

# Mixing and decorrelation in infinite measure: The case of the periodic Sinai billiard

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**Abstract.** We investigate the question of the rate of mixing for observables of a  $\mathbb{Z}^d$ -extension of a probability preserving dynamical system with good spectral properties. We state general mixing results, including expansions of every order. The main motivation of this article is the study of mixing rates for smooth observables of the  $\mathbb{Z}^2$ -periodic Sinai billiard, with different kinds of results depending on whether the horizon is finite or infinite. We establish a first order mixing result when the horizon is infinite. In the finite horizon case, we establish an asymptotic expansion of every order, enabling the study of the mixing rate even for observables with null integrals. This result is related to an Edgeworth expansion in the local limit theorem.

**Résumé.** Cet article est une contribution à l'étude du mélange d'observables de systèmes dynamiques préservant une mesure infinie. Nous étudions le cas de  $\mathbb{Z}^d$ -extensions de systèmes dynamiques probabilisés ayant de bonnes propriétés spectrales. Nous établissons des résultats généraux et les illustrons par plusieurs exemples. Notre motivation principale est l'étude de la vitesse de mélange pour des observables régulières du billard de Sinai  $\mathbb{Z}^2$ -périodique, pour lequel nous obtenons des résultats de types différents selon que l'horizon soit fini ou infini. Nous établissons un résultat de mélange du premier ordre lorsque l'horizon est infini. Dans le cas où l'horizon est fini, nous établissons un développement asymptotique de tout ordre, permettant l'étude de la vitesse de mélange pour des observables d'intégrale nulle. Ce dernier résultat est relié à un développement de Edgeworth dans le théorème limite local.

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## Introduction

Let  $(M, \nu, T)$  be a dynamical system, that is a measure space  $(M, \nu)$  endowed with a measurable transformation  $T : M \rightarrow M$  which preserves the measure  $\nu$ . The mixing properties deal with the asymptotic behaviour, as  $n$  goes to infinity, of integrals of the following form

$$C_n(f, g) := \int_M f \cdot g \circ T^n \, d\nu,$$

for suitable observables  $f, g : M \rightarrow \mathbb{C}$ .

Mixing properties of probability preserving dynamical systems have been studied by many authors. It is a way to measure how chaotic the dynamical system is. A probability preserving dynamical system is said to be mixing if  $C_n(f, g)$  converges to  $\int_M f \, d\nu \int_M g \, d\nu$  for every pair of square integrable observables  $f, g$ . When a probability preserving system is mixing, a natural question is to study the decorrelation rate, i.e. the rate at which  $C_n(f, g)$  converges

to zero when  $f$  or  $g$  have null expectation. This crucial question is often a first step before proving probabilistic limit theorems (such as central limit theorem and its variants). The study of this question has a long history. Such decays of covariance have been studied for wide classes of smooth observables  $f, g$  and for many probability preserving dynamical systems. In the case of the Sinai billiard, such results and further properties have been established in [5–10, 33–35,37].

We are interested here in the study of mixing properties when the invariant measure  $\nu$  is  $\sigma$ -finite. In this context, as noticed in [20], there is no satisfactory notion of mixing. Nevertheless the question of the rate of mixing for smooth observables is natural. A first step in this direction is to establish results of the following form:

$$\lim_{n \rightarrow +\infty} \alpha_n C_n(f, g) = \int_M f d\nu \int_M g d\nu. \tag{1}$$

Such results have been proved in [4,16,21,22,36] for a wide class of models and for smooth functions  $f, g$ , using induction on a finite measure subset of  $M$ . We emphasize on the fact that these works, done in other contexts, provided just an estimate of the form (1), with possibly an expansion of  $1/\alpha_n$  and an estimate of the rate of convergence, but not an expansion of the form of the one we obtain here in our particular context.

An alternative approach, specific to the case of  $\mathbb{Z}^d$ -extensions of probability preserving dynamical system, has been pointed out in [28]. The idea therein is that, in this particular context, (1) is related to an accurate local limit theorem. In the particular case of the  $\mathbb{Z}^2$ -periodic Sinai billiard with finite horizon, it has been proved in [28] that

$$C_n(f, g) = \frac{c_0}{n} \int_M f d\nu \int_M g d\nu + o(n^{-1}),$$

for some explicit constant  $c_0$ , for some dynamically Lipschitz functions, including functions with full support in  $M$ .

This paper is motivated by the question of high order expansions of mixing and by the study of the mixing rate for observables with null integrals. This last question can be seen as decorrelation rate in infinite measure. Let us mention the fact that it has been proved in [30], for the billiard in finite horizon, that sums  $\sum_{k \in \mathbb{Z}} \int_M f \cdot f \circ T^k d\nu$  are well defined for some observables  $f$  with null expectation. In the present paper, we use the approach of [28] to establish, in the context of the  $\mathbb{Z}^2$ -periodic Sinai billiard with finite horizon, a high order mixing result of the following form:

$$C_n(f, g) = \sum_{m=0}^{K-1} \frac{c_m(f, g)}{n^{1+m}} + o(n^{-K}). \tag{2}$$

Our estimates are linked with Edgeworth expansions in the local limit theorem (see for example [3,14,15] for such results in a probabilistic context and also [11] for such a result in a dynamical context).

The classical probabilistic context of  $\mathbb{Z}^d$ -random walks, that is of partial sums  $S'_n := \sum_{k=1}^n X_k$  of a sequence  $(X_n)_{n \geq 1}$  of independent identically distributed (iid for short)  $\mathbb{Z}^d$ -valued random variables corresponds, in terms of dynamical systems, to a  $\mathbb{Z}^d$ -extension of the full shift, by considering  $M = (\mathbb{Z}^d)^{\mathbb{N}^*} \times \mathbb{Z}^d$ ,  $T((x_n)_n, \ell) = ((x_{n+1})_n, \ell + x_1)$  and  $\nu = (\mathbb{P}_{X_1})^{\otimes \mathbb{N}^*} \otimes m_d$  with  $\mathbb{P}_{X_1}$  the distribution of  $X_1$  and  $m_d$  the counting measure on  $\mathbb{Z}^d$ . In this context, if  $f(x, \ell) = F(\ell)$ , then the quantity  $C_n(f, g)$  for  $n \geq 1$  can be rewritten

$$\sum_{\ell \in \mathbb{Z}^d} \mathbb{E}[F(\ell)g((X_{n+k})_k, \ell + S'_n)] = \sum_{\ell, \ell' \in \mathbb{Z}^d} F(\ell)G(\ell')\mathbb{P}(S'_n = \ell' - \ell), \quad \text{with } G(\ell) := \mathbb{E}[g((X_k)_k, \ell)] \tag{3}$$

and thus its expansion is directly related to expansions of  $\mathbb{P}(S'_n = \ell)$ . As for the Green-Kubo formula, the expressions we obtain for the coefficients  $c_m(f, g)$  for  $m \geq 1$  in the general case (general dynamical systems, or even in the iid case when  $f(x, \ell)$  depends on an infinite number of coordinates of  $x$ ) are complicated by the presence of sums of covariances that vanish in (3) (see namely Section 3.2 for the case of  $\mathbb{Z}^d$ -extensions of subshifts of finite type).

An important fact, that motivates our study, is that estimate (2) enables the study of the rate of convergence of  $n C_n(f, g)$  to  $c_0 \int_M f d\nu \int_M g d\nu$  and, most importantly, it enables the study of the rate of decay of  $C_n(f, g)$  for functions  $f$  or  $g$  with integral 0. In general, if  $f$  or  $g$  have zero integral we have

$$C_n(f, g) \sim \frac{c_1(f, g)}{n^2},$$

but it may happen that

$$C_n(f, g) \sim \frac{c_2(f, g)}{n^3},$$

and even that  $C_n(f, g) = o(n^{-3})$ . For example, (2) gives immediately that, if  $\int_M f \, dv \int_M g \, dv \neq 0$ , then

$$\begin{aligned} C_n(f - f \circ T, g) &= C_n(f, g) - C_{n-1}(f, g) \\ &\sim -c_0 \frac{\int_M f \, dv \cdot \int_M g \, dv}{n^2} = \frac{c_1(f - f \circ T, g)}{n^2} \end{aligned} \tag{4}$$

and

$$\begin{aligned} C_n(2f - f \circ T - f \circ T^{-1}, g) &= C_n(f - f \circ T, g - g \circ T) \\ &= 2C_n(f, g) - C_{n-1}(f, g) - C_{n+1}(f, g) \\ &\sim -\frac{2c_0}{n^3} \int_M f \, dv \int_M g \, dv = \frac{c_2(f - f \circ T, g - g \circ T)}{n^3}. \end{aligned}$$

General formulas for the coefficients will be given in Theorems 3.2, 3.7. In particular  $c_1(f, g)$  and  $c_2(f, g)$  will be made precise in Theorem 1.3 and Remark 4.5 (see also Propositions A.3 and A.4).

We point out the fact that the method we use is rather general in the context of  $\mathbb{Z}^d$ -extensions over dynamical systems with good spectral properties, and that, to our knowledge, these are the first results of this kind for dynamical systems preserving an infinite measure.

We establish moreover an estimate of the following form for smooth observables of the  $\mathbb{Z}^2$ -periodic Sinai billiard with infinite horizon:

$$C_n(f, g) = \frac{c_0}{n \log n} \int_M f \, dv \int_M g \, dv + o((n \log n)^{-1}).$$

The paper is organized as follows. In Section 1, we present the model of the  $\mathbb{Z}^2$ -periodic Sinai billiard and we state our main results for this model (finite/infinite horizon). In Sections 2 and 3, we state our general mixing results for  $\mathbb{Z}^d$ -extensions of probability preserving dynamical systems for which the Nagaev–Guivarc’h perturbation method can be implemented, with applications to  $\mathbb{Z}^d$ -extensions of Gibbs–Markov maps and of two-sided subshifts of finite type. Whereas, in Section 2, we state a first order result, we establish, in Section 3, higher order expansions, which are particularly useful in particular when at least one of the observables has null integral. In Section 4, we prove our results for the Sinai billiard with finite or with infinite horizon. We complete our study with a computation of the first coefficients in the case of the billiard with finite horizon (see also Appendix A).

### 1. Main results for $\mathbb{Z}^2$ -periodic Sinai billiards

Let us introduce the  $\mathbb{Z}^2$ -periodic Sinai billiard  $(M, \nu, T)$ .

Billiards systems model the behaviour of a point particle moving at unit speed in a domain  $Q$  and bouncing off  $\partial Q$  with respect to the Descartes reflection law (incident angle = reflected angle). We assume here that  $Q := \mathbb{R}^2 \setminus \bigcup_{\ell \in \mathbb{Z}^2} \bigcup_{i=1}^I (O_i + \ell)$ , with  $I \geq 1$  and where  $O_1, \dots, O_I$  are convex bounded open sets (the boundaries of which are  $C^3$ -smooth and have non null curvature). We assume that the closures of the obstacles  $O_i + \ell$  are pairwise disjoint. The billiard is said to have *finite horizon* if every line in  $\mathbb{R}^2$  meets  $\partial Q$ . Otherwise it is said to have *infinite horizon*.

We consider the dynamical system  $(M, \nu, T)$  corresponding to the dynamics at reflection times which is defined as follows. Let  $M$  be the set of reflected vectors off  $\partial Q$ , i.e.

$$M := \{(q, \vec{v}) \in \partial Q \times S^1 : \langle \vec{n}(q), \vec{v} \rangle \geq 0\},$$

where  $\vec{n}(q)$  stands for the unit normal vector to  $\partial Q$  at  $q$  directed inward  $Q$ . We decompose this set into  $M := \bigcup_{\ell \in \mathbb{Z}^2} \mathcal{C}_\ell$ , with

$$\mathcal{C}_\ell := \left\{ (q, \vec{v}) \in M : q \in \bigcup_{i=1}^I (\partial O_i + \ell) \right\}.$$

The set  $\mathcal{C}_\ell$  is called the  $\ell$ -cell. We define  $T : M \rightarrow M$  as the transformation mapping a reflected vector at a reflection time to the reflected vector at the next reflection time. We consider the measure  $\nu$  absolutely continuous with respect to the Lebesgue measure on  $M$ , with density proportional to  $(q, \vec{v}) \mapsto \langle \vec{n}(q), \vec{v} \rangle$  and such that  $\nu(\mathcal{C}_0) = 1$ . We recall that  $\nu$  is  $T$ -invariant.

Because of the  $\mathbb{Z}^2$ -periodicity of the model, there exists a transformation  $\bar{T} : \mathcal{C}_0 \rightarrow \mathcal{C}_0$  and a function  $\kappa : \mathcal{C}_0 \rightarrow \mathbb{Z}^2$  such that

$$\forall ((q, \vec{v}), \ell) \in \mathcal{C}_0 \times \mathbb{Z}^2, \quad T(q + \ell, \vec{v}) = (q' + \ell + \kappa(q, \vec{v}), \vec{v}'), \quad \text{if } \bar{T}(q, \vec{v}) = (q', \vec{v}'). \tag{5}$$

This allows us to define a probability preserving dynamical  $(\bar{M}, \bar{\nu}, \bar{T})$  (the Sinai billiard) by setting  $\bar{M} := \mathcal{C}_0$  and  $\bar{\nu} = \nu|_{\mathcal{C}_0}$ . Note that (5) means that  $(M, \nu, T)$  can be represented by the  $\mathbb{Z}^2$ -extension of  $(\bar{M}, \bar{\nu}, \bar{T})$  by  $\kappa$ . In particular, iterating (5) leads to

$$\forall n \in \mathbb{N}, \forall ((q, \vec{v}), \ell) \in \mathcal{C}_0 \times \mathbb{Z}^2, \quad T^n(q + \ell, \vec{v}) = (q'_n + \ell + S_n(q, \vec{v}), \vec{v}'_n), \tag{6}$$

if  $\bar{T}^n(q, \vec{v}) = (q'_n, \vec{v}'_n)$  and with the notation

$$S_n := \sum_{k=0}^{n-1} \kappa \circ \bar{T}^k.$$

The set of tangent vectors  $\mathcal{S}_0$  given by

$$\mathcal{S}_0 := \{(q, \vec{v}) \in M : \langle \vec{v}, \vec{n}(q) \rangle = 0\}$$

plays a special role in the study of  $T$ . Note that  $T$  defines a  $C^1$ -diffeomorphism from  $M \setminus (\mathcal{S}_0 \cup T^{-1}(\mathcal{S}_0))$  to  $M \setminus (\mathcal{S}_0 \cup T(\mathcal{S}_0))$ . Statistical properties of  $(\bar{M}, \bar{\nu}, \bar{T})$  have been studied by many authors since the seminal article [33] by Sinai.

In the finite horizon case, limit theorems have been established in [6,8,10,37], including the convergence in distribution of  $(S_n/\sqrt{n})_n$  to a centered gaussian random variable  $B$  with nondegenerate variance matrix  $\Sigma^2$  given by:

$$\Sigma^2 := \sum_{k \in \mathbb{Z}} \mathbb{E}_{\bar{\nu}} [\kappa \otimes \kappa \circ \bar{T}^k], \tag{7}$$

where we used the notation  $X \otimes Y$  for the matrix  $((x_i y_j + x_j y_i)/2)_{i,j}$ , for  $X = (x_i)_i, Y = (y_j)_j \in \mathbb{C}^2$ . Moreover a local limit theorem for  $S_n$  has been established in [34] and some of its refinements have been stated and used in [13,26,27,29] for various purposes. Recurrence and ergodicity of this model follow from [12,25,31,32,34].

Concerning the infinite horizon case, to simplify the exposure of the results, we restrict ourself to the case where the horizon is infinite in two directions (i.e. there exist two non parallel lines in  $\mathbb{R}^2$  meeting no obstacle). In the infinite horizon case, results of exponential decay of correlation have been proved in [9,10]. A nonstandard central limit theorem (with normalization  $\sqrt{n \log n}$ ) and a corresponding local limit theorem have been established in [35], ensuring recurrence and ergodicity of the infinite measure system  $(M, \nu, T)$ . This result states in particular that  $(S_n/\sqrt{n \log n})_n$  converges in distribution to a centered Gaussian distribution with variance  $\Sigma_\infty^2$  given by

$$\Sigma_\infty^2 := \sum_{x \in \mathcal{S}_0 \cap \bar{M} | \bar{T}x=x} \frac{d_x^2}{2|\kappa(x)| \sum_{i=1}^I |\partial O_i|} (\kappa(x))^{\otimes 2}, \tag{8}$$

where  $d_x$  is the width of the corridor corresponding to  $x$  (where  $x = (q, \vec{v}) \in \bar{M} \cap \mathcal{S}_0$  is such that the line  $q + \vec{v}$  is contained in  $M$ ), with the notation  $(\kappa(x))^{\otimes 2} := (\kappa_i(x)\kappa_j(x))_{i,j=1,2}$ .

Our main results provide mixing estimates for dynamically Lipschitz observables. Let us introduce this class of observables. Let  $\xi \in (0, 1)$ . We consider the metric  $d_\xi$  on  $M$  given by

$$\forall x, y \in M, \quad d_\xi(x, y) := \xi^{s(x,y)},$$

where  $s$  is the separation time defined as follows:  $s(x, y)$  is the maximum of the integers  $k \geq 0$  such that  $x$  and  $y$  lie in the same connected component of  $M \setminus \bigcup_{j=-k}^k T^{-j}\mathcal{S}_0$ . For every  $f : M \rightarrow \mathbb{C}$ , we write  $L_\xi(f)$  for the Lipschitz constant with respect to  $d_\xi$ :

$$L_\xi(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_\xi(x, y)}.$$

We then set

$$\|f\|(\xi) := \|f\|_\infty + L_\xi(f).$$

Before stating our main result, let us introduce some additional notations.

We will work with symmetric multilinear forms. For any  $A = (A_{i_1, \dots, i_m})_{(i_1, \dots, i_m) \in \{1, \dots, d\}^m}$  and  $B = (B_{i_1, \dots, i_k})_{(i_1, \dots, i_k) \in \{1, \dots, d\}^k}$  with complex entries ( $A$  and  $B$  are identified respectively with a  $m$ -multilinear form on  $\mathbb{C}^d$  and with a  $k$ -multilinear form on  $\mathbb{C}^d$ ), we define  $A \otimes B$  as the element  $C$  of  $\mathbb{C}^{\{1, \dots, d\}^{m+k}}$  (identified with a symmetric  $(m+k)$ -multilinear form on  $\mathbb{C}^d$ ) such that

$$\forall i_1, \dots, i_{m+k} \in \{1, \dots, d\}, \quad C_{i_1, \dots, i_{m+k}} = \frac{1}{(m+k)!} \sum_{\mathfrak{s} \in \mathfrak{S}_{m+k}} A_{i_{\mathfrak{s}(1)}, \dots, i_{\mathfrak{s}(m)}} B_{i_{\mathfrak{s}(m+1)}, \dots, i_{\mathfrak{s}(m+k)}},$$

with  $\mathfrak{S}_{m+k}$  the set of permutations of  $\{1, \dots, m+k\}$ . Note that  $\otimes$  is associative and commutative. For any  $A = (A_{i_1, \dots, i_m})_{(i_1, \dots, i_m) \in \{1, \dots, d\}^m}$  and  $B = (B_{i_1, \dots, i_k})_{(i_1, \dots, i_k) \in \{1, \dots, d\}^k}$  symmetric with complex entries with  $k \leq m$ , we define  $A * B$  as the element  $C$  of  $\mathbb{C}^{\{1, \dots, d\}^{m-k}}$  (identified with a  $(m-k)$ -multilinear form on  $\mathbb{C}^d$ ) such that

$$\forall i_1, \dots, i_{m-k} \in \{1, \dots, d\}, \quad C_{i_1, \dots, i_{m-k}} = \sum_{i_{m-k+1}, \dots, i_m \in \{1, \dots, d\}} A_{i_1, \dots, i_m} B_{i_{m-k+1}, \dots, i_m}.$$

Note that when  $k = m = 1$ ,  $A * B$  is simply the scalar product  $A.B$ . We identify naturally vectors in  $\mathbb{C}^d$  with 1-linear functions and symmetric matrices with symmetric bilinear functions. For any  $C^m$ -smooth function  $F : \mathbb{C}^d \rightarrow \mathbb{C}$ , we write  $F^{(m)}$  for its  $m$ th differential, which is identified with a  $m$ -linear function on  $\mathbb{C}^d$ . We write  $A^{\otimes k}$  for the product  $A \otimes \dots \otimes A$ . Observe that, with these notations, Taylor expansions of  $F$  at 0 are simply written

$$\sum_{k=0}^m \frac{1}{k!} F^{(k)}(0) * x^{\otimes k}.$$

It is also worth noting that  $A * (B \otimes C) = (A * B) * C$ , for every  $A, B, C$  corresponding to symmetric multilinear forms with respective ranks  $m, k, \ell$  with  $m \geq k + \ell$ .

We extend the definition of  $\kappa$  to  $M$  by setting  $\kappa((q + \ell, \vec{v})) = \kappa(q, \vec{v})$  for every  $(q, \vec{v}) \in \bar{M}$  and every  $\ell \in \mathbb{Z}^2$ . For every  $k \in \mathbb{Z}$  and every  $x \in M$ , we write  $\mathcal{I}_k(x)$  for the label in  $\mathbb{Z}^2$  of the cell containing  $T^k x$ , i.e.  $\mathcal{I}_k$  is the label of the cell in which the particle is at the  $k$ th reflection time. It is worth noting that, for  $n \geq 0$ , we have  $\mathcal{I}_n - \mathcal{I}_0 = \sum_{k=0}^{n-1} \kappa \circ T^k$  and  $\mathcal{I}_{-n} - \mathcal{I}_0 = -\sum_{k=-n}^{-1} \kappa \circ T^k$ .

Now let us state our main results, the proofs of which are postponed to Section 4.3. We start by stating our result in the infinite horizon case, and then we will present sharper results in the finite horizon case.

1.1.  $\mathbb{Z}^2$ -periodic Sinai billiard with infinite horizon

**Theorem 1.1.** *Let  $(M, \nu, T)$  be the  $\mathbb{Z}^2$ -periodic Sinai billiard with infinite horizon. Suppose that the set of corridor free flights  $\{\kappa(x), x \in \mathcal{S}_0, \exists n \neq 0, \bar{T}^n x = x\}$  spans  $\mathbb{R}^2$ . Let  $f, g : M \rightarrow \mathbb{C}$  be two dynamically Lipschitz continuous functions (with respect to  $d_\xi$ ) such that*

$$\sum_{\ell \in \mathbb{Z}^2} (\|f \mathbf{1}_{C_\ell}\|_\infty + \|g \mathbf{1}_{C_\ell}\|_\infty) < \infty. \tag{9}$$

Then

$$\int_M f \cdot g \circ T^n \, d\nu = \frac{1}{2\pi \sqrt{\det \Sigma_\infty^2} n \log n} \left( \int_M f \, d\nu \int_M g \, d\nu + o(1) \right).$$

1.2.  $\mathbb{Z}^2$ -periodic Sinai billiard with finite horizon

We consider now the finite horizon case. We recall that a first order expansion has already been stated in [28]. We state here a result providing an expansion of every order for the mixing (see Proposition 4.4 and Theorem 3.7 for more details).

**Theorem 1.2.** *Let  $K$  be a positive integer. Let  $f, g : M \rightarrow \mathbb{C}$  be two dynamically Lipschitz continuous observables such that*

$$\sum_{\ell \in \mathbb{Z}^2} |\ell|^{2K-2} (\|f \mathbf{1}_{C_\ell}\|_{(\xi)} + \|g \mathbf{1}_{C_\ell}\|_{(\xi)}) < \infty, \tag{10}$$

then there exist  $c_0(f, g), \dots, c_{K-1}(f, g)$  such that

$$\int_M f \cdot g \circ T^n \, d\nu = \sum_{m=0}^{K-1} \frac{c_m(f, g)}{n^{1+m}} + o(n^{-K}).$$

Observe that Assumption (10) is a reinforcement of (9) (which is also the assumption appearing in [28]): first there is an additional multiplicative factor  $|\ell|^{2K-2}$  linked to the order of the expansion we obtain, second the Lipschitz constant appears in (10) (we use this fact to ensure existence and summability of some terms, called  $A_m$ , appearing in the  $c_m$ 's). We specify in the following theorem the expansion of order 2.

**Theorem 1.3.** *Let  $f, g : M \rightarrow \mathbb{R}$  be two bounded observables such that*

$$\sum_{\ell \in \mathbb{Z}^2} |\ell|^2 (\|f \mathbf{1}_{C_\ell}\|_{(\xi)} + \|g \mathbf{1}_{C_\ell}\|_{(\xi)}) < \infty.$$

Then

$$\begin{aligned} \int_M f \cdot g \circ T^n \, d\nu &= \frac{1}{2\pi \sqrt{\det \Sigma^2}} \left\{ \frac{1}{n} \int_M f \, d\nu \int_M g \, d\nu + \frac{1}{2n^2} \Sigma^{-2} * \tilde{\mathfrak{A}}_2(f, g) \right. \\ &\quad \left. + \frac{3}{4!n^2} \int_M f \, d\nu \int_M g \, d\nu (\Sigma^{-2})^{\otimes 2} * \Lambda_4 \right\} + o(n^{-2}), \end{aligned} \tag{11}$$

with  $\Sigma^{-2} = (\Sigma^2)^{-1}$  and

$$\tilde{\mathfrak{A}}_2(f, g) := - \int_M f \, d\nu \mathfrak{B}_2^-(g) - \int_M g \, d\nu \mathfrak{B}_2^+(f) - \int_M f \, d\nu \int_M g \, d\nu \mathfrak{B}_0 + 2\mathfrak{B}_1^+(f) \otimes \mathfrak{B}_1^-(g),$$

$$\begin{aligned}\mathfrak{B}_2^+(f) &:= \lim_{m \rightarrow +\infty} \int_M f \cdot (\mathcal{I}_m^{\otimes 2} - m \Sigma^2) d\nu, \\ \mathfrak{B}_2^-(g) &:= \lim_{m \rightarrow -\infty} \int_M g \cdot (\mathcal{I}_m^{\otimes 2} - |m| \Sigma^2) d\nu, \\ \mathfrak{B}_1^+(f) &:= \lim_{m \rightarrow +\infty} \int_M f \cdot \mathcal{I}_m d\nu, & \mathfrak{B}_1^-(g) &:= \lim_{m \rightarrow -\infty} \int_M g \cdot \mathcal{I}_m d\nu, \\ \mathfrak{B}_0 &= \lim_{m \rightarrow +\infty} (m \Sigma^2 - \mathbb{E}_{\bar{\nu}}[\mathcal{S}_m^{\otimes 2}]), & \Lambda_4 &:= \lambda_0^{(4)} - 3(\Sigma^2)^{\otimes 2},\end{aligned}$$

with  $\lambda_0^{(4)}$  given in Proposition A.4.

Observe that we recover (4) since  $\Sigma^2 * \Sigma^{-2} = 2$  (since  $\Sigma^2$  is symmetric),  $\mathfrak{B}_1^+(f - f \circ T) = 0$  and  $\mathfrak{B}_2^+(f - f \circ T) = \Sigma^2 \int_M f d\nu$ . Indeed, on the one hand, by  $T$ -invariance of  $\nu$ ,

$$\begin{aligned}\mathfrak{B}_1^+(f - f \circ T) &= \lim_{m \rightarrow +\infty} \int_M (f - f \circ T) \cdot \mathcal{I}_m d\nu = \lim_{m \rightarrow +\infty} \int_M f \cdot (\mathcal{I}_m - \mathcal{I}_{m-1}) d\nu \\ &= \lim_{m \rightarrow +\infty} \int_M f \cdot \kappa \circ T^m d\nu = \lim_{m \rightarrow +\infty} \int_{\bar{M}} F \cdot \kappa \circ \bar{T}^m d\bar{\nu} \\ &= \int_{\bar{M}} F d\bar{\nu} \cdot \int_{\bar{M}} \kappa d\bar{\nu} = 0,\end{aligned}$$

with  $F(q, \bar{\nu}) := \sum_{\ell \in \mathbb{Z}^2} f(q + \ell, \bar{\nu})$  and where we used exponential mixing (recalled in Proposition A.2). On the other hand

$$\begin{aligned}\mathfrak{B}_2^+(f - f \circ T) &= \lim_{m \rightarrow +\infty} \int_M f \cdot (\mathcal{I}_m^{\otimes 2} - \mathcal{I}_{m-1}^{\otimes 2}) \\ &= \lim_{m \rightarrow +\infty} \int_M f \cdot \left( \kappa^{\otimes 2} \circ T^{m-1} + 2 \sum_{k=0}^{m-2} (\kappa \circ T^k) \otimes (\kappa \circ T^{m-1-k}) \right) d\nu \\ &= \lim_{m \rightarrow +\infty} \int_{\bar{M}} F \cdot \left( \kappa^{\otimes 2} \circ \bar{T}^{m-1} + 2 \sum_{k=0}^{m-2} (\kappa \circ \bar{T}^k) \otimes (\kappa \circ \bar{T}^{m-1-k}) \right) d\bar{\nu} \\ &= \lim_{m \rightarrow +\infty} \int_M f d\nu \mathbb{E}_{\bar{\nu}} \left[ \kappa^{\otimes 2} + 2 \sum_{k=1}^{m-1} \kappa \otimes \kappa \circ \bar{T}^k \right] = \Sigma^2 \int_M f d\nu.\end{aligned}\tag{12}$$

Indeed, using again exponential mixing (Proposition A.2), for  $0 \leq k \leq m-1$  and the remark thereafter,

$$\int_{\bar{M}} \left( F - \int_{\bar{M}} F d\bar{\nu} \right) \cdot \kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^{m-1-k} d\bar{\nu} = O((L_F^+ + \|F\|_\infty) \vartheta_0^k)$$

and

$$\begin{aligned}& \int_{\bar{M}} \left( F - \int_{\bar{M}} F d\bar{\nu} \right) \cdot \kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^{m-1-k} d\bar{\nu} \\ &= \int_{\bar{M}} \left( F - \int_{\bar{M}} F d\bar{\nu} \right) \circ \bar{T}^{-k-1} \cdot \kappa \circ \bar{T}^{-1} \otimes \kappa \circ \bar{T}^{m-k-2} d\bar{\nu} \\ &= O((L_{F \circ T^{-k-1}, \kappa \circ T^{-1}}^+ + \|F\|_\infty) \vartheta_0^{m-k-2}) \\ &= O((L_F^+ + \|F\|_\infty) \vartheta_0^{m-k-2}),\end{aligned}$$

and so

$$\int_{\bar{M}} \left( F - \int_{\bar{M}} F d\bar{v} \right) \cdot \kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^{m-1} d\bar{v} = O((L_F^+ + \|F\|_\infty) \vartheta_0^{\max(m-k-2, k)}) = (\vartheta_0^{\frac{m}{2}}),$$

which, combined with (7), gives (12).

**Remark 1.4.** Note that

$$\begin{aligned} \mathfrak{B}_2^+(f) &= \sum_{j,m \geq 0} \int_M f \cdot (\kappa \circ T^j \otimes \kappa \circ T^m - \mathbb{E}_{\bar{v}}[\kappa \circ \bar{T}^j \otimes \kappa \circ \bar{T}^m]) d\nu \\ &\quad + \int_M f \mathcal{I}_0^{\otimes 2} d\nu + 2 \sum_{m \geq 0} \int_M f \cdot \mathcal{I}_0 \otimes \kappa \circ T^m d\nu - \mathfrak{B}_0 \int_M f d\nu, \\ \mathfrak{B}_2^-(g) &= \sum_{j,m \leq -1} \int_M g \cdot (\kappa \circ T^j \otimes \kappa \circ T^m - \mathbb{E}_{\bar{v}}[\kappa \circ \bar{T}^j \otimes \kappa \circ \bar{T}^m]) d\nu \\ &\quad + \int_M g \cdot \mathcal{I}_0^{\otimes 2} d\nu - 2 \sum_{m \leq -1} \int_M g \cdot \mathcal{I}_0 \otimes \kappa \circ T^m d\nu - \mathfrak{B}_0 \int_M g d\nu, \\ \mathfrak{B}_1^+(f) &= \sum_{m \geq 0} \int_M f \cdot \kappa \circ T^m d\nu + \int_M f \cdot \mathcal{I}_0 d\nu, \\ \mathfrak{B}_1^-(g) &= - \sum_{m \leq -1} \int_M g \cdot \kappa \circ T^m d\nu + \int_M g \cdot \mathcal{I}_0 d\nu, \\ \mathfrak{B}_0 &= \sum_{m \in \mathbb{Z}} |m| \mathbb{E}_{\bar{v}}[\kappa \otimes \kappa \circ \bar{T}^m]. \end{aligned}$$

**Corollary 1.5.** Under the assumptions of Theorem 1.3, if  $\int_M f d\nu = 0$  and  $\int_M g d\nu = 0$ , then

$$\int_M f \cdot g \circ T^n d\nu = \frac{\Sigma^{-2} * (\mathfrak{B}_1^+(f) \otimes \mathfrak{B}_1^-(g))}{n^2 2\pi \sqrt{\det \Sigma^2}} + o(n^{-2}).$$

Two natural examples of zero integral functions are  $\mathbf{1}_{\mathcal{C}_0} - \mathbf{1}_{\mathcal{C}_{e_1}}$  with  $e_1 = (1, 0)$  or  $f \mathbf{1}_{\mathcal{C}_0}$  with  $\int_{\mathcal{C}_0} f d\nu = 0$ . Note that

$$\int_M ((\mathbf{1}_{\mathcal{C}_0} - \mathbf{1}_{\mathcal{C}_{e_1}}) \cdot (\mathbf{1}_{\mathcal{C}_0} - \mathbf{1}_{\mathcal{C}_{e_1}}) \circ T^n) d\nu \sim \frac{\sigma_{2,2}^2}{n^2 2\pi (\det \Sigma^2)^{3/2}},$$

with  $\Sigma^2 = (\sigma_{i,j}^2)_{i,j=1,2}$  and that

$$\int_M (f \mathbf{1}_{\mathcal{C}_0} \cdot (\mathbf{1}_{\mathcal{C}_0} - \mathbf{1}_{\mathcal{C}_{e_1}}) \circ T^n) d\nu \sim - \frac{1}{n^2 2\pi (\det \Sigma^2)^{3/2}} \sum_{m \geq 0} \mathbb{E}_{\bar{v}}[f \cdot (\sigma_{2,2}^2 \kappa_1 - \sigma_{1,2}^2 \kappa_2) \circ T^m],$$

with  $\kappa = (\kappa_1, \kappa_2)$ , provided the sum appearing in the last formula is non null. As noticed in introduction, it may happen that (11) provides only  $\int_M f \cdot g \circ T^n = o(n^{-2})$ . This is the case for example if  $\int_M g d\nu = 0$  and if  $f$  has the form  $f(q + \ell, \vec{v}) = f_0(q, \vec{v}) \cdot h_\ell$  with  $\mathbb{E}_{\bar{v}}[f_0] = 0$  and  $\sum_\ell h_\ell = 0$ . Hence it can be useful to go further in the asymptotic expansion, which is possible thanks to Theorem 3.7. A formula for the term of order  $n^{-3}$  when  $\int_M f d\nu = \int_M g d\nu = \tilde{\mathfrak{A}}_2(f, g) = 0$  is stated in Remark 4.5 and provides the following estimate, showing that, for some observables,  $C_n(f, g)$  has order  $n^{-3}$ .



**Proposition 1.6.** *If  $f$  and  $g$  can be decomposed in  $f(q + \ell, \vec{v}) = \tilde{f}_0(q, \vec{v}).h_\ell$  and  $g(q + \ell, \vec{v}) = \tilde{g}_0(q, \vec{v}).q_\ell$  with  $\mathbb{E}_{\vec{v}}[\tilde{f}_0] = \mathbb{E}_{\vec{v}}[\tilde{g}_0] = 0$  and  $\sum_\ell q_\ell = \sum_\ell h_\ell = 0$  such that  $\sum_{\ell \in \mathbb{Z}^2} |\ell|^4 (\|f \mathbf{1}_{C_\ell}\|_{(\xi)} + \|g \mathbf{1}_{C_\ell}\|_{(\xi)}) < \infty$ . Then*

$$\int_M f.g \circ T^n dv = \frac{3(\Sigma^{-2})^{\otimes 2}}{2\pi \sqrt{\det \Sigma^2} n^3} * \frac{\mathfrak{B}_2^+(f) \otimes \mathfrak{B}_2^-(g)}{4} + o(n^{-3}),$$

with here

$$\frac{\mathfrak{B}_2^+(f) \otimes \mathfrak{B}_2^-(g)}{4} = -\left(\sum_{\ell \in \mathbb{Z}^2} h_\ell.\ell\right) \otimes \left(\sum_{j \geq 0} \mathbb{E}_{\vec{v}}[\tilde{f}_0.\kappa \circ T^j]\right) \otimes \left(\sum_{\ell \in \mathbb{Z}^2} q_\ell.\ell\right) \otimes \left(\sum_{m \leq -1} \mathbb{E}_{\vec{v}}[\tilde{g}_0.\kappa \circ T^m]\right).$$

**2. First order expansions: General results and first examples**

In this section, as in the next one, we state general results in the general context of  $\mathbb{Z}^d$ -extensions over dynamical systems satisfying good spectral properties. We consider a dynamical system  $(M, \nu, T)$  which is the  $\mathbb{Z}^d$ -extension of a probability preserving dynamical system  $(\bar{M}, \bar{\nu}, \bar{T})$  by  $\kappa : \bar{M} \rightarrow \mathbb{Z}^d$ . This means that  $M = \bar{M} \times \mathbb{Z}^d$ ,  $\nu = \bar{\nu} \otimes m_d$  where  $m_d$  is the counting measure on  $\mathbb{Z}^d$  and that  $T$  is given by

$$\forall (x, \ell) \in \bar{M} \times \mathbb{Z}^d, \quad T(x, \ell) = (\bar{T}(x), \ell + \kappa(x)),$$

so that

$$\forall (x, \ell) \in \bar{M} \times \mathbb{Z}^d, \forall n \geq 1, \quad T^n(x, \ell) = (\bar{T}^n(x), \ell + S_n(x)),$$

with  $S_n := \sum_{k=0}^{n-1} \kappa \circ \bar{T}^k$ . Let  $P$  be the transfer operator associated to  $(\bar{M}, \bar{\nu}, \bar{T})$ , i.e. the dual operator of  $f \mapsto f \circ \bar{T}$  with respect to  $L^2(\bar{\nu})$ . Our method is based on the following key formulas:

$$\int_M f.g \circ T^n dv = \sum_{\ell, \ell' \in \mathbb{Z}^2} \mathbb{E}_{\vec{v}}[f(\cdot, \ell). \mathbf{1}_{\{S_n = \ell' - \ell\}}.g(\bar{T}^n(\cdot), \ell')] \tag{13}$$

$$= \sum_{\ell, \ell' \in \mathbb{Z}^d} \mathbb{E}_{\vec{v}}[P^n(\mathbf{1}_{\{S_n = \ell' - \ell\}} f(\cdot, \ell))g(\cdot, \ell')], \tag{14}$$

$$P^n(\mathbf{1}_{\{S_n = \ell\}} f) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{-it.\ell} P^n(e^{it.S_n} f) dt = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{-it.\ell} P_t^n f dt, \tag{15}$$

with  $P_t f := P(e^{it.\kappa} f)$ . Note that (14) makes a link between the mixing properties of  $(M, \nu, T)$  and the local limit theorem for  $(S_n)_n$  and that (15) shows the importance of the study of the family of perturbed operators  $(P_t)_t$  in this study.

2.1. General results

We will make the following general spectral assumptions about the family  $(P_t)_t$  (or about other analogous families of operators).

**Definition 2.1.** Let  $\mathcal{B}, \mathcal{B}_0$  be two complex Banach spaces, let  $(\Theta_n)_n$  be a sequence of  $GL(d, \mathbb{R})$  and  $\Phi$  a probability density function on  $\mathbb{R}^d$ .

We say that a family  $(Q_s)_s$  satisfies Condition  $(H_1)$  with respect to  $(\mathcal{B}, \mathcal{B}_0, (\Theta_n)_n, \Phi)$  if  $(Q_s)_{s \in [-\pi, \pi]^d}$  is a family of linear continuous operators on  $\mathcal{B}$  and if there exists  $b \in [0, \pi]$  such that

- (i)  $\mathcal{B} \hookrightarrow \mathcal{B}_0$ ,

(ii) there exist two constants  $C > 0$  and  $\vartheta \in (0, 1)$  and three functions  $\lambda_\cdot : [-b, b]^d \rightarrow \mathbb{C}$  and  $\Pi_\cdot, R_\cdot : [-b, b]^d \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{B})$  such that  $\lim_{t \rightarrow 0} \lambda_t = 1$  and  $\lim_{s \rightarrow 0} \|\Pi_s - \mathbb{E}_{\bar{\nu}}[\cdot] \mathbf{1}_{\bar{M}}\|_{\mathcal{L}(\mathcal{B}, \mathcal{B}_0)} = 0$  and such that, in  $\mathcal{L}(\mathcal{B}, \mathcal{B})$ ,

$$\forall s \in [-b, b]^d, \quad Q_s = \lambda_s \Pi_s + R_s, \quad \Pi_s R_s = R_s \Pi_s = 0, \quad \Pi_s^2 = \Pi_s, \quad (16)$$

$$\sup_{s \in [-b, b]^d} \|R_s^k\|_{\mathcal{L}(\mathcal{B}, \mathcal{B}_0)} \leq C \vartheta^k, \quad \sup_{s \in [-\pi, \pi]^d \setminus [-b, b]^d} \|Q_s^k\|_{\mathcal{L}(\mathcal{B}, \mathcal{B}_0)} \leq C \vartheta^k, \quad (17)$$

(iii)  $\lim_{n \rightarrow +\infty} |\Theta_n^{-1}| = 0$ ,  $\vartheta^n = o(\det \Theta_n^{-1})$ , and the characteristic function of the distribution of density  $\Phi$  is integrable and has the form  $a_\cdot := e^{-\psi(\cdot)}$  and

$$\forall s, \quad \lambda_{t_{\Theta_n^{-1} s}}^n \sim e^{-\psi(s)} = a_s, \quad \text{as } n \rightarrow +\infty \quad (18)$$

(where  ${}^t \Theta_n^{-1}$  stands for the transpose matrix of  $\Theta_n^{-1}$ ) and, for every  $n$  large enough,  $\forall s \in [-b, b]^d$ ,  $|\lambda_s^n| \leq |e^{-\tilde{\psi}({}^t \Theta_n s)}|$ , with  $e^{-\tilde{\psi}}$  integrable on  $\mathbb{R}$ .

Note that (16) ensures that

$$\forall s \in [-b, b]^d, \quad Q_s^n = \lambda_s^n \Pi_s + R_s^n. \quad (19)$$

Note also that, if  $(P_t)_t$  satisfies Condition  $(H_1)$  with respect to  $(\mathcal{B}, \mathcal{B}_0, (\Theta_n)_n, \Phi)$  with  $\mathcal{B}_0 \hookrightarrow L^1(\bar{M}, \bar{\nu})$  and  $\mathbf{1}_{\bar{M}} \in \mathcal{B}$ , then

$$\forall s \in \mathbb{R}^d, \quad e^{-\psi(s)} = \lim_{n \rightarrow +\infty} \lambda_{t_{\Theta_n^{-1} s}}^n = \lim_{n \rightarrow +\infty} \mathbb{E}_{\bar{\nu}}[P_{t_{\Theta_n^{-1} s}}^n \mathbf{1}] = \lim_{n \rightarrow +\infty} \mathbb{E}_{\bar{\nu}}[e^{is \cdot (\Theta_n^{-1} S_n)}],$$

and so  $(\Theta_n^{-1} S_n)_n$  converges in distribution to a random variable  $Y$  of density  $\Phi$ . If  $Y$  has a non degenerate stable distribution of index  $\alpha \in (0, 2] \setminus \{1\}$ , then  $\psi$  has the following form

$$\psi(s) = \int_{\mathbb{S}^1} |s \cdot t|^\alpha \left( 1 + \tan \frac{\pi}{\alpha} \text{sign}(s \cdot t) \right) d\Gamma(t) \geq c |s|^\alpha, \quad (20)$$

where  $\Gamma$  is a Borel measure on the unit sphere  $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : x \cdot x = 1\}$ . If

$$\lambda_s = e^{-\psi(s)L(|s|^{-1})} + o(|s|^\alpha L(|s|^{-1})), \quad \text{as } s \rightarrow 0,$$

with  $L$  slowly varying at infinity and  $\psi$  as in (20), then Item (iii) of Condition  $(H_1)$  holds true with  $\Theta_n := a_n \text{Id}$  with  $a_n := \inf\{x > 0 : n|x|^{-\alpha} L(x) \geq 1\}$  which is  $1/\alpha$ -regularly varying and with  $\tilde{\psi}(s) := \exp\left(-\frac{c}{4} \max(|s|^{\alpha+\varepsilon}, |s|^{\alpha-\varepsilon})\right)$  (for any  $\varepsilon \in (0, \alpha)$ ). Indeed, there exists  $n_0$  and  $b > 0$  such that for every  $n \geq n_0$  and for every  $s \in [-a_n b, a_n b]^d$ ,

$$\begin{aligned} \left| \lambda_{a_n^{-1} s}^n \right| &\leq \exp\left(-n \frac{c}{2} |s|^\alpha a_n^{-\alpha} L(|s|^{-1} a_n)\right) \leq \exp\left(-\frac{c}{2} |s|^\alpha \frac{L(|s|^{-1} a_n)}{L(a_n)}\right) \\ &\leq \exp\left(-\frac{c}{4} \max(|s|^{\alpha+\varepsilon}, |s|^{\alpha-\varepsilon})\right), \end{aligned}$$

using Karamata's characterization of slowly varying functions.

Note that Condition  $(H_1)$  allows also the study of situations with anisotropic scaling. We start with a simple statement, the proof of which is short and contains the main ideas.

**Theorem 2.2.** *Let  $\mathcal{B}, \mathcal{B}_0$  be two complex Banach spaces of functions  $f : \bar{M} \rightarrow \mathbb{C}$ . Let  $(\Theta_n)_n$  be a sequence of  $GL(d, \mathbb{R})$  and  $\Phi$  a density function on  $\mathbb{R}^d$ . Assume  $(P_s)_s$  satisfies Condition  $(H_1)$  with respect to  $(\mathcal{B}, \mathcal{B}_0, (\Theta_n)_n, \Phi)$  with  $\mathcal{B}_0 \hookrightarrow L^1(\bar{M}, \bar{\nu})$  and  $\mathbf{1}_{\bar{M}} \in \mathcal{B}$ . Let  $f, g : M \rightarrow \mathbb{C}$  be such that*

$$\|f\|_+ := \sum_{\ell \in \mathbb{Z}^d} \|f(\cdot, \ell)\| < \infty \quad \text{and} \quad \|g\|_{+, \mathcal{B}'_0} := \sum_{\ell \in \mathbb{Z}^d} \|g(\cdot, \ell)\|_{\mathcal{B}'_0} < \infty,$$

where  $\|\cdot\|$  is the norm of  $\mathcal{B}$  and with the notation  $\|u\|_{\mathcal{B}'_0} := \sup_{h \in \mathcal{B}_0: \|h\|_{\mathcal{B}_0}=1} |\mathbb{E}_{\bar{\nu}}[u \cdot h]|$ . Then

$$\int_M f \cdot g \circ T^n dv = \frac{\Phi(0)}{\det \Theta_n} \left( \int_M f dv \int_M g dv + o(1) \right), \quad \text{as } n \rightarrow +\infty.$$

**Proof.** For every positive integer  $n$  and every  $\ell \in \mathbb{Z}^d$ , due to (15) and to Condition  $(H_1)$ , for every  $f \in \mathcal{B}$ , the following equalities hold in  $\mathcal{B}_0$ :

$$\begin{aligned} P^n(\mathbf{1}_{\{S_n=\ell\}} f) &= \frac{1}{(2\pi)^d} \int_{[-b,b]^d} e^{-it \cdot \ell} \lambda_t^n \Pi_t f dt + O(\vartheta^n \|f\|) \\ &= \frac{1}{(2\pi)^d \det \Theta_n} \int_{t \in \Theta_n[-b,b]^d} e^{-is \cdot (\Theta_n^{-1} \ell)} \lambda_{t \Theta_n^{-1} s} \Pi_{t \Theta_n^{-1} s} f ds + O(\vartheta^n \|f\|) \\ &= \frac{1}{(2\pi)^d \det \Theta_n} \int_{\mathbb{R}^d} e^{-is \cdot (\Theta_n^{-1} \ell)} e^{-\psi(s)} \Pi_0 f ds + \varepsilon_{n,\ell} f \\ &= \frac{\Phi(\Theta_n^{-1} \ell)}{\det \Theta_n} \Pi_0 f + \varepsilon_{n,\ell} f, \end{aligned} \tag{21}$$

with  $\sup_{\ell} \|\varepsilon_{n,\ell}\|_{\mathcal{L}(\mathcal{B}, \mathcal{B}_0)} = o(\det \Theta_n^{-1})$  due to the dominated convergence theorem applied to  $\|\lambda_{t \Theta_n^{-1} s} \Pi_{t \Theta_n^{-1} s} - e^{-\psi(s)} \Pi_0\|_{\mathcal{L}(\mathcal{B}, \mathcal{B}_0)} \mathbf{1}_{t \Theta_n[-b,b]^d}(s)$ . Setting  $u_{\ell} := f(\cdot, \ell)$  and  $v_{\ell} := g(\cdot, \ell)$  and using (14), we obtain

$$\begin{aligned} \int_M f \cdot g \circ T^n dv &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \left( \frac{\Phi(\Theta_n^{-1}(\ell' - \ell))}{\det \Theta_n} \mathbb{E}_{\bar{\nu}}[u_{\ell}] \mathbb{E}_{\bar{\nu}}[v_{\ell'}] + \mathbb{E}_{\bar{\nu}}[v_{\ell'} \varepsilon_{n,\ell' - \ell}(u_{\ell})] \right) \\ &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \left( \frac{\Phi(\Theta_n^{-1}(\ell' - \ell))}{\det \Theta_n} \mathbb{E}_{\bar{\nu}}[u_{\ell}] \mathbb{E}_{\bar{\nu}}[v_{\ell'}] \right) + O\left( \sum_{\ell, \ell' \in \mathbb{Z}^d} \|v_{\ell'}\|_{\mathcal{B}'_0} \|\varepsilon_{n,\ell' - \ell}\|_{\mathcal{L}(\mathcal{B}, \mathcal{B}_0)} \|u_{\ell}\| \right) \\ &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \frac{\Phi(\Theta_n^{-1}(\ell' - \ell))}{\det \Theta_n} \mathbb{E}_{\bar{\nu}}[u_{\ell}] \mathbb{E}_{\bar{\nu}}[v_{\ell'}] + \tilde{\varepsilon}_n(f, g), \end{aligned} \tag{22}$$

with  $\lim_{n \rightarrow +\infty} \sup_{f, g} \frac{\det \Theta_n \tilde{\varepsilon}_n(f, g)}{\|g\|_{\mathcal{B}'_0} \|f\|_+} = 0$ . Now, due to the dominated convergence theorem and since  $\Phi$  is continuous and bounded (this classical fact comes from the fact that the characteristic function of  $Y$  is integrable combined with the Lebesgue dominated convergence theorem), we obtain

$$\lim_{n \rightarrow +\infty} \sum_{\ell, \ell' \in \mathbb{Z}^d} \Phi(\Theta_n^{-1}(\ell' - \ell)) \mathbb{E}_{\bar{\nu}}[u_{\ell}] \mathbb{E}_{\bar{\nu}}[v_{\ell'}] = \Phi(0) \sum_{\ell, \ell' \in \mathbb{Z}^d} \mathbb{E}_{\bar{\nu}}[u_{\ell}] \mathbb{E}_{\bar{\nu}}[v_{\ell'}] = \Phi(0) \int_M f dv \int_M g dv,$$

which ends the proof. □

We now state a more elaborate result.

**Definition 2.3.** Let  $p_0, q_0 \in [1, +\infty]$ . Let  $\mathcal{V}, \mathcal{W}, \mathcal{V}_0, \mathcal{W}_0$  be four complex Banach spaces of functions  $f : \bar{M} \rightarrow \mathbb{C}$ . Let  $(\Theta_n)_n$  be a sequence of  $GL(d, \mathbb{R})$  and  $\Phi$  be a probability density function on  $\mathbb{R}^d$ .

We say that  $(M, \nu, T)$  satisfies Condition  $(H_1 \text{ bis})$  with respect to  $(p_0, q_0, \mathcal{V}, \mathcal{W}, \mathcal{V}_0, \mathcal{W}_0, (\Theta_n)_n, \Phi)$  if  $\mathcal{V} \hookrightarrow \mathcal{V}_0 \hookrightarrow L^{p_0}(\bar{\nu})$  and  $\mathcal{W} \hookrightarrow \mathcal{W}_0 \hookrightarrow L^{q_0}(\bar{\nu})$ , if  $(\bar{M}, \bar{\nu}, \bar{T})$  is an extension of a dynamical system  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  by  $\mathfrak{p} : \bar{M} \rightarrow \hat{\Delta}$  and if there exist two complex Banach spaces  $\mathcal{B}, \mathcal{B}_0$  of functions  $f : \hat{\Delta} \rightarrow \mathbb{C}$  such that  $\mathcal{B}_0 \hookrightarrow L^1(\hat{\Delta}, \hat{\mu})$  and  $\mathbf{1}_{\hat{\Delta}} \in \mathcal{B}$  and such that:

- there exists an integer  $m_0$  and a function  $\hat{\kappa} : \hat{\Delta} \rightarrow \mathbb{Z}^d$  such that  $\hat{\kappa} \circ \mathfrak{p} = \kappa \circ \bar{T}^{m_0}$ ,

- the family of operators  $(\hat{P}_s : \hat{f} \mapsto \hat{P}(e^{i s \cdot \hat{\kappa}} \hat{f}))_s$  (with  $\hat{P}$  the transfer operator of  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$ ) satisfies Condition  $(H_1)$  with respect to  $(\mathcal{B}, \mathcal{B}_0, (\Theta_n)_n, \Phi)$ ,
- the sequence  $(\det \Theta_n)_n$  is  $\alpha$ -regularly varying at infinity for some  $\alpha > 0$ ,
- there exist  $C_0 > 0$  and  $\beta > \alpha$  such that, for every couple  $(f, g) \in \mathcal{V} \times \mathcal{W}$  and every integer  $n \geq m_0$ , there exists a couple  $(f_n, g_n) \in \mathcal{B} \times \mathcal{B}'_0$  satisfying the following properties:

$$\begin{aligned} \|f \circ \bar{T}^n - f_n \circ \mathfrak{p}\|_{L^{p_0}(\bar{\nu})} &\leq C_0 \|f\|_{\mathcal{V}} n^{-\beta}, & \|g \circ \bar{T}^n - g_n \circ \mathfrak{p}\|_{L^{q_0}(\bar{\nu})} &\leq C_0 \|g\|_{\mathcal{W}} n^{-\beta}, \\ \|f_n \circ \mathfrak{p}\|_{L^{p_0}(\bar{\nu})} &\leq C_0 \|f\|_{\mathcal{V}_0}, & & \\ \forall t \in \mathbb{R}, \quad \|P_t^{2n}(e^{-it\hat{S}_n - m_0} f_n)\|_{\mathcal{B}} &\leq C_0 \|f\|_{\mathcal{V}_0} \quad \text{and} \quad \|g_n e^{it\hat{S}_n - m_0}\|_{\mathcal{B}'_0} &\leq C_0 \|g\|_{\mathcal{W}_0}, \end{aligned} \quad (23)$$

where  $\hat{S}_n := \sum_{k=0}^{n-1} \hat{\kappa} \circ \hat{\tau}^k$ .

**Theorem 2.4.** *Let  $\mathcal{V}, \mathcal{W}, \mathcal{V}_0, \mathcal{W}_0$  be four complex Banach spaces of functions  $f : \bar{M} \rightarrow \mathbb{C}$  and  $p_0, q_0 \in [1, +\infty]$  such that  $\frac{1}{p_0} + \frac{1}{q_0} \leq 1$  and  $p_0 > 1$ . Let  $(\Theta_n)_n$  be a sequence of  $GL(d, \mathbb{R})$  and  $\Phi$  a probability density function on  $\mathbb{R}^d$ . Assume  $(\bar{M}, \bar{\nu}, \bar{T})$  satisfies Condition  $(H_1 \text{ bis})$  with respect to  $(p_0, q_0, \mathcal{V}, \mathcal{W}, \mathcal{V}_0, \mathcal{W}_0, (\Theta_n)_n, \Phi)$ . Let  $f, g : M \rightarrow \mathbb{C}$  be such that*

$$\sup_{\ell \in \mathbb{Z}^d} (\|f(\cdot, \ell)\|_{\mathcal{V}} + \|g(\cdot, \ell)\|_{\mathcal{W}}) < \infty \quad \text{and} \quad \sum_{\ell \in \mathbb{Z}^d} (\|f(\cdot, \ell)\|_{\mathcal{V}_0} + \|g(\cdot, \ell)\|_{\mathcal{W}_0}) < \infty.$$

Then

$$\int_M f \cdot g \circ T^n d\nu = \frac{\Phi(0)}{\det \Theta_n} \left( \int_M f d\nu \int_M g d\nu + o(1) \right), \quad \text{as } n \rightarrow +\infty.$$

**Proof.** Fix some  $\mathfrak{b} \in (\frac{\alpha}{\beta}, 1)$ . Let  $k_n := \lceil n^{\mathfrak{b}} \rceil$ . Let  $\hat{f}_n(\cdot, \ell)$  and  $\hat{g}_n(\cdot, \ell)$  correspond to the functions  $(f(\cdot, \ell))_{k_n}$  and  $(g(\cdot, \ell))_{k_n}$  given by (23). First note that, for  $n$  large enough,  $k_n \geq m_0$  and

$$\begin{aligned} &\int_M f \cdot g \circ T^n d\nu - \sum_{\ell, \ell' \in \mathbb{Z}^d} \mathbb{E}_{\hat{\mu}} [\hat{f}_n(\cdot, \ell) \mathbf{1}_{\{\hat{S}_n \circ \hat{\tau}^{k_n} - m_0 = \ell' - \ell\}} \hat{g}_n(\hat{\tau}^n(\cdot), \ell')] \\ &= \sum_{\ell, \ell' \in \mathbb{Z}^d} (\mathbb{E}_{\bar{\nu}} [f(\cdot, \ell) \mathbf{1}_{\{S_n = \ell' - \ell\}} g(\bar{T}^n(\cdot), \ell')] - \mathbb{E}_{\hat{\mu}} [\hat{f}_n(\cdot, \ell) \mathbf{1}_{\{\hat{S}_n \circ \hat{\tau}^{k_n} - m_0 = \ell' - \ell\}} \hat{g}_n(\hat{\tau}^n(\cdot), \ell')]) \\ &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \mathbb{E}_{\bar{\nu}} [(f(\bar{T}^{k_n} \cdot, \ell) g(\bar{T}^{n+k_n}(\cdot), \ell') - \hat{f}_n(\mathfrak{p}(\cdot), \ell) \hat{g}_n(\mathfrak{p}(\bar{T}^n(\cdot), \ell')) \mathbf{1}_{\{S_n \circ \bar{T}^{k_n} = \ell' - \ell\}})] \\ &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \mathbb{E}_{\bar{\nu}} [(f(\bar{T}^{k_n} \cdot, \ell) - \hat{f}_n(\mathfrak{p}(\cdot), \ell)) g(\bar{T}^{n+k_n}(\cdot), \ell') + \hat{f}_n(\mathfrak{p}(\cdot), \ell) [g(\bar{T}^{n+k_n}(\cdot), \ell') - \hat{g}_n(\mathfrak{p}(\bar{T}^n(\cdot), \ell'))]) \\ &\quad \times \mathbf{1}_{\{S_n \circ \bar{T}^{k_n} = \ell' - \ell\}}]. \end{aligned}$$

Therefore

$$\begin{aligned} &\left| \int_M f \cdot g \circ T^n d\nu - \sum_{\ell, \ell' \in \mathbb{Z}^d} \mathbb{E}_{\hat{\mu}} [\hat{f}_n(\cdot, \ell) \mathbf{1}_{\{\hat{S}_n \circ \hat{\tau}^{k_n} - m_0 = \ell' - \ell\}} \hat{g}_n(\hat{\tau}^n(\cdot), \ell')] \right| \\ &\leq \sup_{\ell \in \mathbb{Z}^d} \|f(\bar{T}^{k_n} \cdot, \ell) - \hat{f}_n(\mathfrak{p}(\cdot), \ell)\|_{p_0} \sum_{\ell, \ell' \in \mathbb{Z}^d} \|g(\bar{T}^{n+k_n}(\cdot), \ell') \mathbf{1}_{\{S_n \circ \bar{T}^{k_n} = \ell' - \ell\}}\|_{q_0} \\ &\quad + \sum_{\ell, \ell' \in \mathbb{Z}^d} \|\mathbf{1}_{\{S_n \circ \bar{T}^{k_n} = \ell' - \ell\}} \hat{f}_n(\mathfrak{p}(\cdot), \ell)\|_{p_0} \sup_{\ell' \in \mathbb{Z}^d} \|g(\bar{T}^{k_n}(\cdot), \ell') - \hat{g}_n(\mathfrak{p}(\cdot), \ell')\|_{q_0} \end{aligned} \quad (24)$$

$$= O\left(\left(\sup_{\ell \in \mathbb{Z}^d} \|f(\cdot, \ell)\|_{\mathcal{V}} \sum_{\ell' \in \mathbb{Z}^d} \|g(\cdot, \ell')\|_{\mathcal{W}_0} + \sup_{\ell' \in \mathbb{Z}^d} \|g(\cdot, \ell')\|_{\mathcal{W}} \sum_{\ell \in \mathbb{Z}^d} \|f(\cdot, \ell)\|_{\mathcal{V}_0}\right) n^{-b\beta}\right) \tag{25}$$

$$= o(\det \Theta_n^{-1}). \tag{26}$$

Moreover,

$$\begin{aligned} & \mathbb{E}_{\hat{\mu}}[\hat{f}_n(\cdot, \ell) \mathbf{1}_{\{\hat{S}_n \circ \hat{\tau}^{kn-m_0} = \ell' - \ell\}} \hat{g}_n(\hat{\tau}^n(\cdot), \ell')] \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{-it(\ell' - \ell)} \mathbb{E}_{\hat{\mu}}[\hat{f}_n(\cdot, \ell) e^{it \cdot \hat{S}_n \circ \hat{\tau}^{kn-m_0}} \hat{g}_n(\hat{\tau}^n(\cdot), \ell')] dt \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{-it(\ell' - \ell)} \mathbb{E}_{\hat{\mu}}[\hat{F}_{n,t}(\cdot, \ell) e^{it \hat{S}_n} \hat{G}_{n,t}(\hat{\tau}^n(\cdot), \ell')] dt \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{-it(\ell' - \ell)} \mathbb{E}_{\hat{\mu}}[\hat{P}_t^n(\hat{F}_{n,t}(\cdot, \ell)) \hat{G}_{n,t}(\cdot, \ell')] dt, \end{aligned}$$

where we used the fact that  $\hat{S}_n \circ \hat{\tau}^{kn-m_0} = \hat{S}_n - \hat{S}_{kn-m_0} + \hat{S}_{kn-m_0} \circ \hat{\tau}^n$  and the notations  $\hat{F}_{n,t}(x, \ell) := \hat{f}_n(x, \ell) \times e^{-it \hat{S}_{kn-m_0}(x)}$  and  $\hat{G}_{n,t}(x, \ell') := \hat{g}_n(x, \ell') e^{it \hat{S}_{kn-m_0}(x)}$ .

Moreover  $\sup_{n,t} \|\hat{P}_t^{2k_n} \hat{F}_{n,t}(\cdot, \ell)\|_{\mathcal{B}} \leq C_0 \|f(\cdot, \ell)\|_{\mathcal{V}_0}$ . Hence, due to Condition  $(H_1 \text{ bis})$ ,

$$\begin{aligned} & \mathbb{E}_{\hat{\mu}}[\hat{f}_n(\cdot, \ell) \mathbf{1}_{\{\hat{S}_n \circ \hat{\tau}^{kn-m_0} = \ell' - \ell\}} \hat{g}_n(\hat{\tau}^n(\cdot), \ell')] \\ &= \frac{1}{(2\pi)^d} \int_{[-b, b]^d} e^{-it(\ell' - \ell)} \mathbb{E}_{\hat{\mu}}[\hat{P}_t^{n-2k_n}(\hat{P}_t^{2k_n} \hat{F}_{n,t}(\cdot, \ell)) \hat{G}_{n,t}(\cdot, \ell')] dt \\ & \quad + O(\|f(\cdot, \ell)\|_{\mathcal{V}_0} \|g(\cdot, \ell')\|_{\mathcal{W}_0} \vartheta^{n-2k_n}) \end{aligned} \tag{27}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^d} \int_{[-b, b]^d} e^{-it(\ell' - \ell)} \mathbb{E}_{\hat{\mu}}[\hat{G}_{n,t}(\cdot, \ell') \lambda_t^{n-2k_n} \Pi_t \hat{P}_t^{2k_n}(\hat{F}_{n,t}(\cdot, \ell))] dt \\ & \quad + O(\|f(\cdot, \ell)\|_{\mathcal{V}_0} \|g(\cdot, \ell')\|_{\mathcal{W}_0} \vartheta^{n-2k_n}). \end{aligned} \tag{28}$$

Moreover

$$\begin{aligned} & \sum_{\ell, \ell' \in \mathbb{Z}^d} \frac{1}{(2\pi)^d} \int_{[-b, b]^d} e^{-it(\ell' - \ell)} \mathbb{E}_{\hat{\mu}}[\hat{G}_{n,t}(\cdot, \ell') \lambda_t^{n-2k_n} \Pi_t \hat{P}_t^{2k_n}(\hat{F}_{n,t}(\cdot, \ell))] dt \\ &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \frac{1}{\det \Theta_{n-2k_n} (2\pi)^d} \int_{\Theta_{n-2k_n}[-b, b]^d} e^{-is \cdot (\Theta_{n-2k_n}^{-1}(\ell' - \ell))} \\ & \quad \times \mathbb{E}_{\hat{\mu}}[\hat{G}_{n,t \Theta_{n-2k_n}^{-1} s}(\cdot, \ell') \lambda_{t \Theta_{n-2k_n}^{-1} s}^{n-2k_n} \Pi_{t \Theta_{n-2k_n}^{-1} s} \hat{P}_{t \Theta_{n-2k_n}^{-1} s}^{2k_n}(\hat{F}_{n,t \Theta_{n-2k_n}^{-1} s}(\cdot, \ell))] ds \\ &= o(\det \Theta_{n-2k_n}^{-1}) + \sum_{\ell, \ell' \in \mathbb{Z}^d} \frac{1}{\det \Theta_{n-2k_n} (2\pi)^d} \int_{\Theta_{n-2k_n}[-b, b]^d} \mathbb{E}_{\hat{\mu}}[\hat{G}_{n,0}(\cdot, \ell') e^{-\psi(s)} \\ & \quad \times \Pi_0 \hat{P}_{t \Theta_{n-2k_n}^{-1} s}^{2k_n}(\hat{F}_{n,t \Theta_{n-2k_n}^{-1} s}(\cdot, \ell))] ds \\ &= o(\det \Theta_{n-2k_n}^{-1}) + \sum_{\ell, \ell' \in \mathbb{Z}^d} \frac{1}{\det \Theta_{n-2k_n} (2\pi)^d} \int_{\Theta_{n-2k_n}[-b, b]^d} \mathbb{E}_{\hat{\mu}}[\hat{g}_n(\cdot, \ell')] e^{-\psi(s)} \mathbb{E}_{\hat{\mu}}[\hat{f}_n(\cdot, \ell)] ds \\ &= o(\det \Theta_{n-2k_n}^{-1}) + \sum_{\ell, \ell' \in \mathbb{Z}^d} \frac{1}{\det \Theta_{n-2k_n} (2\pi)^d} \int_{\mathbb{R}^d} \mathbb{E}_{\hat{\nu}}[g(\cdot, \ell')] e^{-\psi(s)} \mathbb{E}_{\hat{\nu}}[f(\cdot, \ell)] ds \end{aligned}$$

$$= o(\det \Theta_n^{-1}) + \frac{\Phi(0)}{\det \Theta_n} \int_M f \, d\nu \int_M g \, d\nu, \tag{29}$$

where we used the change of variable  $s = {}^t \Theta_{n-2k_n} t$ , and three times the dominated convergence theorem together with the fact that  $\Phi(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}} e^{-\psi(t)} \, dt$ , the fact that  $\det \Theta_{n-2k_n} \sim \det \Theta_n$  as  $n \rightarrow +\infty$  and the fact that

$$\Pi_0 \hat{P}_{t \Theta_{n-2k_n}^{-1} s}^{2k_n} (\hat{F}_{n, t \Theta_{n-2k_n}^{-1} s}(\cdot, \ell)) - \mathbb{E}_{\hat{\mu}}[\hat{f}_n(\cdot, \ell)] = \mathbb{E}_{\hat{\mu}}[(e^{i s \cdot \Theta_{n-2k_n}^{-1} (\hat{S}_{2k_n} - \hat{S}_{k_n - m_0})} - 1) \hat{f}_n(\cdot, \ell)] \rightarrow 0$$

as  $n \rightarrow +\infty$  (the facts that  $(\Theta_n^{-1} \hat{S}_n)_n$  converges in distribution and that  $\|\Theta_n^{-1}\| \rightarrow 0$  imply that  $\Theta_{n-2k_n}^{-1} (\hat{S}_{2k_n} - \hat{S}_{k_n - m_0})$  converges in probability to 0; moreover  $\|\hat{f}_n(\cdot, \ell)\|_{p_0} \leq C_0 \|f(\cdot, \ell)\|_{\mathcal{V}_0}$ ).

We conclude by combining (26), (28) and (29). □

### 2.2. Application to $\mathbb{Z}^d$ -extensions of Gibbs–Markov maps

For Gibbs–Markov maps with respect to some partition  $\pi$ , given  $\theta \in (0, 1)$ , we consider the metric  $d_\theta$  on  $\bar{M}$  defined by:

$$d_\theta(x, y) := \theta^{\inf\{k \geq 0 : \pi(\bar{T}^k(x)) \neq \pi(\bar{T}^k(y))\}},$$

where  $\pi(z)$  denotes the atom of  $\pi$  containing  $z$ .

**Corollary 2.5.** *Let  $\theta \in (0, 1)$ . Assume that  $(\bar{M}, \bar{\nu}, \bar{T})$  is a mixing Gibbs–Markov dynamical system with respect to some partition  $\pi$  and that  $\kappa : \bar{M} \rightarrow \mathbb{Z}^d$  is  $\bar{\nu}$ -centered, aperiodic and uniformly Lipschitz continuous (with respect to  $d_\theta$ ) on each atom of partition  $\pi$ . Assume that the distortion of the jacobian is Lipschitz continuous with respect to  $d_{\theta_0}$  for some  $\theta_0 \in (0, 1)$ . Assume moreover that the distribution of  $\kappa$  (with respect to  $\bar{\nu}$ ) belongs to the domain of attraction of a nondegenerate symmetric stable distribution  $\mathcal{S}$  of index belonging to  $(0, 2]$ , with the normalization  $\Theta_n \in GL(\mathbb{R}^d)$ . Let  $f, g : M \rightarrow \mathbb{C}$  be such that*

$$\sup_{\ell \in \mathbb{Z}^d} \|f(\cdot, \ell)\| < \infty \quad \text{and} \quad \sum_{\ell \in \mathbb{Z}^d} \|f(\cdot, \ell)\|_\infty + \int_M |g| \, d\nu < \infty,$$

where  $\|h\| := \|h\|_\infty + L_h$ , with  $L_h$  the Lipschitz constant of  $h$  (with respect to  $d_\theta$ ). Then we have

$$\int_M f \cdot g \circ T^n \, d\nu = \frac{\Phi(0)}{\det \Theta_n} \left( \int_M f \, d\nu \int_M g \, d\nu + o(1) \right), \quad \text{as } n \rightarrow +\infty, \tag{30}$$

where  $\Phi$  is the density function of  $\mathcal{S}$ .

Observe that, if we apply Theorem 2.2, we obtain (30) under the stronger condition on  $f$  that  $\sum_{\ell \in \mathbb{Z}^d} \|f(\cdot, \ell)\| < \infty$ .

**Proof.** The fact that  $(\bar{M}, \bar{\nu}, \bar{T})$  satisfies Condition  $(H_1)$  with respect to  $(\mathcal{B}, \mathcal{B}, (\Theta_n)_n, \Phi)$  where  $\mathcal{B}$  is the set of Lipschitz continuous functions with respect to  $d_{\max(\theta, \theta_0)}$  is proved in [1,2]. We apply Theorem 2.4 with  $(\hat{M}, \hat{\nu}, \hat{T}) = (\bar{M}, \bar{\nu}, \bar{T})$ ,  $\mathbf{p} = id$ ,  $m_0 = 0$ , with  $\mathcal{V} = \mathcal{B}$ , with  $\mathcal{V}_0 = L^\infty(\bar{\nu})$  and with  $\mathcal{W} = \mathcal{W}_0 = L^1(\bar{\nu})$ ,  $p_0 = \infty$  and  $q_0 = 1$ . For every couple  $(f, g) \in \mathcal{V} \times \mathcal{W}$  for every  $n \in \mathbb{N}$  and every  $x \in \bar{M}$ , we take  $f_n(x)$  as the conditional expectation of  $f \circ \bar{T}^n$  given the  $2n$ -cylinder  $\mathcal{C}_{2n}(x)$  containing  $x$ , that is :

$$\mathcal{C}_{2n}(x) := \{y \in \bar{M} : \forall k = 0, \dots, 2n, y_k = x_k\}$$

and  $g_n(x) := g \circ \bar{T}^n(x)$ . So that,

$$\|f \circ \bar{T}^n - f_n\|_{L^\infty(\bar{\mu})} \leq \|f\| \theta^n, \quad \|g \circ \bar{T}^n - g_n\|_{L^1(\bar{\mu})} = 0, \quad \|f_n\|_\infty \leq \|f\|_\infty,$$

$$\|g_n e^{i t \hat{S}_{n-m_0}}\|_{\mathcal{B}'} \leq \|g_n\|_{L^1(\hat{\mu})} \leq \|g\|_{\mathcal{W}_0}.$$

The fact that  $\|\hat{P}_t^{2n}(e^{-it\hat{S}_{n-m_0}} f_n)\|_{\mathcal{B}} \leq C_0 \|f\|_{\infty}$  follows from the next lemma applied with  $\theta_0$  instead of  $\theta$ . □

**Lemma 2.6.** *Let  $m_0, N \in \mathbb{N}$  with  $N \geq 1$  and  $\theta \in (0, 1)$ . Let  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  be a probability preserving dynamical system endowed with some metric  $d_{\theta} : (x, y) \mapsto \theta^{s(x,y)}$  (with  $s(\hat{\tau}(x), \hat{\tau}(y)) = s(x, y) - 1$  if  $s(x, y) \geq 1$ ). Let  $\phi : \hat{\Delta} \rightarrow (0, +\infty)$  and  $\hat{\kappa} : \hat{\Delta} \rightarrow \mathbb{Z}^d$  be two measurable functions, Lipschitz continuous on every closed ball of radius  $\theta$ . Assume that the transfer operator of  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  has the following form:*

$$\hat{P}g(x) = \sum_{z \in \hat{\tau}^{-1}(\{x\})} e^{-\phi(z)} g(z).$$

Assume also that, for every  $x, y \in \hat{\Delta}$  satisfying  $s(x, y) \geq 1$  there exists a bijection  $\chi_{x,y} : \hat{\tau}^{-1}(\{x\}) \rightarrow \hat{\tau}^{-1}(\{y\})$  such that, for every  $z \in \hat{\tau}^{-1}(\{x\})$ ,  $s(z, \chi_{x,y}(z)) = 1 + s(x, y)$ . Then there exists  $C_0 > 0$  such that, for every  $n$ , for every function  $f : \hat{\Delta} \rightarrow \mathbb{C}$  constant on closed balls of radius  $\theta^{2n+1}$ , for every  $t \in (-\pi, \pi)$ , then

$$\|\hat{P}_t^{2n}(e^{-it\hat{S}_{n-m_0}} f)\| \leq C_0 \|f\|_{\infty}, \quad \text{with } \hat{S}_n := \sum_{k=0}^{n-1} \hat{\kappa} \circ \hat{\tau}^k \text{ and } \hat{P}_t h := \hat{P}(e^{it\hat{\kappa}} h),$$

where  $\|\cdot\|$  stands for the Lipschitz norm. If moreover  $\hat{\kappa}$  is uniformly bounded then, for every  $j = 1, \dots, N$ , for every function  $f : \hat{\Delta} \rightarrow \mathbb{C}$  constant on balls of radius  $\theta^{2n+1}$  and for every  $t \in (-\pi, \pi)$ ,

$$\left\| \frac{\partial^j}{\partial t^j} (\hat{P}_t^{2n}(e^{-it\hat{S}_{n-m_0}} f)) \right\| \leq C_0 n^j \|f\|_{\infty}.$$

**Proof.** Note that

$$\hat{P}^{2n}g(x) = \sum_{z \in \hat{\tau}^{-2n}(\{x\})} e^{-\phi(z) + \dots + \phi(\hat{\tau}^{2n-1}(z))} g(z).$$

Moreover, for every  $x, y \in \hat{\Delta}$  such that  $s(x, y) \geq 1$ , and every  $n \geq 0$ , there exists a bijection  $\chi_{x,y,2n} : \hat{\tau}^{-2n}(\{x\}) \rightarrow \hat{\tau}^{-2n}(\{y\})$  such that  $\forall z \in \hat{\tau}^{-2n}(\{x\})$ ,  $s(z, \chi_{x,y,2n}(z)) = s(x, y) + 2n$ . Note that  $\|\hat{P}_t^{2n}(e^{-it\hat{S}_n} f)\|_{\infty} \leq \|f\|_{\infty}$ . Let  $n$  be fixed and let  $f : \hat{\Delta} \rightarrow \mathbb{C}$  be a function constant on balls of radius  $\theta^{2n+1}$ . Let  $x, y \in \hat{\Delta}$  such that  $s(x, y) \geq 1$ , we have

$$\left| \frac{\partial^j}{\partial t^j} (\hat{P}_t^{2n}(e^{-it\hat{S}_{n-m_0}} f))(x) - \frac{\partial^j}{\partial t^j} (\hat{P}_t^{2n}(e^{-it\hat{S}_{n-m_0}} f))(y) \right| \leq \left| \sum_{z \in \hat{\tau}^{-2n}(\{x\})} F_{t,j}(z) - F_{t,j}(\chi_{x,y,2n}(z)) \right|,$$

with  $F_{t,j}(z) := e^{-\sum_{k=0}^{2n-1} \phi(\hat{\tau}^k z)} (i \sum_{\ell=n-m_0}^{2n-1} \hat{\kappa} \circ \hat{\tau}^{\ell} z)^{\otimes j} e^{it \sum_{\ell=n-m_0}^{2n-1} \hat{\kappa} \circ \hat{\tau}^{\ell} z} f(z)$ . By definition,  $f(z) = f(\chi_{x,y,2n}(z))$  and so

$$\begin{aligned} & |F_{t,j}(z) - F_{t,j}(\chi_{x,y,2n}(z))| \\ & \leq \left( \sum_{k=0}^{2n-1} \tilde{L}_{\phi} \theta^{2n-k+s(x,y)} + \sum_{\ell=n-m_0}^{2n-1} (1 + |t|) \tilde{L}_{\hat{\kappa}} \theta^{2n-\ell+s(x,y)} \right) (1 + (n + m_0) \|\hat{\kappa}\|_{\infty})^j \|f\|_{\infty} \\ & \leq \frac{\theta^{s(x,y)+1}}{1 - \theta} (\tilde{L}_{\phi} + \tilde{L}_{\kappa}) (1 + (n + m_0) \|\hat{\kappa}\|_{\infty})^j (1 + |t|) \|f\|_{\infty}, \end{aligned}$$

with  $\tilde{L}_h := \sup_{x \neq y: s(x,y) \geq 1} \frac{|h(x) - h(y)|}{\theta^{s(x,y)}}$  (and with convention  $\infty^0 = 1$ ), and so

$$\left\| \frac{\partial^j}{\partial t^j} (\hat{P}_t^{2n}(e^{-it\hat{S}_{n-m_0}} f)) \right\| \leq \|f\|_{\infty} \left( 2 + \theta \frac{\tilde{L}_{\phi} + \tilde{L}_{\hat{\kappa}}}{1 - \theta} \right) (1 + (n + m_0) \|\hat{\kappa}\|_{\infty})^j (1 + |t|). \quad \square$$

### 3. Higher expansion: General results and first examples

We keep the context and notations  $(M, \nu, T), \kappa, (\bar{M}, \bar{\nu}, \bar{T}), \mathfrak{m}_d, P, P_t$  of the previous section. We are now interested in higher order expansions. We will consider situations in which the asymptotic variance  $\Sigma_0^2$  of  $(n^{-1/2} \sum_{k=0}^{n-1} \kappa \circ T^k)_n$  exists and in which  $(S_n/\sqrt{n})_n$  converges in distribution to a centered Gaussian random variable with variance  $\Sigma_0^2$ .

#### 3.1. General results

We will reinforce Condition  $(H_1)$  as follows. Notations  $\lambda_0^{(k)}, a_0^{(k)}, \Pi_0^{(k)}$  stand for the  $k$ th derivatives of  $\lambda, a$  and  $\Pi$  at 0.

**Definition 3.1.** Let  $\Sigma_0^2$  be a  $d$ -dimensional positive symmetric matrix. We say that a family  $(Q_s)_s$  satisfies Condition  $(H_2)$  with respect to  $(\mathcal{B}, K, L, J, \Sigma_0^2)$  if it satisfies Condition  $(H_1)$  with respect to  $(\mathcal{B}, \mathcal{B}, (\sqrt{n} \text{Id})_n, \Phi)$  with  $\Phi$  the density function of the centered Gaussian distribution with variance  $\Sigma_0^2$ , with  $\Pi$  and  $R$  both  $C^K$ -smooth, with  $\lambda$   $C^L$ -smooth and such that

$$\lambda_s - 1 \sim -\psi(s) := \frac{1}{2} \Sigma_0^2 * s^{\otimes 2}, \quad \text{as } s \rightarrow 0, \tag{31}$$

and  $\forall k < J, \lambda_0^{(k)} = a_0^{(k)}$  with  $a_t = e^{-\psi(t)}$ .

**Theorem 3.2.** Let  $\Sigma_0^2$  be a  $d$ -dimensional positive symmetric matrix. Let  $K, L, J$  be three integers such that  $K \geq d, 3 \leq J \leq L + 1$  and  $\frac{L}{2} - \frac{J}{2} \geq \frac{K}{2}$ . Assume  $(P_s)_s$  satisfies Condition  $(H_2)$  with respect to  $(\mathcal{B}, K, L + 1, J, \Sigma_0^2)$  with  $\mathcal{B} \hookrightarrow L^1(\bar{M}, \bar{\nu})$  and  $\mathbf{1}_{\bar{M}} \in \mathcal{B}$ . Let  $f, g : M \rightarrow \mathbb{C}$  be such that

$$\sum_{\ell \in \mathbb{Z}^d} (\|f(\cdot, \ell)\|_{\mathcal{B}} + \|g(\cdot, \ell)\|_{\mathcal{B}'}) < \infty. \tag{32}$$

Then

$$\begin{aligned} \int_M f.g \circ T^n d\nu &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \sum_{m=0}^K \frac{1}{m!} \sum_{j=0}^L \frac{i^{m+j}}{(j)!} \frac{\Phi^{(m+j)}(\frac{\ell' - \ell}{\sqrt{n}})}{n^{\frac{d+m+j}{2}}} * (\mathbb{E}_{\bar{\nu}}[g(\cdot, \ell') \Pi_0^{(m)}(f(\cdot, \ell))] \otimes (\lambda^n/a^n)_0^{(j)}) \\ &\quad + o(n^{-\frac{K+d}{2}}), \end{aligned} \tag{33}$$

with  $\Phi$  the density function of the centered Gaussian distribution with variance  $\Sigma_0^2$ . If moreover  $\sum_{\ell \in \mathbb{Z}^d} |\ell|^K (\|f(\cdot, \ell)\|_{\mathcal{B}} + \|g(\cdot, \ell)\|_{\mathcal{B}'}) < \infty$ , then

$$\begin{aligned} \int_M f.g \circ T^n d\nu &= \sum_{m,j,r} \frac{i^{j+m}}{m!r!j!} \left( \frac{\Phi^{(j+m+r)}(0)}{n^{\frac{j+d+m+r}{2}}} * (\lambda^n/a^n)_0^{(j)} \right) \\ &\quad * \sum_{\ell, \ell' \in \mathbb{Z}^d} (\ell' - \ell)^{\otimes r} \otimes \mathbb{E}_{\bar{\nu}}[g(\cdot, \ell') \Pi_0^{(m)}(f(\cdot, \ell))] + o(n^{-\frac{K+d}{2}}), \end{aligned} \tag{34}$$

where the sum is taken over the  $(m, j, r)$  with  $m, j, r$  non negative integers such that  $j + m + r \in 2\mathbb{Z}$  and such that  $\frac{r+m+j}{2} - \lfloor \frac{j}{J} \rfloor \leq K/2$  (so  $m, j, r \leq L$ ).

Observe that

$$(\lambda^n/a^n)_0^{(j)} = \sum_{k_1 m_1 + \dots + k_r m_r = j} \frac{n!}{m_1! \dots m_r! (n - m_1 - \dots - m_r)!} ((\lambda/a)_0^{(k_1)})^{m_1} \dots ((\lambda/a)_0^{(k_r)})^{m_r}, \tag{35}$$

where the sum is taken over  $r \geq 1, m_1, \dots, m_r \geq 1, k_r > \dots > k_1 \geq J$  (this implies that  $m_1 + \dots + m_r \leq j/J$ ). Hence  $(\lambda^n/a^n)_0^{(j)}$  is polynomial in  $n$  with degree at most  $\lfloor j/J \rfloor$ .



**Remark 3.3.** Note that, since  $\Phi$  is even, (34) provides an expansion of the following form:

$$\int_M f.g \circ T^n dv = \sum_{m=0}^{\lfloor K/2 \rfloor} \frac{c_m(f, g)}{n^{\frac{d}{2}+m}} + o(n^{-\frac{K+d}{2}}).$$

Observe that

$$\begin{aligned} \frac{i^N}{N!} \frac{\partial^N}{\partial t^N} [e^{-it \cdot (\ell - \ell')} \lambda_t^n e^{\frac{n}{2} \Sigma_0^2 * t^{\otimes 2}} \Pi_t] &= i^N \sum_{m+j+r=N} \frac{(-i)^r}{m!j!r!} e^{-it \cdot (\ell - \ell')} [(\ell' - \ell)^{\otimes r} \otimes \Pi_t^{(m)} \otimes (\lambda^n/a^n)_t^{(j)}] \\ &= \sum_{m+j+r=N} \frac{i^{j+m}}{m!j!r!} e^{-it \cdot (\ell - \ell')} [(\ell' - \ell)^{\otimes r} \otimes \Pi_t^{(m)} \otimes (\lambda^n/a^n)_t^{(j)}]. \end{aligned}$$

Therefore, we obtain the following.

**Remark 3.4.** If  $\Pi$  is  $C^L$ -smooth, the right hand side of (34) can be rewritten

$$\frac{1}{n^{\frac{d}{2}}} \sum_{\ell, \ell' \in \mathbb{Z}^d} \sum_{N=0}^L \frac{1}{n^{N/2}} i^N \frac{\Phi^{(N)}(0)}{N!} * \frac{\partial^N}{\partial t^N} (\mathbb{E}_{\bar{v}}[g(\cdot, \ell') \cdot e^{-it \cdot (\ell' - \ell)} \cdot \lambda_t^n \Pi_t f(\cdot, \ell)] e^{\frac{n}{2} \Sigma_0^2 * t^{\otimes 2}})_{|t=0} + o(n^{-\frac{K+d}{2}}).$$

If moreover  $\sup_{m=0, \dots, L} \sup_{u \in [-b, b]^d} \|(R_u^n)^{(m)}\|_{(\mathcal{B}, \mathcal{B})} = O(\vartheta^n)$ , due to (19), it can also be rewritten

$$\frac{1}{n^{\frac{d}{2}}} \sum_{\ell, \ell' \in \mathbb{Z}^d} \sum_{N=0}^L \frac{i^N \Phi^{(N)}(0)}{N!} * \frac{\partial^N}{\partial t^N} (\mathbb{E}_{\bar{v}}[f(\cdot, \ell) e^{it \cdot \frac{S_n - (\ell' - \ell)}{\sqrt{n}}} g(\bar{T}^n(\cdot, \ell'))] e^{\frac{1}{2} \Sigma_0^2 * t^{\otimes 2}})_{|t=0} + o(n^{-\frac{K+d}{2}}).$$

**Proof of Theorem 3.2.** Up to reduce the value of  $b$ , we can assume that, for every  $t \in [-b, b]^d$ ,  $\lambda_t \leq a_{t/\sqrt{2}}$ . Due to (15) and to (19), in  $\mathcal{L}(\mathcal{B}, \mathcal{B})$ , we have

$$\begin{aligned} P^n(\mathbf{1}_{\{S_n = \ell\}} \cdot) &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{-it \cdot \ell} P_t^n(\cdot) dt \\ &= \frac{1}{(2\pi)^d} \int_{[-b, b]^d} e^{-it \cdot \ell} \lambda_t^n \Pi_t(\cdot) dt + O(\vartheta^n) \\ &= \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{[-b\sqrt{n}, b\sqrt{n}]^d} e^{-it \cdot \frac{\ell}{\sqrt{n}}} \lambda_{t/\sqrt{n}}^n \Pi_{t/\sqrt{n}}(\cdot) dt + O(\vartheta^n) \\ &= \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{[-b\sqrt{n}, b\sqrt{n}]^d} e^{-it \cdot \frac{\ell}{\sqrt{n}}} \lambda_{t/\sqrt{n}}^n \sum_{m=0}^K \frac{1}{m!} \Pi_0^{(m)}(\cdot) * \frac{t^{\otimes m}}{n^{\frac{m}{2}}} dt + o(n^{-\frac{K+d}{2}}), \end{aligned}$$

due to the dominated convergence theorem since there exists  $x_{t/\sqrt{n}} \in (0, t/\sqrt{n})$  such that  $\Pi_{t/\sqrt{n}}(\cdot) = \sum_{m=0}^{K-1} \frac{1}{m!} \times \Pi_0^{(m)}(\cdot) * \frac{t^{\otimes m}}{n^{\frac{m}{2}}} + \frac{1}{K!} \Pi_{x_{t/\sqrt{n}}}^{(K)}(\cdot) * \frac{t^{\otimes K}}{n^{\frac{K}{2}}}$  and since

$$\lim_{n \rightarrow +\infty} \int_{[-b\sqrt{n}, b\sqrt{n}]^d} |\lambda_{t/\sqrt{n}}^n| \|\Pi_{x_{t/\sqrt{n}}}^{(K)} - \Pi_0^{(K)}\| |t|^K dt = 0,$$

due to the Lebesgue dominated convergence theorem. Since  $n(\lambda/a)_t^{(k)} \leq O(n^{k/J} |t|^{J-k})$  if  $k < J$ , there exists  $L_0 > 0$  such that

$$\left| \lambda_{t/\sqrt{n}}^n - a_t \sum_{j=0}^L \frac{1}{j!} (\lambda^n/a^n)_0^{(j)} * \frac{t^{\otimes j}}{n^{\frac{j}{2}}} \right| \leq C_1 n^{\frac{L+1}{J}} \frac{|t|^{L+1}}{n^{\frac{L+1}{2}}} a_{t/2}.$$

Due to the conditions on  $L, J, K$ , we obtain

$$\begin{aligned} P^n(\mathbf{1}_{S_n=\ell^\cdot}) &= \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{[-b\sqrt{n}, b\sqrt{n}]^d} e^{-it \cdot \frac{\ell}{\sqrt{n}}} e^{-\frac{1}{2} \Sigma_0^2 * t^{\otimes 2}} \sum_{m=0}^K \frac{1}{m!} \Pi_0^{(m)}(\cdot) * \frac{t^{\otimes m}}{n^{\frac{m}{2}}} \\ &\quad \times \left( 1 + \sum_{j=J}^L \frac{1}{j!} (\lambda^n/a^n)_0^{(j)} * \frac{t^{\otimes j}}{n^{\frac{j}{2}}} \right) dt + o(n^{-\frac{K+d}{2}}) \\ &= \sum_{m=0}^K \sum_{j=0}^L \frac{i^{m+j}}{n^{\frac{m+j+d}{2}} m! j!} \Phi^{(m+j)}\left(\frac{\ell}{\sqrt{n}}\right) * (\Pi_0^{(m)}(\cdot) \otimes (\lambda^n/a^n)_0^{(j)}) + o(n^{-\frac{K+d}{2}}). \end{aligned}$$

This combined with (14) and (32) gives (33). We assume from now on that  $\sum_{\ell \in \mathbb{Z}^d} |\ell|^K (\|f(\cdot, \ell)\| + \|g(\cdot, \ell)\|_{\mathcal{B}^r}) < \infty$ . Recall that  $(\lambda^n/a^n)_0^{(j)}$  is polynomial in  $n$  of degree at most  $\lfloor j/J \rfloor$ . Hence, due to the dominated convergence theorem, we can replace  $\Phi^{(m+j)}(\frac{\ell' - \ell}{\sqrt{n}})$  in (33) by

$$\sum_{r=0}^{K-m-j+2\lfloor \frac{j}{J} \rfloor} \frac{1}{r! n^{\frac{r}{2}}} \Phi^{(m+j+r)}(0) * (\ell' - \ell)^{\otimes r}.$$

Hence we have proved (34). □

**Corollary 3.5.** *Let  $\theta \in (0, 1)$ . Assume that  $(\bar{M}, \bar{\nu}, \bar{T})$  is a mixing Gibbs–Markov dynamical system with respect to some partition  $\pi$  and that  $\kappa : \bar{M} \rightarrow \mathbb{Z}^d$  is  $\bar{\nu}$ -centered, aperiodic, bounded and uniformly Lipschitz continuous (with respect to  $d_\theta$ ) on each atom of partition  $\pi$ . Let  $f, g : M \rightarrow \mathbb{C}$  such that*

$$\sum_{\ell \in \mathbb{Z}^d} |\ell|^K (\|f(\cdot, \ell)\| + \|g(\cdot, \ell)\|_{L^1(\bar{M}, \bar{\nu})}) < \infty$$

with  $\|\cdot\|$  the Lipschitz norm associated to  $d_\theta$ , then there exist  $c_0(f, g), \dots, c_{\lfloor K/2 \rfloor}(f, g)$  such that

$$\begin{aligned} \int_M f \cdot g \circ T^n dv &= \sum_{k=0}^{\lfloor K/2 \rfloor} n^{-k-\frac{d}{2}} c_k(f, g) + o(n^{-\frac{K+d}{2}}) \\ &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \sum_{N=0}^{3K} i^N n^{-\frac{N+d}{2}} \Phi^{(N)}\left(\frac{\ell' - \ell}{\sqrt{n}}\right) * \frac{\partial^N}{\partial t^N} (\mathbb{E}_{\bar{\nu}}[f(\cdot, \ell) e^{it S_n} g(\cdot, \ell')] e^{\frac{1}{2} \Sigma_0^2 * t^{\otimes 2}}) \Big|_{t=0} \\ &\quad + o(n^{-\frac{K+d}{2}}), \end{aligned}$$

where  $\Sigma_0^2$  is the asymptotic variance matrix of  $(S_n/\sqrt{n})_n$ .

**Proof.** This is a direct application of Theorem 3.2 with  $L = 3K$  combined with the Nagaev–Guivarc’h’s method [1, 2, 17, 18]. □

For further applications, it may be useful to consider the case of extensions of systems, the transfer operator of which has good spectral properties. We will make the following assumption.

**Definition 3.6.** Let  $\Sigma_0^2$  be a  $d$ -dimensional positive symmetric matrix. Let  $L, J$  be two positive integers and let  $\mathcal{V}$  be a complex Banach space of functions  $f : \bar{M} \rightarrow \mathbb{C}$ . We say that the dynamical system  $(M, \nu, T)$  satisfies Condition  $(H_2 \text{ bis})$  with respect to  $(\mathcal{V}, L, J, \Sigma_0^2)$  if  $(\bar{M}, \bar{\nu}, \bar{T})$  is an extension, by  $p : \bar{M} \rightarrow \hat{\Delta}$ , of a dynamical system  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  with transfer operator  $\hat{P}$  and if the following conditions hold true:

- there exist a positive integer  $m_0$  and a  $\hat{\mu}$ -centered bounded function  $\hat{\kappa} : \hat{\Delta} \rightarrow \mathbb{Z}^d$  such that  $\hat{\kappa} \circ \mathfrak{p} = \kappa \circ \bar{T}^{m_0}$ ,
- the family of operators  $(\hat{P}_s : \hat{f} \mapsto \hat{P}(e^{is\hat{\kappa}} \hat{f}))_s$  satisfies Condition  $(H_2)$  with respect to  $(\mathcal{B}, L, L, J, \Sigma_0^2)$  with  $\mathcal{B} \hookrightarrow L^1(\hat{\Delta}, \hat{\mu})$ , with  $\mathbf{1}_{\hat{\Delta}} \in \mathcal{B}$ , and with

$$\forall m = 0, \dots, L, \quad \sup_{u \in [-b, b]^d} \|(R_u^n)^{(m)}\|_{(\mathcal{B}, \mathcal{B})} = O(\vartheta^n),$$

- there exist  $C_0 > 0$  and  $\theta \in (0, 1)$  such that, for every  $f \in \mathcal{V}$  and every integer  $n \geq m_0$ , there exists  $f_n \in \mathcal{B}$  satisfying the following properties:

$$\begin{aligned} \|f \circ \bar{T}^n - f_n \circ \mathfrak{p}\|_\infty &\leq C_0 \|f\|_{\mathcal{V}} \theta^n, \quad \forall t \in \mathbb{R}, \forall j = 0, \dots, L, \\ \left\| \frac{\partial^j}{\partial t^j} (\hat{P}_t^{2n}(e^{-it\hat{S}_{n-m_0}} f_n)) \right\|_{\mathcal{B}} &\leq C_0 n^j \|f\|_{\mathcal{V}}, \quad \forall t \in \mathbb{R}, \forall j = 0, \dots, L, \\ \left\| \frac{\partial^j}{\partial t^j} (f_n e^{it\hat{S}_{n-m_0}}) \right\|_{\mathcal{B}'} &\leq C_0 n^j \|g\|_{\mathcal{V}}, \end{aligned} \tag{36}$$

with  $\hat{S}_n := \sum_{k=0}^{n-1} \hat{\kappa} \circ \hat{\tau}^k$ .

**Theorem 3.7.** Let  $\Sigma_0^2$  be a  $d$ -dimensional positive symmetric matrix. Let  $K, L, J$  be three integers such that  $K \geq d$ ,  $3 \leq J \leq L + 1$  and  $\frac{L}{2} - \frac{L}{J} \geq \frac{K}{2}$ . Let  $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$  be a complex Banach space of functions  $f : \bar{M} \rightarrow \mathbb{C}$  such that  $\mathcal{V} \hookrightarrow L^\infty(\bar{\nu})$ . Assume  $(\bar{M}, \bar{\nu}, \bar{T})$  satisfies Condition  $(H_2 \text{ bis})$  with respect to  $(\mathcal{V}, L + 1, J, \Sigma_0^2)$ . Let  $f, g : M \rightarrow \mathbb{C}$  such that  $\sum_{\ell \in \mathbb{Z}^d} (\|f(\cdot, \ell)\|_{\mathcal{V}} + \|g(\cdot, \ell)\|_{\mathcal{V}}) < \infty$ . Then, for every  $\ell, \ell' \in \mathbb{Z}^d$  and every  $N = 0, \dots, L$ , the following quantity is well defined

$$A_N(f(\cdot, \ell), g(\cdot, \ell')) := \lim_{n \rightarrow +\infty} \frac{\partial^N}{\partial t^N} (\mathbb{E}_{\bar{\nu}}[f(\cdot, \ell) \cdot e^{it \cdot S_n} \cdot g(\bar{T}^n(\cdot), \ell')] \lambda_t^{-n})_{|t=0} \tag{37}$$

and

$$\int_M f \cdot g \circ T^n \, d\nu \tag{38}$$

$$= \sum_{\ell, \ell' \in \mathbb{Z}^d} \sum_{N=0}^L i^N \frac{\Phi^{(N)}(\frac{\ell' - \ell}{\sqrt{n}})}{n^{\frac{N+d}{2}}} \sum_{j=0}^N \frac{N!}{j!(N-j)!} A_j(f(\cdot, \ell), g(\cdot, \ell')) (\lambda_t^n e^{\frac{n}{2} \Sigma_0^2 * t^{\otimes 2}})_{|t=0}^{(N-j)} + o(n^{-\frac{K+d}{2}}) \tag{39}$$

$$= \sum_{\ell, \ell' \in \mathbb{Z}^d} \sum_{N=0}^L i^N n^{-\frac{N+d}{2}} \Phi^{(N)}\left(\frac{\ell' - \ell}{\sqrt{n}}\right) * (\mathbb{E}_{\bar{\nu}}[f(\cdot, \ell) e^{it S_n} g(\bar{T}^n(\cdot), \ell')] e^{\frac{n}{2} \Sigma_0^2 * t^{\otimes 2}})_{|t=0}^{(N)} + o(n^{-\frac{K+d}{2}}), \tag{40}$$

where  $\Phi$  is the density of a Gaussian distribution  $\mathcal{N}(0, \Sigma_0^2)$ .

If moreover  $\sum_{\ell \in \mathbb{Z}^d} |\ell|^K (\|f(\cdot, \ell)\|_{\mathcal{V}} + \|g(\cdot, \ell)\|_{\mathcal{V}}) < \infty$ , then there exists  $c_0(f, g), \dots, c_{\lfloor K/2 \rfloor}(f, g)$  such that

$$\begin{aligned} \int_M f \cdot g \circ T^n \, d\nu &= \sum_{m, j, r} \frac{i^{j+m}}{m!r!j!} \left( \frac{\Phi^{(j+m+r)}(0)}{n^{\frac{j+d+m+r}{2}}} * (\lambda_t^n e^{\frac{n}{2} \Sigma_0^2 * t^{\otimes 2}})_{|t=0}^{(j)} \right) \\ &\quad * \sum_{\ell, \ell' \in \mathbb{Z}^d} (\ell' - \ell)^{\otimes r} \otimes A_m(f(\cdot, \ell), g(\cdot, \ell')) + o(n^{-\frac{K+d}{2}}) \end{aligned} \tag{41}$$

$$= \sum_{k=0}^{\lfloor K/2 \rfloor} n^{-k - \frac{d}{2}} c_k(f, g) + o(n^{-\frac{K+d}{2}}), \tag{42}$$

where the sum over  $(m, j, r)$  is taken over the  $(m, j, r)$  with  $m, j, r$  non negative integers such that  $j + m + r \in 2\mathbb{Z}$  and  $\frac{r+m+j}{2} - \lfloor \frac{j}{J} \rfloor \leq K/2$  (and  $j = 0$  or  $j \geq J$ ).

**Proof.** Up to reduce the value of  $b$ , we assume that, for every  $t \in [-b, b]^d$ ,  $\lambda_t \leq a_{t/\sqrt{2}}$ . Let  $k_n := \lceil (L + \frac{L+1+d}{2}) \log n / \log \theta \rceil$ . We consider  $\hat{f}_n, \hat{g}_n : \hat{\Delta} \times \mathbb{Z}^d \rightarrow \mathbb{C}$  such that, for every  $\ell \in \mathbb{Z}^d$ ,  $\hat{f}_n(\cdot, \ell)$  and  $\hat{g}_n(\cdot, \ell)$  correspond to the functions  $(f(\cdot, \ell))_{k_n}$  and  $(g(\cdot, \ell))_{k_n}$  given by (36). Set also  $\phi_n(\cdot, \ell) := \hat{f}_n(\mathfrak{p}(\bar{T}^{-k_n}(\cdot)), \ell)$  and  $\psi_n(\cdot, \ell) := \hat{g}_n(\mathfrak{p}(\bar{T}^{-k_n}(\cdot)), \ell)$ . Recall that, due to (25) and (27), we know that

$$\begin{aligned} \int_M f \cdot g \circ T^n dv &= \sum_{\ell, \ell' \in \mathbb{Z}^d} \frac{1}{(2\pi)^d} \int_{[-b, b]^d} e^{-it(\ell' - \ell)} \mathbb{E}_{\hat{\mu}}[\hat{P}_t^{n-2k_n}(\hat{P}_t^{2k_n} \hat{F}_{n,t}(\cdot, \ell)) \hat{G}_{n,t}(\cdot, \ell')] dt \\ &\quad + O(\theta^{k_n} + \vartheta^{n-2k_n}), \end{aligned} \quad (43)$$

with  $\hat{F}_{n,t}(x, \ell) := \hat{f}_n(x, \ell) e^{-it\hat{S}_{k_n-m_0}(x)}$  and  $\hat{G}_{n,t}(x, \ell) := \hat{g}_n(x, \ell) e^{it\hat{S}_{k_n-m_0}(x)}$ . Moreover, for every  $t \in [-b, b]^d$ ,

$$\begin{aligned} &\mathbb{E}_{\hat{\mu}}[\hat{P}_t^{n-2k_n}(\hat{P}_t^{2k_n} \hat{F}_{n,t}(\cdot, \ell)) \hat{G}_{n,t}(\cdot, \ell')] e^{\frac{n}{2} \Sigma_0^{2*} t^{\otimes 2}} \\ &= \sum_{N=0}^L (\mathbb{E}_{\hat{\mu}}[\hat{P}_t^N(\hat{F}_{n,t}(\cdot, \ell)) \hat{G}_{n,t}(\cdot, \ell')] e^{\frac{n}{2} \Sigma_0^{2*} t^{\otimes 2}})_{|t=0}^{(N)} * t^{\otimes N} \\ &\quad + O((n^{(L+1)/J} + \vartheta^{n-2k_n} n^{L+1}) |t|^{L+1} e^{\frac{n}{3} \Sigma_0^{2*} t^{\otimes 2}} \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}}) \\ &= \sum_{N=0}^L (\mathbb{E}_{\bar{\nu}}[\phi_n(\cdot, \ell) e^{itS_n} \psi_n(\bar{T}^n(\cdot), \ell')] e^{\frac{n}{2} \Sigma_0^{2*} t^{\otimes 2}})_{|t=0}^{(N)} * t^{\otimes N} \\ &\quad + O((n^{(L+1)/J} + \vartheta^{n-2k_n} n^{L+1}) |t|^{L+1} e^{\frac{n}{3} \Sigma_0^{2*} t^{\otimes 2}} \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}}), \end{aligned}$$

since  $\hat{P}_t^n = \lambda_t^n \Pi_t + R_t^n$  with  $(R_t^n)^{(j)} = O(\vartheta^n)$  and  $(\lambda_t^n e^{\frac{n}{2} \Sigma_0^{2*} t^{\otimes 2}})^{(j)}$  is in  $O(n^{j/J} e^{\frac{n}{3} \Sigma_0^{2*} t^{\otimes 2}})$ . Using the fact that

$$\int_{\mathbb{R}^d} |t|^{L+1} e^{-\frac{n}{6} \Sigma_0^{2*} t^{\otimes 2}} dt = n^{-\frac{L+1+d}{2}} \int_{\mathbb{R}^d} |t|^{L+1} e^{-\frac{1}{6} \Sigma_0^{2*} t^{\otimes 2}} dt = O(n^{-\frac{L+1+d}{2}}),$$

and that

$$\begin{aligned} &\int_{[-b, b]^d} e^{-it \cdot (\ell' - \ell)} e^{-\frac{n}{2} \Sigma_0^{2*} t^{\otimes 2}} t^{\otimes N} dt \\ &= n^{-\frac{N+d}{2}} \int_{[-b\sqrt{n}, b\sqrt{n}]^d} e^{-\frac{i}{\sqrt{n}} t \cdot (\ell' - \ell)} t^{\otimes N} e^{-\frac{1}{2} \Sigma_0^{2*} t^{\otimes 2}} dt \\ &= n^{-\frac{N+d}{2}} \int_{\mathbb{R}^d} e^{-\frac{i}{\sqrt{n}} t \cdot (\ell' - \ell)} t^{\otimes N} e^{-\frac{1}{2} \Sigma_0^{2*} t^{\otimes 2}} dt + O(n^{-\frac{L+1+d}{2}}) \\ &= n^{-\frac{N+d}{2}} i^N \Phi^{(N)}\left(\frac{\ell' - \ell}{\sqrt{n}}\right) + O(n^{-\frac{L+1+d}{2}}), \end{aligned}$$

we obtain

$$\begin{aligned} \int_M f \cdot g \circ T^n dv &- \sum_{\ell, \ell' \in \mathbb{Z}^d} \sum_{N=0}^L n^{-\frac{N+d}{2}} i^N \Phi^{(N)}\left(\frac{\ell' - \ell}{\sqrt{n}}\right) * (\mathbb{E}_{\bar{\nu}}[\phi_n(\cdot, \ell) e^{itS_n} \psi_n(\bar{T}^n(\cdot), \ell')] e^{\frac{n}{2} \Sigma_0^{2*} t^{\otimes 2}})_{|t=0}^{(N)} \\ &= O(n^{-\frac{L+1+d}{2} + \lfloor \frac{L+1}{J} \rfloor k_n^{L+1-J} \lfloor \frac{L+1}{J} \rfloor}) = o(n^{-\frac{K+d}{2}}). \end{aligned}$$

But, for every  $N = 0, \dots, L$ ,

$$\begin{aligned} & |(\mathbb{E}_{\bar{v}}[\phi_n(\cdot, \ell) e^{itS_n} \psi_n(\bar{T}^n(\cdot, \ell'))] e^{\frac{n}{2} \Sigma_0^{*t \otimes 2}})_{|t=0}^{(N)} - (\mathbb{E}_{\bar{v}}[f(\cdot, \ell) e^{itS_n} g(\bar{T}^n(\cdot, \ell'))] e^{\frac{n}{2} \Sigma_0^{*t \otimes 2}})_{|t=0}^{(N)}| \\ & \leq 2 \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}} \theta^{k_n} \|(e^{it \cdot S_n} e^{\frac{n}{2} \Sigma_0^{*t \otimes 2}})_{|t=0}^{(N)}\|_{L^\infty(\bar{v})} \\ & \leq 2 \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}} \theta^{k_n} n^N \leq 2 \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}} n^{-\frac{L+1+d}{2}}, \end{aligned}$$

due to the definition of  $k_n$  and so we obtain (40). Let us prove that, for every  $N$ ,

$$(A_{N,n}(f(\cdot, \ell), g(\cdot, \ell'))) := (\mathbb{E}_{\bar{v}}[f(\cdot, \ell) e^{itS_n} g(\bar{T}^n(\cdot, \ell'))] \lambda_t^{-n})_{|t=0}^{(N)}_n$$

is a Cauchy sequence. For  $N = 0, \dots, L$  and  $2k_n \leq n \leq n+m \leq 2n$ , we have

$$\begin{aligned} & |A_{N,n}(f(\cdot, \ell), g(\cdot, \ell')) - A_{N,n+m}(f(\cdot, \ell), g(\cdot, \ell'))| \\ & \leq |A_{N,n}(\phi_n(\cdot, \ell), \psi_n(\cdot, \ell')) - A_{N,n+m}(\phi_n(\cdot, \ell), \psi_n(\cdot, \ell'))| + O(n^N \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}} \theta^{k_n}) \\ & \leq |\mathbb{E}_{\bar{\mu}}[\left((\lambda_t^{-n} \hat{P}_t^{n-2k_n} - \lambda_t^{-n-m} \hat{P}_t^{n+m-2k_n}) (\hat{P}_t^{2k_n} (e^{-it\hat{S}_{k_n-m_0}} \hat{f}_n(\cdot, \ell)) e^{it\hat{S}_{k_n-m_0}} \hat{g}_n(\cdot, \ell'))\right)_{|t=0}^{(N)}]| \\ & \quad + O(n^N \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}} \theta^{k_n}) \\ & \leq O((n^N \theta^{k_n} + \vartheta^{n/2}) \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}}) = O(n^{-\frac{L+1+d}{2}} \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}}), \end{aligned}$$

since  $\lambda_t^{-n} \hat{P}_t^{n-2k_n} = \lambda_t^{-2k_n} \Pi_t + \lambda_t^{-n} R_t^{n-2k_n}$  and  $(R_t^n)_{|t=0}^{(j)} = O(\vartheta^n)$ . Therefore, for every  $N = 0, \dots, L$  and  $k_n \leq n \leq n+m$ , we obtain

$$\begin{aligned} & \sup_{m \geq 0} |A_{N,n}(f(\cdot, \ell), g(\cdot, \ell')) - A_{N,n+m}(f(\cdot, \ell), g(\cdot, \ell'))| \\ & \leq \sum_{p \geq 0} \sup_{m=0, \dots, 2^p n} |A_{N,2^p n}(f(\cdot, \ell), g(\cdot, \ell')) - A_{N,2^p n+m}(f(\cdot, \ell), g(\cdot, \ell'))| \\ & \leq O\left(\sum_{p \geq 0} (2^p n)^{-\frac{L+1+d}{2}} \|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}}\right) = O(\|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}} n^{-\frac{L+d+1}{2}}). \end{aligned}$$

Hence  $A_N(f(\cdot, \ell), g(\cdot, \ell'))$  is well defined and

$$|A_{N,n}(f(\cdot, \ell), g(\cdot, \ell')) - A_N(f(\cdot, \ell), g(\cdot, \ell'))| = O(\|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}} n^{-\frac{L+d+1}{2}}).$$

Hence we have proved (39) since  $(\lambda^n/a^n)_{|t=0}^{(j)}$  is polynomial in  $n$  of degree at most  $\lfloor j/J \rfloor$  (due to (35)). Moreover, fix some  $n_0$  such that  $n_0 \geq 2k_{n_0}$ , then

$$\begin{aligned} |A_{N,n_0}(f(\cdot, \ell), g(\cdot, \ell'))| & \leq |A_{N,n_0}(\phi_{n_0}(\cdot, \ell), \psi_{n_0}(\cdot, \ell'))| + O(\|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}}) \\ & \leq |(\lambda_t^{-n_0} \hat{P}_t^{n_0-2k_{n_0}} (\hat{P}_t^{2k_{n_0}} (\hat{F}_{n_0,t}(\cdot, \ell))) \hat{G}_{n_0,t}(\cdot, \ell'))_{|t=0}^{(N)}| + O(\|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}}) \\ & \leq O(\|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}}). \end{aligned}$$

Thus, for every  $N = 0, \dots, L$ ,  $|A_N(f(\cdot, \ell), g(\cdot, \ell'))| \leq O(\|f(\cdot, \ell)\|_{\mathcal{V}} \|g(\cdot, \ell')\|_{\mathcal{V}})$ .

Assume now  $\sum_{\ell \in \mathbb{Z}^d} |\ell|^K (\|f(\cdot, \ell)\|_{\mathcal{V}} + \|g(\cdot, \ell)\|_{\mathcal{V}}) < \infty$ . The proof of (42) follows exactly the same scheme as the proof of (34) with  $\mathbb{E}_{\bar{v}}[v_{\ell'} \Pi_0^{(m)}(u_{\ell})]$  being replaced by  $A_m(f(\cdot, \ell), g(\cdot, \ell'))$ .  $\square$

### 3.2. Application to $\mathbb{Z}^d$ -extensions of mixing subshifts of finite type

In this subsection, we consider the case of a subshift of finite type on a finite alphabet  $A$ . Given a matrix  $B$  indexed by  $A \times A$  with 0–1 entries (such that there exists  $n_0$  such that  $B^{n_0}$  has positive entries), we consider  $\bar{M}$  as the set of allowed sequences, i.e.

$$\bar{M} := \{x := (x_n)_{n \in \mathbb{Z}} \in A^{\mathbb{Z}} : \forall n \in \mathbb{Z}, B(x_n, x_{n+1}) = 1\}.$$

Given  $\theta \in (0, 1)$ , we consider the metric  $\delta_\theta$  defined on  $\bar{M}$  by:

$$\delta_\theta(x, y) := \theta^{\min\{k \geq 1 : x_k \neq y_k \text{ or } x_{-k} \neq y_{-k}\}}.$$

We define the shift  $\bar{T} : \bar{M} \rightarrow \bar{M}$  by  $\bar{T}((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$ . Let  $\bar{\nu}$  be the Gibbs measure associated to some Lipschitz continuous potential  $h$ . Assume moreover that  $\kappa : \bar{M} \rightarrow \mathbb{Z}^d$  is Lipschitz continuous with respect to  $\delta_\theta$  (so  $\kappa$  is uniformly bounded) and is non-arithmetic.

**Theorem 3.8.** *Let  $f, g : M \rightarrow \mathbb{C}$  be a couple of functions such that*

$$\sum_{\ell \in \mathbb{Z}^d} |\ell|^K (\|f(\cdot, \ell)\| + \|g(\cdot, \ell)\|) < \infty$$

with  $\|\cdot\|$  the Lipschitz norm associated to  $\delta_\theta$ . Then there exist  $c_0(f, g), \dots, c_K(f, g)$  such that

$$\begin{aligned} \int_M f \cdot g \circ T^n \, d\nu &= \sum_{m, j, r} \frac{i^{j+m}}{m!r!j!} \left( \frac{\Phi^{(j+m+r)}(0)}{n^{\frac{j+d+m+r}{2}}} * (\lambda_t^n e^{\frac{n}{2} \Sigma_0^{*t \otimes 2}})^{(j)} \Big|_{t=0} \right) \\ &\quad * \sum_{\ell, \ell' \in \mathbb{Z}^d} (\ell' - \ell)^{\otimes r} \otimes A_m(f(\cdot, \ell), g(\cdot, \ell')) + o(n^{-\frac{K+d}{2}}) \\ &= \sum_{k=0}^{\lfloor K/2 \rfloor} n^{-k - \frac{d}{2}} c_k(f, g) + o(n^{-\frac{K+d}{2}}), \end{aligned}$$

with  $A_m(\cdot)$  given by (37), with  $\lambda_t$  defined for  $t$  small enough by

$$\exists c_t \neq 0, \quad \mathbb{E}_{\bar{\nu}}[e^{it \cdot S_n}] \sim c_t \lambda_t^n, \quad \text{as } n \rightarrow +\infty,$$

and where the sum over  $(m, j, r)$  is taken over the  $(m, j, r)$  with  $m, j, r$  non negative integers such that  $j + m + r \in 2\mathbb{Z}$  and  $\frac{r+m+j}{2} - \lfloor \frac{j}{3} \rfloor \leq K/2$  (and  $j = 0$  or  $j \geq J$ ) and where  $\Phi$  is the density of the Gaussian distribution  $\mathcal{N}(0, \Sigma_0^2)$ , and where  $\Sigma_0^2$  is the asymptotic variance of  $(n^{-1/2} \sum_{k=0}^{n-1} \kappa \circ \bar{T}^k)_n$ .

**Proof.** We apply Theorem 3.7. We consider  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  the one-sided subshift of finite type:

$$\hat{\Delta} := \{(x_n)_{n \geq 0} : \forall n \geq 0, B(x_n, x_{n+1}) = 1\}$$

and  $\mathfrak{p}((x_n)_{n \in \mathbb{Z}}) := (x_n)_{n \geq 0}$ ,  $\hat{\tau}((x_n)_{n \geq 0}) = (x_{n+1})_{n \geq 0}$  and  $\hat{\mu}$  the image measure of  $\bar{\nu}$  by  $\mathfrak{p}$ . We endow  $\hat{\Delta}$  with the metric  $\delta_\theta^{(+)}$  defined by  $\delta_\theta^{(+)}(x, y) := \theta^{\min\{k \geq 1 : x_k \neq y_k\}}$ .

We take  $\mathcal{B} = \mathcal{B}_0$  for the set of functions  $\hat{f} : \hat{\Delta} \rightarrow \mathbb{C}$ , Lipschitz continuous with respect to  $\delta_\theta^{(+)}$ . Observe that there exists  $m_0$  such that  $\kappa((x_n)_{n \in \mathbb{Z}})$  depends only on  $x_{-m_0}, \dots, x_{m_0}$  and so  $\kappa \circ \bar{T}^{m_0}(x)$  only depends on coordinates of  $x$  with non-negative indices. Therefore, there exists  $\hat{\kappa} : \hat{\Delta} \rightarrow \mathbb{Z}^d$  such that  $\hat{\kappa} \circ \mathfrak{p} = \kappa \circ \bar{T}^{m_0}$  with  $\hat{\kappa}((x_k)_{k \geq 0})$  depending only on  $x_0, \dots, x_{2m_0}$ .

Let  $f : \bar{M} \rightarrow \mathbb{C}$  be a Lipschitz continuous function (with respect to  $\delta_\theta$ ). For every  $x \in \bar{M}$  and every  $n \geq 1$ , we set  $\tilde{f}_n(x)$  for the minimum of  $f$  over  $\mathcal{C}_n(x) := \{y \in \bar{M} : \forall |k| \leq n, y_k = x_k\}$ . Observe that  $\|f - \tilde{f}_n\|_\infty \leq \|f\| \theta^n$ .

Since  $\tilde{f}_n \circ \bar{T}^n(x)$  depends only on  $x_0, \dots, x_{2n}$ , there exists  $f_n : \hat{\Delta} \rightarrow \mathbb{C}$  such that  $f_n \circ \mathfrak{p} = \tilde{f}_n \circ \bar{T}^n$ , with  $f_n((x_k)_k)$  depending only on  $x_0, \dots, x_{2n}$ . Let  $\hat{P}$  be the transfer operator associated to  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  and  $\hat{P}_t : f \mapsto \hat{P}(e^{it \cdot \hat{\kappa}} f)$ . The fact that  $\|(\hat{P}_t^{2n}(e^{-it \cdot \hat{S}_{n-m_0}} f_n))^{(N)}\|_{\mathcal{B}} \leq C_0 n^N \|f\|_{\infty}$  comes from Lemma 2.6. Moreover

$$\left\| \frac{\partial^j}{\partial t^j} (f_n e^{it S_{n-m_0}}) \right\|_{\mathcal{B}'_0} \leq \left\| \frac{\partial^j}{\partial t^j} (f_n e^{it S_{n-m_0}}) \right\|_{L^1(\hat{\mu})} \leq n^j \|\kappa\|_{\infty}^j \|f\|_{\infty}.$$

Due to [17] and [18],  $(\hat{P}_t)_t$  satisfies  $(H_2)$  with respect to  $(\mathcal{B}, L+1, L+1, 3, \Sigma_0^2)$ , with  $L = 3K$  and with  $s \mapsto \Pi_s, s \mapsto R_s$  and  $s \mapsto \lambda_s \in C^{\infty}$ , (31) and  $\sup_{u \in [-b, b]^d} \|(R_u^n)^{(m)}\|_{(\mathcal{B}, \mathcal{B})} = O(\vartheta^n)$  for every  $m \geq 0$ . Hence, Theorem 3.7 applies. Finally  $\lambda_s$  satisfies  $\mathbb{E}_{\bar{\nu}}[e^{is \cdot S_n}] = \mathbb{E}_{\hat{\mu}}[e^{is \cdot \hat{S}_n}] = \mathbb{E}_{\hat{\mu}}[\hat{P}_t^n \mathbf{1}] = \mathbb{E}_{\hat{\mu}}[\lambda_s^n \Pi_s \mathbf{1} + R_s^n \mathbf{1}] \sim \lambda_s^n \mathbb{E}_{\hat{\mu}}[\Pi_s \mathbf{1}]$  for every  $s$  small enough, moreover  $\lim_{s \rightarrow 0} \mathbb{E}_{\hat{\mu}}[\Pi_s \mathbf{1}] = \mathbb{E}_{\hat{\mu}}[\Pi_0 \mathbf{1}] = 1 \neq 0$ .  $\square$

## 4. Sinai billiards with finite or infinite horizon

### 4.1. Some facts about young towers for billiards

Now, we come back to the case of  $\mathbb{Z}^2$ -periodic Sinai billiards, with the notations of Section 1. We use the dynamical systems constructed by Young in [37] (Young towers). We won't remind all the construction. We recall just the properties we need. In [37], Young constructed two probability preserving dynamical systems  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  and  $(\Delta, \mu, \tau)$  such that  $(\Delta, \mu, \tau)$  is an extension of both  $(\bar{M}, \bar{\nu}, T)$  and  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  by, respectively,  $q : \Delta \rightarrow \bar{M}$  and  $\mathfrak{p} : \Delta \rightarrow \hat{\Delta}$ . Given (any)  $p > 1$ , Young constructs a Banach space  $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$  of functions  $f : \hat{\Delta} \rightarrow \mathbb{C}$  such that there exist  $c_0 \in (0, 1)$  and  $\beta \in (0, 1)$  such that

$$c_0 \|f\|_{L^q(\hat{\mu})} \leq \|f\|_{\mathcal{B}} \leq \|f\|_{\infty} + \sup_{x, y \in \hat{\Delta} : s_0(x, y) \geq 0} \frac{|f(x) - f(y)|}{\beta^{s_0(x, y)}}, \quad (44)$$

with  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$  and with  $s_0(x, y)$  some dynamical separation time on  $\hat{\Delta}$  satisfying the following property: for every  $x, y \in \hat{\Delta}$  ( $x \neq y$ ),  $s_0(x, y) + 1$  is smaller than the minimal integer  $k \geq 0$  such that the sets  $q(\mathfrak{p}^{-1}(\{x\}))$  and  $q(\mathfrak{p}^{-1}(\{y\}))$  do not lie in same connected component of  $\bar{M} \setminus \bigcup_{m=0}^k \bar{T}^{-m} S_0$ . While the right hand side of (44) may be  $+\infty$  in general on  $\mathcal{B}$ , its upperbound is not useful for every  $f \in \mathcal{B}$ , but only for bounded functions  $f$ , that are Lipschitz continuous with respect to the metric  $\beta^{s_0(\cdot, \cdot)}$ .

We recall also that  $q(\mathfrak{p}^{-1}(\{x\}))$  is a piece of a stable manifold. Hence, for every measurable function  $f : \bar{M} \rightarrow \mathbb{C}$  constant on every stable manifold, there exists  $\hat{f} : \hat{\Delta} \rightarrow \mathbb{C}$  such that  $\hat{f} \circ \mathfrak{p} = f \circ q$ . In particular, there exists  $\hat{\kappa} : \hat{\Delta} \rightarrow \mathbb{Z}^d$  such that  $\kappa \circ q = \hat{\kappa} \circ \mathfrak{p}$ .

We set  $\hat{S}_n := \sum_{k=0}^{n-1} \hat{\kappa} \circ \hat{\tau}^k$ . We recall that the transfer operator  $\hat{P}$  of  $(\hat{\Delta}, \hat{\mu}, \hat{\tau})$  has the following form  $\hat{P}g(x) = \sum_{z \in \hat{\tau}^{-1}(\{x\})} e^{-\phi(z)} g(z)$  with  $|\phi(z) - \phi(y)| \leq C\beta^{s_0(\hat{\tau}(z), \hat{\tau}(y))}$ . Moreover, for every  $x, y \in \hat{\Delta}$  such that  $s_0(x, y) \geq 0$ , there exists a bijection  $\chi_{x, y} : \hat{\tau}^{-1}(\{x\}) \rightarrow \hat{\tau}^{-1}(\{y\})$  such that

$$\forall z \in \hat{\tau}^{-1}(\{x\}), \quad s_0(z, \chi_{x, y}(z)) = 1 + s_0(x, y).$$

We recall that, up to an adaptation of the construction of the tower, the dominating eigenvalue of  $\hat{P}$  on  $\mathcal{B}$  is 1 and is simple. For any  $u \in \mathbb{R}^2$  and  $\hat{f} \in \mathcal{B}$ , we set  $\hat{P}_u(\hat{f}) := \hat{P}(e^{iu \cdot \hat{\kappa}} \hat{f})$ . For the billiard in infinite horizon we take  $\mathcal{B}_0 := L^1(\hat{\mu})$  and for the billiard with finite horizon, we take  $\mathcal{B}_0 := \mathcal{B}$ . For every couple of integers  $k \leq \ell$ , we define  $\mathcal{Z}_k^{\ell}$  as the partition of  $\bar{M} \setminus \bigcup_{j=k}^{\ell} T^{-j} S_0$ . For every couple of dynamically Lipschitz functions  $(f, g)$  (with respect to  $d_{\xi}$ ) and every positive integer  $n$ , we set

$$\tilde{f}_n := \mathbb{E}_{\bar{\nu}}[f \circ \sigma(\mathcal{Z}_{-n}^n)] \circ \bar{T}^n \quad \text{and} \quad \tilde{g}_n := \mathbb{E}_{\bar{\nu}}[g \circ \sigma(\mathcal{Z}_{-n}^n)] \circ \bar{T}^n.$$

It comes that

$$\|f \circ \bar{T}^n - \tilde{f}_n\|_{\infty} \leq L_{\xi}(f) \xi^n \quad \text{and} \quad \|g \circ \bar{T}^n - \tilde{g}_n\|_{\infty} \leq L_{\xi}(g) \xi^n. \quad (45)$$

**Lemma 4.1.** *There exists  $C_1 > 0$  such that, for every bounded  $\mathcal{Z}_{-n}^n$ -measurable function  $f : \bar{M} \rightarrow \mathbb{C}$ , there exists  $\hat{f} \in \mathcal{B}$  such that  $\hat{f} \circ \mathfrak{p} = f \circ \bar{T}^n \circ \mathfrak{q}$ . Moreover,*

$$\|(\hat{P}_t^{2n}(e^{-it\hat{S}_{n-m_0}} \hat{f}))^{(j)}\|_{\mathcal{B}} + \|(f e^{it\hat{S}_{n-m_0}})^{(j)}\|_{\mathcal{B}'_0} \leq C_0 n^j \|f\|_{\infty}$$

hold for  $j = 0$  in any case (finite or infinite horizon) and for every positive integer  $j$  in the finite horizon case.

**Proof.** Observe that the function  $f \circ \bar{T}^n$  is  $\mathcal{Z}_0^{2n}$ -measurable. First, this implies that  $f \circ \bar{T}^n$  is constant on stable manifolds and so that there exists  $\hat{f} : \hat{\Delta} \rightarrow \mathbb{C}$  such that  $\hat{f} \circ \mathfrak{p} = f \circ \bar{T}^n \circ \mathfrak{q}$ . Second, if  $x, y \in \Delta$  are such that  $s_0(x, y) \geq 2n$ , then  $f \circ \bar{T}^n$  is equal to the same constant on  $\mathfrak{q}(\mathfrak{p}^{-1}(\{x\}))$  and on  $\mathfrak{q}(\mathfrak{p}^{-1}(\{y\}))$ . Therefore the function  $\hat{f}$  is in  $\mathcal{B}$  and  $\|\hat{f}\|_{\mathcal{B}} \leq (1 + 2\beta^{-2n})\|f\|_{\infty}$ . Due to (44),

$$\|\hat{f} e^{it\hat{S}_{n-m_0}}\|_{\mathcal{B}'_0} \leq \frac{1}{c_0} \|\hat{f} e^{it\hat{S}_{n-m_0}}\|_{L^\infty(\hat{\mu})} \leq \frac{1}{c_0} \|f\|_{\infty}.$$

Moreover, if  $\kappa$  is bounded (finite horizon case), then, for every positive integer  $j$ , we have:

$$\|(\hat{f} e^{it\hat{S}_{n-m_0}})^{(j)}\|_{\mathcal{B}'_0} \leq \|(\hat{S}_{n-m_0})^{\otimes j} \hat{f} e^{it\hat{S}_{n-m_0}}\|_{L^p(\hat{\mu})} \leq (n\|\kappa\|_{\infty})^j \|f\|_{\infty}.$$

The first point comes from Lemma 2.6. □

#### 4.2. Proof of the mixing result in the infinite horizon case

**Proof of Theorem 1.1.** In [35], Szász and Varjú implemented the Nagaev–Guivarc’h perturbation method via the Keller–Liverani theorem [19] to prove that  $(\hat{P}_s)_s$  satisfies Condition  $(H_1)$  (see Definition 2.1) with respect to  $(\mathcal{B}, \mathcal{B}_0 := \mathbb{L}^1(\hat{\mu}), (\sqrt{n \log n} \text{Id})_n, \Phi)$  with  $\Phi$  the density function of the centered normal distribution of variance  $\Sigma_\infty^2$  (defined in (8)) and with  $\lambda$  having the following expansion  $\lambda_t - 1 \sim \Sigma_\infty^2 * (t^{\otimes 2}) \log |t|$ . We will apply Theorem 2.4 with the previous notations, with  $\mathcal{V} = \mathcal{W}$  the set of functions  $F : \Delta \rightarrow \mathbb{C}$  of the form  $F = f \circ \mathfrak{q}$ , with  $f : \bar{M} \rightarrow \mathbb{C}$  dynamically Lipschitz continuous (wrt.  $d_\xi$ ), with  $\mathcal{B}$  the Young Banach space, with  $\mathcal{B}_0 := L^1(\hat{\mu})$ , with  $\mathcal{V}_0 = \mathcal{W}_0 = L^\infty(\mu)$ ,  $p_0 = q_0 = \infty$ . Let  $(f, g) \in \mathcal{V} \times \mathcal{W}$  and  $n \geq m_0$ . We take  $f_n(x)$  and  $g_n(x)$  as the conditional expectation of respectively  $f \circ \bar{T}^n$  and  $g \circ \bar{T}^n$  given the atom of  $\mathcal{Z}_0^{2n}$  containing  $x$ . Due to (45) and to Lemma 4.1, we obtain (23). Theorem 2.4 applies.

#### 4.3. Proofs of our main results in the finite horizon case

We assume throughout this section that the billiard has finite horizon. The Nagaev–Guivarc’h method [17,23,24] has been applied in this context by Szász and Varjú [34] (see also [26]) to prove that  $(\hat{P}_t)_t$  satisfies Condition  $(H_2)$  with respect to  $(\mathcal{B}, L + 1, L + 1, 4, \Sigma^2)$  (with  $\Sigma^2$  defined in (7)) for every  $L$ . More precisely, we have the following.

**Proposition 4.2 ([26,34]).** *There exist  $b \in (0, \pi)$  and three  $C^\infty$  functions  $t \mapsto \lambda_t$ ,  $t \mapsto \Pi_t$  and  $t \mapsto N_t$  defined on  $[-b, b]^2$  and with values in  $\mathbb{C}$ ,  $\mathcal{L}(\mathcal{B}, \mathcal{B})$  and  $\mathcal{L}(\mathcal{B}, \mathcal{B})$  respectively such that*

- (i) *for every  $t \in [-b, b]^2$ ,  $\hat{P}_t^n = \lambda_t^n \Pi_t + R_t^n$  and  $\Pi_0 = \mathbb{E}_{\hat{\mu}}[\cdot]$ ,  $\Pi_t \hat{P}_t = \hat{P}_t \Pi_t = \lambda_t \Pi_t$ ,  $\Pi_t^2 = \Pi_t$ ;*
- (ii) *there exists  $\vartheta \in (0, 1)$  such that, for every positive integer  $m$ ,*

$$\sup_{t \in [-b, b]^2} \|(R_t^n)^{(m)}\|_{\mathcal{L}(\mathcal{B}, \mathcal{B})} = O(\vartheta^n) \quad \text{and} \quad \sup_{t \in [-\pi, \pi]^2 \setminus [-b, b]^2} \|\hat{P}_t^n\|_{\mathcal{L}(\mathcal{B}, \mathcal{B})} = O(\vartheta^n);$$

- (iii) *we have  $\lambda_t = 1 - \frac{1}{2} \Sigma^2 * t^{\otimes 2} = O(|t|^3)$ , with  $\Sigma^2$  given by (7);*

- (iv) *there exists  $\sigma > 0$  such that, for any  $t \in [-b, b]^2$ ,  $|\lambda_t| \leq e^{-\sigma|t|^2}$  and  $e^{-\frac{1}{2} \Sigma^2 * t^{\otimes 2}} \leq e^{-\sigma|t|^2}$ .*



**Lemma 4.3.** *The sequence  $(\kappa \circ \bar{T}^k)_k$  has same distribution as  $(-\kappa \circ \bar{T}^{-k})_k$ .*

*The function  $t \mapsto \lambda_t$  is even in a neighbourhood of 0. In particular  $\lambda_t = 1 - \frac{1}{2}\Sigma^2 * t^{\otimes 2} + O(|t|^4)$ .*

**Proof.** Let  $\Psi : \bar{M} \rightarrow \bar{M}$  be the map which sends  $(q, \vec{v}) \in \bar{M}$  to  $(q, \vec{v}')$  such that the following angular equality holds true:  $(\widehat{\vec{n}}(q), \vec{v}') = -(\widehat{\vec{n}}(q), \vec{v})$ . Then  $\kappa \circ \bar{T}^k \circ \Psi = -\kappa \circ \bar{T}^{-k-1}$ . Hence,  $S_n$  has the same distribution (with respect to  $\vec{v}$ ) as  $-S_n$  and so, for every  $t$  small enough,

$$\mathbb{E}_{\vec{v}}[e^{-it \cdot S_n}] = \mathbb{E}_{\vec{v}}[e^{it \cdot S_n}] \sim \lambda_t^n \mathbb{E}_{\hat{\mu}}[\Pi_t \mathbf{1}] \sim \lambda_{-t}^n \mathbb{E}_{\hat{\mu}}[\Pi_{-t} \mathbf{1}]$$

as  $n$  goes to infinity, and so  $\lambda$  is even. □

Let  $\Phi$  be the density function of  $B$ , which is given by  $\Phi(x) = \frac{e^{-\frac{(\Sigma^2)^{-1} * x^{\otimes 2}}{2}}}{2\pi \sqrt{\det \Sigma^2}}$ . We set  $a_t := e^{-\frac{1}{2}\Sigma^2 * t^{\otimes 2}}$ . Note that the uneven derivatives of  $\lambda/a$  at 0 are null as well as its three first derivatives. This gives the following result ensuring Theorem 1.2.

**Proposition 4.4.** *The conclusions of Theorem 3.7 hold true provided  $(f, g)$  satisfies the hypotheses stated therein with  $\mathcal{V}$  the set of dynamically Lipschitz continuous functions, with  $J = 4$ , for every  $K, L$  such that  $K \geq 4$  and  $L = 2K$ .*

**Proof.** We apply Theorem 3.7 to the dynamical system  $(\Delta, \nu, \tau)$  with the Banach space  $\tilde{\mathcal{V}}$  of functions of the form  $f \circ q$ , with  $f : \bar{M} \rightarrow \mathbb{C}$  dynamically Lipschitz continuous. The fact that this dynamical system satisfies Condition  $(H_2 \text{ bis})$  with respect to  $(\tilde{\mathcal{V}}, 2K + 1, 4, \Sigma^2)$  for every integer  $K$  such that  $K \geq 4$  comes from the facts contained in Section 4.1 (in particular from (45) and Lemma 4.1) and from Proposition 4.2 and from Lemma 4.3. Theorem 3.7 applies. □

In particular, the following quantity is well defined for every couple  $(u, v)$  of observables in  $\mathcal{V}$ :

$$A_m(u, v) := \lim_{n \rightarrow +\infty} \frac{\partial^m}{\partial t^m} (\mathbb{E}_{\vec{v}}[u \cdot e^{it \cdot S_n} \cdot v \circ \bar{T}^n] \lambda_t^{-n})|_{t=0}, \quad (46)$$

with  $\lambda_t$  being defined in Proposition 4.2.

**Proof of Theorem 1.3.** Due to Proposition 4.4 and to (41), we obtain (11) with

$$\tilde{\mathfrak{A}}_2(f, g) = \mathfrak{a}_{2,0,0}(f, g) + \mathfrak{a}_{0,2,0}(f, g) + \mathfrak{a}_{1,1,0}(f, g),$$

where  $a_{m,r,j}(f, g)$  corresponds to the contribution of the  $(m, r, j)$ -term in the sum of the right hand side of (41). Moreover, using the notations  $f_\ell(q, \vec{v}) := f((q, \vec{v}), \ell)$  and  $g_\ell(q, \vec{v}) := g((q, \vec{v}), \ell)$ , due to Proposition A.3,

$$\begin{aligned} \mathfrak{a}_{2,0,0}(f, g) &= \sum_{\ell, \ell' \in \mathbb{Z}^2} A_2(f_\ell, g_{\ell'}) \\ &= - \lim_{n \rightarrow +\infty} \left\{ \int_M f \, d\nu \sum_{j,m=-n}^{-1} \int_M g \cdot (\kappa \circ T^j \otimes \kappa \circ T^m - \mathbb{E}_{\vec{v}}[\kappa \circ \bar{T}^j \otimes \kappa \circ \bar{T}^m]) \, d\nu \right. \\ &\quad + \int_M g \, d\nu \sum_{j,m=0}^{n-1} \int_M f \cdot [\kappa \circ T^j \otimes \kappa \circ T^m - \mathbb{E}_{\vec{v}}[\kappa \circ \bar{T}^j \otimes \kappa \circ \bar{T}^m]] \, d\nu \\ &\quad \left. + 2 \sum_{r=0}^{n-1} \int_M f \cdot \kappa \circ T^r \, d\nu \otimes \sum_{m=-n}^{-1} \int_M g \cdot \kappa \circ T^m \, d\nu + \int_M f \, d\nu \int_M g \, d\nu (\mathbb{E}_{\vec{v}}[S_n^{\otimes 2}] - n\Sigma^2) \right\}, \\ \mathfrak{a}_{0,2,0}(f, g) &= - \sum_{\ell, \ell' \in \mathbb{Z}^2} A_0(f_\ell, g_{\ell'}) \cdot (\ell' - \ell)^{\otimes 2} = - \sum_{\ell, \ell' \in \mathbb{Z}^2} (\ell' - \ell)^{\otimes 2} \int_{\mathcal{C}_\ell} f \, d\nu \int_{\mathcal{C}_{\ell'}} g \, d\nu, \end{aligned}$$

$$\begin{aligned} \alpha_{1,1,0}(f, g) &= -2i \sum_{\ell, \ell' \in \mathbb{Z}^2} A_1(f_\ell, g_{\ell'}) \otimes (\ell' - \ell) \\ &= 2 \lim_{n \rightarrow +\infty} \left\{ \sum_{\ell, \ell' \in \mathbb{Z}^2} \int_{\mathcal{C}_{\ell'}} g \, d\nu \sum_{r=0}^{n-1} \int_{\mathcal{C}_\ell} f \cdot ((\ell' - \ell) \otimes \kappa \circ T^r) \, d\nu \right. \\ &\quad \left. + \sum_{\ell, \ell' \in \mathbb{Z}^2} \int_{\mathcal{C}_\ell} f \, d\nu \sum_{m=-n}^{-1} \int_{\mathcal{C}_{\ell'}} g \cdot ((\ell' - \ell) \otimes \kappa \circ T^m) \, d\nu \right\}. \end{aligned}$$

For the contribution of the term with  $(m, r, j) = (0, 0, 4)$ , note that

$$(\lambda^n/a^n)_0^{(4)} = n(\lambda/a)_0^{(4)} = n(\lambda_0^{(4)} - 3(\Sigma^2)^{\otimes 2})$$

and apply Proposition A.4. Note that

$$\begin{aligned} \alpha_{2,0,0}(f, g) &= - \lim_{n \rightarrow +\infty} \left\{ \int_M f \, d\nu \int_M g ((\mathcal{I}_0 - \mathcal{I}_{-n})^{\otimes 2} - \mathbb{E}_{\vec{v}}[S_n^{\otimes 2}]) \, d\nu \right. \\ &\quad \left. + \int_M g \, d\nu \int_M f ((\mathcal{I}_n - \mathcal{I}_0)^{\otimes 2} - \mathbb{E}_{\vec{v}}[S_n^{\otimes 2}]) \, d\nu \right. \\ &\quad \left. + 2 \int_M f (\mathcal{I}_n - \mathcal{I}_0) \, d\nu \otimes \int_M g (\mathcal{I}_0 - \mathcal{I}_{-n}) \, d\nu - \int_M f \, d\nu \int_M g \, d\nu \mathfrak{B}_0 \right\}, \end{aligned}$$

$$\alpha_{0,2,0}(f, g) = - \int_M f \mathcal{I}_0^{\otimes 2} \, d\nu \int_M g \, d\nu - \int_M f \, d\nu \int_M g \mathcal{I}_0^{\otimes 2} \, d\nu + 2 \int_M f \mathcal{I}_0 \, d\nu \otimes \int_M g \mathcal{I}_0 \, d\nu$$

and

$$\begin{aligned} \alpha_{1,1,0}(f, g) &= \lim_{n \rightarrow +\infty} \left\{ 2 \int_M g \mathcal{I}_0 \, d\nu \otimes \int_M f (\mathcal{I}_n - \mathcal{I}_0) \, d\nu - 2 \int_M g \, d\nu \int_M f \mathcal{I}_0 \otimes (\mathcal{I}_n - \mathcal{I}_0) \, d\nu \right. \\ &\quad \left. + 2 \int_M f \, d\nu \int_M g \mathcal{I}_0 \otimes (\mathcal{I}_0 - \mathcal{I}_{-n}) \, d\nu - 2 \int_M f \mathcal{I}_0 \, d\nu \otimes \int_M g (\mathcal{I}_0 - \mathcal{I}_{-n}) \, d\nu \right\}. \end{aligned}$$

This ends the proof of the theorem. □

**Remark 4.5.** Let  $f, g : M \rightarrow \mathbb{R}$  be two bounded observables such that

$$\sum_{\ell \in \mathbb{Z}^2} |\ell|^4 (\|f \mathbf{1}_{\mathcal{C}_\ell}\|_{(\xi)} + \|g \mathbf{1}_{\mathcal{C}_\ell}\|_{(\xi)}) < \infty. \tag{47}$$

Assume moreover that  $\int_M f \, d\nu \int_M g \, d\nu = 0$  and that  $\tilde{\mathfrak{A}}_2(f, g) = 0$ . Due to Proposition 4.4 and (41),

$$\begin{aligned} &\int_M f \cdot g \circ T^n \, d\nu \\ &= \frac{3(\Sigma^{-2})^{\otimes 2}}{2\pi \sqrt{\det \Sigma^2 n^3}} * \sum_{\ell, \ell' \in \mathbb{Z}^2} \left( \frac{A_4(f_\ell, g_{\ell'})}{24} + \frac{A_0(f_\ell, g_{\ell'})}{24} (\ell' - \ell)^{\otimes 4} + \frac{i A_1(f_\ell, g_{\ell'})}{6} \otimes (\ell' - \ell)^{\otimes 3} \right. \\ &\quad \left. - \frac{1}{4} A_2(f_\ell, g_{\ell'}) \otimes (\ell' - \ell)^{\otimes 2} - \frac{i}{6} A_3(f_\ell, g_{\ell'}) \otimes (\ell' - \ell) \right) + o(n^{-3}), \end{aligned}$$

where  $f_\ell(q, \vec{v}) := f(q + \ell, \vec{v})$  and  $g_{\ell'}(q, \vec{v}) := g(q + \ell', \vec{v})$ , with  $A_m(u, v)$  defined by (46).

**Proof of Proposition 1.6.** We apply Remark 4.5. Due to Proposition A.3, for every  $\ell, \ell' \in \mathbb{Z}^2$ ,  $A_0(f_\ell, g_{\ell'}) = A_1(f_\ell, g_{\ell'}) = 0$  (since  $\mathbb{E}_{\bar{v}}[f_\ell] = \mathbb{E}_{\bar{v}}[g_{\ell'}] = 0$ ) and

$$\sum_{\ell, \ell' \in \mathbb{Z}^2} A_4(f_\ell, g_{\ell'}) = \sum_{\ell, \ell' \in \mathbb{Z}^2} h_\ell \cdot q_{\ell'} A_4(\tilde{f}_0, \tilde{g}_0) = 0.$$

Moreover

$$\sum_{\ell, \ell' \in \mathbb{Z}^2} A_3(f_\ell, g_{\ell'}) \otimes (\ell' - \ell) = \sum_{\ell, \ell' \in \mathbb{Z}^2} h_\ell q_{\ell'} A_3(\tilde{f}_0, \tilde{g}_0) \otimes (\ell' - \ell) = 0$$

since  $\sum_{\ell \in \mathbb{Z}^2} h_\ell = \sum_{\ell \in \mathbb{Z}^2} q_\ell = 0$ . Therefore

$$\begin{aligned} \int_M f \cdot g \circ T^n dv &= -\frac{3}{4} \frac{(\Sigma^{-2})^{\otimes 2}}{2\pi \sqrt{\det \Sigma^2 n^3}} * \sum_{\ell, \ell' \in \mathbb{Z}^2} A_2(f_\ell, g_{\ell'}) \otimes (\ell' - \ell)^{\otimes 2} + o(n^{-3}) \\ &= \frac{3}{2} \frac{(\Sigma^{-2})^{\otimes 2}}{2\pi \sqrt{\det \Sigma^2 n^3}} * \sum_{\ell, \ell' \in \mathbb{Z}^2} h_\ell q_{\ell'} A_2(\tilde{f}_0, \tilde{g}_0) \otimes \ell \otimes \ell' + o(n^{-3}) \\ &= \frac{3}{2} \frac{(\Sigma^{-2})^{\otimes 2}}{2\pi \sqrt{\det \Sigma^2 n^3}} A_2(\tilde{f}_0, \tilde{g}_0) \otimes \sum_{\ell \in \mathbb{Z}^2} h_\ell \cdot \ell \otimes \sum_{\ell' \in \mathbb{Z}^2} q_{\ell'} \cdot \ell' + o(n^{-3}) \\ &= -\frac{3(\Sigma^{-2})^{\otimes 2}}{2\pi \sqrt{\det \Sigma^2 n^3}} * \left( \sum_{j \geq 0} \mathbb{E}_{\bar{v}}[f_{0, \kappa} \circ \bar{T}^j] \otimes \sum_{m \leq -1} \mathbb{E}_{\bar{v}}[\tilde{g}_{0, \kappa} \circ \bar{T}^m] \otimes \sum_{\ell \in \mathbb{Z}^2} h_\ell \cdot \ell \otimes \sum_{\ell' \in \mathbb{Z}^2} q_{\ell'} \cdot \ell' \right) \\ &\quad + o(n^{-3}). \end{aligned} \quad \square$$

## Appendix A: Billiard with finite horizon: About the coefficients $A_m$

Let us recall some facts on the Sinai billiard with finite horizon. Let  $k_0 \in \mathbb{N}$  be some large enough fixed integer. Let us write  $(\cdot, \cdot)$  for the angular measure in  $(-\pi, \pi]$  between two vectors.

**Definition A.1.** A stable (resp. unstable)  $H$ -manifold (or homogeneous manifold) is a  $C^1$  connected curve which contains no point of  $\bigcup_{k \geq 0} \bar{T}^{-k}(S_0 \cup \mathcal{H})$  (resp.  $\bigcup_{k \geq 0} \bar{T}^k(S_0 \cup \mathcal{H})$ ), with

$$\mathcal{H} := \left\{ x = (q, \vec{v}) \in \bar{M} : \exists k \in \mathbb{N}, |k| \geq k_0, |\widehat{(\vec{n}(q), \vec{v})}| = \frac{\pi}{2} - \frac{1}{k^2} \right\}.$$

Let  $\mathcal{W}^s$  (resp.  $\mathcal{W}^u$ ) be the set of stable (resp. unstable)  $H$ -manifolds. In [10], Chernov defines two separation times  $s_+$  and  $s_-$  which are dominated by  $s$  and are such that, for every positive integer  $k$ ,

$$\forall W^u \in \mathcal{W}^u, \forall x, y \in W^u, \quad s_+(\bar{T}^{-k}x, \bar{T}^{-k}y) = s_+(x, y) + k,$$

$$\forall W^s \in \mathcal{W}^s, \forall x, y \in W^s, \quad s_-(\bar{T}^kx, \bar{T}^ky) = s_-(x, y) + k.$$

More precisely

$$\forall W^u \in \mathcal{W}^u, \forall x, y \in W^u, \quad s_+(x, y) := \inf\{n \geq 0 : \bar{T}^n((W^u)_{x,y}) \text{ is an unstable } H\text{-manifold}\},$$

$$\forall W^s \in \mathcal{W}^s, \forall x, y \in W^s, \quad s_-(x, y) := \inf\{n \geq 0 : \bar{T}^{-n}((W^s)_{x,y}) \text{ is a stable } H\text{-manifold}\},$$

where  $(W^u)_{x,y}$  (resp.  $(W^s)_{x,y}$ ) is the connected part of  $W^u$  (resp.  $W^s$ ) with extremities  $x$  and  $y$ .

**Proposition A.2 ([10], Theorem 4.3 and remark thereafter).** *There exist  $C_0 > 0$  and  $\vartheta_0 \in (0, 1)$  such that, for every positive integer  $n$ , for every bounded measurable functions  $u, v : \bar{M} \rightarrow \mathbb{R}$ ,*

$$|\mathbb{E}_{\bar{v}}[u \cdot v \circ \bar{T}^n] - \mathbb{E}_{\bar{v}}[u]\mathbb{E}_{\bar{v}}[v]| \leq C_0(L_u^+ \|v\|_\infty + L_v^- \|u\|_\infty + \|u\|_\infty \|v\|_\infty) \vartheta_0^n,$$

with

$$L_u^+ := \sup_{W^u \in \mathcal{W}^u} \sup_{x, y \in W^u, x \neq y} (|u(x) - u(y)| \xi^{-s_+(x, y)}),$$

and

$$L_v^- := \sup_{W^s \in \mathcal{W}^s} \sup_{x, y \in W^s, x \neq y} (|v(x) - v(y)| \xi^{-s_-(x, y)}).$$

Note that

$$L_u^+ \leq L_\xi(u \mathbf{1}_{\bar{M}}), \quad L_u^- \leq L_\xi(u \mathbf{1}_{\bar{M}}), \quad L_{u \circ \bar{T}^{-k}}^+ \leq L_u^+ \xi^k \quad \text{and} \quad L_{v \circ \bar{T}^k}^- \leq L_v^- \xi^k.$$

We will set  $\tilde{u} := u - \mathbb{E}_{\bar{v}}[u]$  and  $\tilde{v} := v - \mathbb{E}_{\bar{v}}[v]$ . We will express the terms  $A_m(u, v)$  for  $m \in \{1, 2, 3, 4\}$  in terms of the following quantities:

$$\begin{aligned} B_1^+(u) &:= \sum_{j \geq 0} \mathbb{E}_{\bar{v}}[u \cdot \kappa \circ \bar{T}^j], & B_1^-(v) &:= \sum_{m \leq -1} \mathbb{E}_{\bar{v}}[v \cdot \kappa \circ \bar{T}^m], \\ B_2^+(u) &:= \sum_{j, m \geq 0} \mathbb{E}_{\bar{v}}[\tilde{u} \cdot \kappa \circ \bar{T}^j \otimes \kappa \circ \bar{T}^m], & B_2^-(v) &:= \sum_{j, m \leq -1} \mathbb{E}_{\bar{v}}[\tilde{v} \cdot \kappa \circ \bar{T}^j \otimes \kappa \circ \bar{T}^m], \\ B_0^+(u) &= \sum_{k \geq 0} (k+1) \mathbb{E}_{\bar{v}}[u \cdot \kappa \circ \bar{T}^k], & B_0^-(v) &:= \sum_{k \leq -1} |k| \mathbb{E}_{\bar{v}}[v \cdot \kappa \circ \bar{T}^k], \\ B_0 &:= \sum_{m \in \mathbb{Z}} |m| \mathbb{E}_{\bar{v}}[\kappa \otimes \kappa \circ \bar{T}^m] = \lim_{n \rightarrow +\infty} (n \Sigma^2 - \mathbb{E}_{\bar{v}}[S_n^{\otimes 2}]), \\ B_3^+(u) &:= \sum_{k \geq 0} \mathbb{E}_{\bar{v}}[u \cdot \kappa^{\otimes 3} \circ \bar{T}^k] + 3 \sum_{r > k \geq 0} \mathbb{E}_{\bar{v}}[\tilde{u} \cdot \kappa^{\otimes 2} \circ \bar{T}^k \otimes \kappa \circ \bar{T}^r] \\ &\quad + 3 \sum_{0 \leq k < m, r} \mathbb{E}_{\bar{v}}[(\tilde{u} \cdot \kappa \circ \bar{T}^k - \mathbb{E}_{\bar{v}}[\tilde{u} \cdot \kappa \circ \bar{T}^k]) \otimes \kappa \circ \bar{T}^m \otimes \kappa \circ \bar{T}^r], \\ B_3^-(v) &:= \sum_{k \geq 1} \mathbb{E}_{\bar{v}}[v \cdot \kappa^{\otimes 3} \circ \bar{T}^{-k}] + 3 \sum_{r > k \geq 1} \mathbb{E}_{\bar{v}}[\tilde{v} \cdot \kappa^{\otimes 2} \circ \bar{T}^{-k} \otimes \kappa \circ \bar{T}^{-r}] \\ &\quad + 3 \sum_{1 \leq k < m, r} \mathbb{E}_{\bar{v}}[(\tilde{v} \cdot \kappa \circ \bar{T}^{-k} - \mathbb{E}_{\bar{v}}[\tilde{v} \cdot \kappa \circ \bar{T}^{-k}]) \otimes \kappa \circ \bar{T}^{-m} \otimes \kappa \circ \bar{T}^{-r}]. \end{aligned}$$

**Proposition A.3.** *Let  $u, v : \bar{M} \rightarrow \mathbb{C}$  be two dynamically Lipschitz continuous functions, with respect to  $d_\xi$  with  $\xi \in (0, 1)$ . Then,*

$$A_0(u, v) = \mathbb{E}_{\bar{v}}[u] \cdot \mathbb{E}_{\bar{v}}[v], \tag{49}$$

$$A_1(u, v) = i \lim_{n \rightarrow +\infty} \mathbb{E}_{\bar{v}}[u \cdot S_n \cdot v \circ \bar{T}^n] = i B_1^+(u) \mathbb{E}_{\bar{v}}[v] + i B_1^-(v) \mathbb{E}_{\bar{v}}[u], \tag{50}$$

$$A_2(u, v) = \lim_{n \rightarrow +\infty} (n \mathbb{E}_{\bar{v}}[u] \mathbb{E}_{\bar{v}}[v] \Sigma^2 - \mathbb{E}_{\bar{v}}[u \cdot S_n^{\otimes 2} \cdot v \circ \bar{T}^n]) \tag{51}$$

$$= \mathbb{E}_{\bar{v}}[u] \mathbb{E}_{\bar{v}}[v] B_0 - 2 B_1^+(u) \otimes B_1^-(v) - \mathbb{E}_{\bar{v}}[v] B_2^+(u) - \mathbb{E}_{\bar{v}}[u] B_2^-(v), \tag{52}$$

$$A_3(u, v) = \lim_{n \rightarrow +\infty} (3in\Sigma^2 \otimes \mathbb{E}_{\bar{v}}[u.S_n.v \circ \bar{T}^n] - i\mathbb{E}_{\bar{v}}[u.S_n^{\otimes 3}.v \circ \bar{T}^n]) \quad (53)$$

$$\begin{aligned} &= 3A_1(u, v) \otimes B_0 + 3i\Sigma^2 \otimes (\mathbb{E}_{\bar{v}}[u]B_0^-(v) + \mathbb{E}_{\bar{v}}[v]B_0^+(u)) - i(\mathbb{E}_{\bar{v}}[v]B_3^+(u) + \mathbb{E}_{\bar{v}}[u]B_3^-(v)) \\ &\quad - 3iB_2^-(v) \otimes B_1^+(u) - 3iB_2^+(u) \otimes B_1^-(v). \end{aligned} \quad (54)$$

**Proof.** As in the proof of Theorem 3.7, we set  $A_{m,n}(u, v) := (\mathbb{E}_{\bar{v}}[u.e^{it.S_n}.v \circ \bar{T}^n]/\lambda_t^n)_{|t=0}^{(m)}$ . We will use Proposition A.2 and the fact that  $\lambda_t = 1 - \frac{1}{2}\Sigma^2 * t^{\otimes 2} + \frac{1}{4!}\lambda_0^{(4)} * t^{\otimes 4} + o(|t|^4)$  to compute  $A_m(u, v) = \lim_{n \rightarrow +\infty} A_{m,n}(u, v)$ .

- First we observe that  $A_{0,n}(u, v) = \mathbb{E}_{\bar{v}}[u.v \circ \bar{T}^n]$  and we apply Proposition A.2.
- Second,

$$\begin{aligned} A_{1,n}(u, v) &= i\mathbb{E}_{\bar{v}}[u.S_n.v \circ \bar{T}^n] = i \sum_{k=0}^{n-1} \mathbb{E}_{\bar{v}}[u.\kappa \circ \bar{T}^k.v \circ \bar{T}^n] \\ &= i \sum_{k=0}^{\lfloor n/2 \rfloor} \mathbb{E}_{\bar{v}}[u.\kappa \circ \bar{T}^k]\mathbb{E}_{\bar{v}}[v] + i \sum_{k=\lfloor n/2 \rfloor+1}^{n-1} \mathbb{E}_{\bar{v}}[u]\mathbb{E}_{\bar{v}}[v.\kappa \circ \bar{T}^{-(n-k)}] + O(n\vartheta_0^{n/2}\|u\|_{(\xi)}\|u\|_{(\xi)}) \\ &= i\mathbb{E}_{\bar{v}}[v] \sum_{k \geq 0} \mathbb{E}_{\bar{v}}[u.\kappa \circ \bar{T}^k] + i\mathbb{E}_{\bar{v}}[u] \sum_{m \leq -1} \mathbb{E}_{\bar{v}}[v.\kappa \circ \bar{T}^m] + O(n\vartheta_0^{n/2}\|u\|_{(\xi)}\|u\|_{(\xi)}), \end{aligned} \quad (55)$$

where we used several times Proposition A.2, combined with the fact that  $\mathbb{E}_{\bar{v}}[\kappa] = 0$ .

- Third,

$$\begin{aligned} A_{2,n}(u, v) &= -\mathbb{E}_{\bar{v}}[u.S_n^{\otimes 2}.v \circ \bar{T}^n] + n\Sigma^2\mathbb{E}_{\bar{v}}[u.v \circ \bar{T}^n] \quad (56) \\ &= - \sum_{k,m=0}^{n-1} \mathbb{E}_{\bar{v}}[u.(\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m).v \circ \bar{T}^n] + n\Sigma^2\mathbb{E}_{\bar{v}}[u]\mathbb{E}_{\bar{v}}[v] + O(n\vartheta_0^n\|u\|_{(\xi)}\|u\|_{(\xi)}) \\ &= - \sum_{k,m=0}^{n-1} \mathbb{E}_{\bar{v}}[\tilde{u}.\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m.\tilde{v} \circ \bar{T}^n] \\ &\quad - \sum_{k,m=0}^{n-1} (\mathbb{E}_{\bar{v}}[u]\mathbb{E}_{\bar{v}}[\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m.\tilde{v} \circ \bar{T}^n] + \mathbb{E}_{\bar{v}}[\tilde{u}.\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m]\mathbb{E}_{\bar{v}}[v]) \\ &\quad + \left( n\Sigma^2 - \sum_{k,m=0}^{n-1} \mathbb{E}_{\bar{v}}[\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m] \right) \mathbb{E}_{\bar{v}}[u]\mathbb{E}_{\bar{v}}[v] + O(n\vartheta_0^n\|u\|_{(\xi)}\|u\|_{(\xi)}). \end{aligned} \quad (57)$$

– On the first hand

$$\begin{aligned} n\Sigma^2 - \sum_{k,m=0}^{n-1} \mathbb{E}_{\bar{v}}[\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m] &= n \sum_{k \in \mathbb{Z}} \mathbb{E}_{\bar{v}}[\kappa \otimes \kappa \circ \bar{T}^k] - \sum_{k=-n}^n (n - |k|)\mathbb{E}_{\bar{v}}[\kappa \otimes \kappa \circ \bar{T}^k] \\ &= \sum_{k \in \mathbb{Z}} \min(n, |k|)\mathbb{E}_{\bar{v}}[\kappa \otimes \kappa \circ \bar{T}^k], \end{aligned}$$

which converges to  $\sum_{k \in \mathbb{Z}} |k|\mathbb{E}_{\bar{v}}[\kappa \otimes \kappa \circ \bar{T}^k]$ .

– On the second hand, for  $0 \leq k \leq m \leq n$ , due to Proposition A.2 (treating separately the cases  $k \geq n/3$ ,  $m - k \geq n/3$  et  $n - m \geq n/3$ ),

$$\mathbb{E}_{\bar{v}}[\tilde{u}.\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m.\tilde{v} \circ \bar{T}^n] = \mathbb{E}_{\bar{v}}[\tilde{u}.\kappa \circ \bar{T}^k] \otimes \mathbb{E}_{\bar{v}}[\tilde{v}.\kappa \circ \bar{T}^{n-m}] + O(\|u\|_{(\xi)}\|v\|_{(\xi)}\vartheta_0^{n/3}). \quad (58)$$

Analogously, for  $0 \leq k \leq m \leq n$ ,

$$\mathbb{E}_{\tilde{v}}[\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m \tilde{v} \circ \bar{T}^n] = O(\|v\|_{(\xi)} \vartheta_0^{(n-k)/2}), \quad (59)$$

$$\mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m] = O(\|u\|_{(\xi)} \vartheta_0^{m/2}). \quad (60)$$

Hence  $\sum_{k,m=0}^{n-1} \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m] = B_2^+(u) + O(\vartheta_0^{n/3} \|u\|_{(\xi)})$ ,

$$\sum_{k,m=0}^{n-1} \mathbb{E}_{\tilde{v}}[\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m \tilde{v} \circ \bar{T}^n] = B_2^-(v) + O(\vartheta_0^{n/3} \|v\|_{(\xi)}), \quad (61)$$

and

$$\begin{aligned} & \sum_{k,m=0}^{n-1} \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m \cdot \tilde{v} \circ \bar{T}^n] \\ &= \left( \sum_{k=0}^{n-1} \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa^{\otimes 2} \circ \bar{T}^k \cdot \tilde{v} \circ \bar{T}^n] + 2 \sum_{0 \leq k < m < n} \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m \cdot \tilde{v} \circ \bar{T}^n] \right) \\ &= 2 \sum_{0 \leq k < m < n} \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot (\kappa \circ \bar{T}^k)] \otimes \mathbb{E}_{\tilde{v}}[\tilde{v} \cdot \kappa \circ \bar{T}^{n-m}] + O(\vartheta_0^{n/3} \|u\|_{(\xi)} \|v\|_{(\xi)}) \\ &= 2B_1^+(u) \otimes B_1^-(v) + O(\vartheta_0^{n/3} \|u\|_{(\xi)} \|v\|_{(\xi)}), \end{aligned}$$

where we used the fact that  $\mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa^{\otimes 2} \circ \bar{T}^k \cdot \tilde{v} \circ \bar{T}^n] = O(\|u\|_{(\xi)} \|v\|_{(\xi)} \vartheta_0^{n/2})$ .

Therefore we have proved (52).

- Let us prove (54). By bilinearity, we have

$$A_{3,n}(u, v) = A_{3,n}(\tilde{u}, \tilde{v}) + \mathbb{E}_{\tilde{v}}[u] A_{3,n}(\mathbf{1}, \tilde{v}) + \mathbb{E}_{\tilde{v}}[v] A_{3,n}(\tilde{u}, \mathbf{1}) + \mathbb{E}_{\tilde{v}}[u] \mathbb{E}_{\tilde{v}}[v] A_{3,n}(\mathbf{1}, \mathbf{1}). \quad (62)$$

Note that  $A_{3,n}(\mathbf{1}, \mathbf{1}) = -i \mathbb{E}_{\tilde{v}}[S_n^{\otimes 3}] = 0$  since  $(S_n)_n$  has the same distribution as  $(-S_n)_n$  (due to Lemma 4.3). We will write  $\widetilde{F}$  for  $F - \mathbb{E}_{\tilde{v}}[F]$  when  $F$  is given by a long formula.

– We start with the study of  $A_{3,n}(\tilde{u}, \mathbf{1})$ .

$$\begin{aligned} A_{3,n}(\tilde{u}, \mathbf{1}) &= -i \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot S_n^{\otimes 3}] + 3in \Sigma^2 \otimes \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot S_n] \\ &= -i \sum_{k=0}^{n-1} \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa^{\otimes 3} \circ \bar{T}^k] - 3i \sum_{0 \leq k < r \leq n-1} \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa^{\otimes 2} \circ \bar{T}^k \otimes \kappa \circ \bar{T}^r] \\ &\quad - 3i \sum_{0 \leq k < m, r \leq n-1} \mathbb{E}_{\tilde{v}}[\underbrace{\tilde{u} \cdot \kappa \circ \bar{T}^k}_{\text{}} \otimes \underbrace{\kappa \circ \bar{T}^m \otimes \kappa \circ \bar{T}^r}_{\text{}}] + 3in \Sigma^2 \otimes \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot S_n] \\ &= -3i \sum_{k \geq 0} \sum_{m \in \mathbb{Z}} \max(0, n - |m| - 1 - k) \mathbb{E}_{\tilde{v}}[\tilde{u} \cdot \kappa \circ \bar{T}^k] \otimes \mathbb{E}_{\tilde{v}}[\kappa \otimes \kappa \circ \bar{T}^m], \end{aligned}$$

$$\begin{aligned}
A_{3,n}(\tilde{u}, \mathbf{1}) &= -i \sum_{k=0}^{n-1} \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa^{\otimes 3} \circ \bar{T}^k] - 3i \sum_{0 \leq k < r} \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa^{\otimes 2} \circ \bar{T}^k \otimes \kappa \circ \bar{T}^r] \\
&\quad - 3i \sum_{0 \leq k < m, r} \mathbb{E}_{\tilde{v}}[\overbrace{\tilde{u}.\kappa \circ \bar{T}^k} \otimes \kappa \circ \bar{T}^m \otimes \kappa \circ \bar{T}^r] \\
&\quad + 3i \sum_{k \geq 0} \sum_{m \in \mathbb{Z}} (|m| + 1 + k) \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa \circ \bar{T}^k] \otimes \mathbb{E}_{\tilde{v}}[\kappa \otimes \kappa \circ \bar{T}^m] \\
&\quad + O(\|u\|_{(\xi)} \vartheta^{n/4}), \\
A_{3,n}(\tilde{u}, \mathbf{1}) &= -i \sum_{k \geq 0} \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa^{\otimes 3} \circ \bar{T}^k] - 3i \sum_{0 \leq k < r} \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa^{\otimes 2} \circ \bar{T}^k \otimes \kappa \circ \bar{T}^r] \\
&\quad + 3i \Sigma^2 \otimes \sum_{k \geq 0} (k+1) \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa \circ \bar{T}^k] + 3i B_1^+(\tilde{u}) \otimes B_0 \\
&\quad - 3i \sum_{0 \leq k < m, r} \mathbb{E}_{\tilde{v}}[\overbrace{\tilde{u}.\kappa \circ \bar{T}^k} \otimes \kappa \circ \bar{T}^m \otimes \kappa \circ \bar{T}^r] + O(\|u\|_{(\xi)} \vartheta^{n/4}). \tag{63}
\end{aligned}$$

So

$$A_3(\tilde{u}, \mathbf{1}) = -i B_3^+(u) + 3i \Sigma^2 \otimes B_0^+(u) + 3i B_1^+(u) \otimes B_0. \tag{64}$$

– Analogously

$$A_3(\mathbf{1}, \tilde{v}) = -i B_3^-(v) + 3i \Sigma^2 \otimes B_0^-(v) + 3i B_1^-(v) \otimes B_0. \tag{65}$$

– Finally

$$\begin{aligned}
A_{3,n}(\tilde{u}, \tilde{v}) &= -i \mathbb{E}_{\tilde{v}}[\tilde{u}.S_n^{\otimes 3}.\tilde{v} \circ \bar{T}^n] + 3in \Sigma^2 \otimes \mathbb{E}_{\tilde{v}}[\tilde{u}.S_n.\tilde{v} \circ \bar{T}^n] \\
&= -i \sum_{k,m,r=0}^{n-1} \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m \otimes \kappa \circ \bar{T}^r.\tilde{v} \circ \bar{T}^n] + 3n \Sigma^2 \otimes A_{1,n}(\tilde{u}, \tilde{v}) \\
&= -i \sum_{k,m,r=0}^{n-1} \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m \otimes \kappa \circ \bar{T}^r.\tilde{v} \circ \bar{T}^n] + O(n^2 \vartheta_0^{n/2} \|u\|_{(\xi)} \|v\|_{(\xi)}).
\end{aligned}$$

Assume  $0 \leq k \leq m \leq r \leq n-1$ . Considering separately the cases  $k \geq n/4$ ,  $m-k \geq n/4$ ,  $r-m \geq n/4$  and  $n-r \geq n/4$ , we observe that

$$\begin{aligned}
&\mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m \otimes \kappa \circ \bar{T}^r.\tilde{v} \circ \bar{T}^n] \\
&= \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa \circ \bar{T}^k] \otimes \mathbb{E}_{\tilde{v}}[\tilde{v}.\kappa \circ \bar{T}^{-(n-r)} \otimes \kappa \circ \bar{T}^{-(n-m)}] \\
&\quad + \mathbb{E}_{\tilde{v}}[\tilde{v}.\kappa \circ \bar{T}^{-(n-r)}] \otimes \mathbb{E}_{\tilde{v}}[\tilde{u}.\kappa \circ \bar{T}^k \otimes \kappa \circ \bar{T}^m] + O(\vartheta_0^{n/4} \|v\|_{(\xi)} \|u\|_{(\xi)}). \tag{66}
\end{aligned}$$

And so

$$A_{3,n}(\tilde{u}, \tilde{v}) = -3i B_1^+(\tilde{u}) B_2^-(\tilde{v}) - 3i B_1^-(\tilde{v}) B_2^+(\tilde{u}). \tag{67}$$

This combined with (62), (64) and (65) leads to (54).  $\square$

**Proposition A.4.** *The fourth derivatives of  $\lambda$  at 0 are given by:*

$$\begin{aligned}
\lambda_0^{(4)} &= \mathbb{E}_{\bar{\nu}}[\kappa^{\otimes 4}] + 8 \sum_{n \geq 1} \mathbb{E}_{\bar{\nu}}[\kappa \otimes (\kappa^{\otimes 3} \circ \bar{T}^n)] \\
&\quad + 6 \sum_{n \geq 1} \mathbb{E}_{\bar{\nu}}[(\kappa^{\otimes 2} - \mathbb{E}_{\bar{\nu}}[\kappa^{\otimes 2}]) \otimes (\kappa^{\otimes 2} \circ \bar{T}^n)] \\
&\quad + 24 \sum_{1 \leq n < m} \mathbb{E}_{\bar{\nu}}[(\kappa^{\otimes 2} - \mathbb{E}_{\bar{\nu}}[\kappa^{\otimes 2}]) \otimes (\kappa \circ \bar{T}^n) \otimes (\kappa \circ \bar{T}^m)] \\
&\quad + 12 \sum_{1 \leq n < m} \mathbb{E}_{\bar{\nu}}[\kappa \otimes ((\kappa^{\otimes 2} - \mathbb{E}_{\bar{\nu}}[\kappa^{\otimes 2}]) \circ \bar{T}^n) \otimes \kappa \circ \bar{T}^m] \\
&\quad + 24 \sum_{1 \leq n < m < \ell} \mathbb{E}_{\bar{\nu}}[(\kappa \otimes (\kappa \circ \bar{T}^n) - \mathbb{E}_{\bar{\nu}}[\kappa \otimes (\kappa \circ \bar{T}^n)]) \otimes (\kappa \circ \bar{T}^m) \otimes (\kappa \circ \bar{T}^\ell)] \\
&\quad - 24 \sum_{n \geq 1} \mathbb{E}_{\bar{\mu}}[\kappa \otimes (\kappa \circ \bar{T}^n)] \sum_{m \geq 1} m \mathbb{E}_{\bar{\mu}}[\kappa \otimes (\kappa \circ \bar{T}^m)].
\end{aligned}$$

**Proof.** Let  $h_t \in \mathcal{B}$  be the eigenvector of  $\hat{P}_t$  associated to the eigenvalue  $\lambda_t$  such that  $\mathbb{E}_{\hat{\mu}}[h_t] = 1$ . Note that  $h_0 = \mathbf{1}_{\hat{\Delta}}$ . Recall that  $\lambda_0 = 1$ ,  $\lambda_0' = 0$ ,  $\lambda_0^{(2)} = -\Sigma^2$ ,  $\lambda_0^{(3)} = 0$ . Since  $\lambda_t = \mathbb{E}_{\hat{\mu}}[\hat{P}_t \cdot h_t]$ , it comes

$$\lambda_0^{(4)} = \sum_{k=0}^4 \frac{4!}{k!(4-k)!} \mathbb{E}_{\hat{\mu}}[\hat{P}_0^{(k)} h_0^{(4-k)}]. \quad (68)$$

Derivating four times the equalities:  $\hat{P}_t h_t = \lambda_t \cdot h_t$  and  $\mathbb{E}_{\hat{\mu}}[h_t] = 1$  and taking  $t = 0$  leads to

$$\begin{aligned}
\mathbb{E}_{\hat{\mu}}[h_0'] &= \mathbb{E}_{\hat{\mu}}[h_0^{(2)}] = \mathbb{E}_{\hat{\mu}}[h_0^{(3)}] = \mathbb{E}_{\hat{\mu}}[h_0^{(4)}] = 0, \\
(I - \hat{P})h_0' &= \hat{P}_0' \mathbf{1}_{\hat{\Delta}}, \quad (I - \hat{P})h_0^{(2)} = \hat{P}_0^{(2)} \mathbf{1}_{\hat{\Delta}} + 2P_0' h_0' + \Sigma^2, \\
(I - \hat{P})h_0^{(3)} &= \hat{P}_0^{(3)} \mathbf{1}_{\hat{\Delta}} + 3\hat{P}_0^{(2)} h_0' + 3\hat{P}_0' h_0^{(2)} + 3\Sigma^2 \otimes h_0'.
\end{aligned}$$

Therefore

$$\begin{aligned}
h_0' &= \sum_{n \geq 0} \hat{P}^n (\hat{P}_0' \mathbf{1}_{\hat{\Delta}}) = i \sum_{n \geq 1} \hat{P}^n \hat{\kappa}, \\
h_0^{(2)} &= \sum_{n \geq 0} \hat{P}^n (\hat{P}(-\kappa^{\otimes 2}) + 2\hat{P}(i\hat{\kappa} \otimes h_0') + \Sigma^2) \\
&= \sum_{n \geq 1} \hat{P}^n \left( (-\hat{\kappa}^{\otimes 2} + \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 2}]) - 2 \left( \sum_{m \geq 1} (\hat{\kappa} \otimes \hat{P}^m \hat{\kappa} - \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^m \hat{\kappa}]) \right) \right),
\end{aligned}$$

and, since  $\mathbb{E}_{\hat{\mu}}[\hat{P}(-i\hat{\kappa}^{\otimes 3}) + 3\hat{P}(-\hat{\kappa}^{\otimes 2} \otimes h_0') + 3\hat{P}(i\hat{\kappa} \otimes h_0^{(2)}) + 3\Sigma^2 \otimes h_0'] = 0$ ,

$$\begin{aligned}
h_0^{(3)} &= \sum_{n \geq 0} \hat{P}^n (\hat{P}(-i\hat{\kappa}^{\otimes 3}) + 3\hat{P}(-\hat{\kappa}^{\otimes 2} \otimes h_0') + 3\hat{P}(i\hat{\kappa} \otimes h_0^{(2)}) + 3\Sigma^2 \otimes h_0') \\
&= \sum_{n \geq 0} \hat{P}^n \left( \hat{P}(-i\hat{\kappa}^{\otimes 3}) + 3i\hat{P} \left( -(\hat{\kappa}^{\otimes 2} - \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 2}]) \otimes \sum_{m \geq 1} \hat{P}^m \hat{\kappa} \right) \right. \\
&\quad \left. + 3\hat{P}(i\hat{\kappa} \otimes h_0^{(2)}) + 6i \sum_{k \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^k \hat{\kappa}] \otimes \sum_{m \geq 1} \hat{P}^m \hat{\kappa} \right).
\end{aligned}$$



Combining this with (68) and with  $\mathbb{E}_{\hat{\mu}}[h_0^{(4)}] = 0$ , we obtain

$$\lambda_0^{(4)} = \sum_{k=1}^4 \frac{4!}{k!(4-k)!} \mathbb{E}_{\hat{\mu}}[(i\hat{\kappa})^{\otimes k} \otimes h_0^{(4-k)}].$$

We conclude by computing the four terms of this sum as follows:

$$\begin{aligned} \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 4} \cdot h_0] &= \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 4}], \\ 4\mathbb{E}_{\hat{\mu}}[-i\hat{\kappa}^{\otimes 3} \otimes h_0'] &= 4 \sum_{n \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 3} \otimes \hat{P}^n \hat{\kappa}] = 4 \sum_{n \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes (\hat{\kappa} \circ \hat{\tau}^n)^{\otimes 3}], \\ 6\mathbb{E}_{\hat{\mu}}[-\hat{\kappa}^{\otimes 2} \otimes h_0^{(2)}] &= 6 \sum_{n \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 2} \otimes \hat{P}^n (\hat{\kappa}^{\otimes 2} - \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 2}])] \\ &\quad + 12 \sum_{n \geq 1} \mathbb{E}_{\hat{\mu}} \left[ \hat{\kappa}^{\otimes 2} \otimes \hat{P}^n \left( \sum_{m \geq 1} (\hat{\kappa} \otimes \hat{P}^m \hat{\kappa} - \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^m \hat{\kappa}]) \right) \right], \\ 4\mathbb{E}_{\hat{\mu}}[i\hat{\kappa} \otimes h_0^{(3)}] &= 4 \sum_{n \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^n \hat{\kappa}^{\otimes 3}] \\ &\quad + 12 \sum_{n, m \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^n ((\hat{\kappa}^{\otimes 2} - \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 2}]) \otimes \hat{P}^m \hat{\kappa})] \\ &\quad + 12 \sum_{n, m \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^n (\hat{\kappa} \otimes \hat{P}^m (\hat{\kappa}^{\otimes 2} - \mathbb{E}_{\hat{\mu}}[\hat{\kappa}^{\otimes 2}]))] \\ &\quad + 24 \sum_{n, m, \ell \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^n (\hat{\kappa} \otimes \hat{P}^m (\hat{\kappa} \otimes \hat{P}^\ell \hat{\kappa} - \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^\ell \hat{\kappa}]))] \\ &\quad - 24 \sum_{n \geq 0, \ell \geq 1} \mathbb{E}_{\hat{\mu}} \left[ \hat{\kappa} \otimes \sum_{m \geq 1} \mathbb{E}_{\hat{\mu}}[\hat{\kappa} \otimes \hat{P}^m \hat{\kappa}] \otimes \hat{P}^{n+\ell} \hat{\kappa} \right], \end{aligned}$$

where we used several times the fact that  $\mathbb{E}_{\hat{\mu}}[\hat{\kappa}] = 0$ . We conclude by using the fact that  $(\kappa \circ \bar{T}^k)_k$  (wrt  $\bar{\nu}$ ) has the same distribution as  $(\hat{\kappa} \circ \hat{\tau}^k)_k$  (wrt  $\hat{\mu}$ ) and  $(-\kappa \circ \bar{T}^{-k})_k$  (wrt  $\bar{\nu}$ ) (see Lemma 4.3).  $\square$

**Remark A.5.** The method of the proof of Proposition A.4 can be implemented to compute  $\lambda_0^{(m)}$  for every  $m$ .

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