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Multi-arm incipient infinite clusters in 2D: Scaling limits and winding numbers

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Abstract. We study the alternating k-arm incipient infinite cluster (IIC) of site percolation on the triangular lattice $\mathbb{T}$. Using Camia and Newman's result that the scaling limit of critical site percolation on $\mathbb{T}$ is CLE_6, we prove the existence of the scaling limit of the k-arm IIC for $k = 1, 2, 4$. Conditioned on the event that there are open and closed arms connecting the origin to $\partial\mathbb{D}_R$, we show that the winding number variance of the arms is $(3/2 + o(1)) \log R$ as $R \to \infty$, which confirms a prediction of Wieland and Wilson [*Phys. Rev. E* **68** (2003) 056101]. Our proof uses two-sided radial SLE_6 and coupling argument. Using this result we get an explicit form for the CLT of the winding numbers, and get analogous result for the 2-arm IIC, thus improving our earlier result.

Résumé. Nous étudions le cluster infini conditionné (IIC) à k-bras alternants pour la percolation par site sur le réseau triangulaire $\mathbb{T}$. En utilisant le résultat de Camia et Newman montrant que la limite d'échelle de la percolation par site sur $\mathbb{T}$ est le CLE_6, nous prouvons l'existence de la limite d'échelle de l'IIC à k bras pour $k = 1, 2, 4$. Conditionnellement à l'événement qu'il y ait un bras ouvert et un bras fermé connectant l'origine à $\partial\mathbb{D}_R$, nous montrons que la variance du nombre d'enroulements est $(3/2 + o(1)) \log R$ quand $R \to \infty$, ce qui confirme la prédiction de Wieland et Wilson [*Phys. Rev. E* **68** (2003) 056101]. Notre preuve utilise le SLE_6 radial à deux côtés ainsi que des arguments de couplage. En utilisant ce résultat, nous obtenons une forme explicite pour le CLT sur le nombre d'enroulements, et obtenons des résultats analogues pour le IIC à deux bras, améliorant ainsi notre résultat précédent.

MSC: 60K35; 82B43

Keywords: Percolation; Scaling limit; SLE; CLE; Incipient infinite cluster; Winding number

1. Introduction

Percolation is a central model of probability theory and statistical physics, see [13,40] for background and [14] for a summary of recent progress. For bond percolation on $\mathbb{Z}^d$, there is almost surely no infinite open cluster at the critical point when $d = 2$ or $d > 10$ (see the recent work [11]), and is conjectured that this is the case whenever $d \geq 2$. The term "incipient infinite cluster" (IIC) has been used by physicists to refer to the large-scale connected clusters present in critical percolation, and was defined mathematically by Kesten [22] in two dimensions. Roughly speaking, IIC is obtained by conditioning on the event that there is an open path connecting the origin to the boundary of the box with radius n centered at the origin, and letting $n \to \infty$. Following Kesten's spirit, Damron and Sapozhnikov introduced multi-arm IIC in [8]. We will give the definitions of these IICs later.

In fact, IIC is a very natural and robust object that can be constructed in many different ways. We introduce some natural constructions for dimension two as follows. In [22], Kesten gave an alternative way to construct the IIC: Take $p > p_c$, condition on the cluster of the origin to be infinite, and let $p \to p_c$. Járai [18] showed that if we choose a site uniformly from the largest cluster or the spanning clusters in $[-n, n]^2$, and let $n \to \infty$, then we get the IIC. In [19] Járai also proved that the invasion percolation cluster looks asymptotically like the IIC, when viewed from an invaded site v, in the limit $|v| \to \infty$. Similarly, Damron and Sapozhnikov [8] showed that the invasion percolation cluster looks asymptotically like the 2-arm IIC (resp. 4-arm IIC), when viewed from a site v belonging to the backbone (resp. outlets), in the limit $|v| \to \infty$. Recently, Hammond, Pete and Schramm [15] defined a local time measure on the exceptional set of dynamical percolation, and showed that at a typical time with respect to this measure, the percolation configuration has the law of IIC. For IIC in high dimensions, see [36,38], where it was also shown that several related and natural constructions lead to the same object.

In this paper, we will study the scaling limit of IIC for site percolation on the triangular lattice $\mathbb{T}$ and the winding numbers of the arms. Before giving our main results, we wish to introduce some related works in the literature.

The scaling limit of IIC has been extensively studied in recent years, and it has turned out to be useful in understanding the discrete model. We list a few related works in the following:

- *Percolation in high dimensions.* Van der Hofstad conjectured in [36] that the scaling limit of IIC above 6 dimensions is infinite canonical super-Brownian motion (ICSBM), which corresponds to the canonical measure of super-Brownian motion conditioned on non-extinction. ICSBM consists of a single infinite Brownian motion path together with super-Brownian motions branching off from this path. In [17], it is showed that the scaling limit of the backbone of the high-dimensional IIC is Brownian motion. The scaling limit of another version of high-dimensional IIC is conjectured to be integrated super-Brownian excursion (ISE) by Hara and Slade [16]. Using the lace expansion, they obtained strong evidence for their conjecture in [16].
- *Oriented percolation in high dimensions.* The existence of the IIC for sufficiently spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ above $4+1$ dimensions has been proved by van der Hofstad, den Hollander, and Slade [37]. Van der Hofstad [36] proved that ICSBM is the scaling limit of the IIC.
- *Percolation on a regular tree.* The IIC on a regular tree was constructed by Kesten in [23]. It has a simple structure, and can be viewed as an infinite backbone from the origin with critical percolation clusters attached to it. Very recently, Angel, Goodman and Merle [3] proved that the scaling limit of the IIC (w.r.t. the pointed Gromov-Hausdorff topology) is a random $\mathbb{R}$-tree with a single end.

Motivated by a question from Beffara and Nolin [4], in [41] we proved a CLT for the winding numbers of alternating arms crossing the annulus $A(l,n)$ (as $n \to \infty$ and l fixed) for critical percolation on $\mathbb{T}$ and $\mathbb{Z}^2$. Using this, we also got a CLT for corresponding multi-arm IIC in [41]. However, the exact estimate for the winding number variance was not given in that paper. Based on numerical simulations, Wieland and Wilson [39] made a conjecture on the winding number variance of Fortuin-Kasteleyn contours (and more generally, the winding at points where k paths come together), including the above case. The conjecture seems hard, to our knowledge, it has been verified rigorously on only a few particular cases. For example, conditioned on the event that there are 2 (resp. 3) disjoint loop-erased random walks starting at the neighbors of the origin and ending at the unit circle centered at the origin in $\eta\mathbb{Z}^2$, Kenyon [21] (see also “Remarks on LERW” in [39]) showed that the winding number variance of the paths is $(1/2 + o(1))\log(1/\eta)$ (resp. $(2/9 + o(1))\log(1/\eta)$) as $\eta \to 0$. The interested reader is referred to the Introduction of [41] for a more general discussion and references on winding numbers.

The rest of the paper is organized as follows. Section 1.1 introduces the basic notation used throughout the paper, and gives the definitions of k-arm IIC measure and arm events for CLE_6. Section 1.2 gives our main results, together with the main ideas in their proofs. In Section 2.1, we define the uniform metric, which is related to the convergence in distribution. Section 2.2 collects different versions of coupling arguments that will be used. Section 2.3 gives basic properties of arm events, including a generalized quasi-multiplicativity. Section 3 provides proofs of scaling-limit results for multi-IIC. In Section 4.1, we introduce two-sided radial SLE and give second moment estimate for its winding number. We study convergence of discrete exploration to SLE_6 in Section 4.2, moment bounds on the winding of discrete exploration in Section 4.3, and decorrelation of winding in Section 4.4, which will enable us to translate the winding number result for two-sided radial SLE_6 to percolation. Section 4.5 provides proofs of the winding number results for the arms.

1.1. *The model and notation*

Let $\mathbb{T} = (\mathbb{V}, \mathbb{E})$ denote the triangular lattice, where $\mathbb{V} := \{x + ye^{\pi i/3} \in \mathbb{C} : x, y \in \mathbb{Z}\}$ is the set of sites, and $\mathbb{E}$ is the set of bonds, connecting adjacent sites. Throughout the paper, we will focus on critical site percolation on $\eta\mathbb{T}$ with small mesh size $\eta > 0$, where each site is chosen to be blue (open) or yellow (closed) with probability $1/2$, independently of each other. Let $P = P^\eta$ denote the corresponding product probability measure on the set of configurations. We also represent the measure as a (blue or yellow) random coloring of the faces of the dual hexagonal lattice $\eta\mathbb{H}$, and view the sites of $\eta\mathbb{T}$ as the hexagons of $\eta\mathbb{H}$. Further, let H_v denote the regular hexagon centered at $v \in \mathbb{V}(\mathbb{T})$ with side length $1/\sqrt{3}$ with two of its sides parallel to the imaginary axis.

A *path* is a sequence $v_0, \dots, v_n$ of distinct sites of $\mathbb{T}$ such that v_{i-1} and v_i are neighbors for all $i = 1, \dots, n$. A *boundary path* (or *b-path*) is a sequence $e_0, \dots, e_n$ of distinct edges of $\mathbb{H}$ belonging to the boundary of a cluster and such that e_{i-1} and e_i meet at a vertex of $\mathbb{H}$ for all $i = 1, \dots, n$. A *circuit* is a path whose first and last sites are neighbors. For a circuit $\mathcal{C}$, define

$$\overline{\mathcal{C}} := \mathcal{C} \cup \quad \text{interior sites of } \mathcal{C}.$$

A *color sequence* σ is a sequence $(\sigma_1, \sigma_2, \dots, \sigma_k)$ of "blue" and "yellow" of length k. We use the letters "B" and "Y" to encode the colors. We identify two sequences if they are the same up to a cyclic permutation.

We say that a finite set D of hexagons is *simply connected* if both D and its complement are connected. For a simply connected set D of hexagons, we denote by ΔD its *external site boundary*, or *s-boundary* (i.e., the set of hexagons that do not belong to D but are adjacent to hexagons in D), and by ∂D the topological boundary of D when D is considered as a domain of $\mathbb{C}$. We will call a bounded, simply connected subset D of $\mathbb{T}$ a *Jordan set* if ΔD is a circuit.

Given a Jordan set $D \subset \mathbb{T}$, for any vertex $v \in \mathbb{H}$ that belongs to ∂D, if the edge incident on v that is not in D does not belong to a hexagon in D, we call v an *e-vertex*.

Given a Jordan set D and two e-vertices a, b in ∂D, we denote by $\partial_{a,b} D$ the portion of ∂D traversed counterclockwise from a to b, and call it the *right boundary*; the remaining part of the boundary is denoted by $\partial_{b,a} D$ and is called the *left boundary*. Analogously, the portion of $\Delta_{a,b} D$ of ΔD whose hexagons are adjacent to $\partial_{a,b} D$ is called the *right s-boundary* and the remaining part the *left s-boundary*. Imagine coloring blue all the hexagons in $\Delta_{a,b} D$ and yellow all those in $\Delta_{b,a} D$. Then, for any percolation configuration inside D, there is a unique b-path γ from a to b which separates the blue cluster adjacent to $\Delta_{a,b} D$ from the yellow cluster adjacent to $\Delta_{b,a} D$. We call $\gamma = \gamma_{D,a,b}$ a *percolation exploration path*.

Given a Jordan domain D of the plane, we denote by D^η the largest Jordan set of hexagons of $\eta\mathbb{H}$ that is contained in D. For two distinct points $a, b \in \partial D$, we let $\gamma^\eta_{D,a,b} := \gamma_{D^\eta, a_\eta, b_\eta}$, where a_η (resp. b_η) is the e-vertex in ∂D^η closest to a (resp. b). If there are two such vertices closest to a (resp. b), we choose the first one encountered going clockwise (resp. counterclockwise) along ∂D^η. Further, let $\partial_{a,b} D^\eta := \partial_{a_\eta, b_\eta} D^\eta$ and $\Delta_{a,b} D^\eta := \Delta_{a_\eta, b_\eta} D^\eta$.

For a domain D, let $\overline{D} := D \cup \partial D$. For a topological annulus $A = \overline{D}_2 \setminus D_1$ (D_1 and D_2 are Jordan domains) whose boundary is composed of two simple loops in the plane, we denote by $\partial_1 A$ (resp. $\partial_2 A$) the *inner* (resp. *outer*) *boundary* of A, and let $A^\eta := \overline{D_2^\eta} \setminus D_1^\eta$.

Define the *disc* and *annulus* as follows: for $0 < r < R$, $z \in \mathbb{C}$,

$$\mathbb{D}_R(z) := \{x \in \mathbb{C} : |x - z| < R\}, \qquad \mathbb{D}_R := \mathbb{D}_R(0), \qquad \mathbb{D} := \mathbb{D}_1;$$

$$A(z; r, R) := \overline{\mathbb{D}_R(z)} \setminus \mathbb{D}_r(z), \qquad A(r, R) := A(0; r, R).$$

Now let us define the arm events for percolation. For a topological annulus A whose boundary is composed of two simple loops, denote by $\mathcal{A}^\eta_\sigma(A) = \mathcal{A}^\eta_{k,\sigma}(A)$ the event that there exist $|\sigma| = k$ disjoint monochromatic paths (*arms*) in A^η connecting the two boundary pieces of A^η, whose colors are those prescribed by σ, when taken in counterclockwise order. For $|\sigma| \le 6$, given a Jordan domain D with a point $z \in D$, let $\mathcal{A}^\eta_\sigma(z; D)$ denote the event that there exist $|\sigma|$ disjoint arms connecting ∂D^η and the hexagon in $\eta\mathbb{H}$ whose center is closest to z (if there are more than one such hexagons, we choose a unique one by some deterministic method), whose colors are those prescribed by σ, when taken in counterclockwise order. For any $\eta \le r < R$ and $z \in \mathbb{C}$, write

$$\mathcal{A}^\eta_\sigma(z; r, R) := \mathcal{A}^\eta_\sigma\big(A(z; r, R)\big).$$

For short, let $\mathcal{A}^\eta_\sigma(r, R) = \mathcal{A}^\eta_\sigma(0; r, R)$ and let $\mathcal{A}^\eta_1 = \mathcal{A}^\eta_B$, $\mathcal{A}^\eta_2 = \mathcal{A}^\eta_{BY}$, $\mathcal{A}^\eta_4 = \mathcal{A}^\eta_{BYBY}$.

The IIC was defined by Kesten [22] as follows. It is shown in [22] that the limit

$$\nu^\eta_1(E) := \lim_{R \to \infty} P^\eta\big(E \mid \mathcal{A}^\eta_1(\eta, R)\big)$$

exists for any event E that depends on the state of finitely many sites in $\eta\mathbb{T}$. The unique extension of ν_1^η to a probability measure on configurations of $\eta\mathbb{T}$ exists and we call ν_1^η the *IIC measure* or *1-arm IIC measure*. Then, Damron and Sapozhnikov introduced multi-arm IIC measures in [8]. Let $k=2,4$. For every cylinder event E, it is shown in Theorem 1.6 in [8] the limit

$$\nu_k^\eta(E) := \lim_{R\to\infty} P^\eta\big(E|\mathcal{A}_k^\eta(\eta,R)\big)$$

exists. The unique extension of ν_k^η to a probability measure on the configurations of $\eta\mathbb{T}$ exists. We call ν_k^η the *k-arm IIC measure*. A curve $\gamma[0,1]$ is called a *loop* if $\gamma(0)=\gamma(1)$. All percolation interfaces under ν_k^η induce a probability measure on the loops in the one-point compactification $\hat{\mathbb{C}}$ of $\mathbb{C}$, denoted by μ_k^η. We postpone precise definitions of the space of loops and the topology of weak convergence till Section 2.1. We also call μ_k^η the k-arm IIC measure.

Given a percolation configuration, we assign a direction to each edge of $\eta\mathbb{H}$ belonging to the boundary of a cluster in such a way that the hexagon to the right of the edge with respect to the direction is blue. To each b-path γ, we can associate a direction according to the direction of the edges in the path. Denote by $\Gamma_B(\gamma)$ (resp., $\Gamma_Y(\gamma)$) the set of blue (resp., yellow) hexagons adjacent to γ; we also let $\Gamma(\gamma) := \Gamma_B(\gamma)\cup\Gamma_Y(\gamma)$.

For any Jordan domain D, let P_D^η denote the percolation law in D^η with *monochromatic (blue) boundary condition*, that is, all the sites in ΔD^η are blue. Then the percolation interfaces under P_D^η induce a law on the loops in $\overline{D}$, denoted by μ_D^η.

In Camia and Newman [7], the following theorem is shown:

Theorem 1.1 ([7]). *Let D be a Jordan domain. As $\eta\to 0$, μ_D^η converges in law, under the topology induced by metric* (8), *to a probability distribution μ_D on collections of continuous nonsimple loops in $\overline{D}$.*

The continuum nonsimple loop process in Theorem 1.1 is just the *full scaling limit* introduced by Camia and Newman [5,7]. Since it is also called the *conformal loop ensemble* CLE_6 in [32] (for the general CLE_κ, $8/3\le\kappa\le 8$, see [32,34]), we just call it CLE_6 (in $\overline{D}$) in the present paper.

For simplicity, let $P_R^\eta := P_{\mathbb{D}_R}^\eta$, $\mu_R^\eta := \mu_{\mathbb{D}_R}^\eta$ and $\mu_R := \mu_{\mathbb{D}_R}$.

We need to define arm events for CLE_6 in a way that makes them measurable and equal to the limit of the probability of corresponding arm events for percolation as $\eta\to 0$. Now we express the arm events $\mathcal{A}_k^\eta(r,R)$, $k=1,2,4$ for μ_R^η in terms of loops (cluster interfaces). See Figure 1.

- It is well-known that the complement of $\mathcal{A}_1^\eta(r,R)$ is that there exists a yellow circuit surrounding the origin in $A^\eta(r,R)$. Since μ_R^η has monochromatic blue boundary condition, the outer boundary of the cluster containing this yellow circuit is in $A^\eta(r,R)\setminus\partial_1 A^\eta(r,R)$, and has counterclockwise direction. So, we have

$$\mathcal{A}_1^\eta(r,R) = \left\{\begin{matrix}\text{There exists no counterclockwise loop surrounding}\\ \text{the origin in } A^\eta(r,R)\setminus\partial_1 A^\eta(r,R)\end{matrix}\right\}. \tag{1}$$

In fact, a simple observation leads to that

$$\mathcal{A}_1^\eta(r,R) = \left\{\begin{matrix}\text{There exits neither counterclockwise loop nor clockwise}\\ \text{loop surrounding the origin in } A^\eta(r,R)\setminus\partial_1 A^\eta(r,R)\end{matrix}\right\}.$$

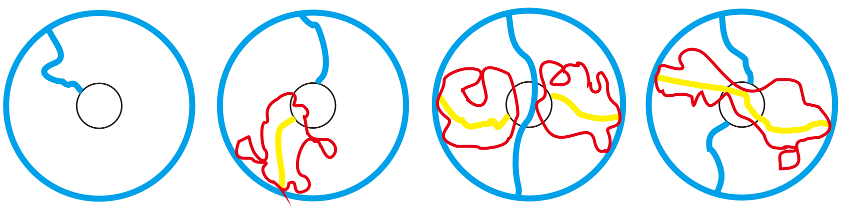

Fig. 1. Illustration of arm events with monochromatic blue boundary condition. The red loops are the outer boundaries of the clusters containing yellow arms. The first panel indicates $\mathcal{A}_1^\eta(r,R)$. The second panel indicates $\mathcal{A}_2^\eta(r,R)$. The last two panels indicate $\mathcal{A}_4^\eta(r,R)$.

- Assume that $\mathcal{A}_2^\eta(r,R)$ holds, then there exist a blue arm and a yellow arm connecting $\partial_1 A^\eta(r,R)$ and $\partial_2 A^\eta(r,R)$. The outer boundary of the cluster containing the yellow arm must intersect with both of the two boundary pieces of $A^\eta(r,R)$. Conversely, if there exists a counterclockwise loop γ in $A^\eta(r,R)$ intersecting both of the two boundary pieces of $A^\eta(r,R)$, we can find a blue arm in $\Gamma_B(\gamma)$ and a yellow arm in $\Gamma_Y(\gamma)$, which connect the two boundary pieces of $A^\eta(r,R)$. Hence,

$$\mathcal{A}_2^\eta(r,R) = \left\{\begin{array}{l}\text{There exists a counterclockwise loop in } \overline{\mathbb{D}_R^\eta}\text{, which}\\ \text{intersects with both } \partial_1 A^\eta(r,R) \text{ and } \partial_2 A^\eta(r,R)\end{array}\right\}. \tag{2}$$

- Denote by $\mathcal{A}_4^{\eta,B}(r,R)$ (resp. $\mathcal{A}_4^{\eta,Y}(r,R)$) the event that there are four alternating arms in $A^\eta(r,R)$ connecting $\partial_1 A^\eta(r,R)$ and $\partial_2 A^\eta(r,R)$, and the two blue (resp. yellow) arms are in the same cluster in $\overline{\mathbb{D}_R^\eta}$. It is clear that $\mathcal{A}_4^\eta(r,R) = \mathcal{A}_4^{\eta,B}(r,R) \cup \mathcal{A}_4^{\eta,Y}(r,R)$. If $\mathcal{A}_4^{\eta,B}(r,R)$ occurs, there exist two counterclockwise loops in $\overline{\mathbb{D}_R^\eta}$, which intersect with both $\partial_1 A^\eta(r,R)$ and $\partial_2 A^\eta(r,R)$; if $\mathcal{A}_4^{\eta,Y}(r,R)$ occurs, there exists a counterclockwise loop in $\overline{\mathbb{D}_R^\eta}$, which is composed of two curves γ_1 and γ_2: γ_1 starts at $a \in \partial_2 A^\eta(r,R)$ and ends at $b \in \partial_2 A^\eta(r,R)$, γ_2 starts at b and ends at a, both γ_1 and γ_2 intersect with $\partial_1 A^\eta(r,R)$. In fact, it is easy to see that

$$\mathcal{A}_4^\eta(r,R) = \left\{\begin{array}{l}\text{There exist two counterclockwise loops in } \overline{\mathbb{D}_R^\eta}\text{, which}\\ \text{intersect with both } \partial_1 A^\eta(r,R) \text{ and } \partial_2 A^\eta(r,R)\text{; or there}\\ \text{exists a counterclockwise loop in } \overline{\mathbb{D}_R^\eta}\text{, which is composed}\\ \text{of two curves } \gamma_1 \text{ and } \gamma_2\text{: } \gamma_1 \text{ starts at } a \in \partial_2 A^\eta(r,R) \text{ and}\\ \text{ends at } b \in \partial_2 A^\eta(r,R)\text{, } \gamma_2 \text{ starts at } b \text{ and ends at } a\text{, both}\\ \gamma_1 \text{ and } \gamma_2 \text{ intersect with } \partial_1 A^\eta(r,R)\end{array}\right\}. \tag{3}$$

This leads us to define arm events $\mathcal{A}_k(r,R)$, $k=1,2,4$ for μ_R as follows:

$$\mathcal{A}_1(r,R) := \{\text{There exists no counterclockwise loop surrounding the origin in } A(r,R)\},$$

$$\mathcal{A}_2(r,R) := \left\{\begin{array}{l}\text{There exists a counterclockwise loop in } \overline{\mathbb{D}}_R\text{, which}\\ \text{intersects with both } \partial_1 A(r,R) \text{ and } \partial_2 A(r,R)\end{array}\right\},$$

$$\mathcal{A}_4(r,R) := \left\{\begin{array}{l}\text{There exist two counterclockwise loops in } \overline{\mathbb{D}}_R\text{, which intersect}\\ \text{with both } \partial_1 A(r,R) \text{ and } \partial_2 A(r,R)\text{; or there exists a counterclockwise}\\ \text{loop in } \overline{\mathbb{D}}_R\text{, which is composed of two curves } \gamma_1 \text{ and } \gamma_2\text{: } \gamma_1 \text{ starts}\\ \text{at } a \in \partial_2 A(r,R) \text{ and ends at } b \in \partial_2 A(r,R)\text{, } \gamma_2 \text{ starts at } b \text{ and ends}\\ \text{at } a\text{, both } \gamma_1 \text{ and } \gamma_2 \text{ intersect with } \partial_1 A(r,R)\end{array}\right\}.$$

Given two Jordan domains D and D' with $\overline{D'} \subset D$, similarly to the definitions of $\mathcal{A}_k(r,R)$ for μ_R, one can define arm events $\mathcal{A}_k(\overline{D} \setminus D')$ for μ_D.

In this paper, we sometimes omit the superscript η of P^η and γ^η when it is clear that we are talking about the the discrete percolation model. $C, C_1, C_2, \ldots$ and α, β denote positive finite constants that may change from line to line or page to page according to the context.

1.2. *Main results*

Our main results include two parts, the first part is about the existence and conformal invariance of the k-arm IIC scaling limit, the second part is about the variance estimate and CLT for the winding numbers of the arms, conditioned on the 2-arm event and under the 2-arm IIC measure, respectively.

1.2.1. *Scaling limit of k-arm IIC*

Theorem 1.2. *Let $k = 1, 2, 4$. Let D be a Jordan domain with a point $z \in D$. Let $\{D_n\}$ be a sequence of Jordan domains such that $z \in D_n$, $\overline{D}_n \subset D$ and the diameter of D_n converges to zero as $n \to \infty$.*

- *As $\eta \to 0$ and $n \to \infty$, $\mu_D^\eta[\cdot|\mathcal{A}_k^\eta(z; D)]$ and $\mu_D[\cdot|\mathcal{A}_k(\overline{D} \setminus D_n)]$ converge in law, under the topology induced by metric (8), to the same probability measure, denoted by $\mu_{k,D,z}$.*
- *Furthermore, let D' be a Jordan domain and let $f : \overline{D} \to \overline{D'}$ a continuous function that maps D conformally onto D'. Let $z' := f(z)$. Then the image of $\mu_{k,D,z}$ under f has the same law as $\mu_{k,D',z'}$.*

We call $\mu_{k,D,z}$ the *scaling limit of k-arm IIC pinned at z in D*, which can be considered as a conditioned version of CLE_6. In [33], the authors constructed *CLE_κ in D conditioned on the event that z is in the gasket* (i.e., the set of points that are not surrounded by any loop in CLE_κ) for $8/3 < \kappa \le 4$. One can view $\mu_{1,D,z}$ as CLE_6 in D conditioned on the event that z is in the gasket. We write $\mu_{k,R} := \mu_{k,\mathbb{D}_R,0}$.

Remark. Using Theorem 1.2 and Theorem 4 in [7], it is not hard to show that $\mu_{k,D,z}$ inherits some domain Markov property from CLE_6. It is expected that analogs of Propositions 4.3 and 4.4 in [33] for domain Markov property of simple CLE in the punctured disc also hold for $\mu_{k,D,z}$.

For a domain D, we denote by I_D the mapping (on Ω or Ω_R, see the definitions in Section 2.1) in which all portions of curves that exit $\overline{D}$ are removed. Let $\hat{I}_D$ be the same mapping lifted to the space of probability measures on Ω or Ω_R.

Theorem 1.3. *There exists a unique probability measure μ_k on the space Ω of collections of continuous curves in $\hat{\mathbb{C}}$ such that $\mu_{k,R} \to \mu_k$ as $R \to \infty$ in the sense that for every bounded domain D, as $R \to \infty$, $\hat{I}_D\mu_{k,R} \to \hat{I}_D\mu_k$. Furthermore, as $\eta \to 0$, μ_k^η converges in law, under the topology induced by metric (10), to μ_k.*

We call μ_k the *scaling limit of k-arm IIC*. In [33], the authors constructed *CLE_κ in the punctured plane* for $8/3 < \kappa \le 4$. One can view μ_1 as CLE_6 in the punctured plane. Note that if one can construct IIC for the discrete $O(n)$ models, it is expected that the scaling limit of the IIC is just the corresponding CLE_κ in the punctured plane. In particular, the scaling limit of IIC of the critical Ising model (which is the $O(1)$ model) is expected to be CLE_3 in the punctured plane.

Remark. From Camia and Newman's construction of the full-plane CLE_6, it is easy to see that full-plane CLE_6 is invariant under scalings, translations, and rotations. However, with their construction, the invariance of full-plane CLE_6 under the inversion $z \mapsto 1/z$ turns out to be not obvious to establish. In [20], using the Brownian loop soup, the authors proved the inversion-invariance of full-plane CLE_κ for $8/3 < \kappa \le 4$. In [33], the inversion-invariance of CLE_κ in the punctured plane for $8/3 < \kappa \le 4$ was also proved. Hence, we propose the following conjecture:

Conjecture 1.4. *The full-plane CLE_6 and μ_k $(k = 1, 2, 4)$ are invariant under $z \mapsto 1/z$.*

1.2.2. *Winding numbers of the arms*

For a curve $\gamma[0, T]$ in the plane with $\gamma(t) \neq 0$ for all $0 \le t \le T$, we define the *winding number* of γ (around 0) by $\theta(\gamma) := \arg(\gamma(T)) - \arg(\gamma(0))$, with arg chosen continuous along γ.

Denote by $\mathcal{A}^\eta$ the event that the percolation exploration path $\gamma_{\mathbb{D},1,-1}^\eta$ intersect with the boundary of the hexagon ηH_0. Note that $\mathcal{A}^\eta$ is the same as the event that there is a blue arm connecting ηH_0 to $\partial_{1,-1}\mathbb{D}^\eta$ and a yellow arm connecting ηH_0 to $\partial_{-1,1}\mathbb{D}^\eta$.

Assume $\mathcal{A}^\eta$ occurs and T is the first hitting time with ηH_0 of $\gamma_{\mathbb{D},1,-1}^\eta$. Let $\theta_\eta := \theta(\gamma_{\mathbb{D},1,-1}^\eta[0, T])$.

Theorem 1.5 establishes a particular case of Wieland and Wilson's conjecture on winding number variance of Fortuin-Kasteleyn contours [39].

Theorem 1.5. *Conditioned on the event $\mathcal{A}^\eta$, we have*

$$\mathrm{Var}[\theta_\eta] = \left(\frac{3}{2} + o(1)\right) \log\left(\frac{1}{\eta}\right) \quad \text{as } \eta \to 0. \tag{4}$$

Furthermore, under the conditional measure $P[\cdot|\mathcal{A}^\eta]$,

$$\frac{\theta_\eta}{\sqrt{\frac{3}{2}\log(\frac{1}{\eta})}} \to_d N(0,1) \quad \textit{as } \eta \to 0. \tag{5}$$

Suppose the 2-arm event $\mathcal{A}_2^\eta(\eta,1)$ happens. We fix a deterministic way to choose a unique blue arm connecting $\partial\mathbb{D}^\eta$ and ηH_0, and denote by $\tilde{\theta}_\eta$ the winding number of this arm (here we consider the arm as a continuous curve by connecting the neighbor sites with line segments).

The following corollary refines [41] for the 2-arm case by giving variance estimates and CLT for winding numbers of the arms in explicit expressions.

Corollary 1.6. *Under the conditional measure* $P[\cdot|\mathcal{A}_2^\eta(\eta,1)]$ *and the* 2*-arm IIC measure* ν_2^η, *as* $\eta \to 0$, *we both have*

$$\operatorname{Var}[\tilde{\theta}_\eta] = \left(\frac{3}{2} + o(1)\right)\log\left(\frac{1}{\eta}\right) \quad \textit{and} \quad \frac{\tilde{\theta}_\eta}{\sqrt{\frac{3}{2}\log(\frac{1}{\eta})}} \to_d N(0,1).$$

Remark. Corollary 1.6 confirms a prediction of Beffara and Nolin [4] for the 2-arm case explicitly. Following [41] (see Theorem 1.1 and Remark 1.2 in [41]), we give the following conjecture for the 4-arm case:

Conjecture 1.7. *Under* $P[\cdot|\mathcal{A}_4^\eta(\eta,1)]$ *and* ν_4^η, *as* $\eta \to 0$ *we both have*

$$\operatorname{Var}[\tilde{\theta}_\eta] = \left(\frac{3}{8} + o(1)\right)\log\left(\frac{1}{\eta}\right) \quad \textit{and} \quad \frac{\tilde{\theta}_\eta}{\sqrt{\frac{3}{8}\log(\frac{1}{\eta})}} \to_d N(0,1).$$

Remark. If one can generalize the results for two-sided radial SLE that we used in this paper to "$2k$-sided radial SLE", it is expected that one can use our method to get precise estimate of the winding number variance for the $2k$-arm case, and get the corresponding CLT.

1.2.3. *Ideas of the proofs*

Let us explain the main ideas in the proofs of our main results.

Scaling limits. First, we use the approach of Aizenman-Burchard [1] to show that the k-arm IIC has subsequential scaling limit. Then, conditioned on the k-arm events for a sequence of annuli, we introduce conditional measures for percolation and CLE_6. Using these measures, by coupling argument introduced in [12] and Theorem 1.1, we establish the uniqueness of the scaling limit. The conformal invariance of the scaling limit can be derived from that of CLE_6 easily.

Winding numbers. The proof can be divided into three main steps as follows.

- First, we use the approach of Schramm [31] to derive the winding number variance of two-sided radial SLE_6.
- Second, conditioned on the event that the percolation exploration path in $\mathbb{D}^\eta$ goes through ηH_0, we show the scaling limit of the path is two-sided radial SLE_6. The key ingredients include a proposition of Green's function for chordal SLE proved by Lawler and Rezaei [26], the coupling argument and the well-know result that the scaling limit of percolation exploration path is SLE_6.
- Third, we divide the unit disk into concentric annuli with large modulus, and show that the sum of winding number variances of the paths in these annuli approximates the variance of θ_η, and the winding number variance corresponding to each annulus can be approximated well by that of two-sided radial SLE_6 as $\eta \to 0$. This step involves many technical issues and uses coupling argument extensively. A key ingredient is the estimate of winding number variance of the arms from [41].

2. Preliminary definitions and results

2.1. *The space of curves*

When taking the scaling limit of percolation on the whole plane, it is convenient to compactify $\mathbb{C}$ into $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \simeq \mathbb{S}^2$ (i.e., the Riemann sphere) as follows. First, we replace the Euclidean metric with a distance function $\Delta(\cdot,\cdot)$ defined on $\mathbb{C} \times \mathbb{C}$ by

$$\Delta(u,v) := \inf_{\varphi} \int \left(1+|\varphi|^2\right)^{-1} ds, \tag{6}$$

where the infimum is over all smooth curves $\varphi(s)$ joining u with v, parameterized by arclength s, and $|\cdot|$ denotes the Euclidean norm. This metric is equivalent to the Euclidean metric in bounded regions. Then, we add a single point ∞ at infinity to get the compact space $\hat{\mathbb{C}}$ which is isometric, via stereographic projection, to the two-dimensional sphere.

Let D be a Jordan domain and denote by $\mathcal{S}_D$ the complete separable metric space of continuous curves in $\overline{D}$ with the metric (7) defined below. Curves are regarded as equivalence classes of continuous functions from the unit interval to $\overline{D}$, modulo monotonic reparametrizations. $\mathcal{F}$ will represent a set of curves (more precisely, a closed subset of $\mathcal{S}_D$). $\mathrm{d}(\cdot,\cdot)$ will denote the uniform metric on curves, defined by

$$\mathrm{d}(\gamma_1,\gamma_2) := \inf \sup_{t\in[0,1]} \left|\gamma_1(t)-\gamma_2(t)\right|, \tag{7}$$

where the infimum is over all choices of parametrizations of γ_1 and γ_2 from the interval $[0,1]$. The distance between two closed sets of curves is defined by the induced Hausdorff metric as follows:

$$\operatorname{dist}\left(\mathcal{F},\mathcal{F}'\right) := \inf\left\{\varepsilon>0 : \forall\gamma\in\mathcal{F}, \exists\gamma'\in\mathcal{F}' \text{ such that } \mathrm{d}\left(\gamma,\gamma'\right)\le\varepsilon \text{ and vice versa}\right\}. \tag{8}$$

The space Ω_D of closed subsets of $\mathcal{S}_D$ (i.e., collections of curves in $\overline{D}$) with the metric (8) is also a complete separable metric space. Write $\Omega_R := \Omega_{\mathbb{D}_R}$.

We will also consider the complete separable metric space $\mathcal{S}$ of continuous curves in $\hat{\mathbb{C}}$ with the distance

$$\mathrm{D}(\gamma_1,\gamma_2) := \inf \sup_{t\in[0,1]} \Delta\left(\gamma_1(t),\gamma_2(t)\right), \tag{9}$$

where the infimum is again over all choices of parametrizations of γ_1 and γ_2 from the interval $[0,1]$. The distance between two closed sets of curves is again defined by the induced Hausdorff metric as follows:

$$\operatorname{Dist}\left(\mathcal{F},\mathcal{F}'\right) := \inf\left\{\varepsilon>0 : \forall\gamma\in\mathcal{F}, \exists\gamma'\in\mathcal{F}' \text{ such that } \mathrm{D}\left(\gamma,\gamma'\right)\le\varepsilon \text{ and vice versa}\right\}. \tag{10}$$

The space Ω of closed sets of $\mathcal{S}$ (i.e., collections of curves in $\hat{\mathbb{C}}$) with the metric (10) is also a complete separable metric space.

It was noted in [5,7] that one should add a "trivial" loop for each z in $\overline{D}$, so that the collection of CLE_6 loops is closed in the appropriate sense [1]. When considering the CLE_6 in $\hat{\mathbb{C}}$, one should also add a trivial loop for each $z\in\hat{\mathbb{C}}$ to make the space of loops closed. In this paper, we will not include these trivial loops to the loop process except for dealing with this technical problem.

2.2. *Coupling argument*

The coupling argument for 1-arm events appeared in [22] for the construction of IIC, and then the coupling argument for multi-arm events appeared in [8] for the construction of multi-arm IIC. Recently, Garban, Pete and Schramm [12] introduced the notion of faces, and gave the coupling argument in a clear and general form, which turns out to be very useful. For example, we used it in [41] to prove a CLT for the winding numbers of the arms with alternating colors. In this paper, we will make extensive use of coupling argument. Being familiar with it in [12] and Lemma 2.3 in [41] will be helpful to the readers. First, let us state the coupling argument that will be used in Section 3 for k-arm IIC. To state the result, we need some definitions.

Let k be an even number. For a circuit $\mathcal{C} = \gamma_1\gamma_2\cdots\gamma_k$ (i.e., the concatenation of $\gamma_1, \dots, \gamma_k$), if $\gamma_1, \dots, \gamma_k$ are monochromatic paths with alternating colors, we call $\mathcal{C}$ a *k-circuit*, and write $\mathcal{C} = (\gamma_1, \dots, \gamma_k)$. We will always assume that γ_1 is blue. For convenience, a monochromatic blue circuit is called a *1-circuit*. For any 4-circuit $\mathcal{C} = (\gamma_1, \dots, \gamma_4)$, denote by $U = U_{\mathcal{C}}$ the indicator function of the event that there exists a blue path connecting γ_1 and γ_3 in $\overline{\mathcal{C}}$ (recall that $\overline{\mathcal{C}} = \mathcal{C} \cup$ interior sites of $\mathcal{C}$). Note that $U_{\mathcal{C}} = 0$ if and only if there exists a yellow path connecting γ_2 and γ_4 in $\overline{\mathcal{C}}$.

The proofs of the following coupling arguments (which are different versions of the coupling arguments in [12]) are essentially the same as those of Proposition 3.1, 3.6 and 5.2 in [12] (see also the sketch of the proof of Lemma 2.3 in [41]), we omit the proofs of Proposition 2.1 and 2.2 except just stating how to deal with an additional issue in the case of $k = 4$ in Proposition 2.1.

Proposition 2.1. *Let $k = 1, 2, 4$. There exists a constant $\alpha = \alpha(k) > 0$, such that for any $10\eta < r < R/100$ and $2r \le r' \le R$, there is a coupling of the measures $P[\cdot|\mathcal{A}_k^\eta(\eta, R)]$ and $P[\cdot|\mathcal{A}_k^\eta(r, R)]$, such that with probability at least $1 - (r/r')^\alpha$ there exists an identical k-circuit $\mathcal{C}$ surrounding the origin in $A^\eta(r, r')$ for both measures, and the configuration outside $\mathcal{C}$ is also identical, and furthermore $U_{\mathcal{C}}$ is identical in the case $k = 4$.*

Proof. As we have said before Proposition 2.1, we only deal with the additional issue for $U_{\mathcal{C}}$ in the case $k = 4$. Similarly to the proofs of Proposition 3.6 in [12] and Lemma 2.3 in [41], one can construct a coupling of the measures $P[\cdot|\mathcal{A}_4^\eta(\eta, R)]$ and $P[\cdot|\mathcal{A}_4^\eta(r, R)]$, such that with probability at least $1 - (r/r')^\alpha$ the following event $\mathcal{B}$ occurs: There exists an identical k-circuit $\mathcal{C}$ surrounding the origin in $A^\eta(r, r')$ for both measures, and the configuration outside $\mathcal{C}$ is also identical. Denote by $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) the k-circuit $\mathcal{C}$ under $P[\cdot|\mathcal{A}_4^\eta(\eta, R)]$ (resp. $P[\cdot|\mathcal{A}_4^\eta(r, R)]$). Further, the above construction is symmetric for the colors, so conditioned on $\mathcal{B}$, the probability of $U_{\mathcal{C}_2} = 1$ equals to that of $U_{\mathcal{C}_2} = 0$. Note that the color of the hexagon ηH_0 under $P[\cdot|\mathcal{A}_4^\eta(\eta, R)]$ is essentially irrelevant to the construction of the coupling. Hence, if $\mathcal{B}$ occurs, one can let the hexagon be blue if $U_{\mathcal{C}_2} = 1$, and yellow if $U_{\mathcal{C}_2} = 0$; otherwise we toss a coin to determine the color. Then under this new coupling one has $U_{\mathcal{C}_1} = U_{\mathcal{C}_2}$. □

Proposition 2.2. *Let $k = 1, 2, 4$. There exists a constant $\alpha = \alpha(k) > 0$, such that for any $100\eta < R_1 < R_2$ and $10\eta < r < R_1/2$, there is a coupling of $P[\cdot|\mathcal{A}_k^\eta(\eta, R_1)]$ and $P[\cdot|\mathcal{A}_k^\eta(\eta, R_2)]$, so that with probability at least $1 - (r/R_1)^\alpha$ there exists an identical k-circuit $\mathcal{C}$ surrounding the origin in $A^\eta(r, R_1)$ for both measures, and the configuration inside $\mathcal{C}$ is also identical.*

Now, we want to give the coupling argument that will be used in Section 4 for winding numbers. Following the terminology of [12,41], we first introduce the notion of faces. Let x_1, x_2 be distinct e-vertices in $\partial\mathbb{D}_R^\eta$. Let γ_1 be a blue path of hexagons joining x_1 to x_2 and let γ_2 be a yellow path of hexagons joining x_2 to x_1. Denote by $\Theta = (\gamma_1, \gamma_2)$ the circuit which is composed of the two paths. We assume furthermore that $\mathbb{D}_R^\eta \subset$ interior of Θ. Then we call the circuit Θ a configuration of *faces* with endpoints x_1, x_2, and say Θ are faces around $\partial\mathbb{D}_R^\eta$. Define the *quality* of a configuration of faces $Q(\Theta)$ to be the distance between the endpoints, normalized by R. That is,

$$Q(\Theta) := \frac{|x_1 - x_2|}{R}.$$

Let $\mathrm{Cone}_1 := \{z \in \mathbb{C} : -3\pi/4 < \arg(z) < 3\pi/4\}$, $\mathrm{Cone}_2 := \{z \in \mathbb{C} : \pi/4 < \arg(z) < 7\pi/4\}$, $\mathrm{Cone}_3 := \{z \in \mathbb{C} : -\pi/4 < \arg(z) < \pi/4\}$, $\mathrm{Cone}_4 := \{z \in \mathbb{C} : 3\pi/4 < \arg(z) < 5\pi/4\}$.

In the annulus $A = A^\eta(R, 2R)$, let $\mathcal{R} = \mathcal{R}(A)$ be the event that there are exactly two disjoint alternating arms crossing A, and the resulting two interfaces are contained respectively in Cone_1 and Cone_2, with the endpoints of the interfaces on the two boundaries of A belonging to Cone_3 and Cone_4, respectively.

Lemma 2.3 is the straightforward 2-arm analog of Lemma 2.2 in [41]. The proof is analogous to the second proof of Lemma 3.4 in [12], we leave it to the reader.

Lemma 2.3. $P(\mathcal{R}(A^\eta(R, 2R))) > C$ *for an absolute constant* $C > 0$.

For $A^\eta(R, 2R)$, if the event $\mathcal{R}$ happens, then the two interfaces induce a natural configuration of faces $\Theta \subset A^\eta(R, 2R)$ around $\partial\mathbb{D}_R^\eta$. We call Θ *good faces* around $\partial\mathbb{D}_R^\eta$. See Figure 2.

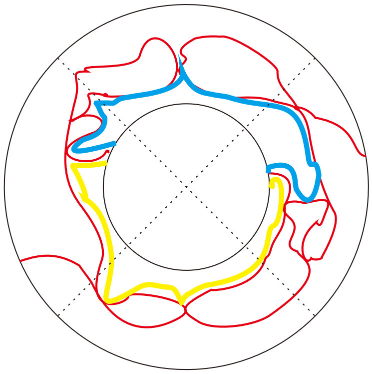

Fig. 2. Two interfaces crossing the annulus induce a natural configuration of good faces.

For $\eta \le r < R$ and faces $\Theta = (\gamma_1, \gamma_2)$ around $\partial\mathbb{D}_R^\eta$, define

$$\mathcal{A}_\Theta^\eta(r, R) := \left\{ \begin{array}{l} \exists \text{ a blue arm connecting } \gamma_1 \text{ to } \partial\mathbb{D}_r^\eta \text{ and} \\ \text{a yellow arm connecting } \gamma_2 \text{ to } \partial\mathbb{D}_r^\eta \end{array} \right\}.$$

Let us now define a measure $P_R^*[\cdot]$ as follows. First, we sample good faces Θ around $\partial\mathbb{D}_R^\eta$ according to the law $P[\cdot|\mathcal{R}]$; then conditioning on Θ, we sample the configuration inside Θ according to $P[\cdot|\mathcal{A}_\Theta(\eta, R)]$. This induces a probability measure on good faces around $\partial\mathbb{D}_R^\eta$ and the configuration inside the good faces, denoted by P_R^*.

For $\eta < R$, denote by $\mathcal{A}^\eta(R)$ the event that the percolation exploration path $\gamma^\eta_{\mathbb{D}_R, R, -R}$ intersects with the hexagon ηH_0. Note that $\mathcal{A}^\eta(1) = \mathcal{A}^\eta$.

The proofs of the following coupling results are very similar to those of Proposition 3.1 and 3.6 in [12] (see also Lemma 2.3 in [41]), which are omitted here.

Proposition 2.4. *There exists a constant $\beta > 0$, such that for all $\eta < 1/100$, $10\eta < r < R/2$ and $R \le 1$, there is a coupling of the measures $P[\cdot|\mathcal{A}^\eta]$, $P[\cdot|\mathcal{A}^\eta(R)]$ and $P[\cdot|\mathcal{A}_2^\eta(\eta, R)]$, so that with probability at least $1 - (r/R)^\beta$ there exist identical good faces $\Theta \subset A^\eta(r, R)$ for these three measures, and the configuration in $\overline{\Theta}$ is also identical.*

Proposition 2.5. *There exist constants $C_0, C_1 > 0$, such that for all $\eta \le r < R/2$, any fixed faces Θ around $\partial\mathbb{D}_R^\eta$ and $N := \lfloor \log_2(R/r) \rfloor$, there is a coupling of $P[\cdot|\mathcal{A}_\Theta^\eta(r, R)]$ and $\{P^*_{(1/2)^j R}[\cdot]\}_{1 \le j \le N}$, so that*

- *for all $1 \le j \le N$, with probability at least $1 - \exp(-C_0 j)$, there exists $1 \le j^* \le j$ such that there exist good faces Θ_{j^*} around $\partial\mathbb{D}^\eta_{(1/2)^{j^*} R}$ under $P[\cdot|\mathcal{A}_\Theta^\eta(r, R)]$, and the configuration in $\overline{\Theta}_{j^*}$ under $P[\cdot|\mathcal{A}_\Theta^\eta(r, R)]$ is the same as the configuration under $P^*_{(1/2)^{j^*} R}[\cdot]$;*
- *for all $1 \le j \le N - 1$, with probability at least $\exp(-C_1(j + 1))$, for all $1 \le j' \le j$ there do not exist good faces around $\partial\mathbb{D}^\eta_{(1/2)^{j'} R}$, but there exist good faces Θ_{j+1} around $\partial\mathbb{D}^\eta_{(1/2)^{j+1} R}$ under $P[\cdot|\mathcal{A}_\Theta^\eta(r, R)]$, and the configuration in $\overline{\Theta}_{j+1}$ under $P[\cdot|\mathcal{A}_\Theta^\eta(r, R)]$ is the same as the configuration under $P^*_{(1/2)^{j+1} R}[\cdot]$.*

For the next proposition we need some additional notation. Let $0 < r < 1$. For the percolation exploration path $\gamma^\eta_{\mathbb{D},1,-1}$, define event

$$\mathcal{A}_r^\eta := \{\gamma^\eta_{\mathbb{D},1,-1} \cap \partial\mathbb{D}_r^\eta \ne \varnothing\}.$$

For a curve γ with $\gamma \cap \partial\mathbb{D}_r^\eta \ne \varnothing$, denote by τ_r^η the first hitting time with $\partial\mathbb{D}_r^\eta$ of γ.

Proposition 2.6. *There exists a constant $\beta > 0$, such that for all $100\eta < 10r < R < 1$, there is a coupling of the measures $P[\cdot|\mathcal{A}^\eta]$ and $P[\cdot|\mathcal{A}_r^\eta]$, so that with probability at lest $1 - (r/R)^\beta$, the stopped percolation exploration path $\gamma^\eta_{\mathbb{D},1,-1}[0, \tau_R^\eta]$ under $P[\cdot|\mathcal{A}^\eta]$ is identical to that under $P[\cdot|\mathcal{A}_r^\eta]$.*

2.3. *Basic properties of arm events*

In this paper, we assume that the reader is familiar with the FKG inequality (see Lemma 13 in [29] for generalized FKG), the BK (van den Berg-Kesten) inequality and Reimer's inequality [30], and the RSW (Russo–Seymour–Welsh) technology. See [13,40]. The following properties of arm events are well known (see [29]) except (12) and (13), where (13) is a generalization of the standard quasi-multiplicativity.

1. *A priori bounds for arm events*: For any color sequence σ, there exist $C_1(|\sigma|)$, $C_2(|\sigma|)$, $\alpha(|\sigma|)$, $\beta(|\sigma|) > 0$ such that for all $\eta \le r < R$,

$$C_1\left(\frac{r}{R}\right)^{\alpha} \le P\left[\mathcal{A}_{\sigma}^{\eta}(r,R)\right] \le C_2\left(\frac{r}{R}\right)^{\beta}. \tag{11}$$

2. There exists a constant $C > 0$, such that for all $\eta \le r < R$ and faces Θ around $\partial\mathbb{D}_R^{\eta}$ with $Q(\Theta) > 1/4$,

$$C P\left[\mathcal{A}_2^{\eta}(r,R)\right] \le P\left[\mathcal{A}_{\Theta}^{\eta}(r,R)\right] \le P\left[\mathcal{A}_2^{\eta}(r,R)\right]. \tag{12}$$

3. *Quasi-multiplicativity*: For any color sequence σ, there is a $C_1(|\sigma|) > 0$, such that for all $\eta \le r_1 < r_2 \le r_3 < r_4$ and $r_3 \le 10r_2$,

$$C_1 P\left[\mathcal{A}_{\sigma}^{\eta}(r_1,r_2)\right]P\left[\mathcal{A}_{\sigma}^{\eta}(r_3,r_4)\right] \le P\left[\mathcal{A}_{\sigma}^{\eta}(r_1,r_4)\right] \le P\left[\mathcal{A}_{\sigma}^{\eta}(r_1,r_2)\right]P\left[\mathcal{A}_{\sigma}^{\eta}(r_3,r_4)\right].$$

Furthermore, there is a $C_2 > 0$, such that for all $\eta \le r_1 < r_2 \le r_3/2$ and any given faces Θ around $\partial\mathbb{D}_{r_3}^{\eta}$,

$$C_2 P\left[\mathcal{A}_2^{\eta}(r_1,r_2)\right]P\left[\mathcal{A}_{\Theta}^{\eta}(r_2,r_3)\right] \le P\left[\mathcal{A}_{\Theta}^{\eta}(r_1,r_3)\right] \le P\left[\mathcal{A}_2^{\eta}(r_1,r_2)\right]P\left[\mathcal{A}_{\Theta}^{\eta}(r_2,r_3)\right]. \tag{13}$$

Proof. We just need to prove (12) and (13). Applying a standard gluing argument with generalized FKG, RSW and Theorem 11 in [29], one gets (12). The details are omitted. Now let us show (13). Conditioned on $\mathcal{A}_{\Theta}^{\eta}(r_2,r_3)$, the two interfaces (or b-paths) starting from the endpoints of $\Theta = (\gamma_1,\gamma_2)$ to reach $\partial\mathbb{D}_{2r_3/3}^{\eta}$ together with Θ induce faces $\Theta' = (\gamma_1',\gamma_2')$ around $\partial\mathbb{D}_{2r_3/3}^{\eta}$. By Lemma 3.3 (Strong Separation Lemma) in [12], there is some absolute constant $C_3 > 0$ such that

$$P\left[Q(\Theta') > \frac{1}{4}\middle|\mathcal{A}_{\Theta}^{\eta}(r_2,r_3)\right] \ge C_3. \tag{14}$$

By a gluing construction with FKG, RSW and Theorem 11 in [29], there is some absolute constant $C_4 > 0$ such that for any given Θ' with $Q(\Theta') > 1/4$ (see an analogous quasi-multiplicativity in [41]),

$$C_4 P\left[\mathcal{A}_2^{\eta}(r_1,r_2)\right]P\left[\mathcal{A}_{\Theta'}^{\eta}(r_2,2r_3/3)\right] \le P\left[\mathcal{A}_{\Theta'}^{\eta}(r_1,2r_3/3)\right]. \tag{15}$$

Define

$$\mathcal{A}_{\Theta,\Theta'} := \left\{\begin{matrix}\exists \text{ a blue arm connecting } \gamma_1 \text{ and } \gamma_1' \text{ and} \\ \text{a yellow arm connecting } \gamma_2 \text{ and } \gamma_2'\end{matrix}\right\}.$$

By (14) and (15), we have

$$\begin{aligned}
&C_3C_4P\left[\mathcal{A}_2^{\eta}(r_1,r_2)\right]P\left[\mathcal{A}_{\Theta}^{\eta}(r_2,r_3)\right] \\
&\quad\le C_4\sum_{Q(\Theta')>1/4} P\left[\Theta',\mathcal{A}_{\Theta,\Theta'}\right]P\left[\mathcal{A}_{\Theta'}^{\eta}(r_2,2r_3/3)\right]P\left[\mathcal{A}_2^{\eta}(r_1,r_2)\right] \\
&\quad\le \sum_{Q(\Theta')>1/4} P\left[\Theta',\mathcal{A}_{\Theta,\Theta'}\right]P\left[\mathcal{A}_{\Theta'}^{\eta}(r_1,2r_3/3)\right] \le P\left[\mathcal{A}_{\Theta}^{\eta}(r_1,r_3)\right].
\end{aligned}$$

By choosing $C_2 = C_3C_4$, we conclude the proof. □

3. Scaling limit of multi-arm IIC

In this section we will prove our main results concerning the scaling limit of k-arm IIC. First we give some lemmas that will be used. The following lemma can be seen as an analog of Lemma 2.9 in [12] for quad-crossing percolation limit.

Lemma 3.1. *For any $0 < r < R$ and $k = 1, 2, 4$, there exists a constant $C_k > 0$ (depending on r/R), such that*

$$\lim_{\eta\to 0} \mu_R^\eta\big[\mathcal{A}_k^\eta(r,R)\big] = \mu_R\big[\mathcal{A}_k(r,R)\big] > C_k. \tag{16}$$

Moreover, in any coupling of the measures $\{\mu_R^\eta\}$ and μ_R on $(\Omega_R, \mathcal{F}_R)$ in which $\operatorname{dist}(\omega_R^\eta, \omega_R) \to 0$ a.s. as $\eta \to 0$, we have

$$\hat{P}\big[\big\{\omega_R^\eta \in \mathcal{A}_k^\eta(r,R)\big\}\Delta\big\{\omega_R \in \mathcal{A}_k(r,R)\big\}\big] \to 0 \quad \text{as } \eta \to 0, \tag{17}$$

where $\hat{P}[\cdot]$ denotes the coupling measure.

Proof. By Theorem 1.1, we can couple the measures $\{\mu_R^\eta\}$ and μ_R on $(\Omega_R, \mathcal{F}_R)$ such that $\operatorname{dist}(\omega_R^\eta, \omega_R) \to 0$ a.s. as $\eta \to 0$. Let us show (17) for $k = 1, 2, 4$ respectively in the following.

By (1) and the definition of $\mathcal{A}_1(r,R)$, it is easy to see that for each small $\varepsilon > 0$ and $\eta < \varepsilon$,

$$\begin{aligned}
&\hat{P}\big[\big\{\omega_R^\eta \in \mathcal{A}_1^\eta(r,R)\big\}\Delta\big\{\omega_R \in \mathcal{A}_1(r,R)\big\}\big] \\
&\quad \le \hat{P}\big[\operatorname{dist}\big(\omega_R^\eta,\omega_R\big) \ge \varepsilon\big] + \hat{P}\left[\begin{array}{l}\exists \text{ counterclockwise loop } \gamma^\eta \in \omega_R^\eta \text{ surrounding the} \\ \text{origin in } A(r-\varepsilon, R), \text{ and } \gamma^\eta \cap A(r-\varepsilon, r+\varepsilon) \neq \varnothing\end{array}\right].
\end{aligned}$$

The first term goes to zero as $\eta \to 0$. The event in the second term produces a half-plane 3-arm event from the 2ε-neighborhood of $\partial\mathbb{D}_r$ to a distance of unit order, whose probability goes to zero as $\varepsilon \to 0$, since the polychromatic half-plane 3-arm exponent is 2; see, e.g., Lemma 6.8 in [35]. Then (17) is proved in the case $k = 1$.

By (2) and the definition of $\mathcal{A}_2(r,R)$, for each small $\varepsilon > 0$ and $\eta < \varepsilon$,

$$\begin{aligned}
&\hat{P}\big[\big\{\omega_R \in \mathcal{A}_2(r,R)\big\} \setminus \big\{\omega_R^\eta \in \mathcal{A}_2^\eta(r,R)\big\}\big] \\
&\quad \le \hat{P}\big[\operatorname{dist}\big(\omega_R^\eta,\omega_R\big) \ge \varepsilon\big] + \hat{P}\left[\begin{array}{l}\exists \text{ counterclockwise loop } \gamma^\eta \in \omega_R^\eta \text{ intersecting with } \partial\mathbb{D}_{r+\varepsilon} \\ \text{and } \partial\mathbb{D}_{R-\varepsilon} \text{ in } \overline{\mathbb{D}_R^\eta}, \text{ and } \gamma^\eta \cap \partial\mathbb{D}_r^\eta = \varnothing \text{ or } \gamma^\eta \cap \partial\mathbb{D}_R^\eta = \varnothing\end{array}\right].
\end{aligned}$$

The event in the second term implies a half-plane 3-arm event from the 2ε-neighborhood of $\partial\mathbb{D}_r$ or $\partial\mathbb{D}_R$ to a distance of unit order, whose probability goes to zero as $\varepsilon \to 0$. Then we get that $\hat{P}[\{\omega_R \in \mathcal{A}_2(r,R)\} \setminus \{\omega_R^\eta \in \mathcal{A}_2^\eta(r,R)\}] \to 0$ as $\eta \to 0$. Now let us show the other direction. Similarly, for each small $\varepsilon > 0$ and $\eta < \varepsilon$, we have

$$\begin{aligned}
&\hat{P}\big[\big\{\omega_R^\eta \in \mathcal{A}_2^\eta(r,R)\big\} \setminus \big\{\omega_R \in \mathcal{A}_2(r,R)\big\}\big] \\
&\quad \le \hat{P}\big[\operatorname{dist}\big(\omega_R^\eta,\omega_R\big) \ge \varepsilon\big] + \hat{P}\left[\begin{array}{l}\exists \text{ counterclockwise loop } \gamma \in \omega_R \text{ intersecting with } \partial\mathbb{D}_{r+\varepsilon} \\ \text{and } \partial\mathbb{D}_{R-2\varepsilon} \text{ in } \overline{\mathbb{D}}_R, \text{ and } \gamma \cap \partial\mathbb{D}_r = \varnothing \text{ or } \gamma \cap \partial\mathbb{D}_R = \varnothing\end{array}\right].
\end{aligned}$$

Clearly the second term goes to zero as $\varepsilon \to 0$. Then (17) is proved in the case $k = 2$.

Similarly to the case $k = 2$, one can prove the case where $k = 4$, and the details are omitted.

(11) and (17) imply (16) immediately. □

A collection of measures is said to be *(weakly) relatively compact* if every sequence has a convergent subsequence. To prove the existence of the scaling limit, we need a lemma on the existence of subsequential scaling limits:

Lemma 3.2. *Let $k = 1, 2, 4$. $\{\mu_R^\eta[\cdot \mid \mathcal{A}_k^\eta(\eta, R)]\}_\eta$ and $\{\mu_k^\eta[\cdot]\}_\eta$ are relatively compact.*

Proof. We use the machinery developed by Aizenman and Burchard (Theorem 1.2 in [1]). Let μ^η denote the probability measure supported on collections of curves that are polygonal paths on the edges of $\eta\mathbb{H}$ in $\overline{\mathbb{D}_R^\eta}$. In our setting, Hypothesis H1 of [1] is as follows.

Hypothesis H1. For all $j \in \mathbb{N}$, $z \in \overline{\mathbb{D}}_R$ and $\eta \le r_1 < r_2 \le 1$, the following bound holds uniformly in η and z:

$$\mu^\eta\big[A(z; r_1, r_2) \text{ is traversed } j \text{ times by a curve}\big] \le K_j (r_1/r_2)^{\phi(j)}$$

for some $K_j < \infty$ and $\phi(j) \to \infty$ as $j \to \infty$.

Observe that the number of segments of a loop crossing an annulus is necessarily even and that, if the annulus is traversed by $j \in 2\mathbb{N}$ separate segments of a loop $\in \omega_R^\eta$, there will be $j/2$ disjoint yellow arms crossing this annulus. Now let us prove that $\{\mu_R^\eta[\cdot|\mathcal{A}_k^\eta(\eta, R)]\}_\eta$ satisfies Hypothesis H1 for $k = 1, 2, 4$. First, we do this in the case $k = 1$, which is the easiest one.

The BK inequality and (11) imply that there exist constants $C > 1$, $\lambda > 0$, such that for all $\eta \le r_1 < r_2$, $z \in \mathbb{C}$ and $j \in \mathbb{N}$,

$$P_R^\eta\big[\mathcal{A}_{j,Y\cdots Y}^\eta(z; r_1, r_2)\big] \le \big\{P_R^\eta\big[\mathcal{A}_Y^\eta(z; r_1, r_2)\big]\big\}^j \le C^j (r_1/r_2)^{\lambda j}. \tag{18}$$

Let $j \in 2\mathbb{N}$, $\eta \le r_1 < r_2$, $z \in \overline{\mathbb{D}}_R$, we have

$$\begin{aligned}
&P_R^\eta\big[A(z; r_1, r_2) \text{ is traversed } j \text{ times by a loop } |\mathcal{A}_1^\eta(\eta, R)\big] \\
&\quad\le \frac{P_R^\eta[\mathcal{A}_1^\eta(\eta, R), \mathcal{A}_{j/2,Y\cdots Y}^\eta(z; r_1, r_2)]}{P_R^\eta[\mathcal{A}_1^\eta(\eta, R)]} \\
&\quad\le P_R^\eta\big[\mathcal{A}_{j/2,Y\cdots Y}^\eta(z; r_1, r_2)\big] \quad \text{by Reimer's inequality} \\
&\quad\le C^{j/2}(r_1/r_2)^{\lambda j/2} \quad \text{by (18)}.
\end{aligned}$$

Now let us consider the cases of $k = 2, 4$. Without loss of generality, we assume $10\eta \le 10r_1 \le r_2 \le R/4$, $j \in 2\mathbb{N}$ and $j \ge k + 2$. Let C_1, C_2, C_3 (just depending on k) be appropriate positive constants. We will distinguish the following four cases (see Figure 3).

Case 1: $R/2 \le |z| \le R$.

$$\begin{aligned}
&P_R^\eta\big[A(z; r_1, r_2) \text{ is traversed } j \text{ times by a loop } |\mathcal{A}_k^\eta(\eta, R)\big] \\
&\quad\le \frac{P_R^\eta[\mathcal{A}_k^\eta(\eta, R/5)]}{P_R^\eta[\mathcal{A}_k^\eta(\eta, R)]} P_R^\eta\big[\mathcal{A}_{j/2,Y\cdots Y}^\eta(z; r_1, r_2)\big] \\
&\quad\le C_1 C^{j/2}(r_1/r_2)^{\lambda j/2} \quad \text{by quasi-multiplicativity and (18)}.
\end{aligned}$$

Case 2: $r_2/3 \le |z| \le R/2$.

$$\begin{aligned}
&P_R^\eta\big[A(z; r_1, r_2) \text{ is traversed } j \text{ times by a loop } |\mathcal{A}_k^\eta(\eta, R)\big] \\
&\quad\le \frac{P_R^\eta[\mathcal{A}_k^\eta(\eta, |z| - r_2/4), \mathcal{A}_k^\eta(|z| + r_2/4, R)]}{P_R^\eta[\mathcal{A}_k^\eta(\eta, R)]} P_R^\eta\big[\mathcal{A}_{j/2,Y\cdots Y}^\eta(z; r_1, r_2/4)\big] \\
&\quad\le C_2 C^{j/2}(r_1/r_2)^{\lambda j/2} \quad \text{by quasi-multiplicativity and (18)}.
\end{aligned}$$

Case 3: $3r_1 \le |z| \le r_2/3$.

$$\begin{aligned}
&P_R^\eta\big[A(z; r_1, r_2) \text{ is traversed } j \text{ times by a loop } |\mathcal{A}_k^\eta(\eta, R)\big] \\
&\quad\le P_R^\eta\big[\mathcal{A}_k^\eta\big(\eta, |z|/2\big), \mathcal{A}_k^\eta\big(\overline{\mathbb{D}(z; r_2)} \setminus \mathbb{D}\big(z/2; |z|\big)\big), \mathcal{A}_k^\eta\big(\overline{\mathbb{D}}_R \setminus \mathbb{D}(z; r_2)\big), \\
&\mathcal{A}_{j/2,Y\cdots Y}^\eta\big(z; r_1, |z|/2\big), \mathcal{A}_{j/2,Y\cdots Y}^\eta\big(\overline{\mathbb{D}(z; r_2)} \setminus \mathbb{D}\big(z/2; |z|\big)\big)\big]/P_R^\eta\big[\mathcal{A}_k^\eta(\eta, R)\big]
\end{aligned}$$

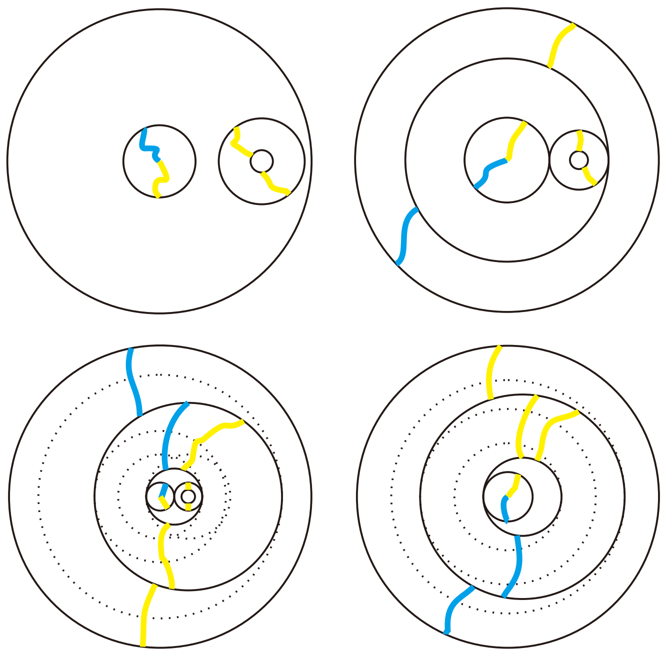

Fig. 3. A sketch of the four cases in the proof of Lemma 3.2 ($k=2$, $j=4$).

$$\leq \frac{P_R^\eta[\mathcal{A}_k^\eta(\eta,|z|/2),\mathcal{A}_k^\eta(3|z|/2,r_2-|z|),\mathcal{A}_k^\eta(r_2+|z|,R)]}{P_R^\eta[\mathcal{A}_k^\eta(\eta,R)]}$$

$$\times P_R^\eta\big[\mathcal{A}^\eta_{j/2,Y\cdots Y}\big(z;r_1,|z|/2\big),\mathcal{A}^\eta_{(j-k)/2,Y\cdots Y}\big(z;3|z|/2,r_2\big)\big] \quad \text{by Reimer's inequality}$$

$$\leq C_3C^{(j-k)/2}(r_1/r_2)^{\lambda(j-k)/2} \quad \text{by quasi-multiplicativity and (18).}$$

Case 4: $|z|\leq 3r_1$.

$$P_R^\eta\big[A(z;r_1,r_2) \text{ is traversed } j \text{ times by a loop } |\mathcal{A}_k^\eta(\eta,R)\big]$$

$$\leq \frac{P_R^\eta[\mathcal{A}_k^\eta(\eta,r_1),\mathcal{A}_k^\eta(z;|z|+r_1,r_2),\mathcal{A}_k^\eta(\overline{\mathbb{D}}_R\setminus\mathbb{D}(z;r_2)),\mathcal{A}^\eta_{j/2,Y\cdots Y}(z;|z|+r_1,r_2)]}{P_R^\eta[\mathcal{A}_k^\eta(\eta,R)]}$$

$$\leq \frac{P_R^\eta[\mathcal{A}_k^\eta(\eta,r_1),\mathcal{A}_k^\eta(2|z|+r_1,r_2-|z|),\mathcal{A}_k^\eta(r_2+|z|,R)]}{P_R^\eta[\mathcal{A}_k^\eta(\eta,R)]}$$

$$\times P_R^\eta\big[\mathcal{A}^\eta_{(j-k)/2,Y\cdots Y}\big(z;|z|+r_1,r_2\big)\big] \quad \text{by Reimer's inequality}$$

$$\leq C_4C^{(j-k)/2}(r_1/r_2)^{\lambda(j-k)/2} \quad \text{by quasi-multiplicativity and (18).}$$

Hence, for $k=1,2,4$, $\{\mu_R^\eta[\cdot|\mathcal{A}_k^\eta(\eta,R)]\}_\eta$ satisfies Hypothesis H1. Then Theorem 1.2 in [1] implies that it is relatively compact.

For the relatively compactness of $\{\mu_k^\eta[\cdot]\}_\eta$, we need to consider $\hat{\mathbb{C}}$ with metric (6). It is noted in the Remark just below Theorem 3.1 in [2], although Theorem 1.2 in [1] was formulated for compact subsets $\Lambda\subset\mathbb{R}^d$, it also applies to this case. By the inequalities above and the definition of μ_k^η, we have that there exists a constant $C_5>0$ depending on k, such that

$$\mu_k^\eta\big(A(z;r_1,r_2) \text{ is traversed } j \text{ times by a loop}\big)$$

$$= \lim_{R\to\infty} P_R^\eta\big[A(z;r_1,r_2) \text{ is traversed } j \text{ times by a loop } |\mathcal{A}_k^\eta(\eta,R)\big] \leq C_5C^j(r_1/r_2)^{\lambda(j-k)/2}.$$

Similarly to the proof of (i) of Theorem 1.1 in [2], by Lemma 3.3 in [2], the corresponding bound on crossing probabilities holds (with the same exponents) also for the system on $\hat{\mathbb{C}}$ with the metric (6). Then Theorem 1.2 in [1] implies that $\{\mu_k^\eta[\cdot]\}_\eta$ is relatively compact for $k = 1, 2, 4$. □

The following lemma is a particular case of the first part of Theorem 1.2, the proof of the general case is essentially the same as for this lemma.

Lemma 3.3. *Let $k = 1, 2, 4$. For each $R > 0$, as $\eta \to 0$ and $\varepsilon \to 0$, $\mu_R^\eta[\cdot|\mathcal{A}_k^\eta(\eta, R)]$ and $\mu_R[\cdot|\mathcal{A}_k(\varepsilon, R)]$ converge in law, under the topology induced by metric (8), to the same probability distribution, denoted by $\mu_{k,R}$.*

Proof. By Theorem 1.1 and Lemma 3.1, for any fixed small $\varepsilon > 0$ and $\delta > 0$, we can couple $\mu_R^\eta[\cdot|\mathcal{A}_k^\eta(\varepsilon, R)]$ and $\mu_R[\cdot|\mathcal{A}_k(\varepsilon, R)]$ for all η small enough, such that with probability at least $1 - \delta$,

$$\operatorname{dist}\bigl(\omega_{k,\varepsilon,R}^\eta, \omega_{k,\varepsilon,R}\bigr) \le \delta, \tag{19}$$

where $\omega_{k,\varepsilon,R}^\eta, \omega_{k,\varepsilon,R}$ are the configurations under these two laws.

By Proposition 2.1, there exists a constant $\alpha > 0$ such that for small $\varepsilon > 10\eta$, we can couple $\mu_R^\eta[\cdot|\mathcal{A}_k^\eta(\varepsilon, R)]$ and $\mu_R^\eta[\cdot|\mathcal{A}_k^\eta(\eta, R)]$, such that with probability at least $1 - \varepsilon^{\alpha/2}$, there exists an identical k-circuit $\mathcal{C} = \mathcal{C}(\eta, \varepsilon)$ surrounding the origin in $A(\varepsilon, \sqrt{\varepsilon})$ for both measures, and the configuration outside $\mathcal{C}$ is also identical, and furthermore $U_{\mathcal{C}}$ is identical in the case $k = 4$. Observe that when the above event happens,

$$\operatorname{dist}\bigl(\omega_{k,\varepsilon,R}^\eta, \omega_{k,\eta,R}^\eta\bigr) \le 2\sqrt{\varepsilon}. \tag{20}$$

Let us now explain (20) separately in the three cases. Assume that the above event holds. If $k = 1$, any loop from $\omega_{k,\varepsilon,R}^\eta$ or $\omega_{k,\eta,R}^\eta$ is either inside or outside $\mathcal{C}$, and the loop configuration outside $\mathcal{C}$ is identical for $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$. Then (20) holds obviously. If $k = 2$, the loops entirely outside $\mathcal{C}$ are identical for $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$. Furthermore, both $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$ have a unique loop crossing the 2-circuit $\mathcal{C}$ which is composed of two curves, one outside $\mathcal{C}$ and the other inside, and the outside one is identical for $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$. From this one gets (20) easily. Suppose $k = 4$, the loops entirely outside $\mathcal{C}$ are identical for $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$. Furthermore, if $U_{\mathcal{C}} = 0$, both $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$ have a unique loop crossing the 4-circuit $\mathcal{C}$ which is composed of four curves, two outside $\mathcal{C}$ and the others inside, and the outside ones are identical for $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$; if $U_{\mathcal{C}} = 1$, both $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$ have exactly two loops crossing $\mathcal{C}$, with each loop composed of two curves, one outside $\mathcal{C}$ and the other inside, and the outside ones are identical for $\omega_{k,\varepsilon,R}^\eta$ and $\omega_{k,\eta,R}^\eta$. Then one obtains (20).

Combining (19) and (20), for each $\delta > 0$, $\varepsilon > 0$, there exists $\eta_0(\delta, \varepsilon) > 0$ such that for each $\eta < \eta_0$, we can couple $\mu_R[\cdot|\mathcal{A}_k(\varepsilon, R)]$ and $\mu_R^\eta[\cdot|\mathcal{A}_k^\eta(\eta, R)]$ such that with probability $1 - \delta - \varepsilon^{\alpha/2}$,

$$\operatorname{dist}\bigl(\omega_{k,\varepsilon,R}, \omega_{k,\eta,R}^\eta\bigr) \le \delta + 2\sqrt{\varepsilon}. \tag{21}$$

Lemma 3.2 says that there exist subsequential limits of $\mu_R^\eta[\cdot|\mathcal{A}_k^\eta(\eta, R)]$ as $\eta \to 0$, (21) implies the uniqueness of the limit, and we denote it by $\mu_{k,R}$. (21) also implies that $\mu_R[\cdot|\mathcal{A}_k(\varepsilon, R)]$ converges in law to $\mu_{k,R}$ as $\varepsilon \to 0$. □

The conformal invariance of CLE_6 is expressed in the following theorem, which will be used in the proof of Theorem 1.2.

Theorem 3.4 ([7]). *Let D, D' be two Jordan domains and let $f : \overline{D} \to \overline{D'}$ be a continuous function that maps D conformally onto D'. Then the CLE_6 in $\overline{D'}$ is distributed like the image under f of the CLE_6 in $\overline{D}$.*

To prove Theorem 1.2, we also need the following lemma about conformal transformations, which is Corollary 3.25 in [24].

Lemma 3.5 ([24]). *Let D, D' be two Jordan domains. If $f : D \to D'$ is a conformal transformation with $z \in D$, then for all $0 < r < 1$ and all $|w - z| \le r\,\mathrm{dist}(z, \partial D)$,*

$$\left|f(w) - f(z)\right| \le \frac{4|w - z|}{(1 - r)^2} \frac{\mathrm{dist}(f(z), \partial D')}{\mathrm{dist}(z, \partial D)}.$$

Proof of Theorem 1.2. The first part of Theorem 1.2 is a generalization of Lemma 3.3. Its proof is basically the same as for Lemma 3.3, and we omit it. Now we show the second part. For any small $\varepsilon > 0$, by the definitions of f and z', it is easy to see that $f(\mathbb{D}(z; \varepsilon))$ is a Jordan domain, $z' \in f(\mathbb{D}(z; \varepsilon))$ and $\overline{f(\mathbb{D}(z; \varepsilon))} \subset D'$. Further, Lemma 3.5 implies that the diameter of $f(\mathbb{D}(z; \varepsilon))$ converges to zero as $\varepsilon \to 0$. By the definitions of arm events and Theorem 3.4, the image of $\mu_D[\cdot|\mathcal{A}_k(\overline{D} \setminus \mathbb{D}(z; \varepsilon))]$ under f has the same law as $\mu_{D'}[\cdot|\mathcal{A}_k(\overline{D'} \setminus f(\mathbb{D}(z; \varepsilon)))]$. Then the first part of Theorem 1.2 implies the second part of Theorem 1.2 immediately. □

Proof of Theorem 1.3. By Proposition 2.2, given any bounded domain D, for each $\varepsilon > 0$, there exists a $R_0(D, \varepsilon) > 0$, such that for any $R_2 > R_1 > R_0$ and any small enough η, we can couple $\mu^{\eta}_{R_1}[\cdot|\mathcal{A}^{\eta}_k(\eta, R_1)]$ and $\mu^{\eta}_{R_2}[\cdot|\mathcal{A}^{\eta}_k(\eta, R_2)]$ such that with probability at least $1 - \varepsilon$, the cluster boundaries or portions of boundaries contained in $\overline{D}$ are identical. Therefore, letting $\eta \to 0$ and using Lemma 3.3, there is a coupling between μ_{k,R_1} and μ_{k,R_2}, such that with probability at least $1 - \varepsilon$ the loops or portions of loops contained in $\overline{D}$ are identical. Taking $\varepsilon \to 0$ and $R = R(\varepsilon) \to \infty$, we get that $\hat{I}_D \mu_{k,R}$ converges in law to a probability measure. For $D = \mathbb{D}_r$, we denote the above limiting measure by $\mu'_{k,r}$. The above argument also implies that $\mu'_{k,r}$ on $(\Omega_r, \mathcal{B}_r)$, for $r > 0$, satisfy the consistency $\mu'_{k,r_1} = \hat{I}_{\mathbb{D}_{r_1}} \mu'_{k,r_2}$ conditions for all $0 < r_1 < r_2$. Then using Kolmogorov's extension theorem (see, e.g., [9]) we conclude that there exists a unique probability measure μ_k on $(\Omega, \mathcal{B})$ with $\mu'_{k,r} = \hat{I}_{\mathbb{D}_r} \mu_k$ for all $r > 0$. For any domain $D \subset \mathbb{D}_r$, the above discussion implies that as $R \to \infty$, $\hat{I}_D \mu_{k,R} \to \hat{I}_D \mu'_{k,r} = \hat{I}_D \mu_k$.

By Lemma 3.2, we let $\{\eta_j\}$ be a convergent subsequence for μ^{η}_k and let μ'_k be the limit in distribution of $\mu^{\eta_j}_k$ as $\eta_j \to 0$. Now we show $\mu'_k = \mu_k$. To achieve this, it is enough to prove that $\hat{I}_{\mathbb{D}_r} \mu'_k = \hat{I}_{\mathbb{D}_r} \mu_k$ for all $r > 0$, which is achieved as follows.

By the definition of μ^{η}_k, for each $\varepsilon > 0$, there exist $\eta_0 > 0$, $R_0 > 0$, such that for all $\eta < \eta_0$ and all $R > R_0$, we can couple $\hat{I}_{\mathbb{D}_r} \mu^{\eta}_k$ and $\hat{I}_{\mathbb{D}_r} \mu^{\eta}_R[\cdot|\mathcal{A}^{\eta}_k(\eta, R)]$ such that with probability at least $1 - \varepsilon$,

$$\mathrm{Dist}\big(\omega^{\eta}_{k,r}, \omega^{\eta}_{k,r,R}\big) \le \varepsilon,$$

where $\omega^{\eta}_{k,r}, \omega^{\eta}_{k,r,R}$ are the configurations under these two laws. Using Lemma 3.3 and the definition of μ'_k, by taking $\eta_j \to 0$, we can couple $\hat{I}_{\mathbb{D}_r} \mu'_k$ and $\hat{I}_{\mathbb{D}_r} \mu_{k,R}$ such that with probability at least $1 - \varepsilon$,

$$\mathrm{Dist}\big(\omega'_{k,r}, \omega_{k,r,R}\big) \le \varepsilon,$$

where $\omega'_{k,r}, \omega_{k,r,R}$ are the configurations under these two laws. Taking $R \to \infty$ and then $\varepsilon \to 0$, by the first part of the proof of Theorem 1.3, we have $\hat{I}_{\mathbb{D}_r} \mu'_k = \hat{I}_{\mathbb{D}_r} \mu_k$. □

4. Winding numbers

In this section we will prove our main results concerning the variance estimate and CLT for winding numbers of the arms in the 2-arm case. We will use two-sided radial SLE_6, which is introduced below. We assume that the reader is familiar with the basic theory of SLE. (See, for instance, Lawler's book [24].) For the basic results regarding two-sided radial SLE, we refer to [10,25,26,28].

4.1. *Winding for two-sided SLE*

To introduce two-sided radial SLE, we need the notion of Green's function for chordal SLE. Roughly speaking, the Green's function gives the normalized probability that the chordal SLE path goes through an interior point. Before stating the precise definition, we set up some notation. If D is a simply connected domain with $z \in D$, we let $\Upsilon_D(z)$ be twice the conformal radius of z in D; that is, if $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = z$, then $\Upsilon_D(z) = 2|f'(0)|$. Suppose $0 < \kappa < 8$, $a, b \in \partial D$, let $\gamma = \gamma_{D,a,b}$ denote chordal SLE$_\kappa$ path from a to b in $\overline{D}$. Let D_∞ denote the component of $D \setminus \gamma$ containing z. The *Green's function* $G_D(z; a, b)$ for γ is defined by

$$\lim_{\varepsilon\to 0} \varepsilon^{d-2} P\big[\Upsilon_{D_\infty}(z) < \varepsilon\big] = C_* G_D(z; a, b),$$

where $d := 1 + \kappa/8$ is the Hausdorff dimension of SLE$_\kappa$ path, $C_* := 2[\int_0^\pi \sin^{8/\kappa} x\,dx]^{-1}$. See, e.g., [28] and Proposition 2.2 in [26]. In fact, for the Euclidean distance, there also exists a constant $\hat{C} > 0$ (the value of $\hat{C}$ is unknown) such that

$$\lim_{\varepsilon\to 0} \varepsilon^{d-2} P\big[\operatorname{dist}(z, \gamma) < \varepsilon\big] = \hat{C} G_D(z; a, b).$$

Furthermore, Lawler and Rezaei proved that the Green's function satisfies the following proposition (Theorem 2.3 in [26], see also Theorem 2.3 in [27]):

Proposition 4.1 ([26]). *Suppose* $0 < \kappa < 8$. *There exist* $0 < \hat{C}, C, u < \infty$ (*depending on* κ) *such that the following holds. Suppose D is a simply connected domain*, $z \in D, a, b, \in \partial D$ *and γ is a chordal SLE$_\kappa$ path from a to b in $\overline{D}$. Then, for all* $0 < \varepsilon < \operatorname{dist}(z, \partial D)/10$,

$$\left| \frac{P[\operatorname{dist}(z, \gamma) \le \varepsilon]}{\varepsilon^{2-d} G_D(z; a, b)} - \hat{C} \right| \le C \left(\frac{\varepsilon}{\operatorname{dist}(z, \partial D)} \right)^u.$$

Assume $0 < \kappa < 8$ and $0 < \alpha < 2\pi$. Roughly speaking, a two-sided radial SLE$_\kappa$ path from 1 to $e^{i\alpha}$ through 0 in $\overline{\mathbb{D}}$ can be thought of as a chordal SLE$_\kappa$ path γ from 1 to $e^{i\alpha}$ in $\overline{\mathbb{D}}$, conditioned to pass through 0 (see Proposition 4.2 below). The curve can be defined by weighting γ in the sense of the Girsanov theorem by Green's function in the slit domain at 0. More precisely, we parametrize γ by the radial parametrization (i.e., $g_t'(0) = e^t$), and let $M_t := G_{\mathbb{D}\setminus\gamma[0,t]}(0; \gamma(t), e^{i\alpha})$, which is a local martingale. Then using Girsanov's theorem, we can define a new probability measure P^* which corresponds to paths "weighted locally by M_t". That is,

$$P^*[V] = M_0^{-1} E[M_t 1_V] \quad \text{for } V \in \mathcal{F}_t,$$

where E denotes expectation with respect to P, $\mathcal{F}_t$ denotes the σ-algebra generated by $\{\hat{W}_s, 0 \le s \le t\}$, and $\hat{W}$ is a standard Brownian motion and is the driving function of γ.

Explicitly, if γ denotes the two-sided radial SLE$_\kappa$ path from 1 to $e^{i\alpha}$ through 0 in $\overline{\mathbb{D}}$ stopped when it reaches 0, D_t denotes the connected component of $\mathbb{D} \setminus \gamma(0, t]$ containing the origin, and g_t (two-sided radial SLE$_\kappa$): $D_t \to \mathbb{D}$ is the conformal transformation with $g_t(0) = 0$, $g_t'(0) = e^t$, then g_t can be obtained from solving the initial value problem

$$\partial_t g_t(z) = g_t(z) \frac{e^{iU_t} + g_t(z)}{e^{iU_t} - g_t(z)}, \tag{22}$$

$$d\Theta_t = 2\cot\left[\frac{\Theta_t}{2}\right] dt + \sqrt{\kappa}\, dW_t, \qquad \Theta_0 = \alpha, \qquad -dU_t = \cot\left[\frac{\Theta_t}{2}\right] dt + \sqrt{\kappa}\, dW_t, \tag{23}$$

where W is a standard Brownian motion with respect to P^*. Further, if we write $g_t(e^{i\alpha}) = e^{iV_t}$, then

$$\Theta_t = V_t - U_t. \tag{24}$$

Note that we write the equation slightly differently than in [10,25], where the authors added a parameter $a = 2/\kappa$ that gives a linear time change, and wrote $2U_t$ in the exponent in (22).

Given $r > 0$ and a curve γ, let $\tau_r = \tau_r(\gamma)$ be the first hitting time with $\partial\mathbb{D}_r$ of γ. The following proposition is the analog of Proposition 2.13 in [28], replacing conformal radius with Euclidean distance. The justification for calling two-sided radial SLE "chordal SLE conditioned to go through an interior point" comes from this proposition.

Proposition 4.2. *Let $0<\kappa<8$ and $0<\alpha<2\pi$. There exist $0<u, C<\infty$ (depending on κ) such that the following is true. Suppose γ is a chordal SLE$_\kappa$ path from 1 to $e^{i\alpha}$. Suppose $0<\varepsilon<1/10$, $0<\varepsilon'<\varepsilon/10$. Let μ', μ^* be the two probability measures on $\{\gamma(t):0\le t\le\tau_\varepsilon\}$ corresponding to chordal SLE$_\kappa$ conditioned on the event $\{\tau_{\varepsilon'}<\infty\}$ and two-sided radial SLE$_\kappa$ through 0, respectively. Then μ', μ^* are mutually absolutely continuous with respect to each other and the Radon–Nikodym derivative satisfies*

$$\left|\frac{d\mu^*}{d\mu'}-1\right|\le C\left(\frac{\varepsilon'}{\varepsilon}\right)^u.$$

Proof. Let $P[\cdot]$ denote the law of the entire γ and let $P_\varepsilon[\cdot]$ denote the law of $\{\gamma(t):0\le t\le\tau_\varepsilon\}$ restricted to the event $\{\tau_\varepsilon<\infty\}$. From the definitions of μ', μ^*, we know that

$$d\mu^*=\frac{M_{\tau_\varepsilon}}{M_0}\,dP_\varepsilon,\qquad d\mu'=\frac{P[\operatorname{dist}(0,\gamma)\le\varepsilon'|\gamma[0,\tau_\varepsilon]]}{P[\operatorname{dist}(0,\gamma)\le\varepsilon']}\,dP_\varepsilon.$$

So μ' and μ^* are mutually absolutely continuous. Denote by E the expectation with respect to P, by $\mathcal{F}_\varepsilon$ the σ-algebra generated by $\gamma[0,\tau_\varepsilon]$, by T the time that γ reaches $e^{i\alpha}$, by $P_{\mathbb{D}\setminus\gamma[0,\tau_\varepsilon]}$ the law of $\gamma[\tau_\varepsilon,T]$. Using Proposition 4.1, we have that for each $V\in\mathcal{F}_\varepsilon$,

$$\begin{aligned}
\mu^*(V)&=M_0^{-1}E[M_{\tau_\varepsilon}1_V]\\
&=G_{\mathbb{D}}\big(0;1,e^{i\alpha}\big)^{-1}E\big[G_{\mathbb{D}\setminus\gamma[0,\tau_\varepsilon]}\big(0;\gamma(\tau_\varepsilon),e^{i\alpha}\big)1_V\big]\\
&=G_{\mathbb{D}}\big(0;1,e^{i\alpha}\big)^{-1}E\big[E\big[G_{\mathbb{D}\setminus\gamma[0,\tau_\varepsilon]}\big(0;\gamma(\tau_\varepsilon),e^{i\alpha}\big)|\mathcal{F}_\varepsilon\big]1_V\big]\\
&=\big(\varepsilon'\big)^{2-d}\hat{C}\big(1+O\big(\big(\varepsilon'\big)^u\big)\big)P\big[\operatorname{dist}(0,\gamma)\le\varepsilon'\big]^{-1}\\
&\qquad\times E\big[E\big[\big(\varepsilon'\big)^{d-2}\hat{C}^{-1}\big(1+O\big(\big(\varepsilon'/\varepsilon\big)^u\big)\big)P_{\mathbb{D}\setminus\gamma[0,\tau_\varepsilon]}\big[\operatorname{dist}\big(0,\gamma[\tau_\varepsilon,T]\big)\le\varepsilon'\big]|\mathcal{F}_\varepsilon\big]1_V\big]\\
&=\big(1+O\big(\big(\varepsilon'/\varepsilon\big)^u\big)\big)P\big[\operatorname{dist}(0,\gamma)\le\varepsilon'\big]^{-1}E\big[E\big[P_{\mathbb{D}\setminus\gamma[0,\tau_\varepsilon]}\big[\operatorname{dist}\big(0,\gamma[\tau_\varepsilon,T]\big)\le\varepsilon'\big]|\mathcal{F}_\varepsilon\big]1_V\big]\\
&=\big(1+O\big(\big(\varepsilon'/\varepsilon\big)^u\big)\big)\mu'(V).
\end{aligned}$$

Then the result follows from the above inequality. □

The following lemma for two-sided radial SLE is an analog of Theorem 7.2 for radial SLE in [31].

Lemma 4.3. *Let $0<\kappa<8$. Suppose γ is a two-sided radial SLE$_\kappa$ path from 1 to -1 through 0 in $\overline{\mathbb{D}}$ stopped when it reaches 0. Let $T\ge0$, and $\theta_\kappa(T)$ be the winding number of the path $\gamma[0,T]$ around 0. Then there exist constants $C_0, C_1>0$ depending only on κ, such that for all $s>0$,*

$$P^*\big[\big|T+\log\big|\gamma(T)\big|\big|>s\big]\le C_0\exp(-C_1s),\tag{25}$$

and

$$P^*\left[\left|\theta_\kappa(T)+\frac{\sqrt{\kappa}}{2}W_T\right|>s\right]\le C_0\exp(-C_1s).\tag{26}$$

Proof. Schwarz Lemma and the Koebe 1/4 Theorem give

$$\operatorname{dist}\big(0,\gamma[0,T]\big)\le e^{-T}=1/g_t'(0)\le4\operatorname{dist}\big(0,\gamma[0,T]\big)\le4\big|\gamma(T)\big|,\tag{27}$$

which implies

$$\log\big|\gamma(T)\big|\ge-T+\log4.\tag{28}$$

By Theorem 3 in [25], there exist $C_2(\kappa), C_3(\kappa) > 0$ such that for all $k, n \in \mathbb{N}$,

$$P^*\big[\gamma[\tau_{e^{-n-k}}, \infty) \cap \partial\mathbb{D}_{e^{-k}} \neq \varnothing\big] \le C_2 \exp(-C_3 n). \tag{29}$$

(27) and (29) imply that there exist $C_4, C_5 > 0$, such that

$$P^*\big[T + \log\big|\gamma(T)\big| > s\big] \le P^*\big[\gamma[\tau_{e^{-T}}, \infty) \cap \partial\mathbb{D}_{e^{-T+s}} \neq \varnothing\big] \le C_4 \exp(-C_5 s).$$

Combining this with (28), we get (25).

The proof of (26) is similar to that of (7.3) in [31], we just sketch it here. For $t \in [0, T]$, let $y(t) := \arg[g_t(\gamma(T))]$, where arg is chosen to be continuous in t. Using the argument in the proof of (7.3) in [31], one can show that

$$\theta_\kappa(T) = U_T - U_0 + y(0) - y(T). \tag{30}$$

By (23), we have

$$-U_t = \frac{\Theta_t}{2} + \frac{\sqrt{\kappa}}{2} W_t. \tag{31}$$

From (24), we have $0 \le \Theta_t \le 2\pi$ for all $t \ge 0$. Then, by (30) and (31), proving (26) boils down to prove the appropriate bound on the tail of $|y(0) - y(T)|$. Let τ_1 be the largest $t \in [0, T]$ such that $\log|g_t(\gamma(T))| \le -1$, and set $\tau_1 = 0$ if such a t does not exist. Analogous to the proof of (7.7) in [31], it can be shown that $|y(0) - y(\tau_1)| < \infty$. Now let us bound $|y(\tau_1) - y(T)|$. Set $t_0 = T$, and inductively, let t_j be the last $t \in [0, t_{j-1}]$ such that $\pi/2 = \min\{|\sqrt{\kappa} U_t - \sqrt{\kappa} U_{t_{j-1}} - 2\pi n| : n \in \mathbb{Z}\}$, and set $t_j = 0$ if no such t exists. Analogous to the proof of (7.8) in [31], one can show that for every $a > 0$ and $n \in \mathbb{N}$,

$$P^*\big[\big|y(\tau_1) - y(T)\big| \ge 2\pi n\big] \le P^*[T - \tau_1 \ge a] + P^*[t_n \ge T - a].$$

Using (25), (31) and the argument at the end of the proof of Theorem 7.2 in [31], choosing a to be n times a very small constant, one can bound the two summands on right hand side appropriately. □

Remark. Theorem 3 in [25] is a result only for two-sided radial SLE$_\kappa$ from 1 to -1 through 0. Adapting the proof of this result, one can get the analog for general two-sided radial SLE$_\kappa$ from 1 to $e^{i\alpha}$ through 0 in $\overline{\mathbb{D}}$, where $0 < \alpha < 2\pi$. Using this, following the proof of Lemma 4.3, one can obtain the analog of Lemma 4.3 for general two-sided radial SLE$_\kappa$. For the general case, it is expected that the corresponding C_0 and C_1 depend only on κ, not on α. Combining Theorem 1.3 in [10] and our proof of Lemma 4.3, one can show this for $0 < \kappa \le 4$.

The following result gives exact second moment estimate for the winding number of the two-sided radial SLE, which will be used to give estimate for the winding number variance of the arms crossing a long annulus in the 2-arm case.

Lemma 4.4. *Let E^* denote the expectation with respect to P^*. γ and θ_κ are as defined in Lemma 4.3. We have*

$$E^*\big[\theta_\kappa(\tau_\varepsilon)^2\big] = \left(\frac{\kappa}{4} + o(1)\right)\log\left(\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0.$$

Proof. Schwarz Lemma and the Koebe 1/4 Theorem give

$$\varepsilon \le e^{-\tau_\varepsilon} = 1/g'_{\tau_\varepsilon}(0) \le 4\varepsilon. \tag{32}$$

Using this, similarly to the proof of (26), one can show that there exist constants $C_0, C_1 > 0$ depending only on κ, such that for all $s > 0$,

$$P^*\left[\left|\theta_\kappa(\tau_\varepsilon) + \frac{\sqrt{\kappa}}{2} W_{\tau_\varepsilon}\right| > s\right] \le C_0 \exp(-C_1 s). \tag{33}$$

Combining (32) and (33), one obtains Lemma 4.4 easily. □

4.2. *Convergence of discrete exploration to SLE_6*

Assume $0 < r < 1$. Similarly to the definition of $\mathcal{A}_r^\eta$ defined above Proposition 2.6, for chordal SLE_6 path $\gamma_{\mathbb{D},1,-1}$ we define event

$$\mathcal{A}_r := \{\gamma_{\mathbb{D},1,-1} \cap \partial\mathbb{D}_r \neq \varnothing\}.$$

The following lemma is a corollary of the well-known result that the percolation exploration path converges in the scaling limit to the chordal SLE_6 path (see, e.g., Theorem 5 in [6]). The proof is standard and easy.

Lemma 4.5. *Let* $0 < r' \le r < 1$. $\gamma^\eta_{\mathbb{D},1,-1}[0, \tau_r^\eta]$ *conditioned on* $\mathcal{A}^\eta_{r'}$ *converges in distribution to stopped chordal* SLE_6 *path* $\gamma_{\mathbb{D},1,-1}[0, \tau_r]$ *conditioned on* $\mathcal{A}_{r'}$ *with respect to the uniform metric* (7) *as* $\eta \to 0$.

Proof. Let P^η and P denote the laws of $\gamma^\eta_{\mathbb{D},1,-1}$ and $\gamma_{\mathbb{D},1,-1}$, respectively. We claim that for each $0 < r < 1$, there exists a constant $C > 0$ (depending on r), such that

$$\lim_{\eta\to 0} P^\eta\big[\mathcal{A}_r^\eta\big] = P[\mathcal{A}_r] > C. \tag{34}$$

Moreover, in any coupling of $\{P^\eta\}$ and P on $(\Omega, \mathcal{F})$ in which $\mathrm{d}(\gamma^\eta_{\mathbb{D},1,-1}, \gamma_{\mathbb{D},1,-1}) \to 0$ a.s. as $\eta \to 0$, we have

$$\hat{P}\big[\big\{\gamma^\eta_{\mathbb{D},1,-1} \in \mathcal{A}_r^\eta\big\}\Delta\{\gamma_{\mathbb{D},1,-1} \in \mathcal{A}_r\}\big] \to 0 \quad \text{as } \eta \to 0, \tag{35}$$

where $\hat{P}[\cdot]$ denotes the coupling measure. The proof of the claim is analogous to that of Lemma 3.1: By Theorem 5 in [6], we can couple $\{P^\eta\}$ and P on $(\Omega, \mathcal{F})$ such that $\mathrm{d}(\gamma^\eta_{\mathbb{D},1,-1}, \gamma_{\mathbb{D},1,-1}) \to 0$ a.s. as $\eta \to 0$. Now let us show (35). For each small $\varepsilon > 0$ and $\eta < \varepsilon$,

$$\begin{aligned}
&\hat{P}\big[\{\gamma_{\mathbb{D},1,-1} \in \mathcal{A}_r\} \setminus \big\{\gamma^\eta_{\mathbb{D},1,-1} \in \mathcal{A}_r^\eta\big\}\big] \\
&\quad\le \hat{P}\big[\mathrm{d}\big(\gamma^\eta_{\mathbb{D},1,-1}, \gamma_{\mathbb{D},1,-1}\big) \ge \varepsilon\big] + \hat{P}\big[\gamma^\eta_{\mathbb{D},1,-1} \cap \partial\mathbb{D}_{r+\varepsilon} \neq \varnothing, \gamma^\eta_{\mathbb{D},1,-1} \cap \partial\mathbb{D}_r^\eta = \varnothing\big].
\end{aligned}$$

The event in the second term implies a half-plane 3-arm event from the 2ε-neighborhood of $\partial\mathbb{D}_r$ to a distance of unit order, whose probability goes to zero as $\varepsilon \to 0$. Then we get that $\hat{P}[\{\gamma^\eta_{\mathbb{D},1,-1} \in \mathcal{A}_r^\eta\}\Delta\{\gamma_{\mathbb{D},1,-1} \in \mathcal{A}_r\}] \to 0$ as $\eta \to 0$. The other direction is easy to prove and the details are omitted. Then we get (35). RSW, FKG and (35) imply (34) immediately.

Let $0 < r' \le r < 1$. Conditioned on $\mathcal{A}^\eta_{r'}$ and $\mathcal{A}_{r'}$, let $\gamma_r^\eta[0,1]$ and $\gamma_r[0,1]$ be the respective reparametrized curve of $\gamma^\eta_{\mathbb{D},1,-1}[0,\tau_r^\eta]$ and $\gamma_{\mathbb{D},1,-1}[0,\tau_r]$. Notice that $\{\gamma_r^\eta\}$ satisfies the conditions in [1] and thus has a scaling limit in terms of continuous curves along subsequence of η. We claim that for every subsequence limit $\widetilde{\gamma}_r[0,1]$, $\widetilde{\gamma}_r[0,1) \subset \overline{\mathbb{D}} \setminus \mathbb{D}_r$ almost surely. Then the fact that γ_r^η converges in distribution to γ_r easily follows from our two claims and Theorem 5 in [6]. It remains to show this claim. Assume that this is not the case for the limit $\widetilde{\gamma}_r$ along some subsequence $\{\eta_k\}_{k\in\mathbb{N}}$. Then with positive probability $\widetilde{\gamma}_r[0,1) \nsubseteq \overline{\mathbb{D}} \setminus \mathbb{D}_r$. Suppose this happens. We can find coupled versions of $\gamma_r^{\eta_k}$ and $\widetilde{\gamma}$ on $(\Omega, \mathcal{B})$ such that $\mathrm{d}(\gamma_r^{\eta_k}, \widetilde{\gamma}_r) \to 0$ a.s. as $k \to \infty$. Using this coupling, for each small $\varepsilon > 0$ and $\eta_k < \varepsilon/10$, we have a half-plane 3-arm event produced by $\gamma_r^{\eta_k}$ from the ε-neighborhood of $\widetilde{\gamma}_r(1)$ to a distance of unit order. As $\eta_k \to 0$, we can let $\varepsilon \to 0$, in which case the probability of the seeing this event goes to zero, leading to a contradiction. $\square$

In order to derive winding number estimates for the arms from the corresponding result for two-sided radial SLE_6, we need the following lemma.

Lemma 4.6. *Suppose* $0 < \varepsilon < 1/10$. *Let* γ^η *and* γ *denote* $\gamma^\eta_{\mathbb{D},1,-1}$ *conditioned on* $\mathcal{A}^\eta$ *and two-sided radial* SLE_6 *path from* 1 *to* -1 *through* 0 *in* $\overline{\mathbb{D}}$, *respectively. Then* $\gamma^\eta[0,\tau_\varepsilon^\eta]$ *converges in distribution to* $\gamma[0,\tau_\varepsilon]$ *with respect to the uniform metric* (7), *as* $\eta \to 0$.

Proof. For $0 < \varepsilon' < 1$, let $\gamma_{\varepsilon'}$ denote $\gamma_{\mathbb{D},1,-1}$ conditioned on $\mathcal{A}_{\varepsilon'}$, and let $\gamma^\eta_{\varepsilon'}$ denote $\gamma^\eta_{\mathbb{D},1,-1}$ conditioned on $\mathcal{A}^\eta_{\varepsilon'}$. By Proposition 2.6, for all $\eta < \varepsilon'/10 < \varepsilon/100$, we can couple γ^η and $\gamma^\eta_{\varepsilon'}$, such that with probability at lest $1 - (\varepsilon'/\varepsilon)^\beta$,

$$\mathrm{d}\big(\gamma^\eta\big[0, \tau^\eta_\varepsilon\big], \gamma^\eta_{\varepsilon'}\big[0, \tau^\eta_\varepsilon\big]\big) = 0. \tag{36}$$

By Lemma 4.5, for each $0 < \delta < 1$, there exists $\eta_0(\delta, \varepsilon')$, such that for each $\eta < \eta_0$ and $0 < \varepsilon' < \varepsilon < 1/10$, there is a coupling of $\gamma^\eta_{\varepsilon'}$ and $\gamma_{\varepsilon'}$, such that with probability at least $1 - \delta$,

$$\mathrm{d}\big(\gamma^\eta_{\varepsilon'}\big[0, \tau^\eta_\varepsilon\big], \gamma_{\varepsilon'}[0, \tau_\varepsilon]\big) \le \delta. \tag{37}$$

By Proposition 4.2, for each $0 < \delta < 1$, there exists $\varepsilon'_0(\delta, \varepsilon)$, such that for each $0 < \varepsilon' < \varepsilon'_0$ there is a coupling of $\gamma_{\varepsilon'}$ and γ, such that with probability at least $1 - \delta$,

$$\mathrm{d}\big(\gamma_{\varepsilon'}[0, \tau_\varepsilon], \gamma[0, \tau_\varepsilon]\big) \le \delta. \tag{38}$$

Combining (36), (37) and (38) gives the desired result. □

4.3. *Moment bounds on the winding of discrete exploration*

Define $R^\eta(r, R) := \{z \in \eta\mathbb{T} : |\arg(z)| < \pi/10\} \cap A^\eta(r, R)$. We say a path $\gamma \subset R^\eta(r, R)$ is a *crossing* of $R^\eta(r, R)$ if the endpoints of γ lie adjacent (Euclidean distance smaller than η) to the rays of argument $\pm\frac{\pi}{10}$ respectively. By Lemma 2.1 in [41], we obtain the following lemma, which implies that it is very unlikely that there is an arm with large winding in an annulus.

Lemma 4.7 ([41]). *There exist constants $C_1, C_2, K_0 > 0$, such that for all $K > K_0$ and $\eta \le r < R$,*

$$P\big[\exists \lfloor K \log(R/r)\rfloor \textit{ disjoint blue crossings of } R^\eta(r, R)\big] \le C_1 \exp\big[-C_2 K \log(R/r)\big].$$

The following three lemmas give moment bounds for the winding numbers of percolation exploration path. Let us define some notation before stating the results.

Suppose $r < R$. For a curve γ hitting with $\partial\mathbb{D}^\eta_R$ before hitting with $\partial\mathbb{D}^\eta_r$, denote by $T^\eta_{R,r}$ the last hitting time with $\partial\mathbb{D}^\eta_R$ of γ before time τ^η_r.

Recall the definition of $P^*_R[\cdot]$ which is defined after the definition of good faces. Denote by E^*_R the expectation with respect to $P^*_R[\cdot]$. Let Θ be the good faces around $\partial\mathbb{D}^\eta_R$ under $P^*_R[\cdot]$. Denote by γ^*_R the percolation exploration path connecting the endpoints of Θ stopped when it reaches ηH_0.

Unless specified otherwise, in the rest of this paper, we denote by $E = E^\eta$ the expectation with respect to $P[\cdot|\mathcal{A}^\eta]$, and by $\gamma = \gamma^\eta$ the percolation exploration path $\gamma^\eta_{\mathbb{D},1,-1}$ conditioned on $\mathcal{A}^\eta$. For simplicity, we will omit the superscript η of γ^η, τ^η_r and $T^\eta_{R,r}$ when it is clear that we are talking about the the discrete percolation model.

Lemma 4.8. *Let $\eta \le r < R \le 1$. We have*

$$\big|E\theta\big(\gamma[0, \tau_r]\big)\big| \le \pi, \qquad \big|E\theta\big(\gamma[T_{R,r}, \tau_r]\big)\big| \le \pi, \qquad \big|E\theta\big(\gamma[\tau_R, \tau_r]\big)\big| \le 2\pi \quad \textit{and} \quad E^*_R\theta\big(\gamma^*_R\big) = 0.$$

Proof. First let us show the first inequality. Conditioned on $\mathcal{A}^\eta$, consider the time-reversal of $\gamma^\eta_{\mathbb{D},1,-1}$ stopped when it reaches ηH_0, denoted by γ'. By the symmetry of the lattice, it is easy to see that $E\theta(\gamma[0, \tau_r]) = -E\theta(\gamma'[0, \tau_r])$. It is obvious that $|\theta(\gamma[0, \tau_r]) - \theta(\gamma'[0, \tau_r])| \le 2\pi$. These two observations immediately imply $|E\theta(\gamma[0, \tau_r])| \le \pi$. Similarly one can show the second inequality. Using the first inequality, we get the third one:

$$\big|E\theta\big(\gamma[\tau_R, \tau_r]\big)\big| \le \big|E\theta\big(\gamma[0, \tau_r]\big) - E\theta\big(\gamma[0, \tau_R]\big)\big| \le 2\pi.$$

Now let us show $E^*_R\theta(\gamma^*_R) = 0$. For any fixed good faces Θ around $\partial\mathbb{D}^\eta_R$, denote by Θ' the mirror image of Θ with opposite colors with respect to the imaginary axis. It is obvious that $\Theta \ne \Theta'$, $P^*_R[\Theta] = P^*_R[\Theta']$ and $E^*_R[\theta(\gamma^*_R)|\Theta] = -E^*_R[\theta(\gamma^*_R)|\Theta']$. Then $E^*_R\theta(\gamma^*_R) = 0$ follows immediately. □

Lemma 4.9. *There exist constants $C_1, C_2, C_3 > 0$, such that for all $\eta \le r \le R/2 \le 1/2$,*

$$E\left|\theta\left(\gamma[T_{R,r}, \tau_r]\right)\right| \le \sqrt{C_1 \log(R/r)}, \tag{39}$$

$$E\left[\theta\left(\gamma[T_{R,r}, \tau_r]\right)^2\right] \le C_1 \log(R/r), \tag{40}$$

$$E\left[\theta\left(\gamma[T_{R,r}, \tau_r]\right)^4\right] \le C_2\left[\log(R/r)\right]^4, \tag{41}$$

$$E\left[\theta\left(\gamma[\tau_R, T_{R,r}]\right)^2\right] \le C_3. \tag{42}$$

Proof. First let us show (40). In [41], conditioned on $\mathcal{A}_2^\eta(\eta, 1)$, we showed that the winding number variance of the arm connecting the two boundary pieces of $A^\eta(\eta, 1)$ is $O(\log(1/\eta))$ (Theorem 1.1 in [41]) by a martingale method. Conditioned on $\mathcal{A}^\eta$, one can use the same method to show that $\mathrm{Var}(\theta_\eta)$ is again $O(\log(1/\eta))$. Furthermore, with a little modification for our setting, one can also use this method to show that there exists a constant $C_0 > 0$, such that for all $\eta \le r \le R/2 \le 1/2$,

$$\mathrm{Var}\left|\theta\left(\gamma[T_{R,r}, \tau_r]\right)\right| \le C_0 \log(R/r). \tag{43}$$

We left the details to the reader. Lemma 4.8 says that

$$\left|E\theta\left(\gamma[T_{R,r}, \tau_r]\right)\right| \le \pi. \tag{44}$$

Then (43) and (44) imply (40). (40) and Cauchy–Schwarz inequality imply (39) immediately.

We show (41) now. We claim that there exist $C_4, C_5, C_6 > 0$, such that for all $\eta \le r \le R/2 \le 1/2$ and $x \ge C_4 \log(R/r)$,

$$P\left[\left|\theta\left(\gamma[T_{R,r}, \tau_r]\right)\right| \ge x | \mathcal{A}^\eta\right] \le C_5 \exp(-C_6 x).$$

Then (41) follows from the claim immediately. The claim is proved as follows. Choosing C_4 large enough, we have

$$\begin{aligned} P\left[\left|\theta\left(\gamma[T_{R,r}, \tau_r]\right)|\right| \ge x | \mathcal{A}^\eta\right] &\le \frac{C_7 P[\exists \lfloor x/2\pi \rfloor - 2 \text{ disjoint blue crossings of } R^\eta(r, R)]}{P[\mathcal{A}_2(r, R)]} \\ &\quad \text{by quasi-multiplicativity and (12)} \\ &\le C_5 \exp(-C_6 x) \text{ by Lemma 4.7 and (11).} \end{aligned}$$

Now let us show (42). Set $N = \max\{\lceil \log_2(1/R) \rceil, \lceil \log_2(R/r) \rceil\}$. For $0 \le j \le N+1$, let $R_j := \min\{1, 2^j R\}$, $r_j := \max\{r, (1/2)^j R\}$. For $0 \le j \le N$, define event

$$\mathcal{B}_j := \left\{\gamma[\tau_R, T_{R,r}] \cap \left(\partial \mathbb{D}_{R_j}^\eta \cup \partial \mathbb{D}_{r_j}^\eta\right) \ne \varnothing, \gamma[\tau_R, T_{R,r}] \subset A^\eta(r_{j+1}, R_{j+1})\right\}.$$

There exist $C_8, C_9, C_{10} > 0$, such that for all $\mathcal{B}_j$, $0 \le j \le N$,

$$\begin{aligned} &P\left[\mathcal{B}_j | \mathcal{A}^\eta\right] \\ &\quad \le \frac{C_8 P[\exists \text{ bichromatic 3-arm crossing } A^\eta(R, R_j) \text{ or } A^\eta(r_j, R), \mathcal{A}_2^\eta(r_j, R_j)]}{P[\mathcal{A}_2^\eta(r_j, R_j)]} \\ &\quad\quad \text{by quasi-multiplicativity and (12)} \\ &\quad \le C_9 \exp(-C_{10} j) \text{ by BK inequality and (11).} \end{aligned} \tag{45}$$

Moreover, using quasi-multiplicativity and a gluing argument with FKG, RSW and Theorem 11 in [29], it is easy to show that there exist $C_{11}, C_{12} > 0$ such that

$$P\left[\mathcal{B}_j, \mathcal{A}^\eta\right] \ge C_{11} \exp(-C_{12} j) P\left[\mathcal{A}_2^\eta(R_{j+1}, 1)\right] P\left[\mathcal{A}_2^\eta(\eta, r_{j+1})\right]. \tag{46}$$

We leave the details to the reader. For simplicity, we let $P[\mathcal{A}_2^\eta(x,x)]=1$ for any $x>0$ in the above inequality and in the rest of the paper.

Hence, we can choose $C_{13}, C_{14}, C_{15}>0$ such that for all $0\le j\le N$ and $x\ge C_{13}(j+1)$,

$$\begin{aligned} &P\big[\big|\theta\big(\gamma[\tau_R, T_{R,r}]\big)\big|\ge x|\mathcal{A}^\eta,\mathcal{B}_j\big] \\ &\quad\le \frac{P[\exists\lfloor x/(2\pi)\rfloor-2 \text{ disjoint blue crossings of } R^\eta(r_{j+1},R_{j+1})]}{C_{11}\exp(-C_{12}j)} \text{ by (46)} \\ &\quad\le C_{14}\exp(-C_{15}x) \text{ by Lemma 4.7.} \end{aligned} \tag{47}$$

Choosing C_3 large enough, (42) follows easily from (45) and (47):

$$\begin{aligned} E\big[\theta\big(\gamma[\tau_R,T_{R,r}]\big)^2\big] &\le \sum_{j=0}^{N} P\big[\mathcal{B}_j|\mathcal{A}^\eta\big]E\big[\theta\big(\gamma[\tau_R,T_{R,r}]\big)^2|\mathcal{B}_j\big] \\ &\le \sum_{j=0}^{N} C_{16}\exp(-C_{10}j)(j+1)^2\le C_3. \end{aligned}$$

□

The following lemma can be considered as a generalization of Lemma 4.8.

Lemma 4.10. *There exists a constant $C>0$, such that for all $\eta\le r<R/2$, any given faces Θ around $\partial\mathbb{D}_R^\eta$, and the percolation exploration path γ connecting the endpoints of Θ stopped when it reaches ηH_0 conditioned on $\mathcal{A}_\Theta^\eta(\eta,R)$, we have*

$$\big|E_\Theta\big[\theta\big(\gamma[0,\tau_r]\big)\big]\big|\le C,$$

where E_Θ is the expectation with respect to $P[\cdot|\mathcal{A}_\Theta^\eta(\eta,R)]$.

Proof. For simplicity, we just show that $|E_\Theta[\theta(\gamma)]|\le C$, the proof of $|E_\Theta[\theta(\gamma[0,\tau_r])]|\le C$ is essentially the same. By Proposition 2.5, there exist $C_0, C_1>0$, such that for all $10\eta<R/2$, any fixed faces Θ around $\partial\mathbb{D}_R^\eta$ and $N:=\lfloor\log_2(R/\eta)\rfloor$, there is a coupling of $P[\cdot|\mathcal{A}_\Theta^\eta(\eta,R)]$ and $\{P^*_{(1/2)^jR}[\cdot]\}_{1\le j\le N}$, so that for all $1\le j\le N$, with probability at least $1-\exp(-C_0j)$, the following event $\mathcal{B}_j$ occurs: There exists $1\le j^*\le j$ such that there exist good faces Θ_{j^*} around $\partial\mathbb{D}^\eta_{(1/2)^{j^*}R}$ under $P[\cdot|\mathcal{A}_\Theta^\eta(\eta,R)]$, and the configuration constraint in $\overline{\Theta}_{j^*}$ under $P[\cdot|\mathcal{A}_\Theta^\eta(\eta,R)]$ is the same as the configuration under $P^*_{(1/2)^{j^*}R}[\cdot]$. Furthermore, under this coupling for all $1\le j\le N-1$, with probability at least $\exp(-C_1(j+1))$ the event $\mathcal{B}_j^c\mathcal{B}_{j+1}$ occurs.

Denote by $\hat{P}$ the coupling law, and by $\hat{E}$ the expectation with respect to $\hat{P}$. By Proposition 2.5 and Lemma 4.8, we have

$$\begin{aligned} \big|E_\Theta\big[\theta(\gamma)\big]\big| &= \big|\hat{E}\big[I_{\mathcal{B}_1}\theta(\gamma)\big]+\hat{E}\big[I_{\mathcal{B}_N^c}\theta(\gamma)\big]+\Sigma_{j=1}^{N-1}\hat{E}\big[I_{\mathcal{B}_j^c\mathcal{B}_{j+1}}\theta(\gamma)\big]\big| \\ &\le \hat{E}\big|\theta\big(\gamma[0,\tau_{R/2}]\big)\big|+\exp(-C_0N)\hat{E}\big[\big|\theta(\gamma)\big||\mathcal{B}_N^c\big] \\ &\quad+\Sigma_{j=1}^{N-1}\exp(-C_0j)\big|\hat{E}\big[\theta\big(\gamma[0,\tau_{(1/2)^{j+1}R}]\big)|\mathcal{B}_j^c\mathcal{B}_{j+1}\big]\big|. \end{aligned}$$

Then $|E_\Theta[\theta(\gamma)]|\le C$ easily follows from the following claim: There exists $C_2>0$, such that for all $1\le j\le N-1$,

$$\hat{E}\big[\big|\theta\big(\gamma[0,\tau_{(1/2)^{j+1}R}]\big)\big||\mathcal{B}_j^c\mathcal{B}_{j+1}\big]\le C_2 j. \tag{48}$$

Furthermore, there exist $C_3, C_4>0$ such that

$$\hat{E}\big|\theta\big(\gamma[0,\tau_{R/2}]\big)\big|\le C_3 \quad\text{and}\quad \hat{E}\big|\theta(\gamma)\big|\mathcal{B}_N^c\big|\le C_4N. \tag{49}$$

Let us show (48) now. By the coupling, there is a constant $C_1 > 0$ such that for all $1 \le j \le N-1$,

$$\hat{P}\big[\mathcal{B}_j^c \mathcal{B}_{j+1}\big] \ge \exp\big(-C_1(j+1)\big). \tag{50}$$

Without loss of generality, for the faces Γ_1 and Γ_2 of $\Theta = (\Gamma_1, \Gamma_2)$ (recall that we always assume that Γ_1 is blue and Γ_2 is yellow), we assume that $|\theta(\Gamma_1)| \le |\theta(\Gamma_2)|$ (we think of the face as a continuous curve by connecting the neighbor sites with line segments). By a gluing construction with RSW and FKG, it is easy to show that

$$P\big[\mathcal{A}_\Theta^\eta(R/2, R)\big] \asymp P\big[\Gamma_1 \overset{\dot{\Theta}}{\leftrightarrow} \partial\mathbb{D}_{R/2}^\eta\big], \tag{51}$$

where $\Gamma_1 \overset{\dot{\Theta}}{\leftrightarrow} \partial\mathbb{D}_r^\eta$ denotes that there exists a blue path connecting Γ_1 and $\partial\mathbb{D}_r^\eta$ in the interior of $\overline{\Theta}$ for $r < R$. Then we know that there exist $C_5, C_6, C_7 > 0$ such that for all $1 \le j \le N-1$,

$$\begin{aligned}
&P\big[\mathcal{A}_\Theta^\eta\big((1/2)^j R, R\big)\big] \\
&\quad \ge C_5 P\big[\mathcal{A}_2^\eta\big((1/2)^j R, R/2\big)\big] P\big[\mathcal{A}_\Theta^\eta(R/2, R)\big] \quad \text{by quasi-multiplicativity} \\
&\quad \ge C_6 \exp(-C_7 j) P\big[\Gamma_1 \overset{\dot{\Theta}}{\leftrightarrow} \partial\mathbb{D}_{R/2}^\eta\big] \quad \text{by (51) and (11).}
\end{aligned} \tag{52}$$

Conditioned on $\mathcal{A}_\Theta^\eta((1/2)^j R, R)$, we let γ_j be the b-path starting at an endpoint of Θ and ending when it reaches $\partial\mathbb{D}_{(1/2)^j R}^\eta$ with yellow hexagons on its left. We can choose C_8 large enough, such that the following inequalities hold:

$$\begin{aligned}
&\hat{P}\big[\big|\theta\big(\gamma[0, \tau_{(1/2)^{j+1}R}]\big)\big| \ge C_8 j \,|\, \mathcal{B}_j^c \mathcal{B}_{j+1}\big] \\
&\quad \le \exp\big(C_1(j+1)\big) P\big[\big|\theta\big(\gamma[0, \tau_{(1/2)^{j+1}R}]\big)\big| \ge C_8 j \,|\, \mathcal{A}_\Theta^\eta(\eta, R)\big] \quad \text{by (50)} \\
&\quad \le C_9 \exp(C_1 j) P\big[\big|\theta(\gamma_{j+1})\big| \ge C_8 j \,|\, \mathcal{A}_\Theta^\eta\big((1/2)^{j+1} R, R\big)\big] \quad \text{by quasi-multiplicativity} \\
&\quad \le C_{10} \exp(C_{11} j) \frac{P[|\theta(\gamma_{j+1})| \ge C_8 j, \mathcal{A}_\Theta^\eta((1/2)^{j+1} R, R)]}{P[\Gamma_1 \overset{\dot{\Theta}}{\leftrightarrow} \partial\mathbb{D}_{R/2}^\eta]} \quad \text{by (52).}
\end{aligned}$$

Observe that if $|\theta(\gamma_{j+1})|$ is very large, then γ_{j+1} will produce many crossings in the "rectangle" $R^\eta((1/2)^{j+1} R, 2R)$, or γ_{j+1} will cross $A^\eta(R, 2R)$ many times and produce many crossings in a longer "rectangle" (it is obvious that if $\Theta \subset \overline{\mathbb{D}_{2R}^\eta}$ this would not happen). This observation and the above inequality lead to

$$\begin{aligned}
&\hat{P}\big[|\theta\big(\gamma[0, \tau_{(1/2)^{j+1}R}]\big)| \ge C_8 j \,|\, \mathcal{B}_j^c \mathcal{B}_{j+1}\big] \\
&\quad \le \frac{C_{10} \exp(C_{11} j)}{P[\Gamma_1 \overset{\dot{\Theta}}{\leftrightarrow} \partial\mathbb{D}_{R/2}^\eta]} \left\{ P\begin{bmatrix} \exists \lfloor C_8 j/4\pi \rfloor - 2 \text{ disjoint yellow arms crossing} \\ A^\eta(R, 2R) \text{ in } \dot{\Theta}, \Gamma_1 \overset{\dot{\Theta}}{\leftrightarrow} \partial\mathbb{D}_{(1/2)^{j+1}R}^\eta \end{bmatrix} \right. \\
&\qquad \left. + P\begin{bmatrix} \exists \lfloor C_8 j/4\pi \rfloor - 2 \text{ disjoint yellow crossings of} \\ R^\eta((1/2)^{j+1} R, 2R) \text{ in } \dot{\Theta}, \Gamma_1 \overset{\dot{\Theta}}{\leftrightarrow} \partial\mathbb{D}_{(1/2)^{j+1}R}^\eta \end{bmatrix} \right\} \\
&\quad \le C_{10} \exp(C_{11} j)(\exp(-C_{12} j) + \exp(-C_{13} j)) \quad \text{by BK inequality, (11) and Lemma 4.7} \\
&\quad \le C_{14} \exp(-C_{15} j).
\end{aligned}$$

Then (48) follows immediately. The proof of (49) is similar to that of (48), the details are left to the reader. $\square$

4.4. *Decorrelation of winding*

To simplify notation, we write $T_j := T_{\varepsilon^j,\varepsilon^{j+1}}$ and $\tau_j := \tau_{\varepsilon^j}$ in the following. The two lemmas below say that $\mathrm{Var}[\theta_\eta]$ is well-approximated by the sum of the second moment of the winding numbers of the paths in annuli on dyadic scales.

Lemma 4.11. *There exists a constant $C > 0$, such that for all $10\eta < \varepsilon < 1/2$, under the conditional law $P[\cdot|\mathcal{A}^\eta]$, we have*

$$\left|\mathrm{Var}[\theta_\eta] - \sum_{j=0}^{\lfloor \log_\varepsilon \eta \rfloor} E\left[\theta\left(\gamma[T_j, \tau_{j+1}]\right)^2\right]\right| \le C\left[\log(1/\varepsilon)\right]^{-\frac{1}{2}} \log(1/\eta).$$

Proof. Lemma 4.8 says that $|E\theta_\eta| \le \pi$. Therefore, in order to prove Lemma 4.11, it is enough to prove that there exists a constant $C_1 > 0$, such that for all $10\eta < \varepsilon < 1/2$,

$$\left|E\left[\theta_\eta^2\right] - \sum_{j=0}^{\lfloor \log_\varepsilon \eta \rfloor} E\left[\theta\left(\gamma[T_j, \tau_{j+1}]\right)^2\right]\right| \le C_1\left[\log(1/\varepsilon)\right]^{-\frac{1}{2}} \log(1/\eta). \tag{53}$$

It is clear that

$$\theta_\eta = \sum_{j=0}^{\lfloor \log_\varepsilon \eta \rfloor} \theta\left(\gamma[\tau_j, \tau_{j+1}]\right).$$

So, to show (53), it suffices to prove that there are $C_2, C_3 > 0$, such that for all $10\eta < \varepsilon < 1/2$,

$$\sum_{j=0}^{\lfloor \log_\varepsilon \eta \rfloor} \left|E\left[\theta\left(\gamma[\tau_j, \tau_{j+1}]\right)^2\right] - E\left[\theta\left(\gamma[T_j, \tau_{j+1}]\right)^2\right]\right| \le C_2\left[\log(1/\varepsilon)\right]^{-\frac{1}{2}} \log(1/\eta) \tag{54}$$

and

$$\left|\sum_{0 \le j < k \le \lfloor \log_\varepsilon \eta \rfloor} E\left[\theta\left(\gamma[\tau_j, \tau_{j+1}]\right)\theta\left(\gamma[\tau_k, \tau_{k+1}]\right)\right]\right| \le C_3 \log_\varepsilon \eta. \tag{55}$$

Let us first show (54). By (40), (42) and Cauchy–Schwarz inequality, there exist $C_4, C_5 > 0$, such that

$$\begin{aligned}
&E\left|\theta\left(\gamma[\tau_j, \tau_{j+1}]\right)^2 - \theta\left(\gamma[T_j, \tau_{j+1}]\right)^2\right| \\
&\quad = E\left|2\theta\left(\gamma[\tau_j, T_j]\right)\theta\left(\gamma[T_j, \tau_{j+1}]\right) + \theta\left(\gamma[\tau_j, T_j]\right)^2\right| \\
&\quad \le 2\left\{E\left[\theta\left(\gamma[\tau_j, T_j]\right)^2\right]\right\}^{\frac{1}{2}}\left\{E\left[\theta\left(\gamma[T_j, \tau_{j+1}]\right)^2\right]\right\}^{\frac{1}{2}} + E\left[\theta\left(\gamma[\tau_j, T_j]\right)^2\right] \\
&\quad \le C_4\left[\log(1/\varepsilon)\right]^{\frac{1}{2}} + C_5.
\end{aligned}$$

Then we get (54).

Now let us show (55). For this, it is enough to show that there are $C_6, C_7 > 0$, such that for any $0 \le j \le \lfloor \log_\varepsilon \eta \rfloor$,

$$\left|E\left[\theta\left(\gamma[\tau_j, \tau_{j+1}]\right)\theta\left(\gamma[\tau_{j+1}, \tau_\eta]\right)\right]\right| \le C_6, \tag{56}$$

and

$$\left|E\left[\theta\left(\gamma[0, \tau_j]\right)\theta\left(\gamma[\tau_j, \tau_{j+1}]\right)\right]\right| \le C_7. \tag{57}$$

We first show (56). Note that $\gamma[0, \tau_j]$ and the b-path $\gamma'[0, \tau_j]$ from $(-1)_\eta$ to $\partial\mathbb{D}^\eta_{\varepsilon^j}$ induce faces Θ_j around $\partial\mathbb{D}^\eta_{\varepsilon^j}$. Denote by E_{Θ_j} the expectation with respect to $P[\cdot|\mathcal{A}^\eta_{\Theta_j}(\eta, \varepsilon^j)]$. By Lemma 4.8 and Lemma 4.10, choosing C_6, C_8

appropriately, we have

$$\begin{aligned}&\left|E\left[\theta\left(\gamma[\tau_j,\tau_{j+1}]\right)\theta\left(\gamma[\tau_{j+1},\tau_\eta]\right)\right]\right|\\&\quad\le\sum_{\Theta_{j+1}}P\left[\Theta_{j+1}|\mathcal{A}^\eta\right]\left|E_{\Theta_{j+1}}\left[\theta\left(\gamma[\tau_j,\tau_{j+1}]\right)\right]\right|\left|E_{\Theta_{j+1}}\left[\theta\left(\gamma[\tau_{j+1},\tau_\eta]\right)\right]\right|\\&\quad\le C_8\left|E\left[\theta\left(\gamma[\tau_j,\tau_{j+1}]\right)\right]\right|\le C_6.\end{aligned}$$

Now let us show (57), which proof is similar to that of (56). By Lemma 4.8 and Lemma 4.10 again, we have

$$\begin{aligned}\left|E\left[\theta\left(\gamma[0,\tau_j]\right)\theta\left(\gamma[\tau_j,\tau_{j+1}]\right)\right]\right|&\le\sum_{\Theta_j}P\left[\Theta_j|\mathcal{A}^\eta\right]\left|E_{\Theta_j}\left[\theta\left(\gamma[0,\tau_j]\right)\right]\right|\left|E_{\Theta_j}\left[\theta\left(\gamma[\tau_j,\tau_{j+1}]\right)\right]\right|\\&\le C_9\left|E\left[\theta\left(\gamma[0,\tau_j]\right)\right]\right|\le C_7.\end{aligned}$$

□

Denote by E_j the expectation with respect to the conditional law $P[\cdot|\mathcal{A}^\eta(\varepsilon^j)]$. Conditioned on $\mathcal{A}^\eta(\varepsilon^j)$, denote by γ_j the percolation exploration path $\gamma^\eta_{\mathbb{D}_{\varepsilon^j},\varepsilon^j,-\varepsilon^j}$ stopped when it reaches $\partial\mathbb{D}^\eta_{\varepsilon^{j+1}}$.

Lemma 4.12. *There exist $C>0$ and $0<\varepsilon_0<1/2$, such that for all $10\eta<\varepsilon<\varepsilon_0$, we have*

$$\left|\sum_{j=0}^{\lfloor\log_\varepsilon\eta\rfloor}E\left[\theta\left(\gamma[T_j,\tau_{j+1}]\right)^2\right]-\sum_{j=0}^{\lfloor\log_\varepsilon\eta\rfloor}E_j\left[\theta(\gamma_j)^2\right]\right|\le C\left[\log(1/\varepsilon)\right]^{-\frac{1}{7}}\log(1/\eta).$$

Proof. Let β be the constant in Proposition 2.4. By Proposition 2.4, we can couple $P[\cdot|\mathcal{A}^\eta]$ and $P[\cdot|\mathcal{A}^\eta(\varepsilon^j)]$ such that with probability at least $1-\varepsilon^{\beta/3}$ there exist identical good faces $\Theta\subset A^\eta(\varepsilon^{j+1/3},\varepsilon^j)$ for both measures, and the configuration in $\overline{\Theta}$ is also identical. Let us denote by $\hat{P}$ the coupling law, by $\hat{E}$ the expectation with respect to $\hat{P}$, and by $\mathcal{B}$ the above event. We write

$$\begin{aligned}&\hat{E}\left|\theta\left(\gamma[T_j,\tau_{j+1}]\right)^2-\theta(\gamma_j)^2\right|\\&\quad=\hat{E}\left[I_{\mathcal{B}^c}\left|\theta\left(\gamma[T_j,\tau_{j+1}]\right)^2-\theta(\gamma_j)^2\right|\right]+\hat{E}\left[I_{\mathcal{B}}\left|\theta\left(\gamma[T_j,\tau_{j+1}]\right)^2-\theta(\gamma_j)^2\right|\right].\end{aligned}$$

Let us estimate the two terms in the r.h.s. of above equality separately. For the first term, with Cauchy–Schwarz inequality and (41), we get

$$\begin{aligned}&\hat{E}\left[I_{\mathcal{B}^c}\left|\theta\left(\gamma[T_j,\tau_{j+1}]\right)^2-\theta(\gamma_j)^2\right|\right]\\&\quad\le\left\{\hat{P}\left[\mathcal{B}^c\right]\right\}^{\frac{1}{2}}\left\{\hat{E}\left[\left|\theta\left(\gamma[T_j,\tau_{j+1}]\right)^2-\theta(\gamma_j)^2\right|^2\right]\right\}^{\frac{1}{2}}\le C_1\varepsilon^{\frac{\beta}{6}}\left[\log(1/\varepsilon)\right]^2.\end{aligned}$$

Now let us bound the second term. For each $x>0$, define event

$$\mathcal{S}_x:=\left\{\exists\lfloor x/2\pi\rfloor-4\text{ disjoint blue crossings of }R^\eta\left(\varepsilon^{j+1/3},\varepsilon^j\right)\right\}.$$

There exist $C_2,C_3,C_4,C_5>0$ such that for all $10\eta<\varepsilon<1/2$ and all $x\ge C_2[\log(1/\varepsilon)]^{\frac{1}{2}-\frac{1}{7}}$,

$$\begin{aligned}&\hat{P}\left[\mathcal{B},\left|\theta\left(\gamma[T_j,\tau_{j+1}]\right)-\theta(\gamma_j)\right|\ge x\right]\\&\quad\le P\left[\mathcal{S}_x\middle|\mathcal{A}^\eta\right]+P\left[\mathcal{S}_x\middle|\mathcal{A}^\eta\left(\varepsilon^j\right)\right]\\&\quad\le\frac{C_3P[\mathcal{S}_x]}{P[\mathcal{A}_2^\eta(\varepsilon^{j+1},\varepsilon^j)]}\quad\text{by quasi-multiplicativity and (12)}\\&\quad\le C_4\exp(-C_5x)\quad\text{by (11) and Lemma 4.7.}\end{aligned}$$

Combining (40), Cauchy–Schwarz inequality and above inequality, we have

$$\begin{aligned}
&\hat{E}\big[I_{\mathcal{B}}\big|\theta\big(\gamma[T_j,\tau_{j+1}]\big)^2-\theta(\gamma_j)^2\big|\big]\\
&\quad\le\big\{\hat{E}\big[\big|\theta\big(\gamma[T_j,\tau_{j+1}]\big)+\theta(\gamma_j)\big|^2\big]\big\}^{\frac{1}{2}}\big\{\hat{E}\big[I_{\mathcal{B}}\big|\theta\big(\gamma[T_j,\tau_{j+1}]\big)-\theta(\gamma_j)\big|^2\big]\big\}^{\frac{1}{2}}\\
&\quad\le C_5\big[\log(1/\varepsilon)\big]^{1-\frac{1}{7}}.
\end{aligned}$$

This, together with the upper bound for the first term completes the proof immediately. □

4.5. *Proofs of Theorem 1.5 and Corollary 1.6*

We now conclude the proof of our main result concerning winding numbers.

Proof of Theorem 1.5. By Lemma 4.11 and Lemma 4.12, to establish (4), it is enough to show that for each $0<\delta<1$, there exists $0<\varepsilon_0(\delta)<1$ such that for each given $0<\varepsilon<\varepsilon_0$, there exists $\eta_0(\varepsilon)>0$, such that for all $\eta<\eta_0$,

$$\left|\frac{3}{2}\lfloor\log_\varepsilon\eta\rfloor\log(1/\varepsilon)-\sum_{j=0}^{\lfloor\log_\varepsilon\eta\rfloor}E_j\big[\theta(\gamma_j)^2\big]\right|\le\delta\lfloor\log_\varepsilon\eta\rfloor\log(1/\varepsilon). \tag{58}$$

By (40), there exists a constant $C>0$, such that for all $\eta<\varepsilon<1/2$ and $0\le j\le\lfloor\log_\varepsilon\eta\rfloor$,

$$E_j\big[\theta(\gamma_j)^2\big]\le C\log(1/\varepsilon). \tag{59}$$

Combining (59) and Lemma 4.6, we have that for any fixed $0<\varepsilon<1/2$, for any j such that $\lfloor\log_\varepsilon\eta\rfloor-j\to+\infty$ as $\eta\to0$,

$$E_j\big[\theta(\gamma_j)^2\big]\to E^*\big[\theta\big(\gamma[0,\tau_\varepsilon]\big)^2\big]\quad\text{as }\eta\to0, \tag{60}$$

where γ is the two-sided radial SLE_6 path from 1 to -1 through 0 in $\overline{\mathbb{D}}$. By the convergence of the Cesàro mean and (60), we have

$$\lim_{\eta\to0}\frac{\sum_{j=0}^{\lfloor\log_\varepsilon\eta\rfloor}E_j[\theta(\gamma_j)^2]}{\lfloor\log_\varepsilon\eta\rfloor}=E^*\big[\theta\big(\gamma[0,\tau_\varepsilon]\big)^2\big].$$

Combining this and Lemma 4.4 gives (58).

Using the approach in the 2-arm case in [41] with a little modification, one can show that under $P[\cdot|\mathcal{A}^\eta]$,

$$\frac{\theta_\eta}{\sqrt{\operatorname{Var}\theta_\eta}}\to_d N(0,1)\quad\text{as }\eta\to0.$$

Then (5) follows from this and (4). □

Proof of Corollary 1.6. Let $\tilde{\theta}_{\eta,1}$ and $\tilde{\theta}_{\eta,\nu}$ denote $\tilde{\theta}_\eta$ under $P[\cdot|\mathcal{A}_2^\eta]$ and ν_2^η, respectively. First we prove the corollary for $\tilde{\theta}_{\eta,1}$. By Lemma 3.4 in [41] and Lemma 4.8, we know $|E'\tilde{\theta}_{\eta,1}|\le2\pi$ and $|E\theta_\eta|\le\pi$, where E' is the expectation with respect to $P[\cdot|\mathcal{A}_2^\eta]$. Combining this, Theorem 1.1 in [41] and Theorem 1.5, to show the corollary for $\tilde{\theta}_{\eta,1}$, it is enough to show that there exists a constant $C>0$, such that for all small η,

$$\big|E\big[\theta_\eta^2\big]-E'\big[\tilde{\theta}_{\eta,1}^2\big]\big|\le C\big[\log(1/\eta)\big]^{\frac{6}{7}}. \tag{61}$$

The proof of (61) is analogous to that of Lemma 4.12, we just sketch it here: By Proposition 2.4, one can couple $P[\cdot|\mathcal{A}^\eta]$ and $P[\cdot|\mathcal{A}_2^\eta]$ such that with probability at least $1-\eta^{\beta/3}$ there exist identical good faces $\Theta\subset A^\eta(\eta^{1/3},1)$ for both measures, and the configuration in $\overline{\Theta}$ is also identical. Denote by $\hat{P}$ the coupling law, by $\hat{E}$ the expectation with respect to $\hat{P}$, and by $\mathcal{B}$ the above event. Then one can show that there exist $C_1, C_2 > 0$ such that

$$\hat{E}\big[I_{\mathcal{B}^c}\big|\theta_\eta^2 - \tilde{\theta}_{\eta,1}^2\big|\big] \le C_1 \eta^{\frac{\beta}{6}}\big[\log(1/\eta)\big]^2,$$

$$\hat{E}\big[I_{\mathcal{B}}\big|\theta_\eta^2 - \tilde{\theta}_{\eta,1}^2\big|\big] \le C_2\big[\log(1/\eta)\big]^{\frac{6}{7}},$$

which imply (61) immediately.

Now let us show the corollary for $\tilde{\theta}_{\eta,\nu}$, which proof is similar to that for $\tilde{\theta}_{\eta,1}$. It is easy to show that $|E_\nu \tilde{\theta}_{\eta,\nu}| \le 2\pi$, where E_ν is the expectation with respect to ν_2^η. Combining this, $|E\theta_\eta| \le \pi$, Corollary 1.5 in [41] and Theorem 1.5, to show the corollary for $\tilde{\theta}_{\eta,\nu}$, it is enough to show that there exists a constant $C_3 > 0$, such that for all small η,

$$\big|E\big[\theta_\eta^2\big] - E_\nu\big[\tilde{\theta}_{\eta,\nu}^2\big]\big| \le C_3\big[\log(1/\eta)\big]^{\frac{6}{7}}. \tag{62}$$

For $n \ge 1$, let $\tilde{\theta}_{\eta,n}$ denote $\tilde{\theta}_\eta$ under $P[\cdot|\mathcal{A}_2^\eta(\eta, n)]$. Similar to the proof of (61), one can show that there is a $C_3 > 0$ such that for all $n \ge 1$ and all small η,

$$\big|E\big[\theta_\eta^2\big] - E_n\big[\tilde{\theta}_{\eta,n}^2\big]\big| \le C_3\big[\log(1/\eta)\big]^{\frac{6}{7}},$$

where E_k is the expectation with respect to $P[\cdot|\mathcal{A}_2^\eta(\eta, n)]$. Then one obtains (62) from the above inequality by taking $n \to \infty$. $\square$

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