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# Sharpness of the phase transition and lower bounds for the critical intensity in continuum percolation on $\mathbb{R}^d$

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**Abstract.** We consider the Boolean model $Z$ on $\mathbb{R}^d$ with random compact grains of bounded diameter, i.e. $Z := \bigcup_{i\in\mathbb{N}}(Z_i + X_i)$ where $\{X_1, X_2, \dots\}$ is a Poisson point process of intensity $t$ and $(Z_1, Z_2, \dots)$ is an i.i.d. sequence of compact grains (not necessarily balls) with diameters a.s. bounded by some constant. We will show that exponential decay holds in the sub-critical regime, that means the volume and radius of the cluster of the typical grain in $Z$ have an exponential tail. To achieve this we adapt the arguments of (A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$ (2015) Preprint) and apply a new construction of the cluster of the typical grain together with arguments related to branching processes.

In the second part of the paper, we obtain new lower bounds for the Boolean model with deterministic grains. Some of these bounds are rigorous, while others are obtained via simulation. The simulated bounds come with confidence intervals and are much more precise than the rigorous ones. They improve known results (*J. Chem. Phys.* **137** (2012) 074106) in dimension six and above.

**Résumé.** Nous considérons le modèle Booléen $Z$ sur $\mathbb{R}^d$ avec des grains compacts aléatoires de diamètres bornés, c'est-à-dire $Z := \bigcup_{i\in\mathbb{N}}(Z_i + X_i)$ où $\{X_1, X_2, \dots\}$ est un processus de Poisson d'intensité $t$ et $(Z_1, Z_2, \dots)$ est une suite i.i.d. de grains compacts (non nécessairement des boules) de diamètres p.s. bornés par une constante. Nous montrons une décroissance exponentielle dans le régime sous-critique, ce qui veut dire que le volume et le rayon du cluster d'un grain typique dans $Z$ a une queue exponentielle. Pour cela, nous adaptons des résultats de (A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$ (2015) Preprint) et appliquons une nouvelle construction du cluster d'un grain typique avec des arguments issus des processus de branchement. Dans la seconde partie du papier, nous obtenons de nouvelles bornes inférieures pour le modèle booléen avec grains déterministes. Certaines des ces bornes sont rigoureuses, alors que d'autres sont obtenues par simulation. Les bornes obtenues par simulation sont fournies avec des intervalles de confiance et sont beaucoup plus précises que celles obtenues rigoureusement. Elles améliorent les résultats connus (*J. Chem. Phys.* **137** (2012) 074106) en dimension 6 et plus.

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## 1. Introduction

Percolation was introduced by Broadbent and Hammersley in the late fifties as a model on the $d$-dimensional lattice. During the eighties Menshikov [11] and Aizenman and Barsky [2] proved that the model exhibits a sharp phase transition in all dimensions $d \geq 2$. However, both proofs where a little involved and it was difficult to adapt them to other percolation models. Recently Duminil-Copin and Tassion [5] found an astonishingly simple and elegant proof of the sharp phase transition. We will transfer this proof to continuum percolation.

In the continuum percolation model introduced by Hall [6] we attach a random compact set (grain) to each point of a Poisson process on $\mathbb{R}^d$ and study the connectivity properties of the covered region. There are two books, by Penrose [13] and by Meester and Roy [10], that cover a lot of the theory, but deal mostly with ball shaped grains. In the setup

where all grains are unit balls, a discretization argument was applied in Penrose's book to obtain the sharp phase transition. The generalization to random grains of arbitrary shape is a novelty in this paper. We will now define the model rigorously.

Let $\mathcal{C}^d$ be the set of nonempty compact subsets of $\mathbb{R}^d$ that contain the origin equipped with the Hausdorff metric and $\mathbb{Q}$ a probability measure on $\mathcal{C}^d$. Let $\xi_t := \{(X_1, Z_1), (X_2, Z_2), \dots\}$ be a Poisson point process on the *particle space* $\Pi := \mathbb{R}^d \times \mathcal{C}^d$ with intensity measure $t\lambda^d \otimes \mathbb{Q}$, where $\lambda^d$ is the $d$-dimensional Lebesgue measure and $t \geq 0$. This corresponds to an independently marked Poisson process on $\mathbb{R}^d$ with intensity $t$ and mark distribution $\mathbb{Q}$. We will say that a particle $(x, K) \in \Pi$ has *location* $x$ and *grain* $K$. The well known properties of the Poisson process imply that we do not lose generality by considering only grains that contain the origin.

However, we will always assume that there is a radius $R \in \mathbb{R}$ such that $\mathbb{Q}$ is concentrated on grains contained in the ball $B_R$ of radius $R$ centered at the origin (in the 2-dimensional case, this assumption might be relaxed; see [1]). This is crucial for our proof to work.

We write $\mathbf{N}(\Pi)$ for the set of simple locally finite counting measures on $\Pi$. Each element $\eta = \sum_{i=1}^{\tau} \delta_{(x_i, K_i)} \in \mathbf{N}(\Pi)$, $\tau \in \mathbb{N} \cup \{\infty\}$ is identified with its support $\{(x_i, K_i) | i \in \mathbb{N}, i \leq \tau\}$. That means the notations $\eta \cap A$ and $\eta|_A$ are employed simultaneously ($\eta|_A$ shall denote the restriction of the measure to the set $A$).

The union of particles

$$\eta^{\cup} := \bigcup_{i \in \mathbb{N}} (K_i + x_i) \tag{1.1}$$

is closely related to the graph $G(\eta) := (V, E)$ with vertex set $V := \{x_i | i \in \mathbb{N}\}$ and edge set $E$ such that two distinct locations $x_i, x_j$ are adjacent iff $(K_i + x_i) \cap (K_j + x_j) \neq \varnothing$, i.e. iff the corresponding particles overlap. In this way the connected components of the *Boolean model* $\xi_t^{\cup}$ correspond to the connected components of $G(\xi_t)$ if $\mathbb{Q}$ is concentrated on the connected sets (this is true in most cases, but we will not need this assumption).

We want to study the connected component (cluster) of a typical particle of $\xi_t$ in $G(\xi_t)$ which is (due to the well known properties of the Palm distribution of Poisson processes, which can be found in [14]) the same as studying the cluster of the origin in $G(\xi_t + \delta_{(0, Z_0)})$ where $Z_0 \sim \mathbb{Q}$ is independent of $\xi_t$. Hence we define the *zero cluster* $C_0 := C_0(t)$ as the set of particles $(x, K) \in \xi_t$ for which $x$ is connected to 0 in $G(\xi_t + \delta_{(0, Z_0)})$. For $D_1, D_2 \subset \mathbb{R}^d$ and $\mu = \{(x_n, K_n) | n \in \mathbb{N}\} \in \mathbf{N}(\Pi)$, we denote by $\{D_1 \leftrightarrow D_2 \text{ in } \mu\}$ the event that there is a path $(x_{a_1}, \dots, x_{a_n})$ in $G(\mu)$ with $K_{a_1} + x_{a_1} \cap D_1 \neq \varnothing$ and $K_{a_n} + x_{a_n} \cap D_2 \neq \varnothing$. We write $\{D_1 \leftrightarrow \infty \text{ in } \xi_t\}$ for the event that $D_1$ is intersected by an infinite cluster of $G(\xi_t)$ and observe that this is a.s. equal to $\bigcap_{n \in \mathbb{N}} \{D_1 \leftrightarrow B_n^c \text{ in } \xi_t\}$.

For a finite set of particles $C \subset \Pi$ we call

$$\rho(C) := \max_{x \in C^{\cup}} \|x\|_2 \tag{1.2}$$

the *radius* of $C$. We denote by $B_r(x)$ the Euclidean ball centered in $x \in \mathbb{R}^d$ of radius $r$ and define $B_r := B_r(0)$.

We recall the definition of the critical intensity

$$t_c := \sup\{t \geq 0 | \mathbb{P}[|C_0(t)| = \infty] = 0\} \tag{1.3}$$

and point out that under our assumption $Z_0 \subset B_R$ a.s. it is easy to show that $t_c > 0$. This is due to a simple coupling of the model with the model where a.s. $Z_0 = B_R$. However, if the grains do not contain a small ball with positive probability, it is possible that $t_c = \infty$ (see [6] for a more elaborate version of these statements). But as there are also interesting grain distributions with lower dimensional grains (e.g. randomly rotated line segments in the plane), we do not want to exclude this case. Our following main result is unaffected by that.

**Theorem 1.1.** *For $t < t_c$ there are constants $c_1, c_2 > 0$ depending on $t$, such that*

$$\mathbb{P}[\rho(C_0(t)) \geq r] \leq e^{-c_1 r}, \quad r \geq 0,$$
$$\mathbb{P}[|C_0(t)| \geq r] \leq e^{-c_2 r}, \quad r \geq 0.$$

*Moreover,* $\mathbb{E}|C_0(t_c)| = \infty$ *and for* $t > t_c$

$$\mathbb{P}[B_R \leftrightarrow \infty \text{ in } \xi_t] \geq \frac{t - t_c}{t},$$

$$\mathbb{P}\big[\big|C_0(t)\big| = \infty\big] \geq \frac{t - t_c}{t^2 \lambda^d(B_{2R})}.$$

The structure of the paper is as follows. In Section 2 we recall the tools needed to work with Poisson processes and discuss a basic algorithm for the construction of $C_0$. In Section 3 we show the exponential decay of the tail of $\rho(C_0)$ and $|C_0|$ in the subcritical regime by constructing $C_0$ in a new way and using a comparison with a branching process. We will also show the mean-field lower bound $\mathbb{P}[|C_0(t)| = \infty] \geq \frac{t-t_c}{t^2\lambda^d(B_{2R})}$ for the percolation function in the supercritical regime. This is done by an adaptation of the arguments in [5]. In the fourth and fifth section we use these results to obtain new lower bounds for the critical intensity $t_c$ in the model where the grains are a.s. balls of radius one.

## 2. Preliminaries

To work with the Poisson process $\xi_t$ we need two well known tools. The first one is the Mecke-equation for Poisson processes which can be found in [8, Thm 4.1]. Let $f : \mathbf{N}(\Pi) \times \Pi \to [0, \infty)$ be measurable. Then

$$\mathbb{E} \sum_{(x,K)\in\xi_t} f(\xi_t, x, K) = t \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{E}\big[f(\xi_t + \delta_{(x,K)}, x, K)\big]\, dx \mathbb{Q}(dK). \tag{2.1}$$

Let $A \subset \mathbf{N}(\Pi)$ be measurable with respect to the usual $\sigma$-algebra (see [14, Chapter 3]). We say that the event $A$ is *determined by a set* $D \subset \Pi$ if for all $\eta_1, \eta_2 \in \mathbf{N}(\Pi)$ with $\eta_1|_D = \eta_2|_D$ we have

$$\eta_1 \in A \quad \Leftrightarrow \quad \eta_2 \in A. \tag{2.2}$$

The second important relation is a Margulis–Russo type formula for Poisson processes, see e.g. [7]. Let $D \subset \Pi$ be such that $(\lambda^d \otimes \mathbb{Q})(D) < \infty$ and let $A$ be an event that is determined by $D$. Then

$$\frac{\partial \mathbb{P}[\xi_t \in A]}{\partial t} = \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{E}\big[\mathbb{1}_A(\xi_t + \delta_{(x,K)}) - \mathbb{1}_A(\xi_t)\big]\, dx \mathbb{Q}(dK). \tag{2.3}$$

A thorough treatment of the Poisson process can also be found in [8].

In addition to these tools, we will heavily use the independence properties of the Poisson process which we will recall briefly. Let $D, D^{(1)}, D^{(2)} \subset \Pi$ such that $D^{(1)} \cup D^{(2)} = D$ and let $(\xi_t^{(n)})_{n\in\mathbb{N}_0}$ be an i.i.d. sequence of Poisson processes with the same distribution as, but independent of $\xi_t$. Then clearly $\xi_t^{(1)}|_{D^{(1)}} + \xi_t^{(2)}|_{D^{(2)}} \stackrel{d}{=} \xi_t|_D$. This remains true, even if $D^{(1)} \subset D$ is a random set independent of $(\xi_t^{(1)}, \xi_t^{(2)}, \xi_t)$ and $D^{(2)} := D \setminus D^{(1)}$.

Hence, given $D \subset \Pi$ and a measurable function $f$ mapping counting measures on $\Pi$ to measurable subsets of $\Pi$, we obtain that

$$\xi_t \stackrel{d}{=} \xi_t^{(1)}|_D + \xi_t^{(2)}|_{D^c} \tag{2.4}$$

$$\stackrel{d}{=} \xi_t^{(1)}|_D + \xi_t^{(2)}\big|_{f(\xi_t^{(1)}|_D)\setminus D} + \xi_t^{(3)}\big|_{(f(\xi_t^{(1)}|_D)\cup D)^c}. \tag{2.5}$$

This corresponds to an algorithmic construction of the Poisson process in the following way. First we generate all the particles in the region $D$ and afterwards we generate the particles in $D^c$. We might do the latter generation again in two steps by splitting $D^c$ into two parts. However, this splitting may depend on the particles already generated (in $D$). Clearly this procedure may also be iterated, i.e. $(f(\xi_t^{(1)}|_D) \cup D)^c$ in (2.5) maybe split again depending on the first two summands.

We will use this reasoning (including lots of iterations) to justify several ways of constructing $\xi_t$. All these ways are described algorithmically rather than by writing down $f$ explicitly for each iteration. The most basic and important way has already been used by Penrose [12] and we recall it to prepare the reader for the more refined ones that come up later.

Assume for the moment (and for simplicity) that all grains are a.s. equal to the ball $B_1$ and that $t < \lambda^d(B_2)^{-1}$. In this case it is enough to construct $\eta_t := \{X_i | i \in \mathbb{N}\} \subset \mathbb{R}^d$. We start by generating $\eta_t$ on $B_2(0)$. This set is chosen, since it contains the locations $x$ (and only those) for which $B_1(x)$ intersects the typical ball at the origin $B_1(0)$. Afterwards, we pick one of the generated particles, call it $(x_1, B_1)$ and generate $\eta_t$ on $B_2(x_1) \cap B_2(0)^c$ which is the part of $B_2(0)^c$ where balls would intersect $B_1(x_1)$. Then we pick again one of the in $B_2(0)$ or $B_2(x_1)$ generated particles $(x_2, B_1)$ with $x_2 \neq x_1$ and generate $\eta_t$ on $B_2(x_2) \cap B_2(x_1)^c \cap B_2(0)^c$. We go on with this until all generated particles have been picked once. This happens a.s. as the expected number of new particles in each iteration is dominated by a Poisson random variable with expected value $t\lambda^d(B_2) < 1$. Say, that we had $n$ iterations so far, then in the last step, we generate $\eta_t$ everywhere else, i.e. on the set $(\bigcup_{i=0}^n B_2(x_i))^c$ where $x_0 := 0$. It is clear, that by this algorithm $C_0 = \bigcup_{i=0}^n (x_i, B_1)$. Hence

$$\eta_t \stackrel{d}{=} \eta_t^{(0)}|_{B_2} + \sum_{i=1}^{n} \left(\eta_t^{(i)}|_{B_2(x_i)\setminus(\bigcup_{j=0}^{i-1} B_2(x_j))}\right) + \eta_t^{(n+1)}|_{(\bigcup_{j=0}^{n} B_2(x_j))^c} \tag{2.6}$$

and

$$\xi_t \cup (0, B_1) \stackrel{d}{=} C_0(t) \cup \left\{(x, B_1) | x \in \eta_t^{(n+1)}, B_1(x) \cap C_0(t)^{\cup} = \varnothing\right\}. \tag{2.7}$$

It is not hard to derive the exponential tails of $|C_0|$ and $\rho(C_0)$ for $t < \lambda^d(B_2)^{-1}$ with this algorithm. To this end, we observe the natural tree structure imposed on the locations $x_0, \ldots, x_n$. If a location $x_i$ is generated by the point process $\eta_t^{(j)}$, i.e. if $x_i \in \eta_t^{(j)}|_{B_2(x_j)\setminus(\bigcup_{k=0}^{j-1} B_2(x_k))}$, then we say $x_i$ is a child of $x_j$. This results in a tree $T$ with root $x_0$. The size of $T$ is equal to $|C_0|$ and the depth of $T$ times two is an upper bound for $\rho(C_0)$ as a parent and a child node of $T$ are at a distance of at most 2 due to the overlap of the corresponding balls.

To obtain a bound on the size and depth of $T$ we observe, that

$$\eta_t^{(i)}|_{B_2(x_i)\setminus(\bigcup_{j=0}^{i-1} B_2(x_j))} \subset \eta_t^{(i)}|_{B_2(x_i)}. \tag{2.8}$$

Hence there is a coupling between $T$ and a Galton–Watson tree $T_{\mathrm{GW}}$ where the number of offsprings is Poisson distributed with parameter $\mathbb{E}[\eta_t(B_2)] = t\lambda^d(B_2) < 1$ such that $T$ is a subtree of $T_{\mathrm{GW}}$. It is well known, that the depth and size of such a Galton–Watson tree have exponential tails (see [3] for a comprehensive treatment of branching processes).

## 3. A sharp phase transition

For $D \subset \mathbb{R}^d$ we write

$$[D] := \left\{(x, K) \in \Pi | (K + x) \cap D \neq \varnothing\right\} \tag{3.1}$$

for the set of particles that intersect $D$ (this is a subset of $\Pi$ not of the Poisson process). We also define the set

$$\partial D := [D] \cap \left[D^c\right] \tag{3.2}$$

that contains the particles that intersect $D$ as well as $D^c$ and the set

$$D^{\circ} := \Pi \setminus \left[D^c\right] \tag{3.3}$$

of particles that are contained in $D$.

The heart of the proof in [5] is the study of a functional $\varphi$ on subsets $S$ of $\mathbb{Z}^d$ containing the origin. The functional is equal to the expected number of open edges in the edge-boundary of $S$ that are connected to the origin in $S$. The proper counterpart of $\varphi$ in our model is defined for each measurable, bounded $S \subset \mathbb{R}^d$ with $B_R \subset S$ by

$$\varphi_t(S) := \mathbb{E} \sum_{(x,K)\in\xi_t} \mathbb{1}\big\{(x,K) \in \partial S, B_R \leftrightarrow K + x \text{ in } \xi_t \cap S^\circ\big\} \tag{3.4}$$

$$= t \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{1}\big\{(x,K) \in \partial S\big\} \mathbb{P}\big[B_R \leftrightarrow K + x \text{ in } \xi_t \cap S^\circ\big]\, dx\mathbb{Q}(dK). \tag{3.5}$$

This is the expected number of particles that "cross the boundary of $S$" and are connected to $B_R$ by particles contained in $S$. We proceed as in [5] by defining a "new" critical intensity

$$\tilde{t}_c := \sup\big\{t \geq 0 | \exists S \in \mathcal{B}\big(\mathbb{R}^d\big) \text{ bounded} : B_R \subset S, \varphi_t(S) < 1\big\}. \tag{3.6}$$

**Theorem 3.1.** *The "new" critical intensity has the following properties*:

(i) *For $t < \tilde{t}_c$ there are constants $c_1, c_2 > 0$ depending on $t$ such that*

$$\mathbb{P}\big[\rho\big(C_0(t)\big) \geq r\big] \leq e^{-c_1 r}, \quad r \geq 0, \tag{3.7}$$

$$\mathbb{P}\big[\big|C_0(t)\big| \geq r\big] \leq e^{-c_2 r}, \quad r \geq 0, \tag{3.8}$$

(ii) *for $t = \tilde{t}_c$,*

$$\mathbb{E}\big|C_0(t)\big| = \infty, \tag{3.9}$$

(iii) *for $t > \tilde{t}_c$,*

$$\mathbb{P}[B_R \leftrightarrow \infty \text{ in } \xi_t] \geq \frac{t - \tilde{t}_c}{t}, \tag{3.10}$$

$$\mathbb{P}\big[\big|C_0(t)\big| = \infty\big] \geq \frac{t - \tilde{t}_c}{t^2 \lambda^d(B_{2R})}. \tag{3.11}$$

*In particular, $t_c = \tilde{t}_c$.*

**Proof.** To prove (i), let $t < \tilde{t}_c$ and let $S \subset \mathbb{R}^d$ be measurable, bounded, with $\varphi_t(S) < 1$ and $B_R \subset S$.

When we explained Penrose's algorithm in Section 2 in the special case of fixed balls $B_1$ as grains, we observed that $|C_0|$ is a.s. finite for $t < \lambda(B_2)^{-1}$. We will now refine this approach such that it is applicable to the whole subcritical regime $t < \tilde{t}_c$ and random grains instead of fixed balls.

Before going into technical details, we want to give an informal description how we construct the cluster $C_0$ in our modification of Penrose's algorithm. After generating the grain $Z_0$ we explore the cluster of $Z_0$ in $S^\circ$ (in the same way as in the original algorithm). Then we generate all particles on $\partial S$ that are connected to $Z_0$ in $S^\circ$. We pick one of these grains $(x, K)$ that wasn't picked before and repeat the procedure with $S$ being replaced by $S + x$ except that we generate our Poisson process only once everywhere. We repeat this until all grains had been picked once (see Figure 1).

To formalize this let $(\xi_t^{(n)})_{n\in\mathbb{N}_0}$ be an i.i.d. sequence of Poisson processes on $\Pi$ with intensity measure $t\lambda^d \otimes \mathbb{Q}$. We start by explaining, how to adapt Penrose's algorithm to random grains. Our starting point is the typical particle $(0, Z_0)$ at the origin. First we generate the particles intersecting $Z_0$. That means we make the split $\xi_t \overset{d}{=} \xi_t^{(0)}|_{[Z_0]} + \xi_t^{(1)}|_{[Z_0]^c}$. If we generated no new particle in $[Z_0]$ we only have to generate $\xi_t$ on $[Z_0]^c$. Otherwise, we pick one of the generated particles $(x_1, K_1)$ and generate $\xi_t$ on $[K_1 + x_1] \cap [Z_0]^c$. If there are particles left that have not been picked yet, we pick another one $(x_2, K_2)$, $x_2 \neq x_1$ and generate $\xi_t$ on $[K_2 + x_2] \cap [K_1 + x_1]^c \cap [Z_0]^c$. We repeat this procedure until

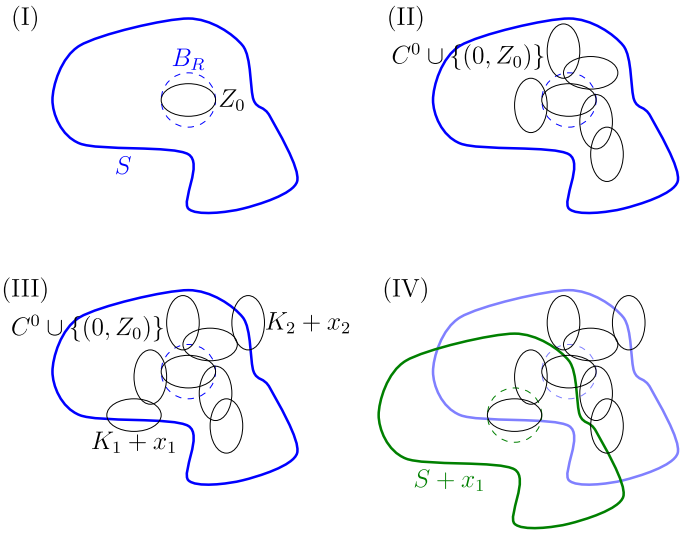


Fig. 1. The first steps of the new algorithm to construct $C_0$ where $\mathbb{Q}$ has probability mass .5 on each of two ellipses. In (I) the grain $Z_0$ at the origin is constructed. In (II) the cluster of $(0, Z_0)$ in $S^\circ$ is explored and called $C^0$. In (III) the particles in $\partial S$ that intersect $C^0$ are generated and added to $C^0$. (IV) shows in green the translated version of $S$ where the steps (b) and (c) of the algorithm are repeated.

every particle has been picked once. Then we generate $\xi_t$ on the remaining part of $\Pi$. Defining $K_0 := Z_0$ and $x_0 := 0$, this leads to

$$\xi_t \stackrel{d}{=} \xi_t^{(0)}|_{[Z_0]} + \sum_{i=1}^{n} \left(\xi_t^{(i)}|_{[K_i+x_i]\setminus[\bigcup_{j=0}^{i-1} K_j+x_j]}\right) + \xi_t^{(n+1)}|_{[\bigcup_{j=0}^{n} K_j+x_j]^c} \tag{3.12}$$

and

$$\xi_t \cup (0, Z_0) \stackrel{d}{=} C_0(t) \cup \left\{(x, K) \in \xi_t^{(n+1)} | (x, K) \notin \left[C_0(t)^\cup\right]\right\} \tag{3.13}$$

in the very same way we obtained (2.6) and (2.7).

We define, what it means "to explore the cluster of a particle $(x_0, K_0)$ in a set $D \subset \Pi$" (this happens several times during the course of the algorithm). Assume we already generated particles on $E \subset \Pi \setminus D$ during the course of the algorithm, one of which was $(x_0, K_0)$. Then we do the above expansion starting with $(x_0, K_0)$ instead of $(0, Z_0)$ but we only add particles from $D$ to the cluster. Formally this leads to

$$\xi_t \stackrel{d}{=} \xi_E + \xi_t^{(0)}|_{[K_0+x_0]\cap D\cap E^c} + \sum_{i=1}^{n} \left(\xi_t^{(i)}|_{[K_i+x_i]\cap[\bigcup_{j=0}^{i-1} K_j+x_j]^c\cap D\cap E^c}\right) \tag{3.14}$$

$$+ \xi_t^{(n+1)}|_{([\bigcup_{j=0}^{n} K_j+x_j]^c\cup D^c)\setminus E}, \tag{3.15}$$

where $\xi_E$ stands for the particles generated up to this point. Applying this, we use the following refined version of the algorithm explained above:

1. Let $\mathcal{Q}$ be an empty queue of particles.
2. Let $\mathcal{R} := \Pi$ be the remaining part of $\Pi$, i.e. the part where $\xi_t$ has not been generated yet.
3. Generate $Z_0$ and append $(0, Z_0)$ to the end of $\mathcal{Q}$.
4. If $\mathcal{Q}$ is not empty, then do the following:
   (a) Remove the first particle $(y, L)$ from $\mathcal{Q}$.
   (b) Explore the cluster of $(y, L)$ in $(S + y)^\circ \cap \mathcal{R}$ and write $C^y$ for the set of all newly generated particles in this step.

(c) Set $\mathcal{R} = \mathcal{R} \setminus [L + y \cup C^{y\cup}]$.
(d) Generate $\xi_t$ on $[C^{y\cup}] \cap \partial(S + y) \cap \mathcal{R}$. Hence, all particles are generated that "cross the boundary of $S + y$" and that intersect a particle of $C^y$.
(e) Append all newly generated particles of step (d) to $\mathcal{Q}$ and also add them to $C^y$.
(f) Set $\mathcal{R} = \mathcal{R} \setminus ([C^{y\cup}] \cap \partial(S + y))$.

5. Generate $\xi_t$ on $\mathcal{R}$.

We write $(y_0, L_0), \dots, (y_n, L_n)$ for the particles that are removed from $\mathcal{Q}$ during the algorithm. The definition of $C^y$ implies that for all $i = 0, \dots, n$ the particle $(y_i, L_i)$ is not contained in $C^{y_i}$. We claim that

$$C_0 = \big\{(0, Z_0)\big\} \cup \bigcup_{i=0}^{n} C^{y_i}. \tag{3.16}$$

On the one hand, it is obvious from the algorithm that any particle which is generated in any but the very last step (step 5) is connected to $Z_0$. On the other hand, for each particle $(z, M)$ that is generated we eventually generate $\xi_t$ on $[M + z]$ even though this might be split into $[M + z] \cap D^\circ$ (during an exploration in 4(b)) and $[M + z] \cap \partial D$ (during the generation of crossing particles in 4(d)) for some $D$ which is a translate of $S$. The property $B_R \subset S$ ensures in this case that $[M + z] \cap [D]^c$ is always empty, hence we do not forget to generate a neighbour of $(z, M)$.

Equation (3.16) enables an analysis similar to the one at the end of Section 2. There is a natural tree structure on the particles $(y_i, L_i)$, $i = 0, \dots, n$. If a particle $(y_i, L_i)$ was removed from $\mathcal{Q}$ in step 4(a) then all particles generated in step (d) of this iteration are the children of $(y_i, L_i)$. This results in a tree $T$ with root $(y_0, L_0)$. The depth and the size of $T$ relate nicely to $|C_0|$ and $\rho(C_0)$. If $(y_i, L_i)$ and $(y_j, L_j)$ are a mother and child pair of $T$, then $\|y_i - y_j\| \le \sup_{x\in S} \|x\| + R$ as $L_j + y_j$ has to intersect $S + y_i$. Hence

$$\rho(C_0) \le \operatorname{depth}(T)\Big(\sup_{x\in S} \|x\| + R\Big). \tag{3.17}$$

Moreover, we have $C^y \subset \xi_t|_{[S+y]}$ and hence (3.16) implies that $|C_0|$ is stochastically dominated by

$$1 + \sum_{i=1}^{\operatorname{size}(T)} \big|\xi_t^{(i)}|_{[S+y_i]}\big|.$$

Hence, the proof of (i) is easy, if we can ensure exponential tails for the size and the depth of $T$. To this end, we observe that the number of children of a particle $(y, L)$ is less than $|\xi_t|_{\partial(S+y)}|$. Further observe, that the particles generated in step (d) are a subset of

$$\big\{(x, K) \in \xi_t | (x, K) \in \partial(S + y), B_R(y) \leftrightarrow K + x \text{ in } \xi_t \cap (S + y)^\circ\big\}.$$

This implies, that the expected number of children is less or equal to $\varphi_t(S)$ which is less than 1. Hence, like in Penrose's algorithm, $T$ may be seen as a subtree of a Galton–Watson tree with an offspring distribution that has exponential tails and an expected value of less than 1. As mentioned in Section 2 the basic theory of branching processes implies that in this case the size and the depth of $T$ have exponential tails too.

*To prove* (ii) we observe that due to [7, Theorem 3.1] $\varphi_t(S)$ is an analytic and hence continuous function in $t$ for a fixed and bounded $S$. It follows that the set of parameters $t$ where there is a bounded set $S \in \mathcal{B}(\mathbb{R}^d)$ such that $B_R \subset S$ and $\varphi_t(S) < 1$ is open in the interval $[0, 1]$. We deduce that for any such $S$ we have $\varphi_{\tilde{t}_c}(S) \ge 1$ and hence

$$\begin{aligned}\mathbb{E}\big|C_0(\tilde{t}_c)\big| &= \mathbb{E} \sum_{(x,K)\in\xi_{\tilde{t}_c}} \mathbb{1}\{Z_0 \leftrightarrow K + x \text{ in } \xi_{\tilde{t}_c}\} \\ &\ge \sum_{n=1}^{\infty} \mathbb{E} \sum_{(x,K)\in\xi_{\tilde{t}_c}} \mathbb{1}\big\{(x, K) \in \partial B_{3nR}, Z_0 \leftrightarrow K + x \text{ in } \xi_{\tilde{t}_c}\big\}\end{aligned}$$

$$\geq \sum_{n=1}^{\infty} \varphi_{\tilde{t}_c}(B_{3nR})$$
$$= \infty.$$

*We prove assertion* (iii) with the help of equation (2.3) applied to the event $\{B_R \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]\}$ for $r > R$. This yields

$$\frac{\partial \mathbb{P}[B_R \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]]}{\partial t}$$
$$= \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{E}\big[\mathbb{1}\big\{B_R \leftrightarrow B_r^c \text{ in } (\xi_t + \delta_{(x,K)}) \cap [B_r] \text{ but not in } \xi_t \cap [B_r]\big\}\big]\, dx\mathbb{Q}(dK).$$

Let $U_r$ be the set of particles in $\xi_t$ that are connected to $B_r^c$ in $\xi_t \cap [B_r]$. Then a simple case-by-case analysis shows, that

$$\frac{\partial \mathbb{P}[B_R \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]]}{\partial t}$$
$$= \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{E}\big[\mathbb{1}\big\{B_R \leftrightarrow K + x \text{ in } \xi_t \cap \big[B_r^c \cup U_r^{\cup}\big]^c,\, B_R \notin \big[U_r^{\cup}\big],\, K + x \in \partial\big(B_r^c \cup U_r^{\cup}\big)\big\}\big]\, dx\mathbb{Q}(dK).$$

Applying the algorithm, that lead to (3.13) starting with $\partial B_r$ instead of $[Z_0]$, we obtain

$$\xi_t \cap [B_r] \stackrel{d}{=} U_r \cup \big\{(x, K) \in \xi_t^{(n+1)} \cap [B_r] | (x, K) \notin \big[U_r^{\cup}\big]\big\}. \tag{3.18}$$

Hence, we condition on $U_r^{\cup}$ (denoting its distribution by $\mathbb{P}_{U_r^{\cup}}$) and observe that conditioned on the event $\{U_r^{\cup} = A\}$ the particles in $\xi_t \cap [B_r^c \cup U_r^{\cup}]^c$ are distributed like $\xi_t^{(1)} \cap [B_r^c \cup A]^c$. Using this, the resulting independence and the definition of $\varphi_t$, we obtain

$$\frac{\partial \mathbb{P}[B_R \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]]}{\partial t}$$
$$= \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \int \mathbb{E}\big[\mathbb{1}\big\{B_R \leftrightarrow K + x \text{ in } \xi_t^{(1)} \cap \big[B_r^c \cup A\big]^c,\, B_R \notin [A],$$
$$K + x \in \partial\big(B_r^c \cup A\big)\big\} | U_r^{\cup} = A\big] \mathbb{P}_{U_r^{\cup}}(dA)\, dx\mathbb{Q}(dK)$$
$$= \int \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{P}\big[B_R \leftrightarrow K + x \text{ in } \xi_t^{(1)} \cap \big[B_r^c \cup A\big]^c\big]$$
$$\times \mathbb{1}\big\{K + x \in \partial(B_r^c \cup A)\big\}\, dx\mathbb{Q}(dK)\mathbb{1}\big\{B_R \notin [A]\big\}\mathbb{P}_{U_r^{\cup}}(dA)$$
$$= \frac{1}{t} \int \varphi_t\big(\big(B_r^c \cup A\big)^c\big)\mathbb{1}\big\{B_R \notin [A]\big\}\mathbb{P}_{U_r^{\cup}}(dA).$$

For any $t \geq \tilde{t}_c$ the functional $\varphi_t$ is greater or equal to 1. This yields

$$\frac{\partial \mathbb{P}[B_R \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]]}{\partial t} \geq \frac{\mathbb{P}[B_R \notin [U_r^{\cup}]]}{t} \tag{3.19}$$
$$= \frac{1 - \mathbb{P}[B_R \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]]}{t} \tag{3.20}$$

for such $t$. Dividing the inequality by $1 - \mathbb{P}[B_R \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]]$, integrating it from $\tilde{t}_c$ to some $t > \tilde{t}_c$ and using the trivial inequality $\mathbb{P}[B_R \leftrightarrow B_r^c \text{ in } \xi_{\tilde{t}_c} \cap [B_r]] \geq 0$ we obtain for $t \geq \tilde{t}_c$ that

$$\mathbb{P}\big[B_R \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]\big] \geq \frac{t - \tilde{t}_c}{t}. \tag{3.21}$$

Taking the limit $r \to \infty$ yields that

$$\mathbb{P}[B_R \leftrightarrow \infty \text{ in } \xi_t] \geq \frac{t - \tilde{t}_c}{t}.$$

The last step is to relate the Palm probability $\mathbb{P}[|C_0(t)| = \infty]$ to the probability that $B_R$ intersects the infinite cluster. We use the well known formula for Palm probabilities (see [14, Theorems 3.3.2 and 3.5.3]) that implies

$$\begin{aligned}
\mathbb{P}\big[\big|C_0(t)\big| = \infty\big] &= \frac{1}{t\lambda^d(B_{2R})}\mathbb{E}\sum_{(x,K)\in\xi_t} \mathbb{1}\{x \in B_{2R}, K + x \leftrightarrow \infty\} \\
&\geq \frac{1}{t\lambda^d(B_{2R})}\mathbb{P}\big[\exists (x, K) \in \xi_t : x \in B_{2R}, K + x \leftrightarrow \infty\big] \\
&\geq \frac{1}{t\lambda^d(B_{2R})}\mathbb{P}[B_R \leftrightarrow \infty \text{ in } \xi_t] \\
&\geq \frac{t - \tilde{t}_c}{t^2\lambda^d(B_{2R})}.
\end{aligned}$$

□

It is an open problem to adapt the argument to grains of arbitrary size. In the current algorithmic approach large grains lead to a bias as they are reached more often when they intersect the boundary of $S$ and would also have larger clusters than small grains in the following step. At the moment, this bias is controlled by the use of $B_R$ instead of a random grain in the definition of $\varphi_t$. With arbitrarily large grains this does not work anymore.

## 4. Rigorous lower bounds for the critical intensity

Theorem 3.1 opens up several ways to obtain lower bounds for the critical intensity. We get the first one, by choosing a certain $S$ and calculating or estimating $\varphi_t(S)$ depending on $t$. If we are able to choose $t$ such that $\varphi_t(S) < 1$, the chosen $t$ is a lower bound for $t_c$.

In the special case, where $\mathbb{Q}$ is concentrated on $B_1$ we retrieve the Penrose bound, by setting $S := B_1$. Then $\varphi_t(S) = \mathbb{E}[|\xi_t \cap [B_1]|] = t\lambda^d(B_2)$ and hence $t_c \geq \lambda^d(B_2)^{-1}$.

But this exact lower bound can be sharpened by choosing $S := B_3$. In this case we have $\partial S = \{B_1(x) | 2 < \|x\|_2 \leq 4\}$, $S^\circ = \{B_1(x) | \|x\|_2 \leq 2\}$ and $R = 1$. Moreover, any ball $B_1(y) \in S^\circ$ intersects $B_1$ and hence for any $x$ such that $B_1(x) \in \partial S$,

$$\begin{aligned}
\mathbb{P}\big[B_1 \leftrightarrow B_1(x) \text{ in } \xi_t \cap S^\circ\big] &= \mathbb{P}\big[\xi_t\big([B_1] \cap \big[B_1(x)\big]\big) \geq 1\big] \\
&= 1 - e^{-t\lambda^d(B_2 \cap B_2(x))}.
\end{aligned}$$

It follows from definition (3.5) that

$$\begin{aligned}
\varphi_t(S) &= t\int_{\mathbb{R}^d} \mathbb{1}\big\{B_1(x) \in \partial S\big\}\mathbb{P}\big[B_1 \leftrightarrow B_1(x) \text{ in } \xi_t \cap S^\circ\big]\,dx \\
&= t\lambda^d(B_1)\int_2^4 r^{d-1}\big(1 - e^{-t\lambda^d(B_2 \cap B_2(r\mathbf{e}_1))}\big)\,dr,
\end{aligned}$$

where $\mathbf{e}_1$ is an arbitrary vector of unit length. The integral may be evaluated numerically to almost arbitrary precision and hence it is easy to find a good approximation of the value $t$ where $\varphi_t(B_3) = 1$. The corresponding lower bounds are listed in Table 1. These two approaches obviously work with other fixed grain shapes too, but the numerical calculations might become significantly more involved.

When working on lower bounds for the critical intensity, it has to be mentioned that the first rigorous lower bound was given by Hall in [6] for dimension $d = 2$. To our knowledge the approach has never been applied to higher

Table 1
Lower bounds for the critical intensity $t_c$ in dimension 2 to 11 when $\mathbb{Q}$ is concentrated on $B_1$

| d | Bound via $\varphi_t(B_3)$ | Penrose bound |
|---|---|---|
| 2 | 0.135802... | 0.0795774... |
| 3 | 0.0433691... | 0.0298415... |
| 4 | 0.0167131... | 0.0126651... |
| 5 | 0.00734445... | 0.00593678... |
| 6 | 0.00357261... | 0.00302358... |
| 7 | 0.00188850... | 0.00165352... |
| 8 | 0.00107117... | 0.000962435... |
| 9 | 0.000645942... | 0.000592123... |
| 10 | 0.000411202... | 0.000382941... |
| 11 | 0.000274803... | 0.000259158... |

dimensions and it was stated in the book by Meester and Roy [10] that it was untractable there. We want to take a short moment to show that this is not the case and even gives bounds that are better than the ones in Table 1.

The idea of Hall may be seen today as a refinement of the Penrose bound. Hall also modified the algorithm from Section 3 to construct $C_0$ by creating more Poisson points. In the algorithm the children of each grain $(x, K)$ are determined by the Poisson process of grains that intersect $K + x$ but not any previous grain in the cluster $C$. Halls modification was, to take all grains as children of $(x, K)$ that intersect $(x, K)$ but not the ancestor of $(x, K)$. In the case where all grains are a.s. equal to $B_1$ the number of children of the grain $B_1(x)$ only depends on the distance of $x$ to the center $y$ of its ancestor grain $B_1(y)$. In this way the cluster $C$ generated by the algorithm can be compared with a multitype branching process, where the type of each grain is the distance to the center of its ancestor grain. Hall showed that the expected size of $C$ was given by

$$1 + \sum_{n=1}^{\infty} t^n T^n(\mathbb{1}_{(0,2)})(1),$$

where $T$ is an operator from the set of continuous functions on the interval $(0, 2)$ onto itself that is defined by

$$T(f)(x) := \int_0^2 f(y)g(y, x)\,dx. \tag{4.1}$$

The function $g(y, x)$ is given by the $d-1$ dimensional Hausdorff measure of the set $\{z \in \mathbb{R}^d | \|z\|_2 > 2, \|z - x\mathbf{e}_1\|_2 = y\}$. Hence to calculate $g$ in higher dimensions, we have to determine the surface area of a spherical cap in higher dimensions. The formulas for this can be found in the literature (see [9]) and we obtain

$$g(y, x) = \begin{cases} (d-1)\lambda^d(B_1)y^{d-1}\int_0^{\arccos(\frac{4-x^2-y^2}{2xy})} \sin^{d-2}(\varphi)\,d\varphi, & y \in (2-x, 2), \\ 0, & y \in (0, 2-x). \end{cases}$$

Hall concludes that the expected size of $C$ is finite if the largest eigenvalue of $T$ is less than $1/t$. For fixed $d$ the integral defining $g$ can be solved analytically. Afterwards the largest eigenvalue of $T$ can be found numerically with very high precision. The results have been collected in Table 2.

## 5. Highly probable lower bounds for the critical intensity

The second way to obtain lower bounds from Theorem 3.1 is, to use the mean-field lower bound. It follows from (3.21) by an easy calculation that for any $t \in [0, \infty)$ and $r > 1$

$$t_c \geq t\big(1 - \theta_t(r)\big), \tag{5.1}$$

Table 2
Rigorous lower bounds for $t_c$ from Section 4 for the Boolean model where $\mathbb{Q}$ is concentrated on $\{B_1\}$

| d | Penrose | Via $\varphi_t(B_3)$ | Hall |
|---|---|---|---|
| 2 | 0.0795774 | 0.135802 | 0.174746 |
| 3 | 0.0298415 | 0.0433691 | 0.0534187 |
| 4 | 0.0126651 | 0.0167131 | 0.0198296 |
| 5 | 0.00593678 | 0.00734445 | 0.00845546 |
| 6 | 0.00302358 | 0.00357261 | 0.00401478 |
| 7 | 0.00165352 | 0.00188850 | 0.00208114 |
| 8 | 0.000962436 | 0.00107117 | 0.00116176 |
| 9 | 0.000592124 | 0.000645943 | 0.000691455 |
| 10 | 0.000382941 | 0.000411203 | 0.000435437 |
| 11 | 0.000259158 | 0.000274804 | 0.000288394 |

Table 3
Simulation results for lower bounds of the critical intensity $t_c$ in dimension 2 to 11 in the Boolean model where $\mathbb{Q}$ is concentrated on $B_1$

| $d$ | $r$ | $t$ | Runs | Successes | 99% CI for $\theta_t(r)$ | Lower bound |
|---|---|---|---|---|---|---|
| 2 | 16000 | 0.357 | 10000 | 0 | 0.00063692 | 0.356772 |
| 3 | 2000 | 0.0814 | 10000 | 0 | 0.00063692 | 0.0813481 |
| 4 | 500 | 0.0261 | 10000 | 10 | 0.002119993 | 0.0260445 |
| 5 | 500 | 0.0101 | 10000 | 0 | 0.00063692 | 0.0100935 |
| 6 | 200 | 0.00456 | 10000 | 1 | 0.000813077 | 0.00455628 |
| 7 | 200 | 0.00228 | 10000 | 18 | 0.003154537 | 0.00227278 |
| 8 | 150 | 0.00124 | 10000 | 21 | 0.003529665 | 0.00123560 |
| 9 | 150 | 0.000725 | 10000 | 6 | 0.001571485 | 0.000723859 |
| 10 | 120 | 0.000450 | 10000 | 4 | 0.001282615 | 0.000449422 |
| 11 | 120 | 0.0002955 | 10000 | 8 | 0.001849554 | 0.000294952 |

where

$$\theta_t(r) := \mathbb{P}\big[B_1 \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]\big]. \tag{5.2}$$

Hence to obtain a lower bound for $t_c$, it suffices to choose an arbitrary $t$ and $r > 1$ and estimate $\theta_t(r)$ by simulation. The event $\{B_1 \leftrightarrow B_r^c \text{ in } \xi_t \cap [B_r]\}$ can be simulated exactly and hence we may compute a rigorous one-sided confidence interval for the true value of $\theta_t(r)$ and hence a confidence interval for the lower bound of $t_c$.

We also want to point out that for any $t < t_c$ the limit $\lim_{r\to\infty} \theta_t(r) = 0$. Hence by investing enough computing time it is in principle possible to approximate the value of $t_c$ arbitrarily well.

We did our simulations in the following way. We fixed a dimension, chose $t$ slightly below the best known value for $t_c$ from the literature and picked $r$ such that our simulations could finish in reasonable time. We simulated 10,000 times the cluster $C_0$ with the algorithm stated before the proof of Theorem 3.1 and counted the number of times it intersected $B_r^c$. In higher dimensions we also terminated the algorithm when the size of $C$ exceeded some large threshold. In this case we counted this run as if $C$ had intersected $B_r^c$ and hence had a conservative estimate. After 10,000 runs, we computed the corresponding confidence interval for $\theta_t(r)$ with the `prop.test` method of the statistical language R and calculated the lower bound for $t_c$. Depending on how fast this was done and on how many times the boundary was reached, we increased $t$ and started another 10,000 runs. This lead to the results in Table 3.

We want to briefly discuss the chosen parameters and results. It can be observed that the precision never exceeds three significant digits. This is due to the fact that we only get a high precision if the confidence interval is small. The size of the confidence interval, however, depends on the number of runs and successes. It turned out in practice that choosing 10,000 runs and a $t$ such that not more than about 20 runs succeed, gave the best tradeoff between time and precision. The few runs, where the cluster actually reaches $B_r^c$ are extremely time consuming, hence it is

Table 4
Simulated lower bounds for $t_c$ for the Boolean model where $\mathbb{Q}$ is concentrated on $\{B_1\}$ compared to the best known values in the literature

| d | Sim. $\theta_r(t)$, 99% CI for lower bound | Sim. [15] lower bound | Sim. [15] upper bound |
|---|---|---|---|
| 2 | 0.356772 | 0.359076 | 0.359085 |
| 3 | 0.0813481 | 0.081854 | 0.081858 |
| 4 | 0.0260445 | 0.02632 | 0.02642 |
| 5 | 0.0100935 | 0.01032 | 0.01034 |
| 6 | 0.00455628 | 0.004516 | 0.004526 |
| 7 | 0.00226708 | 0.002218 | 0.002272 |
| 8 | 0.00123560 | 0.001206 | 0.001208 |
| 9 | 0.000722539 | 0.0007121 | 0.0007133 |
| 10 | 0.000449422 | 0.0004450 | 0.0004462 |
| 11 | 0.000294952 | 0.0002933 | 0.0002935 |

more efficient, to chose $t$ slightly below the expected "true" $t_c$. Nevertheless, it can be seen in Table 4 that for high dimensions our lower bounds exceed the upper error bound (upper end of the 1-$\sigma$ band) of best simulation results in the literature (see [15]).

A few words concerning the implementation of the algorithm. It is very useful to save the approximate position of the grains in the cluster to have a faster access when comparing if the current grain intersects $C$. Due to the fact that only a tiny fraction of the space is covered by the cluster, when $t$ is close to critical, we preferred a hashmap over an array for this task. Another important issue is the ball-picking method, i.e. the method to generate a random vector in $B_1$. This can either be done by generating points in $[-1, 1]^d$ and throwing away the points that don't lie in $B_1$ or it can be done by the formula proposed in [4]. The first approach is faster than the second one in low dimensions. We found that the second method was faster for $d \geq 7$.

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## References

[1] D. Ahlberg, V. Tassion and A. Teixeira. Sharpness of the phase transition for continuum percolation in $\mathbb{R}^2$. Preprint, 2016. Available at arXiv:1605.05926.
[2] M. Aizenman and D. J. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.* **108** (3) (1987) 489–526. MR0874906
[3] K. B. Athreya and P. E. Ney. *Branching Processes*, *Die Grundlehren der mathematischen Wissenschaften* **Band 196**. Springer-Verlag, New York-Heidelberg, 1972. MR0373040
[4] F. Barthe, O. Guédon, S. Mendelson and A. Naor. A probabilistic approach to the geometry of the $L_p$-ball. *Ann. Probab.* **33** (2) (2005) 480–513. MR2123199
[5] H. Duminil-Copin and V. Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$. *Enseign. Math.* **62** (2016) 199–206. MR3605816
[6] P. Hall. On continuum percolation. *Ann. Probab.* **13** (4) (1985) 1250–1266. MR0806222
[7] G. Last. Perturbation analysis of Poisson processes. *Bernoulli* **20** (2) (2014) 486–513. MR3178507
[8] G. Last and M. Penrose. Lectures on the Poisson Process. To be published by Cambridge University Press, 2017. Available at http://www.math.kit.edu/stoch/~last/page/lehrbuch_poissonp/en.
[9] S. Li. Concise formulas for the area and volume of a hyperspherical cap. *Asian J. Math. Stat.* **4** (1) (2011) 66–70. MR2813331
[10] R. Meester and R. Roy. *Continuum Percolation*. *Cambridge Tracts in Mathematics* **119**. Cambridge University Press, Cambridge, 1996. MR1409145
[11] M. V. Menshikov. Coincidence of critical-points in the percolation problems. *Dokl. Akad. Nauk SSSR* **288** (6) (1986) 1308–1311. MR0852458

[12] M. D. Penrose. Continuum percolation and Euclidean minimal spanning trees in high dimensions. *Ann. Appl. Probab.* **6** (2) (1996) 528–544. MR1398056
[13] M. D. Penrose. *Random Geometric Graphs*. *Oxford Studies in Probability* **5**. Oxford University Press, Oxford, 2003. MR1986198
[14] R. Schneider and W. Weil. *Stochastic and Integral Geometry*. *Probability and its Applications (New York)*. Springer-Verlag, Berlin, 2008. MR2455326
[15] S. Torquato and Y. Jiao. Effect of dimensionality on the continuum percolation of overlapping hyperspheres and hypercubes. II. Simulation results and analyses. *J. Chem. Phys.* **137** (7) 2012 074106.