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Distributional limits of positive, ergodic stationary processes and infinite ergodic transformations

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Abstract. In this note we identify the `distributional limits` of non-negative, ergodic stationary processes, showing that all are possible. Consequences for infinite ergodic theory are also explored and new examples of `distributionally stable` – and α-`rationally ergodic` – transformations are presented.

Résumé. Dans cette note, on identifie les limites distributionnelles des processus stationaires, ergodiques et positives. On montre que toutes se produisent. Les conséquences pour la théorie ergodique infinie sont également explorées et nouveaux exemples de transformations `distributionnellement stables` – et α-`rationnellement ergodiques` – sont présentées.

MSC: 37A; 28D; 60F

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0. Short introduction

Classical central limit theory is concerned with the distributional convergence of normalized partial sums $\frac{1}{a_n}\sum_{k=1}^n X_n$ of independent, identically distributed random variables $(X_1, X_2, \dots)$.

Here, we consider this asymptotic distributional behavior of normalized partial sums $\frac{1}{a_n}\sum_{k=1}^n X_n$ of random variables $(X_1, X_2, \dots)$ generated by a *stationary process* (SP) by which we mean a quintuple $(\Omega, \mathcal{F}, P, T, f)$ where $(\Omega, \mathcal{F}, P, T)$ is a probability preserving transformation (PPT) and $f : \Omega \to \mathbb{R}$ is measurable; the "generated random variables" being the sequence of random variables $(X_n = f \circ T^n)_{n\geq 0}$ defined on the sample space $(\Omega, \mathcal{F}, P)$.

The stationary process $(\Omega, \mathcal{F}, P, T, f)$ is *non-negative* if $f \geq 0$; and *ergodic* (ESP) if the underlying PPT $(\Omega, \mathcal{F}, P, T)$ is an ergodic PPT (EPPT).

For independent processes, the possible probability distributions (or laws) occurring as limits were determined by Paul Lévy in [21]. They are the `stable laws` (including the normal distribution of the central limit theorem).

For a general ESP, it was shown in [28] that any probability distribution on $\mathbb{R}$ is a possible limit.

This paper is about what happens when the stationary process is non-negative.

Our main result on stationary processes is

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Theorem 2. *Let* $(\Omega, \mathcal{F}, P, T)$ *be a* EPPT *and let* $Y \in \mathtt{RV}(\mathbb{R}_+)$, *then* $\exists$ 1*-regularly varying function* $b : \mathbb{R}_+ \to \mathbb{R}_+$ *and a positive measurable function* $f : \Omega \to \mathbb{R}_+$ *so that*

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{\mathfrak{d}} Y. \qquad (🚂)$$

Here and throughout,

- $\mathbb{R}_+ := (0, \infty)$,
- for a metric space Z, $\mathtt{RV}(Z)$ denotes the collection of Z-valued random variables, and
- $\xrightarrow[n\to\infty]{\mathfrak{d}}$ denotes `strong distributional convergence` as defined in Section 1 below.

Given a random variable, we'll first construct (Theorem 1) a specific ESP satisfying *inter alia* (🚂). This will be done by `stacking`. We'll then show that a general EPPT induces an extension of the given underlying EPPT and that this enables transference of (🚂).

Previous work on distributional limits of stochastic processes over arbitrary EPPTs can be found in [14,28,30].

We then apply our results to give new examples of distributionally stable MPTs (measure preserving transformations).

In Theorem 3 we show (*inter alia*) that: for any $Y \in \mathtt{RV}(\mathbb{R}_+)$, $\exists$ a MPT $(X, \mathcal{B}, m, T)$ and a 1-regularly varying function $a : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\frac{1}{a(n)} \sum_{k=1}^{n} f \circ T^k \xrightarrow[n\to\infty]{\mathfrak{d}} Y \int_X f\,dm \quad \forall f \in L^1(m)_+.$$

A full statement of Theorem 3 is given in Section 1 below.

Remarks.

(1) It is natural to ask what would be the possible limit laws of the the partial sums of nonnegative ESP which are scaled and also centered by positive constants.

That is, what are the possible limit laws of $\frac{S_n - a(n)}{b(n)}$ where S_n is the nth partial sum of a nonnegative ESP, and $b(n), a(n) > 0$ $(n \geq 1)$ are constants?

Our result shows that any probability distribution with support bounded from below can be obtained in this fashion. It is likely that our proof can be modified so as to obtain all distributions as limits of these normalized and "centered" sums. We thank the referee for raising this issue.

(2) It is also natural to ask about the stochastic processes ocurring as distributional limits of the random step functions $\Phi_n \in D([0, 1])$ (as in [11], Chapter 3) generated by the partials sums of an ESP and defined by $\Phi_n(t) := \frac{S_{[nt]}}{b(n)}$.

For example, if $\frac{S_n}{b(n)} \xrightarrow[n\to\infty]{\mathfrak{d}} Y$ as in Theorem 2, then, due to the 1-regular variation of b, $\Phi_n \xrightarrow[n\to\infty]{\mathfrak{d}} \mathcal{L}_Y$ in $D([0, 1])$ where $\mathcal{L}_Y(t) := tY$.

Glossary of abbreviations

The following abbreviations are used throughout the paper: SP for `stationary process`, ESP for `ergodic, stationary process`, PPT for `probability preserving transformation`, EPPT for `ergodic, probability preserving transformation`, MPT for `measure preserving transformation` and CEMPT for `conervative, ergodic, measure preserving transformation`.

1. Longer introduction

Distributional convergence

Consider the compact metric space $([0,\infty],\rho)$ with

$$\rho(x,y) := \left|\tan^{-1}(x) - \tan^{-1}(y)\right|.$$

For $x, y \in \mathbb{R}_+$, $\rho(x,y) \le |x-y|$. We'll use the

- ρ-*uniform* distance on $\mathtt{RV}(\mathbb{R}_+)$ defined by

$$\mathfrak{u}(Y_1,Y_2) := \min\left\{\sup \rho(Z_1,Z_2) : Z=(Z_1,Z_2) \in \mathtt{RV}(\mathbb{R}_+ \times \mathbb{R}_+),\, Z_i \stackrel{\mathfrak{d}}{=} Y_i \ (i=1,2)\right\};$$

  and the
- ρ-*Vasershtein* distance on $\mathtt{RV}(\mathbb{R}_+)$ defined (as in [29]) by

$$\mathfrak{v}(Y_1,Y_2) := \min\left\{E\left(\rho(Z_1,Z_2)\right) : Z=(Z_1,Z_2) \in \mathtt{RV}(\mathbb{R}_+ \times \mathbb{R}_+),\, Z_i \stackrel{\mathfrak{d}}{=} Y_i \ (i=1,2)\right\}.$$

Evidently $\mathfrak{v}(Y_1,Y_2) \le \mathfrak{u}(Y_1,Y_2)$ and, if $\mathfrak{v}(Y_1,Y_2) < \varepsilon$, then $\exists Z=(Z_1,Z_2) \in \mathtt{RV}(\mathbb{R}_+ \times \mathbb{R}_+)$, $Z_i \stackrel{\mathfrak{d}}{=} Y_i$ $(i=1,2)$ so that

$$\mathtt{Prob}\left(\rho(Z_1,Z_2) > \sqrt{\varepsilon}\right) < \sqrt{\varepsilon}.$$

For $Y_n, Y \in \mathtt{RV}(\mathbb{R}_+)$,

$$E\left(g(Y_n)\right) \xrightarrow[n\to\infty]{} E\left(g(Y)\right) \quad \forall g \in C_B(\mathbb{R}_+) \qquad \Longleftrightarrow \qquad \mathfrak{v}(Y_n,Y) \xrightarrow[n\to\infty]{} 0.$$

See the `Skorohod representation theorem` in [26] and [11].

Strong distributional convergence

For $(X,\mathcal{B})$ be a measurable space, we denote the collection of probability measures on $(X,\mathcal{B})$ by $\mathcal{P}(X,\mathcal{B})$.

Now let $(X,\mathcal{B},m)$ be a measure space, Z be a metric space, $F_n : X \to Z$ be measurable, $Y \in \mathtt{RV}(Z)$ and $P \in \mathcal{P}(X,\mathcal{B})$, $P \ll m$. We'll write

$$F_n \xrightarrow[n\to\infty]{P-\mathfrak{d}} Y$$

if

$$\int_X g(F_n)\,dP \xrightarrow[n\to\infty]{} E\left(g(Y)\right) \quad \forall g \in C_B(Z)$$

and say (as in [3,4] and [27]) that F_n *converges strongly in distribution* (written $F_n \xrightarrow[n\to\infty]{\mathfrak{d}} Y$) if

$$F_n \xrightarrow[n\to\infty]{P-\mathfrak{d}} Y \quad \forall P \in \mathcal{P}(X,\mathcal{B}),\, P \ll m.$$

This is called *mixing distributional convergence* in [22] and [17].

In ergodic situations, strong distributional convergence of normal partial sums is an automatic consequence of distributional convergence. Namely:

Eagleson's theorem ([17], see also [3,9] and [4]). *If $(X, \mathcal{B}, m, T, f)$ is an $\mathbb{R}$-valued, $\mathtt{ESP}$, $a(n) \to \infty$ and $\exists P \in \mathcal{P}(X, \mathcal{B})$, $P \ll m$ so that*

$$\int_X g\left(\frac{S_n}{a(n)}\right) dP \xrightarrow[n\to\infty]{} E\big(g(Y)\big) \quad \forall g \in C\big([0,\infty]\big),$$

where $S_n := \sum_{k=1}^n f \circ T^k$, then $\frac{S_n}{a(n)} \xrightarrow[n\to\infty]{\mathfrak{d}} Y$.

Examples.

(1) Let $\gamma \in (0, 1]$ and let $(\Omega, \mathcal{A}, P, S, f)$ be a positive $\mathtt{SP}$ where $(f \circ S^n : n \geq 1)$ are independent random variables satisfying

$$E(f \wedge t) \underset{t\to\infty}{\propto} \frac{t}{A(t)},$$

where $A(t)$ γ-regularly varying in the sense that $\frac{A(xt)}{A(t)} \xrightarrow[t\to\infty]{} x^\gamma$ $\forall x > 0$ (see [12]).

By the stable limit theorem ([21], also e.g. XIII.6 in [18])

$$\frac{1}{A^{-1}(n)} \sum_{k=1}^{n} f \circ S^k \xrightarrow[n\to\infty]{\mathfrak{d}} Z_\gamma, \tag{SLT}$$

where Z_γ is *normalized, γ-stable* in the sense that $E(e^{-pZ_\gamma}) = e^{-c_\gamma p^\gamma}$ where $c_\gamma > 0$ and $E(Z_\gamma^{-\gamma}) = 1$. Note that $Z_1 \equiv 1$. For generalizations of this to weakly dependent $\mathtt{SP}$s, see [7] and references therein.

(2) In [5] positive $\mathtt{ESP}$s $(\Omega, \mathcal{F}, P, R, f)$ were constructed so that

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} f \circ R^k \xrightarrow[n\to\infty]{\mathfrak{d}} e^{\frac{1}{2}\mathcal{N}(0,1)^2},$$

where $b(n) \propto n\sqrt{\log n}$ and $\mathcal{N}(0, 1)$ is standard normal. For example $R = \tau^f$ where τ is the dyadic adding machine on $\{0, 1\}^{\mathbb{N}}$ and $f(x) := \min\{n \geq 1 : \sum_{k\geq 1}[(\tau^n x)_k - x_k] = 0\}$ is the *exchangeability waiting time*.

The following is the main construction enabling Theorem 2. It is a specific construction tailored to the target random variable.

Theorem 1. *Let $Y \in \mathtt{RV}(\mathbb{R}_+)$, then $\exists$*

- *an odometer $(X, \mathcal{B}, m, T)$,*
- *an increasing, 1-regularly varying function $b : \mathbb{R}_+ \to \mathbb{R}_+$,*
- *a positive measurable function $f : X \to \mathbb{R}_+$ so that*

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{\mathfrak{d}} Y \tag{🚗}$$

$\exists M > 1, r > 0$ and $N_0 \geq 1$ such that

$$P\left(\left[\sum_{k=0}^{n-1} f \circ T^k < x b(n)\right]\right) \leq P(Y \leq Mx) \quad \forall x \in (0, r), n \geq N_0. \tag{🚲}$$

The (🚗) condition (repeated from page 880) is used in the proofs of Theorems 2 and 3. The (🚲) condition will be used in Theorem 3 in Section 6 to obtain examples of α-$\mathtt{rational\ ergodicity}$.

The next proposition explains why the normalizing constants are necessarily 1-regularly varying when the support of Y is compact in $\mathbb{R}_+$.

Normalizing constant proposition. *Suppose that* $(\Omega, \mathcal{F}, P, R, f)$ *is a positive* ESP, $b(n) > 0$, *and* $Y \in \mathrm{RV}(\mathbb{R}_+)$ *with* $\min \mathrm{supp}\, Y =: a > 0$ *and* $\max \mathrm{supp}\, Y =: b < \infty$.

If $\frac{S_n}{b(n)} \xrightarrow[n\to\infty]{\mathfrak{d}} Y$ *where* $S_n := \sum_{k=1}^{n} f \circ T^k$, *then* b *is* 1*-regularly varying.*

Proof. It suffices to show that $\frac{b(2n)}{b(n)} \xrightarrow[n\to\infty]{} 2$. To see this, suppose otherwise, then there exist $\varepsilon > 0$ and a subsequence $K \subset \mathbb{N}$, so that

$$\left|\frac{b(2n)}{b(n)} - 2\right| \geq \varepsilon \quad \forall n \in K. \tag{‡}$$

Next, by compactness, there is a further subsequence $K' \subset K$ and a random variable $Z = (Z_1, Z_2) \in \mathrm{RV}([0,\infty]^2)$ so that

$$\left(\frac{S_n}{b(n)}, \frac{S_n \circ T^n}{b(n)}\right) \xrightarrow[n\to\infty]{\mathfrak{d}} Z.$$

By assumption, we have that $\mathtt{dist}\, Z_i = \mathtt{dist}\, Y$ $(i = 1, 2)$. Thus,

$$2a \leq Z_1 + Z_2 \leq 2b.$$

Now fix $K'' \subset K'$ so that $\frac{b(2n)}{b(n)} \xrightarrow[n\to\infty, n\in K'']{} c \in [0, \infty]$.

By assumption,

$$\begin{aligned} Y &\xleftarrow[n\to\infty, n\in K'']{\mathfrak{d}} \frac{S_{2n}}{b(2n)} \\ &= \frac{b(n)}{b(2n)}\left(\frac{S_n}{b(n)} + \frac{S_n \circ T^n}{b(n)}\right) \\ &\xrightarrow[n\to\infty, n\in K'']{\mathfrak{d}} c^{-1}(Z_1 + Z_2). \end{aligned}$$

It follows that $c \in \mathbb{R}_+$ and that $Z_1 + Z_2 \overset{\text{dist}}{=} cY$. So on the one hand $\min \mathrm{supp}\, cY = ca$ and $\max \mathrm{supp}\, cY =: cb < \infty$ and on the other hand,

$$ca = \min \mathrm{supp}(Z_1 + Z_2) \geq 2a \quad \text{and} \quad cb = \max \mathrm{supp}(Z_1 + Z_2) \leq 2b$$

with the conclusion that $c = 2$ which contradicts (‡). □

Distributional convergence in infinite ergodic theory

Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic MPT (CEMPT) and let $Y \in \mathrm{RV}([0,\infty])$. Let $n_k \uparrow \infty$ be a subsequence and let $d_k > 0$ be constants. As in [3] and [4], we'll write

$$\frac{S_{n_k}^{(T)}}{d_k} \xrightarrow[k\to\infty]{\mathfrak{d}} Y$$

if

$$\frac{S_{n_k}^{(T)}(f)}{d_k} \xrightarrow[k\to\infty]{\mathfrak{d}} Y \int_X f\, dm \quad \forall f \in L_+^1.$$

Call the random variable $Y \in \mathrm{RV}([0,\infty])$ appearing a *subsequence distributional limit of* T and let

$$\mathcal{L}_T := \{\mathtt{subsequence\ distributional\ limits\ of}\ T\}.$$

The collection

$$\big\{T \in \texttt{MPT}(\mathbb{R}) : \mathcal{L}_T = \texttt{RV}\big([0,\infty]\big)\big\}$$

is residual in $\texttt{MPT}(\mathbb{R})$, the group of invertible transformations of $\mathbb{R}$ preserving Lebesgue measure, equipped with the weak topology (see [6]).

We call the $\texttt{CEMPT}$ $(X, \mathcal{B}, m, T)$ *distributionally stable* if there are constants $a(n) = a_{n,Y}(T) > 0$ and a random variable Y on $(0,\infty)$ (called the *ergodic limit*) so that

$$\frac{S_n^{(T)}}{a(n)} \xrightarrow[n\to\infty]{\mathfrak{d}} Y. \qquad (⚽)$$

The sequence of constants $(a_{n,Y}(T) : n \geq 1)$ is determined up to asymptotic equality and we call it the *Y-distributional return sequence*. Note that $a_{n,cY}(T) \sim \frac{1}{c} a_{n,Y}(T)$. For distributionally stable $\texttt{CEMPT}$s which are also weakly rationally ergodic, we have that $a_{n,Y}(T) \propto a_n(T)$ the usual return sequence (see [1]).

Classic examples of distributionally stable $\texttt{CEMPT}$s are obtained via the Darling–Kac theorem ([16]): pointwise dual ergodic transformations (e.g. Markov shifts) with regularly varying return sequences are distributionally stable with Mittag–Leffler ergodic limits (see also [3,4]).

More recently, it has been shown that certain "random walk adic" transformations have exponential chi-square distributional limits (see [5,10] and [13]).

Our main result about infinite, ergodic transformations is

Theorem 3. *For each $Y \in \texttt{RV}(\mathbb{R}_+)$, there is a distributionally stable $\texttt{CEMPT}$ $(X, \mathcal{B}, m, T)$ with ergodic limit Y with $a_{n,Y}(T)$ 1-regularly varying and $\Omega \in \mathcal{B}$, $m(\Omega) = 1$ so that*

$$m\big(\Omega \cap \big[S_n(1_\Omega) \geq x a(n)\big]\big) \leq 2P(Y \geq x) \quad \forall x > 1 \text{ and } n \geq 1 \text{ large}. \qquad (✈)$$

The (✈) condition (which is an inversion of the (🚲) condition on page 882) will be used in the construction of α-rationally ergodic $\texttt{MPT}$s in Section 6.

By Proposition 3.6.3 in [4], distributional stability of a $\texttt{CEMPT}$ entails existence of a `law of large numbers` (as in [3] and [4]) for it. An example in Section 6 shows it does not entail α-rational ergodicity.

Plan of the paper

In Section 2, we recall the `stacking method` used to construct the odometer in Theorem 1. This odometer is constructed together with a sequence of `step functions` and in Section 3, we formulate the `step function extension lemma` needed for the proof of Theorem 1 where the limit is a `rational random variable` (taking finitely many values, each with rational probability). In Section 4 we prove the step function extension lemma and Theorem 1 in this (rational random variable) case. In Section 5, we prove Theorem 1 in general, developing the necessary approximations of random variables by rational ones. We conclude in Section 6 by proving Theorem 3 and considering some of its consequences in infinite ergodic theory.

2. The stacking constructions

Stacking as in [15] (aka the *stacking method* [19] and *cutting and stacking* in [24,25]) is a construction procedure yielding a piecewise translation of an almost open subset $X \subset \mathbb{R}$. This transformation is invertible and preserves Lebesgue measure.

As in [15] and [19], a *column* is a finite sequence of disjoint intervals $W = (I_1, I_2, \dots, I_h)$. with equal lengths. The *width* of the column is the length of I_k. The *height* of the column is h and we'll sometimes call $W = (I_1, I_2, \dots, I_h)$ an *h-column*.

The *base* of the column $W = (I_1, I_2, \dots, I_h)$ is $B(W) := I_1$, its *top* is $A(W) := I_h$ and its *union* is $U(W) = \bigcup_{k=1}^h I_k$. The *measure* of a column is the length of its union. Columns W and W' are disjoint if their unions are disjoint.

The column W is equipped with the periodic map $T = T_W : U(W) \to U(W)$ defined by the translations $T : I_k \to I_{k+1}$ $(1 \le k \le h-1)$ and $T : I_h \to I_1$.

A *castle* (*tower* in [15] and [19]) is a finite collection of disjoint columns.

A castle consisting of a single column is known as a *Rokhlin tower*.

A castle is called *homogeneous* if all the columns have the same height and width. As before, an homogeneous castle consisting of h-columns is called an *h-castle*.

The *base* of the castle $\mathfrak{W} = \{W_1, W_2, \ldots, W_n\}$ is $B(\mathfrak{W}) = \biguplus_{k=1}^{n} B(W_k)$, its top is $A(\mathfrak{W}) = \biguplus_{k=1}^{n} A(W_k)$ and its union is $U(\mathfrak{W}) = \biguplus_{k=1}^{n} U(W_k)$.

It is equipped with the periodic transformation $T_{\mathfrak{W}} : U(\mathfrak{W}) \to U(\mathfrak{W})$ defined by

$$T_{\mathfrak{W}}|_{U(W_k)} \equiv T_{W_k}.$$

Refinements of castles

The castle $\mathfrak{W}'$ *refines* the castle $\mathfrak{W}$ (written $\mathfrak{W}' \succ \mathfrak{W}$) if

(i) each interval of $\mathfrak{W}$ is a union of intervals of $\mathfrak{W}'$;
(ii) $A(\mathfrak{W}') \subset A(\mathfrak{W})$ and $B(\mathfrak{W}') \subset B(\mathfrak{W})$;
(iii) $T_{\mathfrak{W}'}|_{U(\mathfrak{W}) \setminus A(\mathfrak{W})} \equiv T_{\mathfrak{W}}$.

If $\mathfrak{W}' \succ \mathfrak{W}$, then $U(\mathfrak{W}') \supset U(\mathfrak{W})$.

All castle refinements $\mathfrak{W}' \succ \mathfrak{W}$ considered here are mass preserving in the sense that $U(\mathfrak{W}') = U(\mathfrak{W})$ (no "spacers" are added).

Call the refinement $\mathfrak{W}' \succ \mathfrak{W}$ *transitive* if

$$m\big(U\big(W'\big) \cap U(W)\big) > 0 \quad \forall W' \in \mathfrak{W}' \text{ and } W \in \mathfrak{W}.$$

A sequence $(\mathfrak{W}_n)_{n\ge1}$ of castles is a *nested sequence* if each $\mathfrak{W}_{n+1}$ refines $\mathfrak{W}_n$.

Let $(\mathfrak{W}_n)_{n\ge1}$ be a nested sequence of castles and consider the measure space $(X, \mathcal{B}, m)$ with $X := \bigcup_{n=1}^{\infty} U(\mathfrak{W}_n)$ equipped with Borel sets $\mathcal{B}$ and Lebesgue measure m.

As shown in [15] and [19],

☺ There is a measure preserving transformation $(X, \mathcal{B}, m, T)$ defined by

$$T(x) = \lim_{n\to\infty} T_{\mathfrak{W}_n}(x) \quad \text{for } m\text{-a.e. } x$$

iff $m(A(\mathfrak{W}_n)) \xrightarrow[n\to\infty]{} 0$.

It is standard to show that if infinitely many of the refinements $\mathfrak{W}_{n+1} \succ \mathfrak{W}_n$ are transitive, then $(X, \mathcal{B}, m, T)$ is ergodic.

The transformation $(X, \mathcal{B}, m, T)$ is aka the *inverse limit* of $(\mathfrak{W}_n)_{n\ge1}$ and denoted $T = \varprojlim_{n\to\infty} \mathfrak{W}_n$.

Odometers

An *odometer* is an inverse limit of a (mass preserving) nested sequence of Rokhlin towers. Odometers are ergodic because if $\mathfrak{W}', \mathfrak{W}$ are Rokhlin towers and $\mathfrak{W}' \succ \mathfrak{W}$, then the refinement is clearly transitive. The odometers are the ergodic transformations with rational, pure point spectrum.

Induced transformation (as in [20]). Let $(X, \mathcal{B}, m, T)$ be a CEMPT and let $\Omega \in \mathcal{B}, 0 < m(\Omega) < \infty$. The *first return time* to Ω is the function $\varphi_\Omega : \Omega \to \mathbb{N} \cup \{\infty\}$ defined by $\varphi_\Omega(x) := \min\{n \ge 1 : T^n x \in \Omega\}$ which is finite for a.e. $x \in \Omega$ by conservativity.

The *induced transformation* is $(\Omega, \mathcal{B} \cap \Omega, m_\Omega, T_\Omega)$ where $T_\Omega : \Omega \to \Omega$ is defined by $T_\Omega(x) := T^{\varphi_\Omega(x)}$ and $m_\Omega(\cdot) := m(\cdot \| \Omega)$. It is a PPT.

Odometer factor proposition. *Let R be an odometer and let $(X, \mathcal{B}, m, T)$ be an aperiodic* PPT, *then $\exists \Omega \in \mathcal{B}, m(\Omega) > 0$ so that R is a factor* PPT *of T_Ω.*

Proof. Let $R = \varprojlim_{n\to\infty} \mathfrak{W}_n$ where $(\mathfrak{W}_n)_{n\geq 1}$ is a nested sequence of Rokhlin towers. Let the height of $\mathfrak{W}_n$ be H_n, then there is a sequence $a_1, a_2, \ldots \in \mathbb{N}$, $a_n \geq 2$ so that $H_1 = a_1$, $H_{n+1} = a_{n+1}H_n$.

By the basic Rokhlin lemma, for any $\varepsilon_1 \in (0,1)$ there is some B_1 of positive measure such that the sets $\{T^i(B_1) : i = 0, 1, \ldots, a_1 - 1\}$ are disjoint and

$$X = \biguplus_{i=0}^{a_1-1} T^i(B_1) \uplus E_1,$$

where $E_1 \in \mathcal{B}$ and $m(E_1) = \varepsilon_1 m(B_1)$.

Next apply the Rokhlin lemma again to the induced transformation T_{B_1} with $\varepsilon_2 \in (0,1)$ to get a base $B_2 \subset B_1$ with the sets $\{T^i_{B_1} B_2 : 0 \leq i < a_2\}$ disjoint and

$$B_1 = \biguplus_{i=0}^{a_2-1} T^i_{B_1}(B_2) \uplus E_2,$$

where $E_2 \in \mathcal{B}(B_1)$ and $m(E_2) = \varepsilon_2 m(B_2)$.

This process is continued to obtain $B_k \in \mathcal{B}$, $B_k \subset B_{k-1}$ with the sets $\{T^i_{B_{k-1}} B_k : 0 \leq i < a_k\}$ disjoint and

$$B_{k-1} = \biguplus_{i=0}^{a_k-1} T^i_{B_{k-1}}(B_k) \uplus E_k,$$

where $E_k \in \mathcal{B}(B_{k-1})$ and $m(E_k) = \varepsilon_k m(B_{k-1})$. If $\sum_{k\geq 1} \varepsilon_k < 1$, then

$$\Omega := \bigcap_{k\geq 1} \bigcup_{i=0}^{H_k-1} T^i(B_k)$$

is as advertised. □

We'll need a condition for an inverse limit of castles to be isomorphic to an odometer.

If $W = (I_1, I_2, \ldots, I_k)$ and $W' = (I'_1, I'_2, \ldots, I'_{k'})$ are disjoint columns of intervals with equal width, the *stack* of W and W' is the column

$$W \circledcirc W' := \big(I_1, I_2, \ldots, I_k, I'_1, I'_2, \ldots, I'_{k'}\big).$$

Let $q \in \mathbb{N}$. The column W can be sliced into q subcolumns

$${}^qW_1, {}^qW_2, \ldots, {}^qW_q$$

of equal width and the same height.

For a column W and $q \in \mathbb{N}$, $W^{\circledast q}$ denotes the column obtained from W by slicing the column into q disjoint subcolumns of equal width and then stacking. That is

$$W^{\circledast q} = \bigcircledcirc_{k=1}^{q} {}^qW_k.$$

Let $\mathfrak{W} = \{W_k : 1 \leq k \leq K\}$ and $\mathfrak{W}' = \{W'_\ell : 1 \leq \ell \leq L\}$ be homogeneous castles.

The refinement $\mathfrak{W}' \succ \mathfrak{W}$ is *uniform* if $\exists Q \geq 1, \kappa_1, \kappa_2, \ldots, \kappa_Q \in \{1, 2, \ldots, K\}$ with $\{\kappa_q : 1 \leq q \leq Q\} = \{1, 2, \ldots, K\}$ and $s_1, s_2, \ldots, s_Q \in \mathbb{N}$ so that

$$W'_\ell = {}^L\Bigg(\bigcircledcirc_{q=1}^{Q} W_{\kappa_q}^{\circledast s_q}\Bigg)_\ell.$$

Note that a uniform refinement is transitive.

The nested sequence of homogeneous castles $(\mathfrak{W}_n)_{n\ge 1}$ is called *uniformly nested* if each refinement $\mathfrak{W}_{n+1} \succ \mathfrak{W}_n$ is uniform.

Proposition. *Let $(\mathfrak{W}_n)_{n\ge 1}$ be a uniformly nested sequence of homogeneous castles, then the* EPPT $(X, \mathcal{B}, m, T) := \varprojlim_{n\to\infty} \mathfrak{W}_n$ *is an odometer.*

Proof. Let $\mathfrak{W}_n = \{W_j^{(n)} : 1 \le j \le k_n\}$ and suppose that

$$W_\ell^{(n+1)} = {}^{k_{n+1}}\left(\bigodot_{q=1}^{Q_{n+1}} W_{\kappa_q}^{(n)\circledast s_q^{(n+1)}}\right)_\ell,$$

then

$$W_\ell^{(n+1)} = {}^{k_{n+1}}\big(\widetilde{W}^{(n)}\big)_\ell,$$

where

$$\widetilde{W}^{(n)} := \bigodot_{q=1}^{Q_{n+1}} W_{\kappa_q}^{(n)\circledast s_q^{(n+1)}}.$$

The Rokhlin tower $\widetilde{\mathfrak{W}}^{(n)} := \{\widetilde{W}^{(n)}\}$ is refined by $\widetilde{\mathfrak{W}}^{(n+1)}$ and

$$(X, \mathcal{B}, m, T) = \varprojlim_{n\to\infty} \widetilde{\mathfrak{W}}^{(n)}.$$

$\square$

3. Step functions, labeled castles and block arrays

Here we introduce the framework for the proof of Theorem 1.

We'll construct recursively a nested sequence of homogeneous, unit measure castles $(\mathfrak{W}_n)_{n\ge 1}$ and set $(X, \mathcal{B}, m, T) = \varprojlim_{n\to\infty} \mathfrak{W}_n$.

The advertised function $f : X \to \mathbb{R}_+$ will be defined as $f = \lim_{n\to\infty} f^{(n)}$ where $f^{(n)} : \mathfrak{W}_n \to \mathbb{R}_+$ is a *step function* in the sense that it is constant on each of the intervals making up each column in the castle $\mathfrak{W}_n$.

If $\mathfrak{W}_n = \{W_j^{(n)} : 1 \le j \le k_n\}$ where each $W_j^{(n)} = (I_{j,k}^{(n)})_{1\le k\le h_n}$ is a column of height h_n, then

$$f^{(n)} \cong \big(w_j^{(n)} : 1 \le j \le k_n\big) \subset \big(\mathbb{R}_+^{h_n}\big)^{k_n},$$

where

$$f^{(n)} \equiv w_j^{(n)}(k) \quad \text{on } I_{j,k}^{(n)}.$$

Formally, let a *J-block* be a positive vector $w \in \mathbb{R}_+^J$ (where $J \in \mathbb{N}$). The *length* of J-block w is $|w| := J$.

A block $w \in \mathbb{R}_+^J$ determines a *labeled column*: an *underlying column* $W = (I_1, I_2, \dots, I_J)$ together with a step function $F_W : U(W) \to \mathbb{R}_+$ defined by

$$F_W = \sum_{k=1}^{J} w_k 1_{I_k}.$$

A *block array* is an ordered collection of blocks of the same length (called *J-block array* when all the blocks have length J).

The block array $\mathfrak{w} = (w_1, w_2, \dots, w_N) \in (\mathbb{R}_+^h)^N$ determines a *labeled castle*:

an *underlying castle* $\mathfrak{W} = (W_1, W_2, \dots, W_N)$ of height h, together with a step function $F_{\mathfrak{w}} : U(\mathfrak{w}) \to \mathbb{R}_+$ defined by

$$F_{\mathfrak{w}} := \sum_{k=1}^{N} 1_{U(W_k)} F_{W_k}.$$

We'll say that the block array $\mathfrak{y}$ *refines* the block array $\mathfrak{x}$ written $\mathfrak{y} \succ \mathfrak{x}$ if the castle determined by $\mathfrak{y}$ refines that determined by $\mathfrak{x}$.

Blocks can be `concatenated`. If $w \in \mathbb{R}^J$ and $w' \in \mathbb{R}^{J'}$, the *concatenation* of w and w' is

$$w \odot w' := \left(w_1, w_2, \dots, w_J, w'_1, w'_2, \dots, w'_{J'}\right) \in \mathbb{R}^{J+J'}.$$

The concatenation of blocks corresponds to the stacking of their underlying columns.

If W and W' are columns of height J and J' respectively and with the same width, and $w \in \mathbb{R}^J$ and $w' \in \mathbb{R}^{J'}$, then

$$F_{w\odot w'} \equiv F_{\{w,w'\}} \quad \text{on } U\left(W \circledcirc W'\right) = U\left(\left\{W, W'\right\}\right) = U(W) \uplus U\left(W'\right).$$

Similarly, self concatenation $w^{\odot q}$ of the same block w corresponds to cutting and stacking $W^{\circledast q}$ of the corresponding column W.

We call a sequence of block arrays *nested* if the underlying sequence of castles is nested.

We'll obtain the required `ESP` by producing a nested sequence $(\mathfrak{w}_n)_{n\geq 1}$ of block arrays whose associated sequence of step functions $(F_{\mathfrak{w}_n})_{n\geq 1}$ is convergent.

Block statistics

Distributional convergence will be achieved by controlling the empirical distributions of the various short-term partial sums over the tall block arrays.

Given a block $w \in \mathbb{R}_+^h$, define

$$S_k(F_w) := \sum_{j=0}^{k-1} F_w \circ T_w^j,$$

where T_w is the periodic transformation defined on the column underlying w. We have

$$S_k(F_w) = \sum_{\nu=1}^{h} S_k(w)(\nu) 1_{I_\nu},$$

where, for $1 \leq \nu \leq h$,

$$S_k(w)(\nu) := \sum_{j=0}^{k-1} w_{\nu+j}.$$

Here translation is considered $\bmod h$ that is $\nu + j := \nu + j \bmod h$.

For a block array $\mathfrak{w} = \{w_j : 1 \leq j \leq K\}$, set

$$S_k(F_{\mathfrak{w}}) = \sum_{j=1}^{K} 1_{U(w_j)} F_{w_j}$$

and $S_k(\mathfrak{w})(\nu, j) := S_k(w_j)(\nu)$.

We study the distributions of $S_k(w)$ and $S_k(\mathfrak{w})$ considered as $\mathbb{R}_+$-valued random variables on the symmetric probability spaces $\{1, 2, \dots, h\}$ and $\{1, 2, \dots, h\} \times \{1, 2, \dots, K\}$ respectively.

If $w \in \mathbb{R}^h$ and $m \in \mathbb{N}$, then

$$S_k\big(w^{\odot m}\big)(\nu) = S_k(w)(\nu \bmod h)$$

whence $S_k(w^{\odot m})$ and $S_k(w)$ are equidistributed.

In a similar manner, we consider partial sums on a block array $\mathfrak{w} = \{w_k : 1 \le k \le n\} : \{1, \ldots, h\} \times \{1, \ldots, n\} \to \mathbb{R}_+$:

$$S_k(\mathfrak{w})(j, \ell) := S_k(w_\ell)(j).$$

Before starting the construction, we need some notions of `block normalization.`

Block normalizations

Suppose that $h \in \mathbb{N}$ and $w \in \mathbb{R}_+^h$ is a block.

Write

$$|h| := h, \qquad M(w) := \max_{1 \le j \le h} w_j, \qquad \Sigma(w) := \sum_{1 \le j \le h} w_j \quad \text{and} \quad E(w) := \frac{\Sigma(w)}{|w|}.$$

Note that

$$E(w) = \int_{[1,h] \cap \mathbb{N}} w \, dP_{[1,h]\cap\mathbb{N}}.$$

The block $w \in \mathbb{R}_+^h$ is *ε-normalized* if

$$S_k(w) = kE(w)(1 \pm \varepsilon) \quad \forall k \ge \frac{\varepsilon \Sigma(w)}{M(w)}.$$

We call the block array $\mathfrak{w} \subset \mathbb{R}_+^h$ *ε-normalized* if each block $w \in \mathfrak{w}$ is ε-normalized.

Block array distributions

Let X be a metric space. We'll identify the collection $\mathcal{P}(X)$ of Borel probabilities on X with

$$\mathtt{RV}(X) := \{\mathtt{random\ variables\ with\ values\ in}\ X\}$$

by

$$Y \in \mathtt{RV}(X) \quad \leftrightarrow \quad \mathtt{dist}(Y) \in \mathcal{P}(X),$$

where

$$\mathtt{dist}(Y) := P \circ Y^{-1} \in \mathcal{P}(X)$$

in case Y is defined on the probability space $(\Omega, \mathcal{F}, P)$.

A *symmetric representation* of $Y \in \mathtt{RV}(X)$ is an ordered pair (Ω, f) where Ω is a finite set and $f : \Omega \to X$ is so that

$$\mathtt{Prob}(Y = x) = \frac{1}{|\Omega|} \#\big\{\omega \in \Omega : f(\omega) = x\big\} \quad \forall x \in X.$$

Evidently, the random variable $Y \in \mathtt{RV}(X)$ has a symmetric representation iff Y is *rational* in the sense that there is a finite set $V \subset X$ so that $Y \in V$ a.s. and

$$\mathtt{Prob}(Y = x) \in \mathbb{Q}_+ \quad \forall x \in F.$$

Let $Y \in \mathrm{RV}(\mathbb{R}_+)$ be rational. A *Y-distributed, h-block array* is a h-block array of form

$$\mathfrak{w} \subset \mathbb{R}_+^h$$

with respect to which, block averaging is a symmetric representation for $c \cdot Y$ for some $c = c(\mathfrak{w}) \in \mathbb{R}_+$. Specifically,

$$\mathrm{Prob}(c \cdot Y = x) = \frac{1}{|\mathfrak{w}|} \#\big\{w \in \mathfrak{w} : E(w) = c \cdot x\big\} \quad \forall x \in \mathbb{R}_+.$$

Definition: Relative Y-distribution

Let $Y \in \mathrm{RV}(\mathbb{R}_+)$ be rational, let $\Delta > \mathcal{E} > 0$, $h, Q \in \mathbb{N}$ and let $\mathfrak{w} \subset \mathbb{R}_+^h$ and $\mathfrak{w}' \subset \mathbb{R}_+^{Qh}$ be Y-distributed block arrays with $\mathfrak{w}'$ refining $\mathfrak{w}$, $\mathfrak{w}$ Δ-normalized and $\mathfrak{w}'$ $\mathcal{E}$-normalized.

We'll say that the pair $(\mathfrak{w}, \mathfrak{w}')$ is *relatively, $Y - (\Delta, \mathcal{E})$-distributed* if

(i) $m([F_{\mathfrak{w}'} \neq F_{\mathfrak{w}}]) < \Delta$,
(ii) $\exists c(\mathfrak{w}) = \gamma(h) \leq \gamma(h+1) \leq \cdots \leq \gamma(h') = c(\mathfrak{w}')$ and $\Delta \geq \varepsilon_h > \varepsilon_{h+1} > \cdots > \varepsilon_{Qh} = \mathcal{E}$ so that $\gamma(k+1) - \gamma(k) \leq \Delta$ and

$$\mathfrak{u}\left(\frac{S_k(\mathfrak{w}')}{k\gamma(k)}, Y\right) < \varepsilon_k \quad \forall h \leq k \leq Qh.$$

The proof of Theorem 1 for rational random variables is based on the:

Step function extension lemma. *Let $Y \in \mathrm{RV}(\mathbb{R}_+)$ be rational, let $\Delta > 0$ and $h \in \mathbb{N}$. If $\mathfrak{w} \subset \mathbb{R}_+^h$ is a Δ-normalized, Y-distributed block array, then for any $0 < \mathcal{E} < \Delta$ and $Q \in \mathbb{N}$ large enough, there is a homogeneous Qh-block array $\mathfrak{w}'$ refining $\mathfrak{w}$ uniformly so that $F_{\mathfrak{w}'} \geq F_{\mathfrak{w}}$ and so that $(\mathfrak{w}, \mathfrak{w}')$ is relatively $Y - (\Delta, \mathcal{E})$-distributed.*

4. Proof of Theorem 1 in the rational case

We first prove this case of Theorem 1 assuming the step function extension lemma.

Fix $Y \in \mathrm{RV}(\mathbb{R}_+)$. Given $\Delta_n \downarrow 0$, with $\Delta_1 < \frac{1}{9} \min Y$, we build using the step function extension lemma iteratively, a refining sequence of block arrays $(\mathfrak{w}_n)_{n \geq 1}$ with each refinement transitive and each $(\mathfrak{w}_n, \mathfrak{w}_{n+1})$ is relatively, $Y - (\Delta_n, \Delta_{n+1})$-distributed. This gives an ESP with distributional limit Y establishing (🚗) as on page 880.

To see (🚲) as on page 882, we note that by the extension lemma, for $|\mathfrak{w}| \leq k \leq |\mathfrak{w}_{n+1}|$, we have a coupling of

$$\frac{S_k(\mathfrak{w}_{n+1})}{k\gamma(k)} \quad \text{and} \quad Y$$

so that

$$\frac{S_k(\mathfrak{w}_{n+1})}{k\gamma(k)} \geq Y - \frac{1}{9} \min Y \geq \frac{8}{9} Y.$$

By monotonicity,

$$\frac{S_k(\mathfrak{w}_\nu)}{k\gamma(k)} \geq \frac{8}{9} Y \quad \forall \nu \geq n+1$$

whence

$$\frac{S_k(f)}{k\gamma(k)} \geq \frac{8}{9} Y,$$

where $F_{\mathfrak{w}_\nu} \to f$ a.s. Thus

$$P\left(\left[\frac{S_k(f)}{k\gamma(k)} < t\right]\right) \leq P\left(Y \leq \frac{9}{8} t\right) \quad \forall t > 0. \qquad \square$$

The rest of this section is a proof of the step function extension lemma.
The proof is via block concatenation and perturbation.

Basic Lemma I. *Let* $0 < \Delta < 1$ *and let* $w \in \mathbb{R}_+^h$ *be* Δ*-normalized. For each*

$$0 \le \kappa \le \Delta E(w), \qquad \delta > 0 \quad \textit{and} \quad q > \frac{1}{\Delta},$$

then for $\mu \in \mathbb{N}$ *large enough*: *if* $m := \mu q$ *and* $w' \in \mathbb{R}_+^{mh}$ *is defined by*

$$w' = w^{(\mu)} := w^{\odot m} + \kappa q h 1_{[1,mh]\cap qh\mathbb{Z}},$$

then

$$w' \textit{ is } \delta\textit{-normalized}; \tag{i}$$

$$E\big(w'\big) = E(w) + \kappa; \tag{ii}$$

$$P\big(S_J\big(w'\big) \ne S_J\big(w^{\odot m}\big)\big) \le \frac{J}{qh} \quad \forall 1 \le J \le qh; \tag{iii}$$

$$P\big(S_k\big(w'\big) = S_k\big(w^{\odot m}\big)\ \forall 1 \le k \le \sqrt{\Delta}qh\big) \ge 1 - \sqrt{\Delta}; \tag{iii$'$}$$

$$S_k\big(w'\big) = kE(w)(1 \pm 2\sqrt{\Delta}) \quad \forall \sqrt{\Delta}qh \le k \le qh; \tag{iv}$$

$$S_k\big(w'\big) = k\big(E(w) + \kappa\big)\left(1 \pm \left(\Delta \wedge \frac{1}{k} + \frac{\Delta qh}{k}\right)\right) \quad \forall k > qh. \tag{v}$$

Remarks.

(a) Note that $F_{w'} \ge F_w$.
(b) There is no contradiction between (iv) and (v) for $k \sim qh$ as the error in (iv) is at least $\frac{\kappa}{E(w)}$ which is the increment in (v).

Proof for $\kappa > 0$**.**

Proof of (i). Let $v \in \mathbb{R}_+^H$ be a block. We claim that

$$\frac{S_k(v)}{kE(v)} \xrightarrow[k\to\infty]{} 1. \tag{☕}$$

To see this, let $k = JH + r$ where $J \ge 1$ and $0 \le r < H$, then

$$S_k(v) = S_{JH}(v) \pm HM(v) = JE(v) \pm HM(v) = kE(v) \pm 2HM(v)$$

whence

$$\frac{S_k(v)}{kE(v)} = 1 \pm \frac{2HM(v)}{kE(v)} \xrightarrow[k\to\infty]{} 1.$$

We have,

$$w' = w^{(\mu)} := \big(w''\big)^{\odot\mu},$$

where

$$w'' := w^{\odot q} + \kappa q h 1_{\{qh\}}.$$

It follows that

$$E\big(w^{(\mu)}\big) = E\big(w''\big) \quad \text{and} \quad M\big(w^{(\mu)}\big) = M\big(w''\big).$$

By (☕), δ-normalization of w' is obtained by enlarging μ. □

Proof of (ii). We have

$$S_k\big(w'\big)(\nu) = S_k\big(w^{\odot m}\big)(\nu) + \kappa q h \#\big([\nu, \nu + k - 1] \cap qh\mathbb{Z}\big) \quad \forall \nu \in [1, mh].$$

Therefore

$$S_{Jqh}\big(w'\big) = Jq\Sigma(w) + J\kappa q h, \qquad \Sigma\big(w'\big) = m\Sigma(w) + \mu\kappa q h \quad \text{and} \quad E\big(w'\big) = E(w) + \kappa. \qquad \square$$

Also

$$S_k\big(w'\big) \le S_k\big(w^{\odot m}\big) + \kappa q h \left\lceil \frac{k}{qh} \right\rceil \le S_k\big(w^{\odot m}\big) + k\kappa\left(1 + \frac{qh}{k}\right);$$

and

$$S_k\big(w'\big) \ge S_k\big(w^{\odot m}\big) + \kappa q h \left\lfloor \frac{k}{qh} \right\rfloor \ge S_k\big(w^{\odot m}\big) + k\kappa\left(1 - \frac{qh}{k}\right).$$

Proof of (iii) and (iii′).

$$S_k\big(w'\big) = S_k\big(w^{\odot m}\big) \quad \text{on } [1, mh] \setminus \bigcup_{1 \le J \le \frac{m}{q}} (Jhq - k, Jhq] \quad \therefore$$

$$P\big(S_K\big(w'\big) \ne S_K\big(w^{\odot m}\big)\big) \le \frac{K}{qh}; \quad \text{and}$$

$$P\big(S_k\big(w'\big) = S_k\big(w^{\odot m}\big)\ \forall 1 \le k \le \sqrt{\Delta} qh\big) \ge 1 - \sqrt{\Delta}. \qquad \square$$

Proof of (iv) and (v). We begin with an estimate of $S_k(w^{\odot m})$ for $k \ge \Delta h$.

$$S_k\big(w^{\odot m}\big) = kE(w)\left(1 \pm \Delta \wedge \frac{h}{k}\right) \quad \forall k \ge \Delta h. \tag{§}$$

Proof of (§). For $\Delta h \le k \le h$, we have $\Delta \wedge \frac{h}{k} = \Delta$ and (§) follows from the Δ-normalization of w.
Let $h \le k$, then $k = Jh + r$ with $J \ge 1$ and $r < h$ and

$$\begin{aligned} S_k\big(w^{\odot m}\big)(\nu) &= JhE(w) + \sum_{i=\nu+Jh}^{\nu+Jh+r-1} w_i \\ &= kE(w) - rE(h) + \sum_{i=\nu+Jh}^{\nu+Jh+r-1} w_i \\ &=: kE(w) + \mathcal{E}. \end{aligned}$$

Thus

$$-\Sigma(w) < -rE(h) \le \mathcal{E} \le S_r(w)(\nu \bmod h) \le \Sigma(w)$$

and

$$\frac{|\mathcal{E}|}{kE(w)} \le \frac{\Sigma(w)}{kE(w)} = \frac{h}{k}.$$

To see the other estimation, we use the Δ-normalization of w.

If $r \le \frac{\Delta h E(w)}{M(w)}$, then

$$|\mathcal{E}| \le Mr \le \Delta h E(w);$$

and if $r > \frac{\Delta h E(w)}{M(w)}$, then by Δ-normalization of w,

$$\mathcal{E} = -rE(w) + S_r(\nu + Jh) = -rE(w) + rE(w)(1 \pm \Delta) = \pm \Delta E(w). \qquad \square$$

We have

$$S_k\big(w'\big)(\nu) - S_k\big(w^{\odot m}\big)(\nu) = \kappa q h \#\big([\nu, \nu + k - 1] \cap qh\mathbb{Z}\big).$$

For $\sqrt{\Delta} qh \le k < qh$, $\#([\nu, \nu + k - 1] \cap qh\mathbb{Z}) = 0, 1$

$$S_k\big(w'\big) - S_k\big(w^{\odot m}\big) \le \kappa q h \le \Delta E(w) qh < \sqrt{\Delta} \cdot kE(w)$$

and by (§),

$$S_k\big(w'\big) = kE(w)\left(1 \pm \left(\Delta \wedge \frac{h}{k} + \sqrt{\Delta}\right)\right) = kE(w)(1 \pm 2\sqrt{\Delta}).$$

For $k \ge qh$,

$$\begin{aligned} S_k\big(w'\big)(\nu) - S_k\big(w^{\odot m}\big)(\nu) &= \kappa q h \#\big([\nu, \nu + k - 1] \cap qh\mathbb{Z}\big) \\ &= \kappa q h \left(\frac{k}{qh} \pm 1\right) \\ &= \kappa k \pm \kappa q h. \end{aligned}$$

Therefore

$$\begin{aligned} S_k\big(w'\big) &= S_k\big(w^{\odot m}\big) + \kappa k \pm \kappa q h \\ &= kE(w)\left(1 \pm \Delta \wedge \frac{h}{k}\right) + \kappa k \pm \kappa q h \\ &= k\big(E(w) + \kappa\big)\left(1 \pm \left(\Delta \wedge \frac{h}{k} + \frac{\kappa q h}{kE(w)}\right)\right) \\ &= k\big(E(w) + \kappa\big)\left(1 \pm \left(\Delta \wedge \frac{h}{k} + \frac{\Delta q h}{k}\right)\right). \end{aligned} \qquad \square$$

This proves the basic lemma. $\square$

Example 1. *Constant limit random variable*

To see how the basic lemma works, we build a sequence of (trivial) block arrays $(\mathfrak{w}_n)_{n \ge 1}$ with each $\mathfrak{w}_n = \{w^{(n)}\}$ a single block. This will give $Y \equiv 1$ as distributional limit.

We'll define $f^{(n)} := w^{(n)} : \mathbb{Z}_{b_n} \to \mathbb{R}_+$ where $b_n = |w^{(n)}|$.

Suppose that each block $w^{(n)}$ is constructed from $w^{(n-1)}$ using the basic lemma with parameters

$$\Delta_n, \kappa_n, q_n, \mu_n, m_n, \delta_n = \Delta_{n+1}.$$

(1) $\exists \lim_{n\to\infty} f^{(n)} =: f \in \mathbb{R}_+$ a.s.

Proof.

$$P\big(\big[w^{(n)} \neq w^{(n-1)}\big]\big) = \frac{1}{q_n |w^{(n-1)}|}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{q_n |w^{(n-1)}|} < \infty$, $\exists N : \Omega \to \mathbb{N}$ so that a.s., $f^{(k)} \equiv f^{(N)}$ $\forall k \geq N$. □

(2) If $\sum_{n=1}^{\infty} \kappa_n = \infty$, then as $n \to \infty$,

$$E\big(w^{(n)}\big) \sim \sum_{k=1}^{n} \kappa_k.$$

Now let $(\Omega, \mathcal{F}, P, T)$ be the corresponding odometer and let $f := \lim_{n\to\infty} f^{(n)} : \Omega \to \mathbb{R}_+$.
Define $b : \mathbb{N} \to \mathbb{R}_+$ by

$$b(N) := N E\big(w^{(n)}\big) \quad \text{for } \big|w^{(n-1)}\big| < N \leq \big|w^{(n)}\big|, n \geq 1.$$

(3) If $\kappa_n \to 0$ and $\sum_{n=1}^{\infty} \kappa_n = \infty$, then

$$\frac{b(n)}{n} \uparrow \infty, \qquad \frac{b(2n)}{b(n)} \xrightarrow[n\to\infty]{} 2$$

and

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{\delta} 1.$$

In Example 1, the normalizing constants were directly determined by the sequence $(E(w^{(n)}))_{n\geq 1}$ of block expectations, which increased slowly.

For more complicated limit random variables (e.g. $Y \in \mathrm{RV}(\mathbb{R}_+)$ given by $P(Y = 1) = P(Y = 2) = \frac{1}{2}$) this is no longer the case as the distributions of the block expectations need to be considered. A more elaborate construction procedure is necessary.

We'll need the following simultaneous version of Basic Lemma I which is an immediate consequence of it.

Basic Lemma II. *Let $\mathfrak{w} \subset \mathbb{R}_+^h$ be a Δ-normalized h-block array and let $\kappa : \mathfrak{w} \to \mathbb{R}_+$ satisfy $0 \leq \kappa(w) \leq \Delta E(w)$.*

For each $\delta > 0$ and $q > \frac{1}{\Delta}$, and $\mu \in \mathbb{N}$ large enough: if $m := \mu q$ and the mh-block array $\mathfrak{w}' := \{v(w) \in \mathbb{R}_+^{mh} : w \in \mathfrak{w}\}$ is defined by

$$v(w) = w^{(\mu)} := w^{\odot m} + \kappa(w) q h 1_{[1,mh]\cap qh\mathbb{Z}} \quad (w \in \mathfrak{w}),$$

then $\mathfrak{w}' \succ \mathfrak{w}$ and $F_{\mathfrak{w}'} \geq F_{\mathfrak{w}}$ and for $w \in \mathfrak{w}$,

$$v(w) \text{ is } \delta\text{-normalized}; \tag{i}$$

$$E\big(v(w)\big) = E(w) + \kappa(w); \tag{ii}$$

$$P\big(S_J\big(v(w)\big) \neq S_J\big(w^{\odot m}\big)\big) \leq \frac{J}{qh} \quad \forall 1 \leq J \leq qh; \tag{iii}$$

$$P\big(S_k\big(v(w)\big) = S_k\big(w^{\odot m}\big)\ \forall 1 \le k \le \sqrt{\Delta} q h\big) \ge 1 - \sqrt{\Delta}; \tag{iii$'$}$$

$$S_k\big(v(w)\big) = kE(w)(1 \pm 2\sqrt{\Delta}) \quad \forall \sqrt{\Delta} q h \le k \le qh; \tag{iv}$$

$$S_k\big(v(w)\big) = k\big(E(w) + \kappa(w)\big)\left(1 \pm \left(\Delta \wedge \frac{1}{k} + \frac{\Delta q h}{k}\right)\right) \quad \forall k > qh. \tag{v}$$

The next lemma is an iteration of the procedure in Basic Lemma II to achieve larger, but gradual changes of the block averages $E(w)$. We'll use it to prove both the step function extension lemma and the step function straightening lemma.

Compound lemma. *Let* $0 < \Delta < 1, h \in \mathbb{N}$ *and let* $\mathfrak{w} \subset \mathbb{R}_+^h$ *be a* Δ*-normalized* h*-block array. Let* $\mathfrak{t} : \mathfrak{w} \to (1, \infty)$, *then* $\forall \beta > 0$ *and* $\mathcal{E} > 0$, *and* $Q \in \mathbb{N}$ *large enough, there is an* $\mathcal{E}$*-normalized,* Qh*-block array*

$$\mathfrak{w}' := \big\{v(w) : w \in \mathfrak{w}\big\} \subset \mathbb{R}_+^{Qh},$$

numbers

$$\delta_k \ge \delta_{k+1}, \delta_{Qh} < \mathcal{E} \quad \textit{and} \quad 0 = p_h < p_{h+1} < \cdots < p_{Qh} = 1, \quad 0 \le p_{k+1} - p_k \le \beta$$

so that $\mathfrak{w}' \succ \mathfrak{w}$ *and* $F_{\mathfrak{w}'} \ge F_{\mathfrak{w}}$ *for each* $w \in \mathfrak{w}$,

$$E\big(v(w)\big) = \mathfrak{t}(w)E(w); \tag{ii}$$

$$P\big(S_k\big(v(w)\big) = S_k\big(w^{\odot Q}\big)\ \forall 1 \le k \le \sqrt{\Delta} h\big) > 1 - 2\sqrt{\Delta}; \tag{iii}$$

$$\forall k > \Delta h, \quad S_k\big(v(w)\big) \ge kE(w)\big((1 - p_k) + p_k \mathfrak{t}(w)\big)(1 - \delta_k) \quad \textit{and} \tag{iv}$$

$$P\big(\big[S_k\big(v(w)\big) = kE(w)\big((1 - p_k) + p_k \mathfrak{t}(w)\big)(1 \pm \delta_k)\big]\big) \ge 1 - \delta_k.$$

Proof of the step function extension lemma. Suppose that that $Y \in \mathrm{RV}(\mathbb{R}_+)$ is rational. Let:

- (Ω, f) be a symmetric representation of Y with $|\Omega| \ge 2$,
- $\mathfrak{w} = \{w^{(\omega)} : s \in \Omega\} \subset \mathbb{R}_+^h$ be a Δ-normalized block array, where $\Delta > 0$ and $h \in \mathbb{N}$ so that

$$E\big(w^{(\omega)}\big) = c \cdot f(\omega) \quad (\omega \in \Omega),$$

where $c = c(\mathfrak{w}) > 0$.

Fix $0 < \mathcal{E} < \Delta$. We'll construct for any $Q \in \mathbb{N}$ large enough, a Qh-block array $\mathfrak{w}' = \{w'^{(s)} : s \in \Omega\} \subset \mathbb{R}_+^{Qh}$ so that

$$E\big(w'^{(s)}\big) = c' \cdot f(s) \quad (\omega \in \Omega),$$

where $c' = c(\mathfrak{w}') > c(\mathfrak{w})$; $\mathfrak{w}' \succ \mathfrak{w}$ is a transitive, homogeneous extension and $(\mathfrak{w}, \mathfrak{w}')$ is relatively, $Y - (\Delta, \mathcal{E})$-distributed.

The construction is via auxiliary, intermediary block arrays $\mathfrak{w}^{(1)}, \mathfrak{w}^{(2)}, \ldots, \mathfrak{w}^{(N)}$ where $N > \frac{1}{\mathcal{E}}$ is arbitrary and fixed.

Let $V \subset \mathbb{R}_+$ be the value set of Y and let

$$K > \frac{2 \max V}{\min V} \quad \text{and} \quad N' := 2\big(|\Omega| - 1\big)N.$$

We have that $\min_{s,t} \frac{Kf(t)}{f(s)} > 1$ and so, using the compound lemma, we can find $J_1 > 1$ and for each $s, t \in \Omega$ find

$\mathcal{E}$-normalized $w^{(s,t)}(1) \in \mathbb{R}_+^{J_1 h}$ so that

$$E\big(w^{(s,t)}(1)\big) = Kcf(t) = \frac{Kf(t)}{f(s)} E\big(w^{(s)}\big); \tag{o}$$

$$P\big(S_k\big(w^{(s,t)}(1)\big) = S_k\big(w^{(s)\odot J_1}\big)\ \forall 1 \le k \le \Delta J_1 h\big) > 1 - \Delta; \tag{i}$$

$$c = \gamma(k_0) \le \gamma(k_0+1) \le \cdots \le \gamma(qh) = Kc; \tag{ii}$$

$$P\big(\big[S_k\big(w^{(s,s)}(1)\big) = k\gamma(k) f(s)(1 \pm \Delta)\big]\big) \ge 1 - \Delta \quad \forall k > k_0. \tag{iii}$$

Here $\gamma(k) = E(w^{(s)})((1-p_k) + p_k K)$ is as in the compound lemma with $\mathfrak{t} \equiv K$.

The first intermediary block array is

$$\mathfrak{w}^{(1)} = \big\{w^{(s,s)}(1,k) : 1 \le k \le |\Omega|\big(N' - |\Omega| + 1\big), s \in \Omega\big\} \cup \big\{w^{(u,v)}(1) : u, v \in \Omega, u \ne v\big\},$$

where $w^{(s,s)}(1,k)$ $(1 \le k \le N-1)$ is a copy of $w^{(s,s)}(1)$.

Next, find $J_2 \ge 1$ and for each $s, t, u \in \Omega$, $s \ne t$ find $w^{(s,t,u)}(2) \in \mathbb{R}_+^{J_2 J_1 h}$ so that

$$E\big(w^{(s,t,u)}(\nu)\big) = cK^2 f(u) = \frac{Kf(u)}{f(t)} E\big(w^{(s,t)}(1)\big); \tag{iii$'$}$$

$$P\big(S_k\big(w^{(s,t,u)}(2)\big) = S_k\big(w^{(s,t)}(1)^{\odot J_2}\big)\ \forall 1 \le k \le \Delta J_2 J_1 h\big) > 1 - \Delta; \tag{iv}$$

$$Kc = \gamma(k_0) \le \gamma(k_0+1) \le \cdots \le \gamma(qh) = K^2 c; \tag{v}$$

$$P\big(\big[S_k\big(w^{(s,t,t)}(\nu)\big) = k\gamma(k) f(t)(1 \pm \Delta)\big]\big) \ge 1 - \Delta \quad \forall k > k_0. \tag{vi}$$

The second intermediary block array is

$$\mathfrak{w}^{(2)} = \big\{w^{(s,s,s)}(2,k) : 1 \le k \le |\Omega|\big(N' - 2\big(|\Omega| - 1\big)\big), s \in \Omega\big\} \cup \big\{w^{(s,t,t)}(2), w^{(s,s,t)}(2) : s, t \in \Omega, s \ne t\big\},$$

where $w^{(s,s,s)}(2,k)$ $(1 \le k \le N-2)$ is a copy of $w^{(s,s,s)}(2)$.

Recurse this, to find $J_2, J_3, \ldots, J_N$ and for each $2 \le \nu \le N, s_1, s_2, \ldots, s_\nu \in \Omega$, $w^{(s_1,s_2,\ldots,s_\nu)}(\nu) \in \mathbb{R}_+^{h^{(\nu-1)}}$ where $h^{(\nu)} := hJ_1 J_2 \cdots J_\nu$; so that

$$E\big(w^{(s_1,s_2,\ldots,s_\nu)}(\nu)\big) = cK^\nu f(s_\nu) = \frac{Kf(s_\nu)}{f(s_{\nu-1})} E\big(w^{(s_1,s_2,\ldots,s_{\nu-1})}(\nu-1)\big), \tag{iii$'$}$$

$$P\big(S_k\big(w^{(s_1,s_2,\ldots,s_\nu)}(\nu)\big) = S_k\big(w^{(s_1,s_2,\ldots,s_{\nu-1})}(\nu-1)\big)^{\odot J_\nu}\ \forall 1 \le k \le \Delta h^{(\nu)}\big) > 1 - \Delta; \tag{iv}$$

$$K^{\nu-1} c = \gamma(k_0) \le \gamma(k_0+1) \le \cdots \le \gamma(qh) = K^\nu c; \tag{v}$$

$$P\big(\big[S_k\big(w^{(s_1,s_2,\ldots,s_{\nu-2},t,t)}(\nu)\big) = f(t) k\gamma(k)(1 \pm \Delta)\big]\big) \ge 1 - \Delta \quad \forall k > k_0. \tag{vi}$$

The νth intermediary block array is

$$\mathfrak{w}^{(\nu)} = \big\{w^{(s^\nu)}(\nu,k) : 1 \le k \le |\Omega|\big(N' - \nu\big(|\Omega| - 1\big)\big), s \in \Omega\big\} \cup \bigcup_{j=1}^{\nu-1} \big\{w^{(s^j, t^{\nu-j})}(\nu) : s, t \in \Omega, s \ne t\big\},$$

where $w^{(s^\nu)}(\nu,k)$ $(1 \le k \le N-\nu)$ is a copy of $w^{(s^\nu)}(\nu)$. In particular,

$$\mathfrak{w}^{(N)} = \big\{w^{(s^N)}(N,k) : 1 \le k \le |\Omega|\big(N' - N\big(|\Omega| - 1\big)\big), s \in \Omega\big\} \cup \bigcup_{j=1}^{N-1} \big\{w^{(s^j, t^{N-j})}(N) : s, t \in \Omega, s \ne t\big\},$$

where $w^{(s^N)}(N,k)$ $(1 \le k \le N-N)$ is a copy of $w^{(s^N)}(N)$.

Now set $\mathfrak{w}' = \{w'^{(s)} : s \in \Omega\}$ where

$$w'^{(s)} := \left(\bigodot_{k=1}^{N(|\Omega|-1)} w^{(s^N)}(N,k) \odot \bigodot_{t\in\Omega\setminus\{s\}} \bigodot_{j=1}^{N} w^{(t^{N-j},s^j)}(N) \right)^{\odot T},$$

where T is chosen large enough to ensure $\mathcal{E}$-normalization.

This is as advertised. □

5. General case of Theorem 1 and Theorem 2

We now complete the proof of Theorem 1 by constructing an ESP with an arbitrary $Y \in \mathrm{RV}(\mathbb{R}_+)$ as distributional limit.

For this, we need to approximate an arbitrary $Y \in \mathrm{RV}(\mathbb{R}_+)$ with rational random variables in a controlled manner.

Splittings

A *splitting* of the finite set Ω is a surjection $\pi : \Xi \to \Omega$ defined on another finite set Ξ so that $P_\Omega = P_\Xi \circ \pi^{-1}$.

Equivalently, $\#\pi^{-1}\{x\} = \frac{\#\Xi}{\#\Omega}\ \forall x \in \Omega$.

Let the compact metric space $([0,\infty], \rho)$ be as before, let $\pi : \Xi \to \Omega$ be a splitting and let (Ω, f), (Ξ, g) be symmetric representations.

We'll say, for $\varepsilon > 0$, that (Ξ, g) ε*-splits* (Ω, f) via $\pi : \Xi \to \Omega$ if

$$E_\Xi\big(\rho(g, f\circ\pi)\big) := \frac{1}{\#\Xi}\sum_{u\in\Xi} \rho\big(g(u), f\big(\pi(u)\big)\big) < \varepsilon$$

and we'll call $\pi : \Xi \to \Omega$ the (associated) ε*-splitting*.

Note that if Z has a symmetric representation which ε-splits some symmetric representation of Y, then $\mathfrak{v}(Y, Z) < \varepsilon$.

Splitting approximation lemma. *Let $Y \in \mathrm{RV}(\mathbb{R}_+)$, then $\forall \varepsilon_k \downarrow 0$ there is a sequence $(Y_1, Y_2, \dots)$ of rational random variables on $\mathbb{R}_+$ with a nested sequence of symmetric representations (Ω_k, f_k) so that*

(o) $\mathfrak{v}(Y_k, Y) < \varepsilon_k\ \forall k \geq 1$;
(i) (Ω_{k+1}, f_{k+1}) ε_k*-splits* (Ω_k, f_k) $\forall k \geq 1$.
(ii) $\exists R > 0$ *so that* $P_{\Omega_k}(Y_k < t) \leq \mathrm{Prob}(Y < t)\ \forall t \in (0, R), k \geq 1$.

Proof. Considering Y as a random variable on the compact metric space $([0,\infty], \rho)$, we let $\mu := \mathrm{dist}(Y) \in \mathcal{P}([0,\infty])$. There is a non-decreasing map $\Phi : [0,1] \to [0,\infty]$ so that $\mu = \lambda \circ \Phi^{-1}$ where λ is Lebesgue measure on $[0,1]$. Let $\Gamma \subset [0,1]$ be the collection of discontinuity points of Φ. By monotonicity, this set is at most countable.

Let $Z := \{0,1\}^{\mathbb{N}}$ equipped with the product, discrete topology, and let $B : Z \to [0,1]$ be the "binary expansion map"

$$B\big((x_1, x_2, \dots)\big) := \sum_{k=1}^{\infty} \frac{x_k}{2^k}.$$

It follows that the collection of discontinuity points of $\Psi := \Phi \circ B : Z \to [0,\infty]$ is $\widetilde{\Gamma} = B^{-1}\Gamma$. This set is also at most countable.

We have

$$\mu = \nu \circ \Psi^{-1},$$

where $\nu = \prod(\frac{1}{2}, \frac{1}{2}) \in \mathcal{P}(Z)$.

By the above,

$$\Phi\left(\sum_{k=1}^{n-1}\frac{x_k}{2^k}+\frac{1}{2^n}\right)\xrightarrow[n\to\infty]{}\Psi(x_1,x_2,\dots)\quad\text{for }\nu\text{-a.e. }(x_1,x_2,\dots)\in Z$$

(indeed $\forall(x_1,x_2,\dots)\notin\widetilde{\Gamma}$).

Now, for $n\ge 1$, let $Z_n:=\{0,1\}^n$, define $\psi_n:Z_n\to[0,1]$ by

$$\psi_n(x_1,x_2,\dots,x_n):=\Phi\left(\sum_{k=1}^{n-1}\frac{x_k}{2^k}+\frac{1}{2^n}\right).$$

We have that for ν-a.e. $(x_1,x_2,\dots)\in Z$,

$$\psi_n(x_1,x_2,\dots,x_n)\xrightarrow[n\to\infty]{}\Psi(x_1,x_2,\dots).$$

Define the restriction maps $\pi_n:Z\to Z_n$ and $\pi_n^{n+m}:Z_{n+m}\to Z_n$ by

$$\pi_n(x_1,x_2,\dots)=(x_1,x_2,\dots,x_n)\quad\text{and}\quad\pi_n^{n+m}(x_1,x_2,\dots,x_{n+m})=(x_1,x_2,\dots,x_n),$$

then $\pi_n^{n+m}:Z_{n+m}\to Z_n$ is a splitting and, along a sufficiently sparse subsequence $n_k\uparrow\infty$, we have

$$\int_Z\rho(\psi_{n_k}\circ\pi_{n_k},\Psi)\,d\nu<\frac{\varepsilon_k}{2}$$

whence

$$E_{Z_{n_{k+1}}}\big(\rho\big(\psi_{n_k}\circ\pi_{n_k}^{n_{k+1}},\psi_{n_{k+1}}\big)\big)<\varepsilon_k.$$

Thus

$$\Omega_k:=Z_{n_k},\qquad f_k:=\psi_{n_k}\quad\text{and}\quad\texttt{dist}(Y_k):=P_{\Omega_k}\circ f_k^{-1}\in\mathcal{P}(\mathbb{R}_+)$$

are as required for (i), which entails (o).

To see (ii) we note that

$$\psi_n(x_1,x_2,\dots,x_n)\ge\Psi(x_1,x_2,\dots)$$

whenever $(x_1,x_2,\dots,x_n)\ne\mathbb{1}$. Let

$$R:=\Phi\left(\sum_{j=1}^{n_1-1}\frac{1}{2^j}\right)=\Phi\left(1-\frac{1}{2^{n_1}}\right)\le\Phi\left(\sum_{j=1}^{n_k-1}\frac{1}{2^j}\right)\quad\forall k\ge 1.$$

If $k\ge 1$ and $\psi_{n_k}(x_1,x_2,\dots,x_{n_k})<R$ then $(x_1,x_2,\dots,x_{n_k})\ne\mathbb{1}$ and $\psi_{n_k}(x_1,x_2,\dots,x_{n_k})\ge\Psi(x_1,x_2,\dots)$.

Since $f_k=\psi_{n_k}$, for $t\in(0,R)$

$$P_{\Omega_k}\big([f_k\le t]\big)\le\nu\big([\Psi\le t]\big)=P(Y\le t).\qquad\square$$

Step function straightening lemma. *Let $Y,Z\in\mathrm{RV}(\mathbb{R}_+)$ be rational with symmetric representations (Ω,f) and (Ξ,g) respectively.*

Suppose that $\mathcal{E},\Delta>0$ and that (Ξ,g) $\mathcal{E}$-splits (Ω,f) with $\mathcal{E}$-splitting $\Phi:\Xi\to\Omega$.

Let $\mathfrak{w}=\{w(\omega):\omega\in\Omega\}\subset\mathbb{R}_+^h$ be a Δ-normalized, Y-distributed, h-block array with $E(w(\omega))=c(\mathfrak{w})f(\omega)$ $\forall\omega\in\Omega$.

Then for each $Q \in \mathbb{N}$ large enough and $\eta > 0$, $\exists$ a $\mathcal{E}$-normalized, (Ξ, g)-distributed, Qh-block array

$$\mathfrak{b} = \{b(\xi) : \xi \in \Xi\} \subset \mathbb{R}_+^{Qh},$$

so that

$$F_{\mathfrak{b}} \geq F_{\mathfrak{w}} \quad \text{and} \quad m([F_{\mathfrak{b}} \neq F_{\mathfrak{w}}]) < \mathcal{E},$$

and

$$\beta(h) \leq \beta(h+1) \leq \cdots \leq \beta(Qh), \qquad \beta(k+1) - \beta(k) \leq \eta,$$

$$0 = q_h < q_{h+1} < \cdots < q_{Qh} = 1, \qquad \delta_h \geq \delta_{k+1} \geq \cdots \geq \delta_{Qh}, \qquad \delta_{Qh} < \mathcal{E}$$

so that for $h \leq k \leq Qh$,

$$S_k(b(\xi)) \geq k\beta(k)\big((1-q_k) f(\Phi(\xi)) + q_k g(\xi)\big)(1-\delta_k),$$
$$P\big(\big[S_k(b(\xi)) = k\beta(k)\big((1-q_k) f(\Phi(\xi)) + q_k g(\xi)\big)(1 \pm \delta_k)\big]\big) \geq 1 - \delta_k,$$
$$\mathfrak{v}\left(\frac{S_k(\mathfrak{b})}{k\beta(k)}, Z\right) < \mathcal{E} + \Delta.$$

Proof. Let $\Phi : \Xi \to \Omega$ be so that

$$P_\Xi \circ \Phi^{-1} = P_\Omega \quad \text{and} \quad E_\Xi(\rho(f \circ \Phi, g)) < \mathcal{E}.$$

For $\xi \in \Xi$, let $v(\xi) := w(\Phi(\xi)) \in \mathfrak{w}$ and consider the block array

$$\widetilde{\mathfrak{w}} := \{v(\xi) : \xi \in \Xi\}.$$

Note that $E(v(\xi)) = cf(\Phi(\xi))$. In order to use the compound lemma, define $\mathfrak{t} : \Xi \to (1, \infty)$ by

$$\mathfrak{t}(\xi) := \frac{Kg(\xi)}{f(\Phi(\xi))} \quad \text{where } K > \max_{\xi \in \Xi} \frac{f(\Phi(\xi))}{g(\xi)}$$

so that $\mathfrak{t} > 1$.

By the compound lemma for $Q \geq 1$ large enough, there is an $\mathcal{E}$-normalized, Qh-block array

$$\mathfrak{b} = \{b(\xi) : \xi \in \Xi\} \subset \mathbb{R}_+^{Qh},$$

numbers

$$\delta_k \geq \delta_{k+1}, \qquad \delta_{Qh} < \mathcal{E} \quad \text{and} \quad 0 = p_h < p_{h+1} < \cdots < p_{Qh} = 1, \qquad p_{k+1} - p_k < \eta$$

so that for each $\xi \in \Xi$,

$$E(b(\xi)) = \mathfrak{t}(\xi) E(v(\xi)) = c(\mathfrak{w}) f(\Phi(\xi));$$
$$P\big(S_k(b(\xi)) = S_k(v(\xi)^{\odot Q})\ \forall 1 \leq k \leq \Delta h\big) > 1 - 2\Delta$$

and $\forall k > \Delta h$,

$$P\big(\big[S_k(b(\xi)) = kE(b(\xi))\big((1-p_k) + p_k \mathfrak{t}(\xi)\big)(1 \pm \delta_k)\big]\big) \geq 1 - \delta_k.$$

Next, for $\xi \in \Xi$,

$$E(b(\xi))\big((1-p_k) + p_k \mathfrak{t}(\xi)\big) = c(\mathfrak{w})((1-p_k) f(\Phi(\xi)) + K p_k g(\xi)).$$

Let

$$\beta(k) := c(\mathfrak{W})\big(p_k + (1-p_k)K\big),$$
$$q_k := \frac{Kp_k}{p_k + (1-p_k)K},$$

then

$$0 = q_h < q_{h+1} < \cdots < q_{Qh} = 1$$

and

$$E\big(b(\xi)\big)\big((1-p_k) + p_k\mathfrak{t}(\xi)\big) = \beta(k)\big((1-q_k)f\big(\Phi(\xi)\big) + q_k g(\xi)\big).$$

Thus, with probability $\geq 1 - \delta_k$,

$$\rho\bigg(\frac{S_k(b(\xi))}{k\gamma(k)}, (1-q_k)f\big(\Phi(\xi)\big) + q_k g(\xi)\bigg) < \delta_k$$

and

$$E_\Xi\bigg(\rho\bigg(\frac{S_k(b(\xi))}{k\gamma(k)}, g(\xi)\bigg)\bigg) \leq 2\delta_k + E_\Xi\big(\rho(f\circ\Phi, g)\big)$$
$$\leq \delta_k + \mathcal{E}.$$

The inequality $F_{\mathfrak{b}} \geq F_{\mathfrak{w}}$ follows from monotonicity. □

It is not hard to see that the refinement $\mathfrak{w} \prec \mathfrak{b}$ above has the property that if $\mathfrak{b} \prec \mathfrak{w}'$ is a uniform refinement, then so is $\mathfrak{w} \prec \mathfrak{w}'$.

Proof of Theorem 1. Fix $\varepsilon_n \downarrow 0$, $\sum_{n=1}^\infty \varepsilon_n < \infty$ and use the splitting approximation lemma to obtain a sequence $(Y_1, Y_2, \dots)$ of rational random variables on $\mathbb{R}_+$ with a nested sequence of symmetric representations (Ω_k, f_k) so that

(o) $\mathfrak{v}(Y_k, Y) < \varepsilon_k\ \forall k \geq 1$;
(i) (Ω_{k+1}, f_{k+1}) ε_k-splits (Ω_k, f_k) $\forall k \geq 1$.
(ii) $\exists R > 0$ so that $P_{\Omega_k}(Y_k < t) \leq \mathtt{Prob}(Y < t)\ \forall t \in (0, R), k \geq 1$.

Using the step function extension- and straightening lemmas (respectively), we next, construct sequences $(\mathfrak{v}_n)_n$ and $(\mathfrak{e}_n)_n$ of Y_n-distributed h_n- and k_n-block arrays (respectively) so that

$$\mathfrak{v}_n \prec \mathfrak{w}_n \prec \mathfrak{v}_{n+1} \quad \text{and} \quad F_{\mathfrak{v}_n} \leq F_{\mathfrak{w}_n} \leq F_{\mathfrak{v}_{n+1}}$$

with each $\mathfrak{w}_n \prec \mathfrak{w}_{n+1}$ a uniform refinement;

and a slowly varying sequence $(\gamma(k))_k$, $\gamma(k+1) - \gamma(k) \to 0$ so that with $b(k) := k\gamma(k)$, for some $r > 0$

(iii) $m([F_{\mathfrak{v}_n} \neq F_{\mathfrak{w}_n}]) < \varepsilon_n$ and $m([F_{\mathfrak{w}_n} \neq F_{\mathfrak{v}_{n+1}}]) < \varepsilon_{n+1}$;
(iv) $\frac{S_k(\mathfrak{w}_n)(\xi)}{b(k)} \geq r f_n(\xi)\ \forall h_n < k \leq h_{n+1}$ where $\mathfrak{w}_n = \{w(\xi) : \xi \in \Omega_n\}$,
(v) $\mathfrak{v}(\frac{S_k(\mathfrak{w}_n)}{b(k)}, g) < \varepsilon_n\ \forall h_n < k \leq h_{n+1}$.

Let

$$(X, \mathcal{B}, m, T) := \varprojlim_{n\to\infty} \mathfrak{W}_n \quad \text{and} \quad f := \lim_{n\to\infty} F_{\mathfrak{W}_n, \mathfrak{w}_n},$$

then $(X, \mathcal{B}, m, T, f)$ is an $\mathtt{ESP}$ over an odometer with distributional limit Y.

Moreover, if $h_n < k \le h_{n+1}$, and $t \in (0, R)$ then $S_k(f) \ge S_k(F_{\mathfrak{w}_n})$ whence

$$\big[S_k(f) \le tb(k)\big] \subset \big[S_k(F_{\mathfrak{w}_n}) \le tb(k)\big]$$

whence by (iv),

$$\begin{aligned} P\big(\big[S_k(f) \le tb(k)\big]\big) &\le P\left(\left[\frac{S_k(\mathfrak{w}_n)(\xi)}{b(k)} \le t\right]\right) \\ &\le P\left(Y_n \le \frac{t}{r}\right) \\ &\le P\left(Y \le \frac{t}{r}\right). \end{aligned}$$

□

Proof of Theorem 2. We use the odometer construction of Theorem 1 to prove Theorem 2.

Let $Y \in \mathtt{RV}(\mathbb{R}_+)$ and let $(\Omega, \mathcal{F}, P, \tau)$ be an EPPT. We must exhibit a measurable function $\phi : \Omega \to \mathbb{R}_+$ so that the ESP $(\Omega, \mathcal{F}, P, \tau, \phi)$ has distributional limit Y.

Now fix as above, an odometer $(X, \mathcal{B}, m, T)$ with $f : X \to \mathbb{R}_+$ measurable so that $(X, \mathcal{B}, m, T, f)$ satisfies (🚜) in Theorem 1 (on page 880) with distributional limit Y and 1-regularly varying normalizing constants $b(n)_{n\ge1}$.

By the odometer factor proposition, there is a set $\Omega_0 \in \mathcal{F}$, $P(\Omega_0) > 0$ so that the induced EPPT $(\Omega_0, \mathcal{F} \cap \Omega_0, P_{\Omega_0}, \tau_{\Omega_0})$ has $(X, \mathcal{B}, m, T)$ as a factor.

Let $\pi : (\Omega_0, \mathcal{F} \cap \Omega_0, P_{\Omega_0}, \tau_{\Omega_0}) \to (X, \mathcal{B}, m, T)$ be the factor map and define $\phi : \Omega \to \mathbb{R}$ by

$$\phi = f \circ \pi \quad \text{on } \Omega_0 \quad \text{and} \quad \phi \equiv 0 \quad \text{off } \Omega_0.$$

We have that

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} \phi \circ \tau_{\Omega_0}^k \xrightarrow[n\to\infty]{P_{\Omega_0}-\mathfrak{d}} Y.$$

Now let $\kappa : \Omega_0 \to \mathbb{N}$ be the first return time of τ to Ω_0 and let $\kappa_n := \sum_{j=0}^{n-1} \kappa \circ \tau_{\Omega_0}^j$ (the nth return time of τ to Ω_0), then on Ω_0,

$$\sum_{k=0}^{n-1} \phi \circ \tau_{\Omega_0}^k \equiv \sum_{j=0}^{\kappa_n - 1} \phi \circ \tau^j.$$

By Birkhoff's theorem, $\kappa_n \sim \frac{n}{P(\Omega_0)}$ a.s. on Ω_0 and so by monotonicity and 1-regular variation of $b : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} \phi \circ \tau^k \xrightarrow[n\to\infty]{P_{\Omega_0}-\mathfrak{d}} P(\Omega_0)Y$$

whence by Eagleson's theorem,

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} \phi \circ \tau^k \xrightarrow[n\to\infty]{\mathfrak{d}} P(\Omega_0)Y.$$

□

6. New examples in infinite ergodic theory

We begin by reviewing:

Kakutani skyscrapers and inversion

As in [20], the *skyscraper* over the $\mathbb{N}$-valued SP $(\Omega, \mathcal{F}, P, S, f)$ is the MPT $(X, \mathcal{B}, m, T)$ defined by

$$X = \{(x,n) : x \in \Omega, 1 \le n \le f(x)\},$$
$$\mathcal{B} = \sigma\{A \times \{n\} : n \in \mathbb{N}, A \in \mathcal{F} \cap [f \ge n]\}, \qquad m(A \times \{n\}) = P(A),$$

and

$$T(x,n) = \begin{cases} (Sx, f) & \text{if } n = f(x), \\ (x, n+1) & \text{if } 1 \le n \le f(x) - 1. \end{cases}$$

The skyscraper MPT is always conservative as $\bigcup_{n\ge 1} T^{-n}\Omega \times \{1\} = X$ and its ergodicity is equivalent to that of $(\Omega, \mathcal{F}, P, S)$. Any invertible CEMPT $(X, \mathcal{B}, m, T)$ is isomorphic to the skyscraper over a *first return time* SP $(\Omega, \mathcal{B} \cap \Omega, m_\Omega, T_\Omega, \varphi_\Omega)$ where $\varphi_\Omega(x) := \min\{n \ge 1 : T^n x \in \Omega\}$ is the *first return time* which is finite for a.e. $x \in \Omega$ by conservativity, $T_\Omega(x) := T^{\varphi_\Omega(x)}$ is the *induced transformation* on Ω which is a PPT.

Let $(X, \mathcal{B}, m, T)$ be an invertible CEMPT let $\Omega \in \mathcal{B}, m(\Omega) = 1$ and consider the `return time stochastic process` on Ω:

$$(\Omega, \mathcal{B} \cap \Omega, m_\Omega, T_\Omega, \varphi_\Omega) \quad \text{where } \varphi_\Omega(x) := \min\{n \ge 1 : T^n x \in \Omega\}.$$

Distributional limits with regularly varying normalizing constants are transferred between the return time SP and the Kakutani skyscraper by means of the following

Inversion proposition ([3]). *Let $a(n)$ be γ-regularly varying with $\gamma \in (0,1]$ and fix $\Omega \in \mathcal{F}$, then for Y a rv on* $(0,\infty)$:

$$\frac{1}{a(n)} S_n(1_\Omega) \xrightarrow{\mathfrak{d}} Y m(\Omega) \quad \iff \quad \frac{\varphi_n}{a^{-1}(n)} \xrightarrow{\mathfrak{d}} \left(\frac{1}{m(\Omega) Y}\right)^{\frac{1}{\gamma}},$$

where $\varphi_n = \sum_{k=0}^{n-1} \varphi_\Omega \circ T_\Omega^k$.

Proof of Theorem 3. Fix $Y \in \mathrm{RV}(\mathbb{R}_+)$, let $(\Omega, \mathcal{F}, P, S, f)$ be a $\mathbb{N}$-valued ESP and let $b(n)$ be 1-regularly varying so that

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{\mathfrak{d}} \frac{1}{Y},$$
$$P\left(\left[\sum_{k=0}^{n-1} f \circ T^k < x b(n)\right]\right) \le P\left(\frac{1}{Y} \le t\right) \quad \forall t > 0 \text{ small} \quad \text{and} \quad n \ge 1 \text{ large}.$$

These exist by Theorem 1. Now let $(X, \mathcal{B}, m, T)$ be the Kakutani skyscraper over $(\Omega, \mathcal{F}, P, S, f)$. By inversion,

$$\frac{S_n^{(T)}}{b^{-1}(n)} \xrightarrow[n\to\infty]{\mathfrak{d}} Y \quad \text{and} \tag{⚽}$$
$$m_\Omega([S_n^{(T)}(1_\Omega) > x b^{-1}(n)]) \le P(Y \ge x) \quad \forall y > 1, n \ge 1 \text{ large}. \tag{✗}$$

□

Rational ergodicity properties

Now let $\alpha > 0$ and let $K \subset \mathbb{N}$ be a subsequence.

We'll say that the CEMPT $(X, \mathcal{B}, m, T)$ is *α-rationally ergodic along K* if for some $\Omega \in \mathcal{B}, 0 < m(\Omega) < \infty$, we have

$$\int_A \left(\frac{S_n(1_B)}{a(n)}\right)^\alpha dm \xrightarrow[n\to\infty, n\in K]{} m(A)m(B)^\alpha \quad \forall A, B \in \mathcal{B}(\Omega), \tag{α-RE$_K$}$$

where $a(n) = a_{\alpha,\Omega}(n) := \frac{1}{m(\Omega)^{1+\frac{1}{\alpha}}}(\int_\Omega S_n(1_\Omega)^\alpha\, dm)^{\frac{1}{\alpha}}$.

We'll say that $(X, \mathcal{B}, m, T)$ is *α-rationally ergodic* if it is α-rationally ergodic along $\mathbb{N}$ and *subsequence α-rationally ergodic* if it is α-rationally ergodic along some $K \subset \mathbb{N}$.

Properties like this have been considered in [8] and [23].

Standard techniques show that $\Omega \in \mathcal{B}, 0 < m(\Omega) < \infty$ satisfies (α-RE$_K$) iff

$$\left\{\left(\frac{S_n(1_\Omega)}{a_{\alpha,\Omega}(n)}\right)^\alpha : n \in K\right\}$$

is uniformly integrable on Ω, and, if nonempty, the collection

$$R_{\alpha,K}(T) := \left\{\Omega \in \mathcal{B} : 0 < m(B) < \infty \text{ satisfying } (\alpha\text{-RE}_K)\right\}$$

is a dense T-invariant hereditary ring.

Moreover $a_{\alpha,\Omega}(n) \sim a_{\alpha,\Omega'}(n)$ along K whenever $\Omega, \Omega' \in R_{\alpha,K}(T)$.

We'll call the CEMPT $(X, \mathcal{B}, m, T)$ *∞-rationally ergodic along K* if for some $\Omega \in \mathcal{B}, 0 < m(\Omega) < \infty$, we have

$$\sup_{n\in K}\left\|\frac{S_n(1_\Omega)}{a_{1,\Omega}(n)}\right\|_{L^\infty(\Omega)} < \infty. \tag{BRE$_K$}$$

Analogously to as above, if nonempty, the collection

$$R_{\infty,K}(T) := \left\{\Omega \in \mathcal{B} : 0 < m(B) < \infty \text{ satisfying } (\text{BRE}_K)\right\}$$

is a dense T-invariant hereditary ring. It is contained in $R_{\alpha,K}(T)\ \forall \alpha > 0$.

The condition ∞-rational ergodicity along $\mathbb{N}$ is aka bounded rational ergodicity. For more information and examples, see [2].

α-return sequence

We define the *α-return sequence* of an α-rationally ergodic CEMPT $(X, \mathcal{B}, m, T)$ as the growth rate

$$a_{n,\alpha}(T) \sim a_{\alpha,\Omega}(n), \quad \Omega \in R_\alpha(T).$$

It is also possible to define "subsequence α-return sequence" for a subsequence α-rationally ergodic CEMPT.

Note that

- 1-rational ergodicity is equivalent to weak rational ergodicity as in [1] with $R_1(T) = R(T)$ and $a_{n,1}(T) \sim a_n(T)$;
- 2-rational ergodicity implies rational ergodicity;
- for $0 < \alpha \le \infty$, α-rational ergodicity implies β-rational ergodicity for each $\beta \in (0, \alpha)$;
- pointwise dual ergodic transformations are α-rationally ergodic $\forall 0 < \alpha < \infty$ (this follows from the existence of moment sets).

Let $(X, \mathcal{B}, m, T)$ be distributionally stable with limit $Y \in \mathrm{RV}(\mathbb{R}_+)$.

For $\alpha \in \mathbb{R}_+$, set $\|Y\|_\alpha := E(Y^\alpha)^{\frac{1}{\alpha}} \le \infty$ and

$$\|Y\|_\infty := \sup\{t > 0 : P(Y > t) > 0\} = \lim_{\alpha\to\infty} \|Y\|_\alpha \le \infty.$$

- For $0 < \alpha \le \infty$, if T is α-rationally ergodic, then $\|Y\|_\alpha < \infty$ and if $\alpha \in \mathbb{R}_+$, then $a_{n,\alpha}(T) \sim \|Y\|_\alpha a_{n,Y}(T)$.
- If $\|Y\|_\alpha = \infty$, then T is not subsequence, α-rationally ergodic.

Example: *Distributional stability* $\not\Rightarrow$ *α-rational ergodicity*

Let $Y \in \mathrm{RV}(\mathbb{R}_+)$ be so that $E(Y^\alpha) = \infty\ \forall \alpha > 0$. By Theorem 3, there is a distributionally stable CEMPT $(X, \mathcal{B}, m, T)$ with ergodic limit Y with $a_{n,Y}(T)$ 1-regularly varying. By the above $\forall \alpha > 0, T$ is not subsequence, α-rationally ergodic.

For a given CEMPT $(X, \mathcal{B}, m, T)$, we consider the collection

$$I(T) := \{\alpha > 0 : T \text{ is } \alpha\text{-rationally ergodic}\}.$$

It follows from the above that $I(T)$ must be an interval, either empty, or $\mathbb{R}$, or of form $(0, a)$ or $(0, a]$ for some $a \in (0, \infty]$.

We conclude this paper by showing that all possibilities occur.

Lemma. *Let $(X, \mathcal{B}, m, T)$ be distributionally stable with ergodic limit $Y \in \mathrm{RV}(\mathbb{R}_+)$ and $a_{n,Y}(T)$ 1-regularly varying. Suppose that $\Omega \in \mathcal{B}, m(\Omega) = 1$ satisfies (✈) as on page 884, then T is α-rationally ergodic iff $\|Y\|_\alpha < \infty$ and in this case, when $\alpha < \infty$, $a_{n,\alpha}(T) \sim E(Y^\alpha)\frac{1}{a} a_{n,Y}(T)$.*

Proof of $\|Y\|_\alpha < \infty \Longrightarrow \alpha$-RE. We only consider the case $0 < \alpha < \infty$. The case where $\alpha = \infty$ is easy. We claim first that

$$\left\{\Phi_n := \left(\frac{S_n(1_\Omega)}{a_{n,Y}(T)}\right)^\alpha : n \ge 1\right\}$$

is a uniformly integrable family in $L^1(\Omega)$.

Now, since $E(Y^\alpha) < \infty$, we have by monotone convergence and Fubini's theorem that

$$\rho(t) := \int_t^\infty P\big(Y^\alpha > s\big)\, ds = E\big(1_{[Y^\alpha > t]} Y^\alpha\big) \xrightarrow[t\to\infty]{} 0.$$

By (✈) (page 884),

$$\begin{aligned}\int_\Omega 1_{[\Phi_n > t]} \Phi_n\, dm &= \int_t^\infty m\big([\Phi_n > s]\big)\, ds \\ &\le 28 \int_t^\infty P\big(Y^\alpha > s\big)\, ds \\ &=: \rho(t)\end{aligned}$$

whence

$$\sup_{n\ge1} \int_\Omega 1_{[\Phi_n > t]} \Phi_n\, dm \le \rho(t) \xrightarrow[t\to\infty]{} 0$$

and the family is uniformly integrable.

Next by (⚽) as on page 884, for $A, B \in \mathcal{B}(\Omega)$ and $x > 0$,

$$\int_A \left(\frac{S_n(1_B)}{a_{n,Y}(T)}\right)^\alpha \wedge x\, dm \xrightarrow[n\to\infty]{} m(A) E\Big(\big(m(B)Y\big)^\alpha \wedge x\Big).$$

Moreover, $E((m(B)Y)^\alpha \wedge x) \xrightarrow[x\to\infty]{} m(B)^\alpha E(Y^\alpha)$. To estimate the error,

$$\begin{aligned}
0 &\le \int_A \left(\frac{S_n(1_B)}{a_{n,Y}(T)}\right)^\alpha dm - \int_A \left(\frac{S_n(1_B)}{a_{n,Y}(T)}\right)^\alpha \wedge x\, dm \\
&\le \int_A \left(\frac{S_n(1_B)}{a_{n,Y}(T)}\right)^\alpha 1_{[(\frac{S_n(1_B)}{a_{n,Y}(T)})^\alpha > x]}\, dm \\
&\le \int_\Omega 1_{[\Phi_n > x]} \Phi_n\, dm \\
&\le \rho(x) \xrightarrow[x\to\infty]{} 0.
\end{aligned}$$

Standard arguments now show that

$$\int_A \left(\frac{S_n(1_B)}{a_{n,Y}(T)}\right)^\alpha dm \xrightarrow[n\to\infty]{} m(A)m(B)^\alpha E\big(Y^\alpha\big). \qquad \square$$

Note that a boundedly rationally ergodic transformation T has $I(T) = (0, \infty]$ and a pointwise, dual ergodic transformation T with return sequence which is regularly varying with index $\gamma < 1$ has as ergodic limit a γ-Mittag–Leffler random variable (see [3]) which is unbounded but has moments of all orders, whence $I(T) = (0, \infty)$.

The following completes the picture (and is also a strengthening of [8]):

Proposition. *For each $a \in \mathbb{R}_+$ there are distributionally stable* MPT*s T_o and T_c with $I(T_o) = (0, a)$ or $I(T_c) = (0, a]$.*

Proof. To construct T_o with $I(T_o) = (0, \alpha)$ fix a $Y \in \mathrm{RV}(\mathbb{R}_+)$ so that $E(Y^t) < \infty$ $\forall t < \alpha$ but $E(Y^\alpha) = \infty$ and construct T as in Theorem 3.

To construct T_c with $I(T_c) = (0, \alpha]$ the same but using a $Z \in \mathrm{RV}(\mathbb{R}_+)$ so that $E(Z^\alpha) < \infty$ but $E(Z^t) = \infty$ $\forall t > \alpha$. $\square$

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