}{2}}} \frac{1}{(\alpha q_j - \delta_i)} \frac{\prod_{l=1}^N (\alpha q_l - \delta_i)(\alpha q_j - \delta_l)}{\prod_{k=1; k \neq i}^N (\delta_k - \delta_i) \prod_{l=1; l \neq j}^N (\alpha q_j - \alpha q_l)}. \quad (3.30)$$

Consequently, inserting this expression together with (1.10) into (3.6) we obtain the following explicit expression for the kernel at $\nu = 0$:

$$\begin{aligned} K_N(x, y) &= \left(\frac{x}{y}\right)^{\frac{\kappa}{2}} \sum_{i=1}^N \sum_{j=1}^M I_\kappa(2\sqrt{\delta_i x}) K_\kappa(2\sqrt{\alpha q_j y}) \frac{2\left(\frac{\alpha q_j}{\delta_j}\right)^{\frac{\kappa}{2}}}{(\alpha q_j - \delta_i)} \frac{\prod_{l=1}^N (\alpha q_l - \delta_i)(\alpha q_j - \delta_l)}{\prod_{k=1; k \neq i}^N (\delta_k - \delta_i) \prod_{l=1; l \neq j}^N (\alpha q_j - \alpha q_l)} \\ &= \left(\frac{x}{y}\right)^{\frac{\kappa}{2}} 2 \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} \oint_{\gamma_q} \frac{d\zeta}{2\pi i} \left(\frac{\zeta}{\eta}\right)^{\frac{\kappa}{2}} \frac{I_\kappa(2\sqrt{\eta x}) K_\kappa(2\sqrt{\zeta y})}{\eta - \zeta} \frac{\prod_{l=1}^N (\zeta - \delta_l)(\eta - \alpha q_l)}{\prod_{l=1}^N (\eta - \delta_l)(\zeta - \alpha q_l)}. \end{aligned} \quad (3.31)$$

Here, the contours are defined analogously to Theorem 3.5, with the difference that only the *two* choices are possible that are depicted in Figure 1. After the replacement $-\sigma_l \rightarrow \delta_l$ and $q_l \rightarrow \alpha q_l$, the closed contour γ_δ is encircling $\delta_1, \dots, \delta_N \geq 0$ counter-clockwise, including or excluding all q_l . Note that $\eta^{-\frac{\kappa}{2}} I_\kappa(2\sqrt{\eta x})$ does not have a branch cut in η , cf. (A.4). The closed contour γ_q is encircling $\alpha q_1, \dots, \alpha q_N > 0$ counter-clockwise and, in contrast, excludes the origin, because $K_\kappa(2\sqrt{\zeta y})$ has a logarithmic singularity there. Thus the contour γ_q may *not* include all $\delta_l \geq 0$. The requirement of non-intersecting contours is always possible, due to the condition (1.3) that $\alpha q_i - \delta_j > 0 \forall i, j$. The last equality in (3.31) is easy to see with the help of the Residue Theorem, where the order of integration may depend on the nesting of the contours. The prefactor $(x/y)^{\frac{\kappa}{2}}$ before the two integrals can be dropped as it cancels out in the determinant (3.5), leading to an equivalent kernel (see also the remark after Theorem 1.3). Let us present the proof of this theorem now for general $M \geq N$.

Proof of Theorem 1.3. In view of the Gram matrix $I_{i,j}$ (2.41), it is advantageous for $\nu > 0$ to slightly modify the Gram type matrix (3.2) by including the appropriate powers of α :

$$\tilde{A} = \begin{pmatrix} 1 & \alpha q_1 & \dots & (\alpha q_1)^{\nu-1} & I_{1,1} & \dots & I_{1,N} \\ 1 & \alpha q_2 & \dots & (\alpha q_2)^{\nu-1} & I_{2,1} & \dots & I_{2,N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \alpha q_M & \dots & (\alpha q_M)^{\nu-1} & I_{M,1} & \dots & I_{M,N} \end{pmatrix}. \quad (3.32)$$

This can be trivially achieved by multiplying numerator and denominator of (3.1) by $\alpha^{\nu(\nu-1)/2}$. Its inverse is now denoted by \tilde{C} , with $\tilde{A}\tilde{C} = \mathbb{1}_M$. Following the ideas of the proof of Theorem 3.5 from the previous subsection, it is then not difficult to relate the inversion of the corresponding full Gram type matrix (3.32),

$$\sum_{k=1}^{\nu} (\alpha q_j)^{k-1} \tilde{C}_{k,l} + \sum_{k=1}^N \frac{\delta_j^{\frac{\kappa}{2}}}{2\alpha^\kappa} \frac{1}{(\alpha q_j - \delta_k)} \tilde{C}_{k+\nu,l} = \delta_{j,l} \quad \text{for } 1 \leq j, l \leq M, \quad (3.33)$$

to a set of $l = 1, \dots, M$ meromorphic functions

$$f_l(z) = \sum_{k=1}^{\nu} z^{k-1} \tilde{C}_{k,l} + \sum_{k=1}^N \frac{\delta_j^{\frac{\kappa}{2}}}{2\alpha^\kappa} \frac{1}{(z - \delta_k)} \tilde{C}_{k+\nu,l} = \prod_{i=1, i \neq l}^M \frac{z - \alpha q_i}{\alpha q_l - \alpha q_i} \prod_{k=1}^N \frac{\alpha q_l - \delta_k}{z - \delta_k}. \quad (3.34)$$

Its zeros at $z = \alpha q_i \neq l$, poles at $z = \delta_j$, the condition $f_l(\alpha q_l) = 1$ and checking its correct behaviour at infinity completely fixes the right hand side. With only little more thought we can also write the analogue of the conditions (3.24) and (3.25) for the respective function ψ_i from (1.10):

$$\psi_i(x) = x^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\delta_i x}) = 2 \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} \frac{1}{\eta - \delta_i} \left(\frac{\delta_i x}{\eta}\right)^{\frac{\kappa}{2}} \frac{1}{2} I_\kappa(2\sqrt{\eta x}), \quad i = 1, \dots, N, \quad (3.35)$$

and, due to (A.4)

$$0 = 2 \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} \eta^{j-1} \alpha^\kappa \left(\frac{x}{\eta} \right)^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\eta x}), \quad j = 1, 2, \dots \quad (3.36)$$

We can then rewrite the kernel (3.6) as in the previous subsection:

$$\begin{aligned} K_N(x, y) &= \sum_{j=1}^M \left[\sum_{i=1}^N \tilde{C}_{i+v, j} 2 \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} \frac{1}{\eta - \delta_i} \left(\frac{\delta_i x}{\eta} \right)^{\frac{\kappa}{2}} \frac{1}{2} I_\kappa(2\sqrt{\eta x}) \right. \\ &\quad \left. + \sum_{i=1}^v \tilde{C}_{i, j} 2 \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} \eta^{i-1} \alpha^\kappa \left(\frac{x}{\eta} \right)^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\eta x}) \right] \varphi_j(y) \\ &= \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} 2\alpha^\kappa \left(\frac{x}{\eta} \right)^{\frac{\kappa}{2}} \sum_{j=1}^M I_\kappa(2\sqrt{\eta x}) \left[\sum_{i=1}^v \eta^{i-1} \tilde{C}_{i, j} + \sum_{i=1}^N \frac{\delta_i^{\frac{\kappa}{2}}}{2\alpha^\kappa (\eta - \delta_i)} \tilde{C}_{i+v, j} \right] \varphi_j(y) \\ &= \left(\frac{x}{y} \right)^{\frac{\kappa}{2}} 2 \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} \sum_{j=1}^M I_\kappa(2\sqrt{\eta x}) \left[\prod_{m=1, m \neq j}^M \frac{\eta - \alpha q_m}{\alpha q_j - \alpha q_m} \prod_{k=1}^N \frac{\alpha q_j - \delta_k}{\eta - \delta_k} \right] \left(\frac{\alpha q_j}{\eta} \right)^{\frac{\kappa}{2}} \\ &\quad \times K_\kappa(2\sqrt{\alpha q_j y}). \end{aligned} \quad (3.37)$$

We have inserted $\varphi_j(y)$ from (1.10) and in the last step used the Residue Theorem, leading to a kernel equivalent to (1.12). As discussed previously we have two choices for the contours not to intersect, being either nested or separated. In case they are nested the inner integration has to be done first. \square

Along the same lines as in the previous subsection we can derive the following equivalent factorised form of the kernel, using the identity (3.27). For this factorised form we have to choose again the contours to be non-nested (cf. Figure 1 top), for the integrals to factorise and become independent. Here we also removed the prefactor $(x/y)^{\frac{\kappa}{2}}$ in (3.37).

Corollary 3.7. *The kernel $K_N(x, y)$ given by Theorem 1.3 is equal to the following kernel*

$$K_N(x, y) = - \int_0^1 \frac{du}{u} F_1(x; u) F_2(y; u). \quad (3.38)$$

The functions $F_1(x; u)$ and $F_2(y; u)$ are defined by the formulae

$$F_1(x; u) = \oint_{\gamma_\delta} \frac{d\eta}{2\pi i} u^{-\eta} \eta^{-\frac{\kappa}{2}} I_\kappa(2\sqrt{\eta x}) \frac{\prod_{l=1}^M (\eta - \alpha q_l)}{\prod_{l=1}^N (\eta - \delta_l)}, \quad (3.39)$$

where γ_δ encloses all δ_l in a counter-clockwise way, and

$$F_2(y; u) = 2 \oint_{\gamma_q} \frac{d\zeta}{2\pi i} u^\zeta \zeta^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\zeta y}) \frac{\prod_{l=1}^N (\zeta - \delta_l)}{\prod_{l=1}^M (\zeta - \alpha q_l)}. \quad (3.40)$$

The contour γ_q encloses all αq_l in a counter-clockwise way, excludes the origin and all δ_l .

3.3. Kernel of the product of two independent correlated matrices

We turn to the kernel of the ensemble (1.5) of two independent matrices, one of which has correlated entries. Rather than trying to first invert the Gram matrix for $N = M$, we immediately turn to the procedure from the previous two subsections, that directly leads to the following double contour integral representation.

Theorem 3.8. *The correlation kernel $K_N^{(2)}(x, y)$ of the ensemble (1.5) permits the following double contour integral representation*

$$K_N^{(2)}(x, y) = \left(\frac{x}{y}\right)^{\frac{\kappa}{2}} 2 \oint_{\gamma_0} \frac{d\eta}{2\pi i} \oint_{\gamma_q} \frac{d\zeta}{2\pi i} \left(\frac{\zeta}{\eta}\right)^{N+\frac{\kappa}{2}} \frac{I_\kappa(2\sqrt{\eta x}) K_\kappa(2\sqrt{\zeta y})}{\eta - \zeta} \prod_{l=1}^M \frac{\eta - \alpha q_l}{\zeta - \alpha q_l}, \quad (3.41)$$

where γ_0 is a closed contour encircling the origin in counter-clockwise direction, and γ_q is a closed contour encircling $\alpha q_1, \dots, \alpha q_M > 0$ counter-clockwise, excluding the origin and not intersecting γ_0 .

Proof. In view of the Gram matrix (2.32), $I_{i,j}^{(2)} = \Gamma(\kappa + j)\Gamma(j)/(2\alpha^{\kappa+j}q_i^j)$, it is again useful to start with the modified Gram type matrix (3.32) as in the previous subsection. The corresponding equation $\tilde{A}\tilde{C} = \mathbb{1}_M$ thus reads:

$$\sum_{k=1}^{\nu} (\alpha q_j)^{k-1} \tilde{C}_{k,l} + \sum_{m=1}^N \frac{\Gamma(\kappa + m)\Gamma(m)}{2\alpha^{\kappa+m}q_j^m} \tilde{C}_{m+\nu,l} = \delta_{j,l} \quad \text{for } 1 \leq j, l \leq M. \quad (3.42)$$

Once again this can be used to define the following set of $l = 1, \dots, M$ meromorphic functions

$$f_l(z) = \sum_{k=1}^{\nu} z^{k-1} \tilde{C}_{k,l} + \sum_{m=1}^N \frac{\Gamma(\kappa + m)\Gamma(m)}{2\alpha^{\kappa}} z^{-m} \tilde{C}_{m+\nu,l} = \left(\frac{\alpha q_l}{z}\right)^N \prod_{i=1, i \neq l}^M \frac{z - \alpha q_i}{\alpha q_l - \alpha q_i}. \quad (3.43)$$

These functions are determined by their $M - 1$ zeros at $z = \alpha q_{i \neq l}$, the poles of up to order N at $z = 0$, the condition $f_l(\alpha q_l) = 1$, and by checking its behaviour at infinity. Given that in this ensemble we have (2.31), we can rewrite (cf. (A.4))

$$\psi_i^{(2)}(x) = x^{\kappa+i-1} = \frac{1}{(2\pi i)^2} \oint_{\gamma_0} d\eta \eta^{-i} \left(\frac{x}{\eta}\right)^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\eta x}) \Gamma(\kappa + i) \Gamma(i), \quad i = 1, 2, \dots, \quad (3.44)$$

where γ_0 is a closed contour encircling the origin counter-clockwise. For the additional condition we can reuse (3.36) which is true also for γ_0 , due to the analyticity of the integrand. We thus obtain from (3.6) in our case

$$\begin{aligned} K_N^{(2)}(x, y) &= \sum_{j=1}^M \left[\sum_{i=1}^N \tilde{C}_{i+\nu,j} \oint_{\gamma_0} \frac{d\eta}{2\pi i} \eta^{-i} \left(\frac{x}{\eta}\right)^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\eta x}) \Gamma(\kappa + i) \Gamma(i) \right. \\ &\quad \left. + \sum_{i=1}^{\nu} \tilde{C}_{i,j} \oint_{\gamma_0} \frac{d\eta}{2\pi i} \eta^{i-1} 2\alpha^\kappa \left(\frac{x}{\eta}\right)^{\frac{\kappa}{2}} I_\kappa(2\sqrt{\eta x}) \right] \varphi_j^{(2)}(y) \\ &= \oint_{\gamma_0} \frac{d\eta}{2\pi i} 2\alpha^\kappa \left(\frac{x}{\eta}\right)^{\frac{\kappa}{2}} \sum_{j=1}^M I_\kappa(2\sqrt{\eta x}) \left[\sum_{i=1}^{\nu} \eta^{i-1} \tilde{C}_{i,j} + \sum_{i=1}^N \frac{\Gamma(\kappa + i)\Gamma(i)}{2\alpha^\kappa} \eta^{-i} \tilde{C}_{i+\nu,j} \right] \varphi_j^{(2)}(y) \\ &= \left(\frac{x}{y}\right)^{\frac{\kappa}{2}} 2 \oint_{\gamma_0} \frac{d\eta}{2\pi i} \sum_{j=1}^M I_\kappa(2\sqrt{\eta x}) \left[\frac{(\alpha q_j)^N}{\eta^N} \prod_{m=1, m \neq j}^M \frac{\eta - \alpha q_m}{\alpha q_j - \alpha q_m} \right] \left(\frac{\alpha q_j}{\eta}\right)^{\frac{\kappa}{2}} K_\kappa(2\sqrt{\alpha q_j y}). \quad (3.45) \end{aligned}$$

Here, we have inserted $\varphi_j^{(2)}(y)$ from (2.29) and used the Residue Theorem, with two possible choices of contours, cf. Figure 2. \square

Note that as a check (3.41) agrees with the kernel of the coupled ensemble (1.12), when setting all $\delta_{k=1, \dots, N} = 0$ there.

Using again the identity (3.27), from (3.41) together with the choice of non-nested contours we obtain the following factorised integral representation of a kernel equivalent to that in Theorem 3.8.

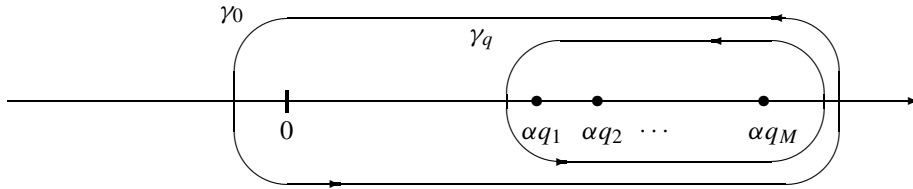


Fig. 2. Nested choice of the contours γ_0 and γ_q . For the second choice of non-intersecting contours that is non-nested γ_0 is including only the origin, but none of the αq_l . Because γ_q is excluding the origin we don't have a third choice here, where γ_0 is lying inside γ_q .

Corollary 3.9. *The kernel $K_N^{(2)}(x, y)$ given by Theorem 3.8 is equivalent to the following kernel*

$$K_N^{(2)}(x, y) = - \int_0^1 \frac{du}{u} F_1^{(2)}(x; u) F_2^{(2)}(y; u). \tag{3.46}$$

The functions $F_1^{(2)}(x; u)$ and $F_2^{(2)}(y; u)$ are defined by the formulae

$$F_1^{(2)}(x; u) = \oint_{\gamma_0} \frac{d\eta}{2\pi i} u^{-\eta} \eta^{-\frac{\kappa}{2}-N} I_\kappa(2\sqrt{\eta x}) \prod_{l=1}^M (\eta - \alpha q_l), \tag{3.47}$$

with γ_0 encircling the origin counter-clockwise, and

$$F_2^{(2)}(y; u) = 2 \oint_{\gamma_q} \frac{d\xi}{2\pi i} u^\xi \xi^{\frac{\kappa}{2}+N} K_\kappa(2\sqrt{\xi y}) \prod_{l=1}^M \frac{1}{\xi - \alpha q_l}, \tag{3.48}$$

with γ_q including all αq_l counter-clockwise and excluding the origin.

This corollary agrees with Corollary 3.7 of the coupled ensemble, after setting all $\delta_{k=1, \dots, N} = 0$ therein.

4. Large- N limit at the origin and integrability

In this section we will study the limit of large matrix size $N \rightarrow \infty$ at the origin of the spectrum, in all three ensembles separately. It turns out that the kernel of the generalised Wishart ensemble (1.6) will lead to the generalised Bessel kernel \mathbb{K}_{III} (1.30) in the large- N limit. This will be shown in the first Section 4.1. The kernel of the second ensemble (1.5) of independent matrices with correlated entries leads to the limiting kernel \mathbb{K}_{I} (1.26). It generalises the limiting Meijer G -kernel [34] obtained for the product of two independent random matrices by adding finite rank perturbations, as will be shown in Section 4.2. In the last Section 4.3 we will show that the kernel of the product of two correlated coupled random matrices leads to three different limiting kernels, depending on the coupling parameter $\mu = \mu(N)$ as a function of N : The kernel \mathbb{K}_{III} follows in limit (III) $\mu(N)N \rightarrow 0$, and kernel \mathbb{K}_{I} in limit (I) $\mu(N)N \rightarrow \infty$. A third kernel \mathbb{K}_{II} given in (1.28) follows in limit (II) when $\mu(N)N \rightarrow \tau/4$, with $\tau > 0$, and interpolates between the kernels obtained in limits (I) and (III). It generalises the interpolating kernel of [6] and of [35], by adding further finite rank perturbations. In that sense all three limiting kernels $\mathbb{K}_{\text{I,II,III}}$ are universal as they follow from different ensembles. For all three kernels we provide their corresponding integrable form, in the sense of [25].

4.1. Bessel kernel with finite rank perturbations from the generalised Wishart ensemble

We begin by recalling the generalised Wishart ensemble (1.6)

$$\mathcal{P}_1(X) = c_1 \exp[-\text{Tr}(X \Sigma X^*) - \text{Tr}(Q X X^*)],$$

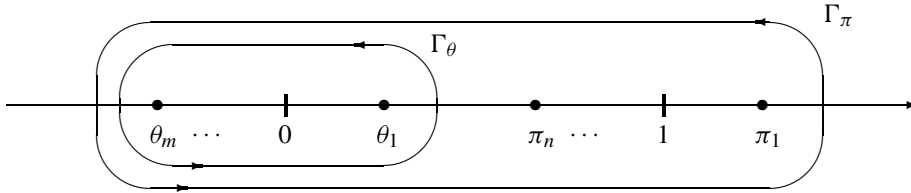


Fig. 3. The contours Γ_θ and Γ_π resulting from the substitution (4.3) are shown. From (4.4) the poles θ_k are centred around the degenerate value σ that has been mapped to the origin, and likewise the poles π_j are centred around the degenerate value q mapped to unity.

with eigenvalues $\sigma_1, \dots, \sigma_N$ of Σ and q_1, \dots, q_M of Q , respectively. In the following we will consider finite rank perturbations of the fully degenerate case, $\Sigma = \sigma \mathbb{1}_N$ and $Q = q \mathbb{1}_M$, by setting

$$\sigma_{n+1} = \dots = \sigma_N = \sigma \quad \text{and} \quad q_{m+1} = \dots = q_M = q, \quad (4.1)$$

with n and m independent of N . Thus we consider a perturbation around the standard Wishart–Laguerre ensemble $\mathcal{P}_1(X) = c_1 \exp[-(q + \sigma) \text{Tr}(XX^*)]$, with $q + \sigma > 0$.

If we want to compare to our most general ensemble (1.2) later, e.g. by integrating out random matrix G there, we would have to identify $-\Sigma = \Omega \Omega^* / \alpha$, or $-\sigma_j = \delta_j / \alpha$ for $j = 1, \dots, N$. In Section 4.3 we will make the parameters α and δ_j there μ - and thus N -dependent, which would lead to identify $q + \sigma = 2/(1 + \mu)$. In this subsection, however, there is no need to introduce such an extra parameter $\mu = \mu(N)$, as the large- N limit at the origin that we will take here does not depend on it.

Inserting the degeneracy (4.1) into the kernel at finite- N (3.20) from Theorem 3.5, we obtain

$$\begin{aligned} K_N^{(1)}(x, y) &= \oint_{\gamma_\sigma} \frac{d\eta}{2\pi i} \oint_{\gamma_q} \frac{d\zeta}{2\pi i} \frac{e^{x\eta - y\zeta}}{\eta - \zeta} \left(\frac{\zeta + \sigma}{\eta + \sigma} \right)^{N-n} \left(\frac{\eta - q}{\zeta - q} \right)^{M-m} \prod_{l=1}^n \frac{\zeta + \sigma_l}{\eta + \sigma_l} \prod_{k=1}^m \frac{\eta - q_k}{\zeta - q_k} \\ &= e^{q(x-y)} (q + \sigma) \oint_{\Gamma_\pi} \frac{dv}{2\pi i} \oint_{\Gamma_\theta} \frac{du}{2\pi i} \frac{e^{(q+\sigma)(yu-xv)}}{u-v} \left(\frac{1 - \frac{1}{u}}{1 - \frac{1}{v}} \right)^{N-n} \left(\frac{v}{u} \right)^{v-m+n} \\ &\quad \times \prod_{l=1}^n \frac{u - \frac{q+\sigma_l}{q+\sigma}}{v - \frac{q+\sigma_l}{q+\sigma}} \prod_{k=1}^m \frac{v - \frac{q-q_k}{q+\sigma}}{u - \frac{q-q_k}{q+\sigma}}. \end{aligned} \quad (4.2)$$

In the second line we have made the following substitution:

$$\zeta = q - (q + \sigma)u \quad \text{and} \quad \eta = q - (q + \sigma)v. \quad (4.3)$$

Starting from the nested contours as in Figure 1 bottom, the integration contours resulting from this substitution are given in Figure 3. Due to $q + \sigma > 0$ the substitution is non-singular.

In order to take the large- N limit let us introduce the following notation for the locations of the nontrivial poles of the integrand (4.2):

$$\pi_l = \frac{q + \sigma_l}{q + \sigma}, \quad l = 1, \dots, n \quad \text{and} \quad \theta_k = \frac{q - q_k}{q + \sigma}, \quad k = 1, \dots, m. \quad (4.4)$$

These will be rescaled as

$$\hat{\pi}_l = \lim_{N \rightarrow \infty} \frac{\pi_l}{N}, \quad \hat{\theta}_k = \lim_{N \rightarrow \infty} \frac{\theta_k}{N}, \quad \hat{\pi}_l - \hat{\theta}_k \geq 0, \quad \hat{\pi}_l \geq 0, \quad \forall k, l, \quad (4.5)$$

as well as the integration variables

$$v = N\hat{v}, \quad u = N\hat{u}, \quad (4.6)$$

and the arguments of the kernel

$$x = \frac{\hat{x}}{(q + \sigma)N}, \quad y = \frac{\hat{y}}{(q + \sigma)N}. \tag{4.7}$$

Note that after this rescaling in Figure 3 unity is mapped to $1/N$ that moves to the origin when $N \rightarrow \infty$. Consequently, the limiting variables $\hat{\pi}_j$ and $\hat{\theta}_k$ may no longer be separated in the large- N limit. Because of the choice of nested contour integrals this is not a problem. This leads to the following result for the limiting kernel at the hard edge.

Theorem 4.1. Consider the finite rank perturbations (4.1) and define the parameters $\hat{\pi}_l, l = 1, \dots, n$ and $\hat{\theta}_k, k = 1, \dots, m$ as in (4.4) and (4.5). Then the following limit of the kernel (3.20) leads to a kernel equivalent to

$$\begin{aligned} \mathbb{K}_{\text{III}}^{(n,m)}(\hat{x}, \hat{y}) &= \lim_{N \rightarrow \infty} \frac{1}{(q + \sigma)N} K_N^{(1)}\left(\frac{\hat{x}}{(q + \sigma)N}, \frac{\hat{y}}{(q + \sigma)N}\right) e^{-\frac{q}{(q+\sigma)N}(\hat{x}-\hat{y})} \\ &= \oint_{\Gamma_{\text{out}}} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{d\hat{u}}{2\pi i} \frac{\exp[-\hat{x}\hat{v} + \hat{y}\hat{u}]}{\hat{u} - \hat{v}} e^{-\frac{1}{\hat{u}} + \frac{1}{\hat{v}}} \left(\frac{\hat{v}}{\hat{u}}\right)^{\nu+n-m} \prod_{l=1}^n \frac{\hat{u} - \hat{\pi}_l}{\hat{v} - \hat{\pi}_l} \prod_{k=1}^m \frac{\hat{v} - \hat{\theta}_k}{\hat{u} - \hat{\theta}_k}, \end{aligned} \tag{4.8}$$

which is a Bessel kernel with finite rank perturbations. The closed integration contour Γ_{in} encircles the $\hat{\theta}_{j=1,\dots,m}$ including the origin counter-clockwise, and the closed contour Γ_{out} contains the $\hat{\pi}_{l=1,\dots,n}$ and encircles them counter-clockwise. It also contains the contour Γ_{in} without intersecting it.

Proof. We take (4.2) as starting point, being equal to (3.20), and insert the finite rank conditions (4.1). The prefactor $\exp[q(x - y)]$ can be removed, as it leads to an equivalent kernel. The scaling variables are defined in (4.6) and (4.7). Clearly the N -dependence drops out everywhere, except in the factor

$$\left(\frac{1 - \frac{1}{\hat{u}N}}{1 - \frac{1}{\hat{v}N}}\right)^{N-n} \sim \exp\left[-\frac{1}{\hat{u}} + \frac{1}{\hat{v}}\right] \text{ as } N \rightarrow \infty. \tag{4.9}$$

Due to Lebesgue’s Dominated Convergence Theorem we can interchange the limit $N \rightarrow \infty$ and the double contour integral to apply this limit. Recalling that n, m and $\nu = M - N$ are fixed in this limit we arrive at (4.8). \square

Note that in this ensemble the domains of the parameters $\hat{\theta}_{k=1,\dots,m} \leq \hat{\pi}_{l=1,\dots,n}$ are not restricted and can be the entire real line. As mentioned already, without parameters $\hat{\theta}_k$ the kernel (4.8) at $m = 0$ was found in [35] for the ensemble (1.2) with $Q \sim \mathbb{1}_M$. The very same kernel was found previously in [15], Theorem 15, for the Wishart ensemble with an external field. There, it was also shown that without any finite rank perturbations, that is when $n = 0$ (and $m = 0$ here), it agrees with the Bessel kernel

$$\mathbb{K}_{\text{III}}^{(0,0)}(x, y) = \mathbb{K}_{\text{Bessel}}(x, y) = \oint_{\Gamma_{\text{out}}} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{d\hat{u}}{2\pi i} \frac{\exp[-x\hat{v} + y\hat{u}]}{\hat{u} - \hat{v}} e^{-\frac{1}{\hat{u}} + \frac{1}{\hat{v}}} \left(\frac{\hat{v}}{\hat{u}}\right)^{\nu}, \tag{4.10}$$

with Γ_{out} and Γ_{in} two nested, non-intersecting contours that both enclose the origin in counter-clockwise direction. The relation to the Bessel kernel can be spelled out more explicitly by bringing the kernel (4.8) to a form that is called integrable. Namely, a kernel is called integrable in the sense of [25], if it can be written as

$$K(x, y) = \sum_{l=1}^L \frac{F_l(x)G_l(y)}{x - y} \quad \text{with} \quad \sum_{l=1}^L F_l(x)G_l(x) = 0, \tag{4.11}$$

holding for some given functions $F_l(x)$ and $G_l(x)$ and fixed value of L .

Corollary 4.2. *The Bessel kernel with finite rank perturbations (4.8) is integrable and can be written in the following form:*

$$\mathbb{K}_{\text{III}}^{(n,m)}(x, y) = \mathbb{K}_{\text{Bessel}}(x, y)|_{v \rightarrow v+n-m} - \sum_{i=1}^m \tilde{\Lambda}_{\text{III}}^{(i)}(x) \tilde{\Xi}_{\text{III}}^{(i)}(y) + \sum_{j=1}^n \Lambda_{\text{III}}^{(j)}(x) \Xi_{\text{III}}^{(j)}(y), \tag{4.12}$$

where we introduce four functions

$$\begin{aligned} \tilde{\Lambda}_{\text{III}}^{(i)}(x) &= \oint_{\Gamma_{\hat{\pi}}} \frac{d\hat{v}}{2\pi i} \exp\left[-x\hat{v} + \frac{1}{\hat{v}}\right] \hat{v}^{v+n-m} \prod_{k=1}^{i-1} (\hat{v} - \hat{\theta}_k), \\ \Lambda_{\text{III}}^{(j)}(x) &= \oint_{\Gamma_{\hat{\pi}}} \frac{d\hat{v}}{2\pi i} \exp\left[-x\hat{v} + \frac{1}{\hat{v}}\right] \hat{v}^{v+n-m} \frac{\prod_{k=1}^m (\hat{v} - \hat{\theta}_k)}{\prod_{l=1}^j (\hat{v} - \hat{\pi}_l)}, \\ \tilde{\Xi}_{\text{III}}^{(i)}(y) &= \oint_{\Gamma_{\hat{\theta}}} \frac{d\hat{u}}{2\pi i} \exp\left[y\hat{u} - \frac{1}{\hat{u}}\right] \hat{u}^{-v-n+m} \prod_{k=1}^i \frac{1}{\hat{u} - \hat{\theta}_k}, \\ \Xi_{\text{III}}^{(j)}(y) &= \oint_{\Gamma_{\hat{\theta}}} \frac{d\hat{u}}{2\pi i} \exp\left[y\hat{u} - \frac{1}{\hat{u}}\right] \hat{u}^{-v-n+m} \frac{\prod_{l=1}^{j-1} (\hat{u} - \hat{\pi}_l)}{\prod_{k=1}^m (\hat{u} - \hat{\theta}_k)}. \end{aligned} \tag{4.13}$$

Here the closed contour $\Gamma_{\hat{\theta}}$ contains the poles at $\hat{\theta}_{k=1, \dots, m}$ and the origin, encircling them counter-clockwise, and the closed contour $\Gamma_{\hat{\pi}}$ contains the poles at $\hat{\pi}_{l=1, \dots, n}$ and the origin, encircling them counter-clockwise.

This corollary solves an open problem stated in [15], Section 7.2, to find such an integrable representation. Note that as mentioned in the Introduction the generalised Bessel kernel (4.8) enjoys a formal duality relation, interchanging the parameters $\hat{\pi}_l \leftrightarrow -\hat{\theta}_l$ and $v \rightarrow -v$, which ultimately amounts to interchange matrices Σ and Q , and N and M in (1.6). Thus this duality holds already for the kernel at finite- N , see (3.20).

Proof. The crucial step for the integrability is the following identity [15], Eq. (5.12),

$$\frac{1}{\hat{u} - \hat{v}} \prod_{l=1}^n \frac{\hat{u} - \hat{\pi}_l}{\hat{v} - \hat{\pi}_l} = \frac{1}{\hat{u} - \hat{v}} + \sum_{k=1}^n \frac{1}{\hat{v} - \hat{\pi}_k} \prod_{l=1}^{k-1} \frac{\hat{u} - \hat{\pi}_l}{\hat{v} - \hat{\pi}_l}, \tag{4.14}$$

which we have to apply twice, in view of the two products in (4.8):

$$\frac{1}{\hat{u} - \hat{v}} \prod_{l=1}^n \frac{\hat{u} - \hat{\pi}_l}{\hat{v} - \hat{\pi}_l} \prod_{k=1}^m \frac{\hat{v} - \hat{\theta}_k}{\hat{u} - \hat{\theta}_k} = \frac{1}{\hat{u} - \hat{v}} - \sum_{i=1}^m \frac{\prod_{k=1}^{i-1} (\hat{v} - \hat{\theta}_k)}{\prod_{k=1}^i (\hat{u} - \hat{\theta}_k)} + \sum_{j=1}^n \frac{\prod_{l=1}^{j-1} (\hat{u} - \hat{\pi}_l)}{\prod_{k=1}^m (\hat{u} - \hat{\theta}_k)} \frac{\prod_{k=1}^m (\hat{v} - \hat{\theta}_k)}{\prod_{l=1}^j (\hat{v} - \hat{\pi}_l)}. \tag{4.15}$$

Inserting this identity into (4.8) the right-hand sides of Equations (4.10) and (4.13) can be read off. Regarding contours, only in the integral (4.10), where the pole $\frac{1}{\hat{u}-\hat{v}}$ remains present, the condition of non-intersection contours remains. In all other integrals the contour Γ_{out} can be deformed to $\Gamma_{\hat{\theta}}$ to contain the poles at $\hat{\theta}_{k=1, \dots, m}$ and the origin, and the contour Γ_{in} to $\Gamma_{\hat{\pi}}$ to contain the poles at $\hat{\pi}_{l=1, \dots, n}$ and the origin.

The kernel is integrable due to two observations. First, as shown in [15] the contour integral (4.10) is equivalent to the more common representation of the Bessel kernel, see e.g. [18]

$$\mathbb{K}_{\text{Bessel}}(x, y) = \frac{-\sqrt{x} J_{v+1}(\sqrt{x}) J_v(\sqrt{y}) + \sqrt{y} J_{v+1}(\sqrt{y}) J_v(\sqrt{x})}{2(x - y)}, \tag{4.16}$$

making the first term in (4.12) integrable. The simple observation in [35] states that any factorising sum, such as the two sums on the right-hand side of (4.12), can be brought to an integrable form,

$$\sum_{l=1}^L f_l(x)g_l(y) = \frac{1}{x-y} \sum_{l=1}^L (xf_l(x)g_l(y) - f_l(x)yg_l(y)). \tag{4.17}$$

Finally we remark that an alternative representation to (4.12) could be obtained, by applying the identity (4.14) first to the product containing $\hat{\theta}_k$'s, and then to the product containing $\hat{\pi}_l$'s. This leads to an alternative identity to (4.15) and different functions in (4.13) that we do not display. \square

It is well known that the Bessel kernel (4.16) is universal for various deformations of the Wishart–Laguerre ensemble, see e.g. [33] for invariant ensembles and [15] for external fields. Theorem 4.1 adds a further ensemble (1.6) to this list, namely when the finite rank perturbations are chosen such that their values vanish, $\hat{\theta}_k, \hat{\pi}_l \rightarrow 0 \forall k, l$. This leads from (4.8) to (4.10).

4.2. Meijer G-kernel with finite rank perturbations from two independent correlated matrices

The ensemble (1.5) of two independent random matrices where one has correlated entries reads

$$\mathcal{P}_2(G, X) = c_2 \exp[-\alpha \text{Tr}(GG^*) - \text{Tr}(QXX^*)].$$

The Hermitian matrix Q has positive eigenvalues $q_1, \dots, q_M > 0$, and $\alpha > 0$ is a constant (that does not depend on μ here). We will consider finite rank perturbations of $Q = q\mathbb{1}_M$ by setting

$$q_{m+1} = \dots = q_M = q > 0, \tag{4.18}$$

with m independent of N .

Putting the degeneracy (4.18) inside the kernel (3.41) for finite- N from Theorem 3.8, we obtain

$$\begin{aligned} K_N^{(2)}(x, y) &= \left(\frac{x}{y}\right)^{\frac{\kappa}{2}} 2 \oint_{\gamma_0} \frac{d\eta}{2\pi i} \oint_{\gamma_q} \frac{d\zeta}{2\pi i} \left(\frac{\zeta}{\eta}\right)^{N+\frac{\kappa}{2}} \frac{I_\kappa(2\sqrt{\eta x})K_\kappa(2\sqrt{\zeta y})}{\eta - \zeta} \left(\frac{\eta - \alpha q}{\zeta - \alpha q}\right)^{M-m} \prod_{l=1}^m \frac{\eta - \alpha q_l}{\zeta - \alpha q_l} \\ &= \left(\frac{x}{y}\right)^{\frac{\kappa}{2}} 2\alpha q \oint_{\Gamma_1} \frac{dv}{2\pi i} \oint_{\Gamma_\theta} \frac{du}{2\pi i} \frac{I_\kappa(2\sqrt{\alpha q(1-v)x})K_\kappa(2\sqrt{\alpha q(1-u)y})}{u-v} \\ &\quad \times \left(\frac{1-\frac{1}{u}}{1-\frac{1}{v}}\right)^{N+\frac{\kappa}{2}} \left(\frac{v}{u}\right)^{v-m-\frac{\kappa}{2}} \prod_{k=1}^m \frac{v - (1-\frac{q_k}{q})}{u - (1-\frac{q_k}{q})}, \end{aligned} \tag{4.19}$$

with the following substitution

$$\zeta = \alpha q(1-u) \quad \text{and} \quad \eta = \alpha q(1-v). \tag{4.20}$$

It is nonsingular due to $\alpha q > 0$, and we define for later the poles in the new integration variables

$$\theta_k = 1 - \frac{q_k}{q}, \quad k = 1, \dots, m. \tag{4.21}$$

The integration contours in the new variables obtained from Figure 2 that we take to be nested here are given in Figure 4.

The integration variables will be rescaled, and the poles take the following limiting values:

$$v = N\hat{v}, \quad u = N\hat{u}, \quad \hat{\theta}_k = \lim_{N \rightarrow \infty} \frac{1}{N}\theta_k, \quad k = 1, \dots, m, \tag{4.22}$$

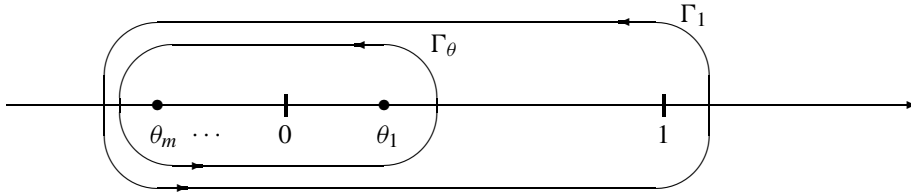


Fig. 4. The contours Γ_θ and Γ_1 resulting from the substitution (4.20), the θ_k are now centred around the origin and the origin has been mapped to unity.

mapping the identity in Figure 4 to $1/N$, and thus to the origin in the limit $N \rightarrow \infty$. This is not a problem, due to the nesting of the contours. Consequently, the $\hat{\theta}_{k=1, \dots, m}$ become non-positive in the limit $N \rightarrow \infty$. The arguments of the kernel will be rescaled as

$$x = \frac{\hat{x}}{\alpha q N}, \quad y = \frac{\hat{y}}{\alpha q N}. \quad (4.23)$$

Putting all together yields the following result for the limiting kernel at the hard edge.

Theorem 4.3. Define the finite rank perturbations (4.18) and the parameters $\hat{\theta}_k$, $k = 1, \dots, m$, as in (4.21) and (4.22). The following limit of the kernel (3.41) leads to a kernel equivalent to

$$\begin{aligned} \mathbb{K}_\Gamma^{(m)}(\hat{x}, \hat{y}) &= \lim_{N \rightarrow \infty} \frac{1}{\alpha q N} K_N^{(2)}\left(\frac{\hat{x}}{\alpha q N}, \frac{\hat{y}}{\alpha q N}\right) \left(\frac{\hat{x}}{\hat{y}}\right)^{-\kappa} \\ &= \oint_{\gamma_0} \frac{ds}{2\pi i} \int_0^\infty dt s^{-\kappa-1} t^{\kappa-1} e^{s-t} \mathbb{K}_{\text{III}}^{(n=0, m)}\left(\frac{\hat{x}}{s}, \frac{\hat{y}}{t}\right). \end{aligned} \quad (4.24)$$

The limiting kernel is a Meijer G -kernel with finite rank perturbations.

Proof. In order to take the large- N limit we use the following integral representations of the modified Bessel functions of the second kind in (4.19):

$$\begin{aligned} I_\kappa(z) &= \left(\frac{z}{2}\right)^\kappa \oint_{\gamma_0} \frac{ds}{2\pi i} s^{-\kappa-1} \exp\left[s + \frac{z^2}{4s}\right], \\ K_\kappa(z) &= K_{-\kappa}(z) = \frac{z^{-\kappa}}{2^{-\kappa+1}} \int_0^\infty dt t^{\kappa-1} \exp\left[-t - \frac{z^2}{4t}\right]. \end{aligned} \quad (4.25)$$

Here, γ_0 is a closed contour encircling the origin in counter-clockwise way. We use Fubini's Theorem to rewrite the rescaled kernel from (4.19) as

$$\begin{aligned} &\frac{1}{\alpha q N} K_N^{(2)}\left(\frac{\hat{x}}{\alpha q N}, \frac{\hat{y}}{\alpha q N}\right) \left(\frac{\hat{x}}{\hat{y}}\right)^{-\kappa} \\ &= \oint_{\gamma_0} \frac{ds}{2\pi i} \int_0^\infty dt \frac{t^{\kappa-1}}{s^{\kappa+1}} \oint_{\Gamma_1} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_\theta} \frac{d\hat{u}}{2\pi i} \frac{e^{s+\frac{1}{s}(\frac{1}{N}-\hat{v})\hat{x}-t-\frac{1}{t}(\frac{1}{N}-\hat{u})\hat{y}}}{\hat{u}-\hat{v}} \\ &\quad \times \left(\frac{\frac{1}{N}-\hat{v}}{\frac{1}{N}-\hat{u}}\right)^{\frac{\kappa}{2}} \left(\frac{1-\frac{1}{N\hat{u}}}{1-\frac{1}{N\hat{v}}}\right)^{N+\frac{\kappa}{2}} \left(\frac{\hat{v}}{\hat{u}}\right)^{\nu-m-\frac{\kappa}{2}} \prod_{k=1}^m \frac{\hat{v}-\frac{\theta_k}{N}}{\hat{u}-\frac{\theta_k}{N}}. \end{aligned} \quad (4.26)$$

The application of Lebesgue's Dominated Convergence Theorem together with the limits of each individual factor inside the integrand (such as (4.9)) leads to (4.24), after comparing with (4.8) at $n = 0$. We recall here that $\kappa = L - N$ is kept fixed in the large- N limit.

Notice that in this ensemble the domain of the parameters $\hat{\theta}_{k=1,\dots,m}$ is non-positive. Furthermore, in [35], Prop. 5.1, it was shown that the kernel (4.24) at $m = 0$ equals the Meijer G -kernel [34] for two independent matrices,

$$\mathbb{K}_I^{(m=0)}(x, y) = \mathbb{K}_{\text{Meijer}}(x, y) = \int_0^1 du G_{0,3}^{1,0} \left(\begin{matrix} - \\ 0, -\nu, -\kappa \end{matrix} \middle| uy \right) G_{0,3}^{2,0} \left(\begin{matrix} - \\ \nu, \kappa, 0 \end{matrix} \middle| ux \right). \tag{4.27}$$

Therefore, the kernel (4.24) represents a finite rank perturbation of the Meijer G -kernel. In the absence of finite rank perturbations the kernel was also shown to agree with (4.27) in [6]. \square

We note that using Fubini's Theorem and (4.25) the integrals over s and t in (4.24) can be done. Using the relations [22], Eqs. 8.406-7,

$$I_\kappa(iz) = i^\kappa J_\kappa(z) \quad \text{and} \quad K_{-\kappa}(iz) = -\frac{\pi}{2} i^{\kappa+1} H_\kappa^{(2)}(z) \tag{4.28}$$

in terms of the Bessel functions of first and third kind (also called Hankel functions), we obtain

$$\begin{aligned} \mathbb{K}_I^{(m)}(\hat{x}, \hat{y}) &= (-1)^{\kappa+1} i\pi \left(\frac{\hat{y}}{\hat{x}} \right)^{\frac{\kappa}{2}} \oint_{\Gamma_{\text{out}}} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{d\hat{u}}{2\pi i} \frac{J_\kappa(2\sqrt{\hat{v}\hat{x}}) H_\kappa^{(2)}(2\sqrt{\hat{u}\hat{y}})}{u-v} e^{-\frac{1}{\hat{u}} + \frac{1}{\hat{v}}} \left(\frac{\hat{v}}{\hat{u}} \right)^{\nu-m-\frac{\kappa}{2}} \\ &\quad \times \prod_{k=1}^m \frac{\hat{v} - \hat{\theta}_k}{\hat{u} - \hat{\theta}_k}. \end{aligned} \tag{4.29}$$

It remains to show that the kernel (4.24) is integrable.

Corollary 4.4. *The Meijer G -kernel with finite rank perturbations (4.24) is integrable and can be written as follows*

$$\mathbb{K}_I^{(m)}(x, y) = \mathbb{K}_{\text{Meijer}}(x, y)|_{\nu \rightarrow \nu-m} - \sum_{i=1}^m \tilde{\Lambda}_I^{(i)}(x) \tilde{\Xi}_I^{(i)}(y), \tag{4.30}$$

where the remaining functions read

$$\begin{aligned} \tilde{\Lambda}_I^{(i)}(x) &= \oint_{\gamma_0} \frac{d\hat{s}}{2\pi i} s^{-\kappa-1} e^s \tilde{\Lambda}_{\text{III}}^{(i)} \left(\frac{x}{s} \right) \Big|_{n=0}, \\ \tilde{\Xi}_I^{(i)}(y) &= \int_0^\infty dt t^{\kappa-1} e^{-t} \tilde{\Xi}_{\text{III}}^{(i)} \left(\frac{y}{t} \right) \Big|_{n=0}, \end{aligned} \tag{4.31}$$

together with (4.13) at $n = 0$.

Proof. Equation (4.30) together with (4.31) immediately follow from Corollary 4.2, by inserting (4.12) at $n = 0$ into (4.24). While the integrability of the Meijer G -kernel (4.27) was shown in [34], the integrability of the sum in (4.30) follows as in the proof of Corollary 4.2, using (4.17). The integrals over s and t in (4.31) can be done in analogy to (4.29) but will not be displayed here. \square

As mentioned already in the [Introduction](#) relatively little was known so far about the universality of the Meijer G -kernel. In the case when the finite rank perturbations are such that their limiting values vanish, $\hat{\theta}_k \rightarrow 0 \forall k$, this leads from (4.24) to (4.27). Thus in this particular case the ensemble of two independent random matrices where one has correlated entries (1.5) leads to the same Meijer G -kernel as the product of two independent matrices without correlations, showing its universality.

4.3. Universal limiting kernels from two correlated coupled matrices

Our most general ensemble of two coupled matrices where one has correlated entries (1.2) is given by

$$\mathcal{P}(G, X) = c \exp[-\alpha \operatorname{Tr}(GG^*) + \operatorname{Tr}(\Omega GX + X^*G^*\Omega^*) - \operatorname{Tr}(QXX^*)].$$

Here $\alpha > 0$, the matrix Ω has squared singular values $\delta_1, \dots, \delta_N$, and the Hermitian matrix Q has positive eigenvalues $q_1, \dots, q_M > 0$. In this subsection we will consider the following parameter dependent finite rank perturbations around $\Omega\Omega^* = \frac{(1-\mu)^2}{4\mu^2}\mathbb{1}_N$ and $Q = \frac{1+\mu}{2\mu}\mathbb{1}_M$, by setting

$$\delta_{n+1} = \dots = \delta_N = \frac{(1-\mu)^2}{4\mu^2}, \quad q_{m+1} = \dots = q_M = \frac{1+\mu}{2\mu}, \quad (4.32)$$

together with

$$\alpha = \frac{1+\mu}{2\mu}, \quad (4.33)$$

with n, m independent of N . The ensemble with $m = n = 0$ of two coupled random matrices was studied in [5,6]. In contrast to the previous two subsections we have the parameter $\mu = \mu(N) \in (0, 1]$ here, that can vary with N . As it was already found in [5,6], depending on the scaling of $\mu(N)$ we will obtain three different limits $N \rightarrow \infty$: (I) $\mu(N)N \rightarrow \infty$, (II) $\mu(N)N \rightarrow \tau/4$ and (III) $\mu(N)N \rightarrow 0$.

Because it was already discussed in detail in the [Introduction](#) after (1.16), what is the range that the various parameters take due to the condition (1.3), in particular after setting some of these equal as in (4.32), we only summarise the findings that we need below. Let us recall the following quantities:

$$\pi_l = \frac{(1+\mu)^2}{4\mu} \left(1 - \frac{4\mu^2\delta_l}{(1+\mu)^2} \right), \quad l = 1, \dots, n, \quad (4.34)$$

$$\theta_j = \frac{(1+\mu)^2}{4\mu} \left(1 - \frac{2\mu q_j}{1+\mu} \right), \quad j = 1, \dots, m, \quad (4.35)$$

which were shown to satisfy

$$0 < \pi_l, \quad \theta_j < \pi_l \quad \text{and} \quad \theta_j < 1, \quad \text{for } j = 1, \dots, m, l = 1, \dots, n. \quad (4.36)$$

Compared to Section 4.1, the natural scale is here $\frac{(1+\mu)^2}{4\mu^2} - \frac{(1-\mu)^2}{4\mu^2} = \frac{1}{\mu}$. We therefore make the following substitutions in the integrals in the kernel at finite- N for this ensemble (1.12), which are non-singular:

$$\zeta = \frac{(1+\mu)^2}{4\mu^2} - \frac{1}{\mu}u, \quad \eta = \frac{(1+\mu)^2}{4\mu^2} - \frac{1}{\mu}v. \quad (4.37)$$

Together with equations (4.32), (4.33), (4.34), and (4.35), the kernel (1.12) results into

$$\begin{aligned} K_N(x, y) &= \frac{2}{\mu} \oint_{\Gamma_\pi} \frac{dv}{2\pi i} \oint_{\Gamma_\theta} \frac{du}{2\pi i} \frac{I_\kappa(2\sqrt{((1+\mu)^2 - 4\mu v)\frac{x}{4\mu^2}}) K_\kappa(2\sqrt{((1+\mu)^2 - 4\mu u)\frac{y}{4\mu^2}})}{u - v} \\ &\quad \times \left(\frac{x}{y}\right)^{\frac{\kappa}{2}} \left(\frac{u - \frac{(1+\mu)^2}{4\mu}}{v - \frac{(1+\mu)^2}{4\mu}}\right)^{\frac{\kappa}{2}} \left(\frac{1 - \frac{1}{u}}{1 - \frac{1}{v}}\right)^{N-n} \left(\frac{v}{u}\right)^{v-m+n} \prod_{l=1}^n \frac{u - \pi_l}{v - \pi_l} \prod_{k=1}^m \frac{v - \theta_k}{u - \theta_k}. \end{aligned} \quad (4.38)$$

For simplicity here and from now on we will suppress the N -dependence of all parameters. Taking (4.38) as a starting point, together with the scaling

$$u = N\hat{u}, \quad v = N\hat{v} \quad (4.39)$$

and

$$\hat{\pi}_l = \lim_{N \rightarrow \infty} \frac{1}{N} \pi_l, \quad \hat{\theta}_j = \lim_{N \rightarrow \infty} \frac{1}{N} \theta_j, \quad \hat{\pi}_l - \hat{\theta}_j \geq 0, \quad \hat{\pi}_l \geq 0, \quad \forall j, l, \quad (4.40)$$

we are in the position to present the proof of Theorem 1.4.

Proof of Theorem 1.4 (I). In limit (I) leading to the Meijer G -kernel with finite rank perturbations we will let $\mu N \rightarrow \infty$, and rescale the arguments of the kernel as

$$x = \frac{\mu \hat{x}}{N}, \quad y = \frac{\mu \hat{y}}{N}. \quad (4.41)$$

From the definition (4.34) and the bounds (4.36) we have that in the limit (4.40) the following parameters vanish, $\hat{\pi}_l = 0$, for all $l = 1, \dots, n$. Furthermore, from (4.35) we have that the parameters $\hat{\theta}_k \leq 0$, for all $k = 1, \dots, m$, will become negative (or zero) in the large- N limit. The contours in the integral (4.38) are thus as in Figure 4, with all $\theta_{k=1, \dots, m}$ to the left of the origin.

Following the previous Section 4.2, we apply the integral representations of the modified Bessel functions (4.25), use Fubini's Theorem to interchange integrals and Lebesgue's Dominated Convergence Theorem to exchange the limit with the integrations. We obtain the following for the kernel in terms of rescaled variables:

$$\begin{aligned} & \frac{\mu}{N} K_N \left(\frac{\mu \hat{x}}{N}, \frac{\mu \hat{y}}{N} \right) \frac{\hat{y}^\kappa}{\hat{x}^\kappa} \\ &= \oint_{\gamma_0} \frac{ds}{2\pi i} \int_0^\infty dt \oint_{\Gamma_1} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_\theta} \frac{d\hat{u}}{2\pi i} \frac{t^{\kappa-1}}{s^{\kappa+1}} \frac{1}{\hat{u} - \hat{v}} \left(\frac{\hat{v}}{\hat{u}} \right)^{v-m+n} \prod_{k=1}^m \frac{\hat{v} - \frac{\theta_k}{N}}{\hat{u} - \frac{\theta_k}{N}} \prod_{l=1}^n \frac{\hat{u} - \frac{\pi_l}{N}}{\hat{v} - \frac{\pi_l}{N}} \\ & \times \frac{(((1+\mu)^2 - 4\mu N \hat{v}) \frac{1}{4\mu N})^{\frac{\kappa}{2}}}{(((1+\mu)^2 - 4\mu N \hat{u}) \frac{1}{4\mu N})^{\frac{\kappa}{2}}} e^{s + \frac{1}{s}((1+\mu)^2 - 4\mu N \hat{v}) \frac{\hat{x}}{4\mu N} - t - \frac{1}{t}((1+\mu)^2 - 4\mu N \hat{u}) \frac{\hat{y}}{4\mu N}} \\ & \times \left(\frac{\hat{u} - \frac{(1+\mu)^2}{4\mu N}}{\hat{v} - \frac{(1+\mu)^2}{4\mu N}} \right)^{\frac{\kappa}{2}} \left(\frac{1 - \frac{1}{\hat{u}N}}{1 - \frac{1}{\hat{v}N}} \right)^{N-n}. \end{aligned} \quad (4.42)$$

The large- N limit $\mu N \rightarrow \infty$ of the third and fourth line are easily taken, and after cancelling several factors we arrive at

$$\lim_{N \rightarrow \infty} \frac{\mu}{N} K_N \left(\frac{\mu \hat{x}}{N}, \frac{\mu \hat{y}}{N} \right) \frac{\hat{y}^\kappa}{\hat{x}^\kappa} = \oint_{\gamma_0} \frac{d\hat{s}}{2\pi i} \int_0^\infty dt s^{-\kappa-1} t^{\kappa-1} e^{s-t} \mathbb{K}_{\text{III}}^{(n=0, m)} \left(\frac{\hat{x}}{s}, \frac{\hat{y}}{t} \right), \quad (4.43)$$

the right hand side of (4.24). This is the limiting kernel as stated in the theorem. Obviously, it is true uniformly for any \hat{x}, \hat{y} in a compact subset of $(0, \infty)$. \square

The fact that this kernel is integrable has already been stated in Corollary 4.4.

Proof of Theorem 1.4 (III). In limit (III) leading to the Bessel kernel with finite rank perturbations we will let $\mu N \rightarrow 0$ and rescale the arguments of the kernel as

$$x = \frac{\hat{x}}{4N^2}, \quad y = \frac{\hat{y}}{4N^2}. \quad (4.44)$$

Following (4.34), the positivity of the parameters $\hat{\pi}_{l=1, \dots, n}$ (4.36) leads to their limiting domain to be $[0, \infty)$. The domain of the limiting parameters $\hat{\theta}_{k=1, \dots, m}$ becomes $\bigcap_{l=1}^n (-\infty, \hat{\pi}_l]$. In the scaling limit the arguments of the modified

Bessel functions in (4.38) become large,

$$2\sqrt{\frac{(1+\mu)^2 - 4\mu N\hat{v}}{(4\mu N)^2}} \frac{\hat{x}}{(4\mu N)^2} = \frac{(1+\mu)\sqrt{\hat{x}}}{2\mu N} \left(1 - \frac{4\mu N}{(1+\mu)^2} \hat{v}\right)^{\frac{1}{2}} = \frac{\sqrt{\hat{x}}}{2\mu N} (1 - 2\mu N\hat{v} + \mathcal{O}((\mu N)^2)),$$

and likewise for the other argument with \hat{u} and \hat{y} . For that reason we need to make use of asymptotic formulas of modified Bessel functions as $z \rightarrow \infty$

$$I_\kappa(z) \sim \frac{e^z}{\sqrt{2\pi z}} \quad \text{for } |\arg(z)| \leq \frac{1}{2}\pi - \beta \quad \text{and} \quad K_\kappa(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad \text{for } |\arg(z)| \leq \frac{3}{2}\pi - \beta, \quad (4.45)$$

uniformly for arbitrary $0 < \beta < \pi/2$, from which simple calculations give us

$$I_\kappa\left(2\sqrt{\frac{(1+\mu)^2 - 4\mu N\hat{v}}{(4\mu N)^2}} \frac{\hat{x}}{(4\mu N)^2}\right) \sim \frac{\sqrt{\mu N}}{\sqrt{\pi}} \hat{x}^{-\frac{1}{4}} e^{\frac{1}{2\mu N}\sqrt{\hat{x}} - \hat{v}\sqrt{\hat{x}}}, \quad \text{as } \mu N \rightarrow 0 \quad (4.46)$$

and

$$K_\kappa\left(2\sqrt{\frac{(1+\mu)^2 - 4\mu N\hat{u}}{(4\mu N)^2}} \frac{\hat{y}}{(4\mu N)^2}\right) \sim \sqrt{\mu N \pi} \hat{y}^{-\frac{1}{4}} e^{-\frac{1}{2\mu N}\sqrt{\hat{y}} + \hat{u}\sqrt{\hat{y}}}, \quad \text{as } \mu N \rightarrow 0. \quad (4.47)$$

Apart from the asymptotic result (4.9) we also need that

$$\left(\frac{4\mu N\hat{u} - (1+\mu)^2}{4\mu N\hat{v} - (1+\mu)^2}\right)^{\frac{\kappa}{2}} \sim 1 \quad \text{as } N \rightarrow \infty. \quad (4.48)$$

Finally, we use Lebesgue's Dominated Convergence Theorem to exchange the limit with the integrations in (4.38). Collecting all the above expansions results into the following limit

$$\lim_{N \rightarrow \infty} \frac{1}{4N^2} K_N\left(\frac{\hat{x}}{4N^2}, \frac{\hat{y}}{4N^2}\right) \frac{\hat{y}^{\frac{\kappa}{2}}}{\hat{x}^{\frac{\kappa}{2}}} e^{\frac{1}{2\mu N}(\sqrt{\hat{y}} - \sqrt{\hat{x}})} = \frac{1}{2} (\hat{x}\hat{y})^{-\frac{1}{4}} \mathbb{K}_{\text{III}}^{(n,m)}(\sqrt{\hat{x}}, \sqrt{\hat{y}}), \quad (4.49)$$

as stated in the theorem. \square

Corollary 4.2 implies that this kernel is integrable.

Proof of Theorem 1.4 (II). In limit (II) leading to an interpolating kernel we will let $\mu N \rightarrow \tau/4$ with fixed $\tau > 0$ of $\mathcal{O}(1)$. The scaling of the arguments of the kernel is given as in (4.44). In this limit, due to (4.36) the domain of $\hat{\pi}_{l=1, \dots, n}$ (4.40) becomes $[0, 1/\tau)$, while the limiting $\hat{\theta}_{k=1, \dots, m}$ remain in the interval $\bigcap_{l=1}^n (-\infty, \hat{\pi}_l]$. Let us write the kernel (4.38) in terms of these scaling variables:

$$\begin{aligned} & \frac{1}{4N^2} K_N\left(\frac{\hat{x}}{4N^2}, \frac{\hat{y}}{4N^2}\right) \frac{\hat{y}^{\frac{\kappa}{2}}}{\hat{x}^{\frac{\kappa}{2}}} \\ &= \frac{2}{4\mu N} \oint_{\Gamma_1} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_\theta} \frac{d\hat{u}}{2\pi i} \frac{1}{\hat{u} - \hat{v}} \left(\frac{\hat{v}}{\hat{u}}\right)^{v-m+n} \prod_{l=1}^n \frac{\hat{u} - \frac{\pi_l}{N}}{\hat{v} - \frac{\pi_l}{N}} \prod_{k=1}^m \frac{\hat{v} - \frac{\theta_k}{N}}{\hat{u} - \frac{\theta_k}{N}} \\ & \times I_\kappa\left(2\sqrt{\frac{(1+\mu)^2 - 4\mu N\hat{v}}{(4\mu N)^2}} \frac{\hat{x}}{(4\mu N)^2}\right) \left(\frac{4\mu N\hat{u} - (1+\mu)^2}{4\mu N\hat{v} - (1+\mu)^2}\right)^{\frac{\kappa}{2}} \\ & \times K_\kappa\left(2\sqrt{\frac{(1+\mu)^2 - 4\mu N\hat{u}}{(4\mu N)^2}} \frac{\hat{y}}{(4\mu N)^2}\right) \left(\frac{1 - \frac{1}{\hat{u}N}}{1 - \frac{1}{\hat{v}N}}\right)^{N-n}. \end{aligned} \quad (4.50)$$

Using once again Lebesgue's Dominated Convergence Theorem, the limit of the two last lines is easily taken, and we obtain

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{4N^2} K_N \left(\frac{\hat{x}}{4N^2}, \frac{\hat{y}}{4N^2} \right) \frac{\hat{y}^{\frac{\kappa}{2}}}{\hat{x}^{\frac{\kappa}{2}}} \\
&= \frac{2}{\tau} \oint_{\Gamma_{\text{out}}} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{d\hat{u}}{2\pi i} \frac{1}{\hat{u} - \hat{v}} \left(\frac{\hat{v}}{\hat{u}} \right)^{\nu - m + n} \prod_{l=1}^n \frac{\hat{u} - \hat{\pi}_l}{\hat{v} - \hat{\pi}_l} \prod_{k=1}^m \frac{\hat{v} - \hat{\theta}_k}{\hat{u} - \hat{\theta}_k} \\
&\quad \times I_\kappa \left(2\sqrt{\frac{\hat{x}}{\tau^2} (1 - \tau \hat{v})} \right) K_\kappa \left(2\sqrt{\frac{\hat{y}}{\tau^2} (1 - \tau \hat{u})} \right) \left(\frac{1 - \tau \hat{u}}{1 - \tau \hat{v}} \right)^{\frac{\kappa}{2}} e^{-\frac{1}{\hat{u}} + \frac{1}{\hat{v}}} \\
&= \mathbb{K}_{\text{II}}^{(n,m)}(\hat{x}, \hat{y}; \tau). \tag{4.51}
\end{aligned}$$

□

We note that the interpolating kernel can also be written as a double integral of the generalised Bessel kernel with finite rank perturbations. Using Fubini's Theorem and the integral representations (4.25) we obtain

$$\mathbb{K}_{\text{II}}^{(n,m)}(\hat{x}, \hat{y}; \tau) = \frac{1}{\tau} \left(\frac{\hat{x}}{\hat{y}} \right)^{\frac{\kappa}{2}} \oint_{\gamma_0} \frac{d\hat{s}}{2\pi i} \int_0^\infty dt s^{-\kappa-1} t^{\kappa-1} e^{s-t} e^{\frac{\hat{x}}{s\tau^2} - \frac{\hat{y}}{t\tau^2}} \mathbb{K}_{\text{III}}^{(n,m)} \left(\frac{\hat{x}}{s\tau}, \frac{\hat{y}}{t\tau} \right). \tag{4.52}$$

We use a different definition here for the interpolating kernel, compared to [35] where $m = 0$. There, the scaling (4.44) was made μ -dependent, leading to $\mathbb{K}_{\text{II}}^{(n,0)}(\hat{x}, \hat{y}; \tau) \rightarrow \frac{1}{\tau^2} \mathbb{K}_{\text{II}}^{(n,0)} \left(\frac{\hat{x}}{\tau^2}, \frac{\hat{y}}{\tau^2}; \tau \right)$. When the finite rank parameters $\hat{\theta}_k \rightarrow 0$, for all $k = 1, \dots, m$, we obtain the same kernel as in [35] (up to rescaling), which is thus universal. The question of universality under further deformations is open.

In analogy to the previous two subsections we can show that the interpolating kernel is integrable.

Corollary 4.5. *The family of limiting kernels with finite rank perturbations (4.51) is integrable and can be written in the following form:*

$$\mathbb{K}_{\text{II}}^{(n,m)}(x, y) = \mathbb{K}_{\text{II}}^{(0,0)}(x, y)|_{\nu \rightarrow \nu + n - m} - \sum_{i=1}^m \tilde{\Lambda}_{\text{II}}^{(i)}(x) \tilde{\Xi}_{\text{II}}^{(i)}(y) + \sum_{j=1}^n \Lambda_{\text{II}}^{(j)}(x) \Xi_{\text{II}}^{(j)}(y), \tag{4.53}$$

where

$$\begin{aligned}
\tilde{\Lambda}_{\text{II}}^{(i)}(x) &= \oint_{\Gamma_{\hat{\pi}}} \frac{d\hat{v}}{2\pi i} I_\kappa \left(2\sqrt{\frac{x}{\tau^2} (1 - \tau \hat{v})} \right) \exp \left[\frac{1}{\hat{v}} \right] (1 - \tau \hat{v})^{-\frac{\kappa}{2}} \hat{v}^{\nu + n - m} \prod_{k=1}^{i-1} (\hat{v} - \hat{\theta}_k), \\
\Lambda_{\text{II}}^{(j)}(x) &= \oint_{\Gamma_{\hat{\pi}}} \frac{d\hat{v}}{2\pi i} I_\kappa \left(2\sqrt{\frac{x}{\tau^2} (1 - \tau \hat{v})} \right) \exp \left[\frac{1}{\hat{v}} \right] (1 - \tau \hat{v})^{-\frac{\kappa}{2}} \hat{v}^{\nu + n - m} \frac{\prod_{k=1}^m (\hat{v} - \hat{\theta}_k)}{\prod_{l=1}^j (\hat{v} - \hat{\pi}_l)}, \\
\tilde{\Xi}_{\text{II}}^{(i)}(y) &= \frac{2}{\tau} \oint_{\Gamma_{\hat{\theta}}} \frac{d\hat{u}}{2\pi i} K_\kappa \left(2\sqrt{\frac{y}{\tau^2} (1 - \tau \hat{u})} \right) \exp \left[-\frac{1}{\hat{u}} \right] (1 - \tau \hat{u})^{\frac{\kappa}{2}} \hat{u}^{-\nu - n + m} \prod_{k=1}^i \frac{1}{\hat{u} - \hat{\theta}_k}, \\
\Xi_{\text{II}}^{(j)}(y) &= \frac{2}{\tau} \oint_{\Gamma_{\hat{\theta}}} \frac{d\hat{u}}{2\pi i} K_\kappa \left(2\sqrt{\frac{y}{\tau^2} (1 - \tau \hat{u})} \right) \exp \left[-\frac{1}{\hat{u}} \right] (1 - \tau \hat{u})^{\frac{\kappa}{2}} \hat{u}^{-\nu - n + m} \frac{\prod_{l=1}^{j-1} (\hat{u} - \hat{\pi}_l)}{\prod_{k=1}^m (\hat{u} - \hat{\theta}_k)}.
\end{aligned} \tag{4.54}$$

Here, the closed contour $\Gamma_{\hat{\theta}}$ contains the poles at $\hat{\theta}_{l=1, \dots, m}$ and the origin, encircling them counter-clockwise, and the closed contour $\Gamma_{\hat{\pi}}$ contains the poles at $\hat{\pi}_{k=1, \dots, n}$ and the origin, encircling them counter-clockwise.

Proof. The fact that the kernel without finite rank perturbations, $\mathbb{K}_{\text{II}}^{(0,0)}(x, y)$, is integrable was already shown in [35], Proposition 5.3. We thus can use the identity (4.15) to split off the remaining parts as written in (4.53) and arrive at (4.54). The integrability of these additional terms follows immediately from (4.17). □

We move to the interpolating property of the kernel as stated in Theorem 1.5.

Proof of Theorem 1.5. Part (a) In this limit $\tau \rightarrow \infty$ the τ -dependent domain $[0, 1/\tau]$ of the finite rank perturbation parameters $\hat{\pi}_{l=1,\dots,n}$ shrinks to the origin, while the $\hat{\theta}_{k=1,\dots,m}$ remain non-positive. This explains why in this limit we obtain a kernel that contains only the second set of parameters, setting all $\hat{\pi}_l = 0$ (or $n = 0$). In order to show that the limit of the kernel $\mathbb{K}_{\text{II}}^{(n,m)}(\hat{x}, \hat{y}; \tau)$ is interpolating to the Meijer G -kernel with finite rank perturbations, $\mathbb{K}_{\text{I}}^{(m)}(\hat{x}, \hat{y})$, when $\tau \rightarrow \infty$, we need again the integral representations of the modified Bessel functions (4.25), yielding

$$\begin{aligned} I_\kappa \left(2\sqrt{\frac{\hat{x}}{\tau}(1-\tau\hat{v})} \right) &= \left(\frac{\hat{x}}{\tau}(1-\tau\hat{v}) \right)^{\frac{\kappa}{2}} \oint_{\gamma_0} \frac{ds}{2\pi i} s^{-\kappa-1} \exp \left[s + \frac{1}{s} \left(\frac{\hat{x}}{\tau} - \hat{x}\hat{v} \right) \right], \\ K_{-\kappa} \left(2\sqrt{\frac{\hat{y}}{\tau}(1-\tau\hat{u})} \right) &= \frac{1}{2} \left(\frac{\hat{y}}{\tau}(1-\tau\hat{u}) \right)^{-\frac{\kappa}{2}} \int_0^\infty dt t^{\kappa-1} \exp \left[-t - \frac{1}{t} \left(\frac{\hat{y}}{\tau} - \hat{y}\hat{u} \right) \right]. \end{aligned} \quad (4.55)$$

Note that compared to (4.51) we have already used the rescaled arguments $\tau\hat{x}$ and $\tau\hat{y}$ of the kernel as in Theorem 1.5 Part (a). With the same arguments as before we exchange integrals and limits to obtain

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \tau \mathbb{K}_{\text{II}}^{(n,m)}(\tau\hat{x}, \tau\hat{y}; \tau) &\frac{\hat{y}^{\frac{\kappa}{2}}}{\hat{x}^{\frac{\kappa}{2}}} \\ &= \oint_{\gamma_0} \frac{ds}{2\pi i} \int_0^\infty dt \frac{t^{\kappa-1}}{s^{\kappa+1}} e^{s-t} \oint_{\Gamma_{\text{out}}} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{d\hat{u}}{2\pi i} \frac{\exp[-\frac{1}{s}\hat{x}\hat{v} + \frac{1}{t}\hat{y}\hat{u}]}{\hat{u} - \hat{v}} e^{-\frac{1}{\hat{u}} + \frac{1}{\hat{v}}} \left(\frac{\hat{v}}{\hat{u}} \right)^{\nu-m} \prod_{k=1}^m \frac{\hat{v} - \hat{\theta}_k}{\hat{u} - \hat{\theta}_k}, \end{aligned} \quad (4.56)$$

which is the statement to be proven.

Part (b) In order to show that the kernel $\mathbb{K}_{\text{II}}^{(n,m)}(\hat{x}, \hat{y}; \tau)$ given in (4.51) is interpolating to the Bessel kernel with finite rank perturbations, $\mathbb{K}_{\text{III}}^{(n,m)}(\hat{x}, \hat{y})$, when $\tau \rightarrow 0+$, we need to expand the Bessel functions in the integrand, due to their argument becoming large. Using (4.45) we obtain in this limit

$$\begin{aligned} I_\kappa \left(2\sqrt{\frac{\hat{x}}{\tau^2}(1-\tau\hat{v})} \right) &\sim \frac{\sqrt{\tau}}{\sqrt{4\pi\hat{x}^{\frac{1}{4}}}} e^{2\sqrt{\hat{x}}\frac{1}{\tau} - \hat{v}\sqrt{\hat{x}}} \quad \text{as } \tau \rightarrow 0+, \\ K_\kappa \left(2\sqrt{\frac{\hat{y}}{\tau^2}(1-\tau\hat{u})} \right) &\sim \sqrt{\frac{\pi}{4}} \frac{\sqrt{\tau}}{\hat{y}^{\frac{1}{4}}} e^{-2\sqrt{\hat{y}}\frac{1}{\tau} + \hat{u}\sqrt{\hat{y}}} \quad \text{as } \tau \rightarrow 0+. \end{aligned} \quad (4.57)$$

Evoking Lebesgue's Dominated Convergence Theorem and removing the divergent prefactors by mapping to an equivalent kernel we obtain

$$\begin{aligned} \lim_{\tau \rightarrow 0} \mathbb{K}_{\text{II}}^{(n,m)}(\hat{x}, \hat{y}; \tau) &e^{2(\sqrt{\hat{y}} - \sqrt{\hat{x}})\frac{1}{\tau}} \\ &= \frac{1}{2(\hat{x}\hat{y})^{\frac{1}{4}}} \oint_{\Gamma_{\text{out}}} \frac{d\hat{v}}{2\pi i} \oint_{\Gamma_{\text{in}}} \frac{d\hat{u}}{2\pi i} \frac{1}{\hat{u} - \hat{v}} \left(\frac{\hat{v}}{\hat{u}} \right)^{\nu-m+n} \prod_{l=1}^n \frac{\hat{u} - \hat{\pi}_l}{\hat{v} - \hat{\pi}_l} \prod_{k=1}^m \frac{\hat{v} - \hat{\theta}_k}{\hat{u} - \hat{\theta}_k} \exp[\hat{u}\sqrt{\hat{y}} - \hat{v}\sqrt{\hat{x}}] e^{-\frac{1}{\hat{u}} + \frac{1}{\hat{v}}}, \end{aligned} \quad (4.58)$$

which is the statement of the theorem. Note that in this limit $\tau \rightarrow 0+$ the limiting domains of the $\hat{\pi}_l$ become $[0, \infty)$, and of the $\hat{\theta}_k$ become $\bigcap_{l=1}^n (-\infty, \hat{\pi}_l]$. \square

Appendix A

In this appendix we offer alternative derivations of Proposition 2.1 and Theorem 2.2 as a check. They will be shown to follow directly from Theorem 1.1 as well. Note that our proof of Theorem 1.1 in Section 2.3 neither uses Proposition 2.1 nor Theorem 2.2, but merely utilises the same matrix parametrisations, Jacobians and group integrals as they already occur in the proofs of Proposition 2.1 and Theorem 2.2.

A.1. Alternative derivation of Proposition 2.1

As it was mentioned in the [Introduction](#), the generalised Wishart ensemble (1.6) can be obtained from our most general ensemble of correlated coupled matrices (1.2) by integrating out the random matrix G and identifying $\Sigma = -\Omega\Omega^*/\alpha$. For the ensemble (1.2) Theorem 1.1 states the joint probability density of squared singular values of the product matrix $Y = GX$ and of matrix X . Consequently, when integrating out the squared singular values y_1, \dots, y_N of Y in Theorem 1.1, we obtain the joint probability density of squared singular values x_1, \dots, x_N of matrix X distributed according to (1.6), upon identification of the eigenvalues $\sigma_i = -\delta_i/\alpha$ of Σ for $i = 1, \dots, N$. The explicit integration of (1.7) leads to the following:

$$\begin{aligned} \prod_{j=1}^N \int_0^\infty dy_j P(x_1, \dots, x_N; y_1, \dots, y_N) &= \frac{N!}{Z} \det \left[x_j^{-\kappa-1} \int_0^\infty dy y^{\frac{\kappa}{2}} e^{-\alpha y/x_j} (-1)^{\frac{\kappa}{2}} J_\kappa(2\sqrt{\alpha\sigma_l y}) \right]_{j,l=1}^N \\ &\quad \times \det \left[1, q_i, \dots, q_i^{\nu-1}, e^{-q_i x_1}, \dots, e^{-q_i x_N} \right]_{i=1}^M \\ &= \frac{N!}{Z} \alpha^{-N(1+\frac{\kappa}{2})} (-1)^{N\frac{\kappa}{2}} \prod_{i=1}^N \sigma_i^{\frac{\kappa}{2}} \det \left[e^{-x_j \sigma_l} \right]_{j,l=1}^N \\ &\quad \times \det \left[1, q_i, \dots, q_i^{\nu-1}, e^{-q_i x_1}, \dots, e^{-q_i x_N} \right]_{i=1}^M. \end{aligned} \quad (\text{A.1})$$

In the first step we replaced the δ_i by $-\sigma_i\alpha$, for all $i = 1, \dots, N$, which turns the modified into an ordinary Bessel function of the first kind, and applied the standard Andréief formula, (2.13) at $\nu = 0$. In the second step, after changing variables, we have used the following integral [22], Eq. 6.631.4

$$2 \int_0^\infty dt t^{\kappa+1} e^{-t^2} J_\kappa(2\sqrt{x_j \sigma_l t}) = (x_j \sigma_l)^{\frac{\kappa}{2}} e^{-x_j \sigma_l}, \quad (\text{A.2})$$

and taken out common factors of the determinant. A comparison of the prefactors of the two determinants in (A.1) together with (1.8) for Z yields the joint probability density (2.3) of Proposition 2.1, with the correct normalisation (2.4).

A.2. Alternative derivation of Theorem 2.2

In this subsection we rederive Theorem 2.2 by taking the limit $\delta_j \rightarrow 0$ for all $j = 1, \dots, N$ in Theorem 1.1, providing an independent derivation. For this purpose we make use of the rule of l'Hôpital which can be formulated for our purposes as

$$\lim_{\delta_1, \dots, \delta_N \rightarrow 0} \frac{\det[f(\delta_k \lambda_l)]_{k,l=1}^N}{\Delta_N(\delta_1, \dots, \delta_N)} = \prod_{n=0}^{N-1} c_n \Delta_N(\lambda_1, \dots, \lambda_N) \quad \text{for } f(x) = \sum_{n=0}^{\infty} c_n x^n. \quad (\text{A.3})$$

With the series representation for the modified Bessel function of the first kind,

$$x^{-\frac{\kappa}{2}} I_\kappa(2\sqrt{x}) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\kappa+1)\Gamma(n+1)} x^n, \quad (\text{A.4})$$

it is not difficult to see that combining the δ_k -dependent parts of (1.7) and (1.8), the rule of l'Hôpital (A.3) can be applied,

$$\begin{aligned} \lim_{\delta_1, \dots, \delta_N \rightarrow 0} \frac{\prod_{j=1}^N y_j^\kappa}{\prod_{i=1}^M \prod_{j=1}^N (\alpha q_i - \delta_j)^{-1}} \frac{\det[(\delta_l y_j)^{-\frac{\kappa}{2}} I_\kappa(2\sqrt{\delta_l y_j})]_{j,l=1}^N}{\Delta_N(\delta_1, \dots, \delta_N)} \\ = \frac{\prod_{j=1}^N y_j^\kappa}{\alpha^{-NM} \prod_{i=1}^M q_i^{-N}} \prod_{l=0}^{N-1} \frac{1}{\Gamma(l+1)\Gamma(\kappa+l+1)} \Delta_N(y_1, \dots, y_N), \end{aligned} \quad (\text{A.5})$$

yielding the joint probability density function of the squared singular values of product of two independent correlated matrices stated in (2.18).

Appendix B: Alternative derivation of Proposition 3.1

In this appendix we give yet another derivation of Proposition 3.1, following the idea of Tracy and Widom [41] that is independent of the map to a standard biorthogonal ensemble. We define for (3.1)

$$Z[f] = \int_0^\infty \cdots \int_0^\infty dy_1 \cdots dy_N \prod_{j=1}^N (1 + f(y_j)) \times \det[\psi_i(y_j)]_{i,j=1}^N \det[1, q_i, \dots, q_i^{v-1}, \varphi_i(y_1), \dots, \varphi_i(y_N)]_{i=1}^M, \quad (\text{B.1})$$

where f is a test function. It follows from a generalisation of Andréief's integration formula derived in [28] that $Z[f]$ can be written as a determinant,

$$Z[f] = N! \det[A + B^f], \quad (\text{B.2})$$

where the matrix A is defined by equation (3.2) and the matrix B^f is given by

$$B^f = \begin{pmatrix} 0 & \cdots & 0 & I_{1,1}^f & \cdots & I_{1,N}^f \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & I_{M,1}^f & \cdots & I_{M,N}^f \end{pmatrix}, \quad I_{i,j}^f = \int_0^\infty dt \varphi_i(t) \psi_j(t) f(t). \quad (\text{B.3})$$

The quotient of $Z[f]$ and $Z[0]$ can thus be expressed through the inverse matrix $C = A^{-1}$ as

$$\frac{Z[f]}{Z[0]} = \det \left[\delta_{i,j} + \sum_{k=1}^M C_{i,k} B_{k,j}^f \right]_{i,j=1}^M. \quad (\text{B.4})$$

Note that the matrix B^f is zero in the first columns. By indicating these zero entries we extend the definition of ψ_j to

$$\Psi(j, t) = \begin{cases} 0 & \text{for } 1 \leq j \leq M - N, \\ \psi_{j-M+N}(t) & \text{for } M - N + 1 \leq j \leq M, \end{cases} \quad (\text{B.5})$$

which yields immediately an integral expression for all entries of B^f as

$$B_{i,j}^f = \int_0^\infty dt \varphi_i(t) \Psi(j, t) f(t). \quad (\text{B.6})$$

The above definition of B^f allows to express the matrix multiplication of C and B as

$$(CB^f)_{i,j} = \int_0^\infty dt \Phi(i, t) \Psi(j, t) \quad \text{with } \Phi(i, t) = \sum_{k=1}^M C_{i,k} \varphi_k(t) f(t). \quad (\text{B.7})$$

We now make use of the notion of Φ and Ψ to rewrite the quotient of $Z[f]$ and $Z[0]$ as

$$\frac{Z[f]}{Z[0]} = \det[\mathbb{1}_M + \Phi\Psi] \quad \text{where } (\Phi\Psi)_{i,j} = \int_0^\infty \Phi(i, t) \Psi(j, t) dt, \quad 1 \leq i, j \leq M. \quad (\text{B.8})$$

Now use the fact that for arbitrary Hilbert–Schmidt operators \mathcal{A} and \mathcal{B} the following general relation holds true: $\det[\mathbb{1} + \mathcal{A}\mathcal{B}] = \det[\mathbb{1} + \mathcal{B}\mathcal{A}]$. This gives

$$\frac{Z[f]}{Z[0]} = \det[\mathbb{1}_M + \Psi\Phi], \quad (\text{B.9})$$

where $\Psi\Phi$ is an operator on $L^2(0, \infty)$ with the kernel

$$K_N(x, y)f(y) = \sum_{i=1}^M \Psi(i, x)\Phi(i, y), \quad (\text{B.10})$$

proving our Proposition 3.1.

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