Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2018, Vol. 54, No. 2, 757–789
https://doi.org/10.1214/17-AIHP820

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Stochastic Ising model with flipping sets of spins and fast decreasing temperature

Roy Cerqueti[a] and Emilio De Santis[b]

[a]University of Macerata, Department of Economics and Law. Via Crescimbeni 20, I-62100, Macerata, Italy. E-mail: roy.cerqueti@unimc.it
[b]University of Rome La Sapienza, Department of Mathematics. Piazzale Aldo Moro, 5, I-00185, Rome, Italy. E-mail: desantis@mat.uniroma1.it

Received 29 September 2015; revised 12 September 2016; accepted 16 January 2017

Abstract. This paper deals with the stochastic Ising model with a temperature shrinking to zero as time goes to infinity. A generalization of the Glauber dynamics is considered, on the basis of the existence of simultaneous flips of some spins. Such dynamics act on a wide class of graphs which are periodic and embedded in $\mathbb{R}^d$. The interactions between couples of spins are assumed to be quenched i.i.d. random variables following a Bernoulli distribution with support $\{-1, +1\}$. The specific problem here analyzed concerns the assessment of how often (finitely or infinitely many times, almost surely) a given spin flips. Adopting the classification proposed in (*Comm. Math. Phys.* **214** (2002) 373–387), we present conditions in order to have models of type $\mathcal{F}$ (any spin flips finitely many times), $\mathcal{I}$ (any spin flips infinitely many times) and $\mathcal{M}$ (a mixed case). Several examples are provided in all dimensions and for different cases of graphs. The most part of the obtained results holds true for the case of zero-temperature and some of them for the cubic lattice $\mathbb{L}_d = (\mathbb{Z}^d, \mathbb{E}_d)$ as well.

Résumé. Cet article est dedié au modèle d'Ising stochastique avec une température decroissante à zéro avec le temps. Une généralisation de la dynamique de Glauber est considérée, basée sur des inversions simultanées des ensembles de spins. La dynamique est considerée sur une large classe de graphes qui sont périodiques et plongés dans un espace euclidien. Les interactions entre les couples de spins sont supposées être des variables aléatoires i.i.d. qui suivent une loi de Bernoulli avec support $\{-1, +1\}$. Le problème particulier analysé ici concerne l'évaluation du nombre d'inversions d'un spin donné (fini ou infini, presque sûrement). En adoptant la classification proposée dans (*Comm. Math. Phys.* **214** (2002) 373–387), nous présentons des conditions pour des modèles de type $\mathcal{F}$ (tout les spins sont sujets à un nombre fini d'inversions), $\mathcal{I}$ (tout les spins sont sujets à un nombre infini d'inversions) et $\mathcal{M}$ (le cas mixte). Plusieurs exemples sont fournis en toutes dimensions et pour plusieurs graphes. La partie majeure des résultats reste vraie à température zéro et certains des résultats sont vrais pour le réseau cubique en dimension d quelconque.

MSC: 60K35; 82B44

Keywords: Ising model; Glauber dynamics; Fast decreasing temperature; Graphs

1. Introduction

In this paper we deal with a class of non homogeneous Markov processes $(\sigma(t) : t \geq 0)$ in the frame of random environment. In particular, we consider a generalization of Glauber dynamics of the Ising model, see e.g. [18], in the case of flipping sets whose cardinality is smaller than or equal to a given $k \in \mathbb{N}$. However, accordingly to Glauber, also our dynamics will satisfy the reversibility property with respect to the Gibbs measure of the Ising model. The Markov process describes the stochastic evolution of spins, which are binary variables ± 1 on the vertices of an infinite periodic graph $G = (V, E)$ with finite degree. Such a class of graphs includes the meaningful standard case of d-dimensional cubic lattices $\mathbb{L}_d = (\mathbb{Z}^d, \mathbb{E}_d)$, for $d \in \mathbb{N}$. The *interactions* $\mathcal{J} = (J_e : e \in E)$ are deterministic or i.i.d. random variables having Bernoulli distribution $\mu_{\mathcal{J}}(J_e = +1) = \alpha$ and $\mu_{\mathcal{J}}(J_e = -1) = 1 - \alpha$, with $\alpha \in [0, 1]$, and they constitute the random environment. The case $\alpha = 1$ (resp. $\alpha = 0$) corresponds to the interactions of the homogeneous ferromagnetic (resp. antiferromagnetic) Ising model.

The *temperature profile* of the model is a function $T : [0, \infty) \to [0, \infty)$, which is measurable with respect to the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ with $T(t)$ denoting the *temperature at time* t, for any $t \in [0, \infty)$. In most of the results it is assumed that $\lim_{t\to\infty} T(t) = 0$ with a specific reference to T fast decreasing to zero, this will be important due to its connections with the well studied case of $T \equiv 0$.

The initial configuration of the Markov process is $\sigma(0) \in \{-1, +1\}^V$. Its components are assumed to be given or i.i.d. and randomly selected by a Bernoulli measure $\nu_{\sigma(0)}$ with parameter $\gamma \in [0, 1]$.

In the case of zero temperature stochastic Ising models with homogeneous interactions, the following question is of particular relevance:

(Q) *Does a given spin on $v \in V$ flip infinitely many times almost surely?*

This problem is indeed paradigmatic in this context (see e.g. [2–5,10–14,16,22,30]).

However, for constant and positive temperature T, question **(Q)** does not make sense because all the spins flip infinitely many times. Moreover, question **(Q)** should be rephrased also in the frame of random interactions. In fact, it is not always meaningful to deal with single sites, because the random environment leads to sites differently behaving.

In the deep contribution of Gandolfi, Newman and Stein [14] the authors propose also a classification for these models, which is a partition of them. Specifically, a model is of type $\mathcal{I}$, $\mathcal{F}$, and $\mathcal{M}$, according to if all the sites flip infinitely many times (a.s.), finitely many times (a.s.), or some sites flip infinitely many times and the others do it finitely many times (a.s.), respectively. We adopt here the same classification. Question **(Q)** becomes:

(Q′) *Is a model of type $\mathcal{I}$, $\mathcal{F}$ or $\mathcal{M}$?*

We notice that $\mathcal{F}$ corresponds to the almost surely convergence with respect to the product topology. Vice versa $\mathcal{I}$ and $\mathcal{M}$ correspond to a.s. no-convergence. In this paper we show that some universal classes can be identified, in the sense that the graph G and the parameter k determine the class of the model under mild conditions on α, γ, T. Indeed, the main part of our results holds true for $\alpha \in (0, 1)$ and $\gamma \in [0, 1]$ under some natural requirements on the decay rate of the temperature profile T.

In a very different context, [1] has shown the recurrence of annihilating random walks on $\mathbb{Z}^d$, with $d \in \mathbb{N}$, under very general conditions. This paper has as a consequence that the one-dimensional stochastic Ising model with $\alpha = 1$, $\gamma \in (0, 1)$ and $T \equiv 0$ is of type $\mathcal{I}$ (see [23]).

In [14] there is an analysis of the zero-temperature case for the cubic lattice $\mathbb{L}_d = (\mathbb{Z}^d, \mathbb{E}_d)$, $\gamma = 1/2$ and different product measures $\mu_{\mathcal{J}}$ over $\mathbb{R}$ for the interactions $\mathcal{J}$. Among the results, the authors provide a complete characterization for $\mathbb{L}_1$ and for all the measures $\mu_{\mathcal{J}}$, i.e. they identify the classes $\mathcal{I}$, $\mathcal{F}$, and $\mathcal{M}$ for each $\mu_{\mathcal{J}}$. In doing so, [14] adapts and extends [1] in this context. Moreover, the authors identify the class $\mathcal{M}$ for $\mathbb{L}_2$ when the measure $\mu_{\mathcal{J}}$ is a product of Bernoulli with parameter $\alpha \in (0, 1)$.

In [23] there is a treatment of $\alpha = 1$ and $\gamma = 1/2$ for $\mathbb{L}_2$, where it is proven that the model is of type $\mathcal{I}$. Under the same conditions, [2] refines [23] in discussing the recurrence and the growth of some geometrical structures.

It is also worth noting that [23] analyzes the framework where $\mu_{\mathcal{J}}$ is a continuous measure over $\mathbb{R}$ with finite mean, and they find $\mathcal{F}$. The finite mean restriction has been eliminated in the same setting by [11], and the identified class remains $\mathcal{F}$.

The contribution of [13] is for $T \equiv 0$, $\alpha = 1$, $\mathbb{L}_d$ with $d \geq 2$ and $\gamma > \gamma_d^\star$, with $\gamma_d^\star \in (0, 1)$. The authors show that, when $\gamma > \gamma_d^\star$, the value of any spin converges to $+1$ a.s., hence leading to a model of type $\mathcal{F}$. The paper [22] extends [13] and shows that $\lim_{d\to\infty} \gamma_d^\star = 1/2$.

In this last context, in order to prove that, for a large value of γ, all the spins converge to $+1$, it is worth mentioning [3] where the stochastic Ising model at zero-temperature on a d-ary regular tree $\mathbb{T}_d$ is analyzed. Analogously to [13,22], the authors prove that there exists $\hat{\gamma}_d \in (0, 1)$ such that for $\gamma > \hat{\gamma}_d$ all the spins converge to $+1$. Moreover it is also shown, along with other results, that $\lim_{d\to\infty} \hat{\gamma}_d = 1/2$.

There are important results for the convergence of the system to the Gibbs state in the case of low temperature (see e.g. [6,7,9,20]) or high temperature (see e.g. [6,8,19,21]). In particular, in [6], in the framework of random interactions, it is shown a different approach to the equilibrium measure in relation to the temperature. We believe that there is a connection between the behaviour of the stochastic Ising model with $\alpha \in (0, 1)$ at low, constant and positive temperature and at zero-temperature. In this respect, it seems that the type of the model, $\mathcal{F}$ or $\mathcal{M}$, at zero-temperature is related to properties of metastability of the same model at low constant temperature. The geometric structure of the graph is also relevant in both the cases of zero- and low constant temperature. In [6,14] some of the main results come out from a combinatorial-geometric interpretation of the underlying graphs.

For cases different from $\mathbb{L}_d$ at zero-temperature, we mention [3] and [12], with tree-related graphs; [30], where trees and cylinders originating by graphs are considered; [16], where the hexagonal lattice is explored. In this latter paper the authors move from [23], where it is proven that sites fixate, and show that the expected value of the cardinality of the cluster containing the origin becomes infinite when time grows.

As already announced above, we aim to provide an answer to **(Q′)** in the framework of general graphs. In particular, we deal with graphs which are periodic and embedded in $\mathbb{R}^d$. In so doing, we contribute to the literature, which is mainly focussed on $\mathbb{L}_1$ (see e.g. [14]), $\mathbb{L}_2$ (see e.g. [2,14,23]) and the hexagonal lattice (see e.g. [16]) with the exceptions of the general dimensional cubic lattices $\mathbb{L}_d$ in [13,22].

Notice that this topic is important either at a purely theoretical level as well as in the applied science. Indeed, we mention [10,24,28,29], where applications of Ising models to social science are presented. In particular, [10] deals with Glauber dynamics at zero-temperature over random graphs, where nodes are social entities and the spatial structure captures the social connections among the nodes. For a review of the relevant contribution on the so-called *sociophysics*, refer to [28].

This paper adds to the literature on the stochastic Ising models as follows:

(i) We allow for simultaneous flips of the spins, hence allowing for flipping regions. This statement has an interest under a theoretical perspective and it seems to be also reasonable for the development of real-world decision processes (think at the changing of opinions process of groups of connected agents rather than of single individuals). The cardinality of the flipping sets is constrained by a parameter k, which will be formalized below. Some material related to this aspect can be found in [30].

(ii) The considered graphs are periodic, infinite and with finite degree. This framework naturally includes $\mathbb{L}_d$, for each $d \in \mathbb{N}$. This generalization allows us to provide theoretical results and physically consistent examples (like crystal lattices) outside the restrictive world of $\mathbb{L}_d$.

(iii) The temperature is taken not necessarily zero. Specifically, we consider a temperature fast decreasing to zero and we require only in one result its positivity. In doing so, we develop a theory on the reasonable situation of a temperature changing continuously in time, without assuming the jump from infinite to zero. The framework of temperature fast decreasing to zero includes also the case of $T \equiv 0$.

In more details, Lemma 1 is a technical result giving the framework we deal with. Specifically, it provides some grounding consequences of the definition of temperature profile fast decreasing to zero, which are useful in the rest of the paper.

Theorem 1 gives sufficient conditions for the temperature profile having a similar or different behaviour of the zero-temperature case. It is shown that the model is of type $\mathcal{I}$ under an asymptotic condition for the temperature profile. This result depends on the Hamiltonian, and can then be rewritten for general Gibbs models endowed with a Glauber-type dynamics. Some conditions of Theorem 1 can be replaced with weaker ones over graphs with even-degree sites (e.g. $\mathbb{L}_d$).

In Theorem 2 we present a result stating that the model is of type $\mathcal{M}$ or $\mathcal{I}$, under some hypotheses. Specifically, it is required that the temperature profile T is fast decreasing to zero, positive and the graph belongs to the rather wide class of *k-stable d-$\mathcal{E}$graphs* (see next section for the formal definition of this concept). Such a class contains also the cubic lattices $\mathbb{L}_d$, with $d \geq 2$. We believe that Theorem 2 represents our main result because it is a relevant step to the identification of the type of stochastic Ising models in dimension $d \geq 2$. We stress that the positivity of the temperature is required only in this result.

Theorem 3 states some conditions to obtain models of type $\mathcal{M}$, and it is used to prove some findings in Section 4.

Theorem 4 shows that the model with ferromagnetic interactions and $\gamma = 1/2$ on the hexagonal lattice is not of type $\mathcal{F}$ when $k \geq 2$. In so doing we complement [23], where it is proven that the same model with $k = 1$ and $T \equiv 0$ is of type $\mathcal{F}$.

In Section 4 we describe conditions leading to fixating sites. Lemmas 4 and 5 highlight the connection between lowering and increasing energy flips. Indeed, under the hypothesis of Lemma 1, the number of such flips is a random variable having finite mean on any given set of flipping spins. These two Lemmas are widely used to prove the results of this section.

Theorem 5 is inspired by [14,23] and generalizes their outcomes to the case of $k > 1$. In particular, it links the parameter k with the properties of the graph to obtain that some sites fixate, i.e. the model is of type $\mathcal{M}$ or $\mathcal{F}$. Theorem 6 formalizes the intuitive fact that if there exists a set of sites strongly interconnected and weakly connected with the complement of the set, then these sites will fixate with positive probability.

In Definition 4 we adapt to our setting the concept of *e-absent* configurations (namely, *k-absent on $\mathcal{J}$* here), which has been introduced for the graph $\mathbb{L}_2$ in [2,14]. We generalize such a definition by including $k > 1$ and the considered class of graphs. Theorems 7 and 8 are based on k-absence. The former result states that configurations which are k-absent on $\mathcal{J}$ can appear only on a random finite time interval (a.s.) when interactions are properly selected; the latter one provides a condition on the graph and the parameter k such that some sites have a positive probability to fixate. An interesting consequence of the definition of k-absence is that if a configuration is k-absent on $\mathcal{J}$, then it is k'-absent on $\mathcal{J}$, for each $k' > k$. Hence, a large value of k seems to facilitate the fixation of the sites (see Theorem 7). Differently, a large value of k is an obstacle for the fixation of the sites in Theorem 8. This contrast leads to a not straightforward link between the value of k and the identification of the model. However, in our setting, we have shown that a model on the hexagonal lattice is of type $\mathcal{F}$ for $k = 1$ (see also [23]) and it is not of type $\mathcal{F}$ for $k \geq 2$ (see Theorem 4). This Theorem suggests a general conjecture that link the value of k with the type of the model (see the Conclusions).

The provided examples illustrate a wide part of the outcomes, and complement the theoretical findings of the paper. In particular, we have introduced the graph $\Gamma_{\ell,m}(G)$, it is constructed by replacing the original edges of a graph $G = (V, E)$ with more complex structures (see Definition 5). For this specific class, we have provided some conditions for which the models over such graphs are of type $\mathcal{M}$ (see Theorem 9) and $\mathcal{I}$ (see Theorem 10). The case $\mathcal{F}$ is left as a conjecture in the Conclusions.

The last section of the paper concludes and offers some conjectures and open problems. In order to assist the reader, we have provided some figures presenting the main contributions of the related literature at zero-temperature, the results obtained in the present paper for the $\Gamma_{\ell,m}(G)$ graphs and for k-stable d-$\mathcal{E}$ graphs (see Figure 6).

2. Definitions and first properties of the model

The main target of this section is to define a dynamical stochastic Ising model. In order to do it we introduce some notation.

2.1. *Graphs*

A graph $G = (V, E)$ with origin O is said to be a *d-graph* if:

1. G is embedded in $\mathbb{R}^d$, i.e. the vertices are points and the edges are line segments;
2. G is translation invariant with respect to the vectors e_i of the canonical basis of $\mathbb{R}^d$;
3. for all $v \in V$, the degree of v, denoted by d_v, is such that $d_v < \infty$;
4. every finite region $S \subset \mathbb{R}^d$ contains finitely many vertices of G.

Thus we can construct a d-graph $G = (V, E)$ doing a tessellation of $\mathbb{R}^d$ with the basic cell $\text{Cell} = [0, 1)^d$, i.e. the space $\mathbb{R}^d$ is seen as the union of disjoint hypercubes $[0, 1)^d + z$ with $z \in \mathbb{Z}^d$. In order to specify the d-graph $G = (V, E)$ it is enough to give the vertices V_{Cell} inside Cell with the edges

$$E_{\text{Cell}} = \big\{\{x, y\} \in E : \{x, y\} \cap \text{Cell} \neq \varnothing\big\}.$$

Set $d_G = \sup\{d_v : v \in V\}$ *the maximal degree* of the d-graph $G = (V, E)$; by periodicity and the fact that Cell contains finitely many vertices, then d_G is finite.

If a d-graph has at least a vertex with even degree we denote it as *d-$\mathcal{E}$ graph*. As an example of a d-$\mathcal{E}$ graph we take the lattice $\mathbb{L}_d = (\mathbb{Z}^d, \mathbb{E}_d)$, where the degree of any vertex is equal to $2d$.

Some standard definitions on graph theory are now recalled. Given $n \in \mathbb{N}$, a *path of length n* starting in $x \in V$ and ending in $y \in V$ is a sequence of vertices $(x_0 = x, x_1, \dots, x_{n-1}, x_n = y)$ having $\{x_{i-1}, x_i\} \in E$ for each $i = 1, \dots, n$. A set $A \subset V$ is *connected* if for any couple $x, y \in A$ there exists a path $(x_0 = x, x_1, \dots, x_{n-1}, x_n = y)$ with $x_i \in A$, for $i = 0, \dots, n$. We say that a graph $G = (V, E)$ is connected if V is connected.

Since, without loss of generality, one can study separately the different connected components, as will be clear below, in the sequel we will consider only connected d-graphs.

The distance $\nu_G(u,v)$ in G of two vertices $u, v \in V$ is the length of the shortest path (not necessarily unique) starting in u and ending in v. For $u \in V$ and $L \in \mathbb{N}$ we define the ball centered in u with radius L as

$$B_L(u) = \{v \in V : \nu_G(u,v) \le L\}.$$

The *boundary* of a set $A \subset V$ is

$$\partial A = \{v \in A : \exists u \notin A \text{ such that } \{u,v\} \in E\},$$

whereas the *external boundary* is

$$\partial^{\text{ext}} A = \{v \notin A : \exists u \in A \text{ such that } \{u,v\} \in E\}.$$

2.2. *Hamiltonian*

Let us consider the space $\{-1,+1\}^V$ equipped with the product topology. For a given *configuration* $\sigma \in \{-1,+1\}^V$ and for any subset A of V, we write $\sigma^{(A)} = (\sigma_v^{(A)} : v \in V)$ to denote the configuration

$$\sigma_y^{(A)} = \begin{cases} -\sigma_y, & \text{if } y \in A; \\ \sigma_y, & \text{if } y \notin A; \end{cases}$$

that corresponds to *flip* the configuration σ on the set A.

The *formal Hamiltonian* associated to the *interactions* $\mathcal{J} \in \{-1,+1\}^E$ is

$$\mathcal{H}_{\mathcal{J}}(\sigma) = -\sum_{e=\{x,y\} \in E} J_e \sigma_x \sigma_y. \tag{1}$$

However, the definition (1) is not well posed for infinite graphs. Thus, we shall work with the *increment of the Hamiltonian at A*, namely for a finite set $A \subset V$

$$\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) = -2 \cdot \sum_{e=\{x,y\} \in E : x \in \partial A, y \in \partial^{\text{ext}} A} J_e \sigma_x^{(A)} \sigma_y^{(A)} = 2 \cdot \sum_{e=\{x,y\} \in E : x \in \partial A, y \in \partial^{\text{ext}} A} J_e \sigma_x \sigma_y. \tag{2}$$

Notice that the value of $\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)$ can be only an even integer. In particular, if $\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) \neq 0$, then $|\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)| \ge 2$.

2.3. *Dynamics*

The dynamics of the system will be a non-homogeneous Markov process depending on the interactions, the temperature profile and the initial configuration.

The process is denoted by $\sigma(\cdot) = (\sigma_v(t) : v \in V, t \in [0,\infty))$. It takes value in $\{-1,+1\}^V$ and has left continuous trajectories.

For $k \in \mathbb{N}$, we call $\mathcal{A}_k$ the collection of the connected subsets $A \subset V$ having cardinality smaller or equal to k. It is important to stress that, for a given $v \in V$, the set $\{A \in \mathcal{A}_k : v \in A\}$ is finite for a d-graph. Therefore, the set $\mathcal{A}_k$ is countable and, following [18], one defines the infinitesimal generator related to k as follows

$$\mathcal{L}_t^{k,\mathcal{J},T}\big(f(\sigma)\big) = \sum_{A \in \mathcal{A}_k} c_t^{\mathcal{J},T,(A)}(\sigma)\big(f\big(\sigma^{(A)}\big) - f(\sigma)\big), \tag{3}$$

where $f : \{-1,+1\}^V \to \mathbb{R}$ is a continuous function, T is the temperature profile, $\mathcal{J} \in \{-1,+1\}^E$ and $\sigma \in \{-1,+1\}^V$. Moreover, when $T(t) > 0$, the rates are

$$c_t^{\mathcal{J},T,(A)}(\sigma) = \frac{e^{-\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)/T(t)}}{e^{\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)/T(t)} + e^{-\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)/T(t)}} = \frac{1}{1 + e^{2\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)/T(t)}}. \tag{4}$$

In the case in which $T(t)=0$ for $t \geq 0$, set

$$c_t^{\mathcal{J},T,(A)}(\sigma) = \lim_{T\to 0^+} \frac{e^{-\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)/T}}{e^{\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)/T} + e^{-\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)/T}}.$$

Hence, when $T(t)=0$, one has

$$c_t^{\mathcal{J},T,(A)}(\sigma) = \begin{cases} 0 & \text{if } \Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) > 0; \\ \frac{1}{2} & \text{if } \Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) = 0; \\ 1 & \text{if } \Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) < 0. \end{cases}$$

We remark that in the case of $T \equiv 0$ and $k=1$ our dynamics is the Glauber dynamics at zero temperature (see for instance [2,11,14,22,23]).

The previous defined process can be constructed by using a collection of independent Poisson processes $(\mathcal{P}_A : A \in \mathcal{A}_k)$ with rate 1, the so-called Harris' graphical representation [15].

Call $\mathcal{T}_A = (\tau_{A,n} : n \in \mathbb{N})$ the arrivals of the Poisson process $\mathcal{P}_A$. The probability that there is a *flip at the set A* (conditioning on the event $\{t \in \mathcal{T}_A\}$), i.e. $\sigma(t^+) = \lim_{s\downarrow t} \sigma(s) = \sigma^{(A)}(t)$, is equal to $c_t^{\mathcal{J},T,(A)}(\sigma(t))$, where we pose $\sigma^{(A)}(t) = (\sigma(t))^{(A)}$. An useful representation of such events can be given through the family of i.i.d. random variables

$$(U_{A,n} : A \in \mathcal{A}_k, n \in \mathbb{N}) \tag{5}$$

which are uniform in [0, 1] and such that if $U_{A,n} < c_{\tau_{A,n}}^{\mathcal{J},T,(A)}(\sigma(\tau_{A,n}))$, then there is a flip at the set A at time $\tau_{A,n}$ (see [15,18]).

The representation of the Markov process based on the Poisson processes is very popular in the framework of zero-temperature dynamics, since it exhibits remarkable advantages with respect to the one based on the generator. Firstly, the spatial ergodicity of the process with respect to the translations related to the canonical basis can be invoked on the ground of a very general theory (see [15,17,23,26]). Secondarily, this representation is the natural setting for the proofs of the results.

We say that a flip at A at time $t \in \mathcal{T}_A$ is in *favour of the Hamiltonian* if $\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma(t)) < 0$; it is *indifferent for the Hamiltonian* if $\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma(t)) = 0$ and it is *in opposition of the Hamiltonian* if $\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma(t)) > 0$.

For $k \in \mathbb{N}$, $t \in [0,\infty]$ and $A \in \mathcal{A}_k$, we define the sets $\mathcal{S}^-_{A,t}$, $\mathcal{S}^0_{A,t}$ and $\mathcal{S}^+_{A,t}$ as

$$\mathcal{S}^-_{A,t} = \{s \in [0,t] \cap \mathcal{T}_A : \text{there is a flip at } A \text{ at time } s \text{ with } \Delta_A \mathcal{H}_{\mathcal{J}}(\sigma(s)) > 0\}, \tag{6}$$

$$\mathcal{S}^0_{A,t} = \{s \in [0,t] \cap \mathcal{T}_A : \text{there is a flip at } A \text{ at time } s \text{ with } \Delta_A \mathcal{H}_{\mathcal{J}}(\sigma(s)) = 0\}, \tag{7}$$

$$\mathcal{S}^+_{A,t} = \{s \in [0,t] \cap \mathcal{T}_A : \text{there is a flip at } A \text{ at time } s \text{ with } \Delta_A \mathcal{H}_{\mathcal{J}}(\sigma(s)) < 0\}, \tag{8}$$

and, for any $x \in V$,

$$\mathcal{N}^-_x = \bigcup_{A\in\mathcal{A}_k : A\ni x} \mathcal{S}^-_{A,\infty}, \qquad \mathcal{N}^0_x = \bigcup_{A\in\mathcal{A}_k : A\ni x} \mathcal{S}^0_{A,\infty}, \qquad \mathcal{N}^+_x = \bigcup_{A\in\mathcal{A}_k : A\ni x} \mathcal{S}^+_{A,\infty}. \tag{9}$$

Moreover, we define *the set of total flips involving the site x* as

$$\mathcal{N}_x = \mathcal{N}^-_x \cup \mathcal{N}^0_x \cup \mathcal{N}^+_x,$$

and *the set of total arrivals involving the site x* as

$$\mathcal{Q}_x = \bigcup_{A\in\mathcal{A}_k : A\ni x} \{t \in \mathcal{T}_A\}.$$

We now provide the definition of the probability measure associated to the dynamical model defined in (3), which can be written as

$$P_{k,\mu_{\mathcal{J}},\nu_{\sigma(0)}} = \mu_{\mathcal{J}} \times \nu_{\sigma(0)} \times P_k,$$

where $\mu_{\mathcal{J}}$ is the probability distribution of the interactions $\mathcal{J}$ forming the quenched random environment; $\nu_{\sigma(0)}$ is the probability distribution of the initial configuration $\sigma(0)$; P_k is the measure of the arrivals of the i.i.d. Poisson processes on the sets $A \in \mathcal{A}_k$ and the sequences of independent U's as in (5) – independent also from the Poisson processes.

Some particular cases are important in our context. The considered measures over the interactions are of two families

$$\mu_{\mathcal{J}} = \prod_{e\in E} \mathrm{Ber}_e(\alpha), \quad \alpha \in [0,1], \tag{10}$$

where $\mathrm{Ber}_e(\alpha)$ is the Bernoulli distribution with parameter α and support $\{-1,+1\}$, labeled by the edge $e \in E$. The deterministic case is

$$\mu_{\mathcal{J}} = \delta_{\mathcal{J}}, \quad \mathcal{J} \in \{-1,+1\}^E. \tag{11}$$

Measure $\mu_{\mathcal{J}}$ in (10) is the case of $\pm J$ model in [14], while (11) is the case of deterministically selecting the interactions on the edges of the graph.

Analogously, for the initial configuration $\sigma(0)$, we use

$$\nu_{\sigma(0)} = \prod_{v\in V} \mathrm{Ber}_v(\gamma), \quad \gamma \in [0,1]; \tag{12}$$

$$\nu_{\sigma(0)} = \delta_{\sigma}, \quad \sigma \in \{-1,+1\}^V. \tag{13}$$

To avoid a cumbersome notation, we will denote the probability measure $P_{k,\mu_{\mathcal{J}},\nu_{\sigma(0)}}$ simply as $\mathbb{P}$, and the identification of it will be clear from the context. Analogously, the expected value associated to the probability measure $P_{k,\mu_{\mathcal{J}},\nu_{\sigma(0)}}$ will be indicated as $\mathbb{E}$.

Let us consider a d-graph $G = (V, E)$. Then,

- when $\mu_{\mathcal{J}}$ and $\nu_{\sigma(0)}$ are as in (10) and (12), respectively, then the process will be denoted as $(k, \alpha, \gamma; T)$-model on G, for a temperature profile T;
- when $\mu_{\mathcal{J}}$ and $\nu_{\sigma(0)}$ are as in (11) and (13), respectively, then the process will be denoted as $(k, \langle\mathcal{J}\rangle, \langle\sigma\rangle; T)$-model on G, for a temperature profile T.

Remark 1. To give the interactions $\mathcal{J}$ and the initial configuration $\sigma(0)$ (hence, leading to the indices $\langle\mathcal{J}\rangle$ and $\langle\sigma\rangle$, respectively) has an important role in the proof of some results, as we will see below. In fact, in the considered framework, when a property is shown for all $\mathcal{J}$ and $\sigma(0)$, then such a property holds true in the Bernoullian case as well.

The definition of the measure associated to the model allows us to introduce the quantities of interest in assessing its type.

By adopting the notation of [14], a model is said to be of type $\mathcal{I}$, $\mathcal{F}$, and $\mathcal{M}$, if all the sites flip infinitely many times (a.s.), finitely many times (a.s.), or some sites flip infinitely many times and the others do it finitely many times (a.s.), respectively.

In particular, using standard ergodic arguments one can see that for a $(k, \alpha, \gamma; T)$-model on a d-graph G the quantity

$$\rho_{\mathcal{I}} = \rho_{\mathcal{I}}(k, \alpha, \gamma; T) = \lim_{\ell\to\infty} \frac{|\{x \in B_\ell(v) : |\mathcal{N}_x| = \infty\}|}{|B_\ell(v)|}$$

does exist and it is constant almost surely, and it does not depend on the vertex v (for details see [14,23]).

We also define

$$\rho_{\mathcal{F}} = \rho_{\mathcal{F}}(k, \alpha, \gamma; T) = \lim_{\ell\to\infty} \frac{|\{x \in B_\ell(v) : |\mathcal{N}_x| < \infty\}|}{|B_\ell(v)|}.$$

Therefore $\rho_{\mathcal{I}} + \rho_{\mathcal{F}} = 1$, and

- if $\rho_{\mathcal{I}} = 1$ ($\rho_{\mathcal{F}} = 1$, resp.) then the $(k, \alpha, \gamma; T)$-model on the d-graph G is of type $\mathcal{I}$ ($\mathcal{F}$, resp.);
- if $0 < \rho_{\mathcal{I}} < 1$ then the $(k, \alpha, \gamma; T)$-model on the d-graph G is of type $\mathcal{M}$.

3. Conditions for $\rho_{\mathcal{I}} > 0$

The first results are given in a general setting, where the interactions $\mathcal{J}$ and the initial configuration $\sigma(0)$ are provided; moreover, they could be presented for a completely general Glauber dynamics associated to a Gibbs measure and, thus, associated to a Hamiltonian.

First, we give the following uniform integrability condition.

Definition 1. Let us consider a d-graph $G = (V, E)$. The temperature profile T is said to be *fast decreasing to zero* if, for a $(k, \langle\mathcal{J}\rangle, \langle\sigma\rangle; T)$-model on G, one has

$$\lim_{t\to\infty} \sup_{\hat{\sigma}\in\{-1,+1\}^V} \mathbb{E}\big(\big|\mathcal{N}_x^- \cap [t, \infty)\big| \big| \sigma(t) = \hat{\sigma}\big) = 0,$$

for any $x \in V$, $k \in \mathbb{N}$, $\mathcal{J} \in \{-1, +1\}^E$.

Notice that the condition of being fast decreasing to zero is an asymptotic property of the temperature profile, and the complete knowledge of the behavior of T is not required.

An immediate consequence of the previous definition is the following

Lemma 1. *Let us take a $(k, \langle\mathcal{J}\rangle, \langle\sigma\rangle; T)$-model on a d-graph $G = (V, E)$ with T fast decreasing to zero. Then*

$$\mathbb{E}\big(\big|\mathcal{N}_x^-\big|\big) < \infty, \quad \text{for any } x \in V. \tag{14}$$

Furthermore, for any finite set $V_0 \subset V$,

$$\lim_{t\to\infty} \inf_{\hat{\sigma}\in\{-1,+1\}^V} \mathbb{P}\bigg(\bigcap_{x\in V_0} \big\{\mathcal{N}_x^- \cap [t, \infty) = \varnothing\big\} \big| \sigma(t) = \hat{\sigma}\bigg) = 1, \tag{15}$$

and

$$\mathbb{P}\bigg(\bigcap_{x\in V_0} \big\{\mathcal{N}_x^- = \varnothing\big\}\bigg) > 0. \tag{16}$$

Proof. Let us define

$$K_x = \big|\{A \in \mathcal{A}_k : A \ni x\}\big|, \qquad K = \max_{x\in V} K_x. \tag{17}$$

Clearly, for any $x \in V$, K_x is finite and also K is finite because it corresponds to take the maximum only inside Cell.

Since T is fast decreasing to zero, we can take t_0 sufficiently large to have

$$\sup_{\hat{\sigma}\in\{-1,+1\}^V} \mathbb{E}\big(\big|\mathcal{N}_x^- \cap [t_0, \infty)\big| \big| \sigma(t_0) = \hat{\sigma}\big) \le 1.$$

One has

$$\begin{aligned}\mathbb{E}\big(\big|\mathcal{N}_x^-\big|\big) &= \mathbb{E}\big(\big|\mathcal{N}_x^- \cap [0, t_0)\big|\big) + \mathbb{E}\big(\big|\mathcal{N}_x^- \cap [t_0, \infty)\big|\big) \\ &\le \mathbb{E}\big(\big|\mathcal{Q}_x \cap [0, t_0)\big|\big) + \sup_{\hat{\sigma}\in\{-1,+1\}^V} \mathbb{E}\big(\big|\mathcal{N}_x^- \cap [t_0, \infty)\big| \big| \sigma(t_0) = \hat{\sigma}\big) \le K t_0 + 1,\end{aligned}$$

and (14) is proved.

To prove (15) we notice that, by continuity of the measure and being T fast decreasing to zero, for each $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $t \geq t_0$ and $x \in V_0$

$$\inf_{\hat{\sigma}\in\{-1,+1\}^V} \mathbb{P}\big(\mathcal{N}_x^- \cap [t,\infty) = \varnothing | \sigma(t) = \hat{\sigma}\big) > 1 - \varepsilon,$$

and thus

$$\pi_0 = \inf_{\hat{\sigma}\in\{-1,+1\}^V} \mathbb{P}\bigg(\bigcap_{x\in V_0} \big\{\mathcal{N}_x^- \cap [t,\infty) = \varnothing\big\} | \sigma(t) = \hat{\sigma}\bigg) \geq 1 - \varepsilon|V_0|.$$

This leads to (15).

Finally, by the Markov property

$$\mathbb{P}\bigg(\bigcap_{x\in V_0} \big\{\mathcal{N}_x^- = \varnothing\big\}\bigg) \geq \mathbb{P}\bigg(\bigcap_{x\in V_0} \big\{\mathcal{N}_x^- \cap [0,t_0) = \varnothing\big\}\bigg)\pi_0.$$

But

$$\mathbb{P}\bigg(\bigcap_{x\in V_0} \big\{\mathcal{N}_x^- \cap [0,t_0) = \varnothing\big\}\bigg) \geq \mathbb{P}\bigg(\bigcap_{x\in V_0} \big\{\mathcal{Q}_x \cap [0,t_0) = \varnothing\big\}\bigg) \geq e^{-K|V_0|t_0} > 0.$$

For $\varepsilon < \frac{1}{|V_0|}$, one obtains (16). □

Theorem 1. *Let us consider a* $(k, \langle\mathcal{J}\rangle, \langle\sigma\rangle; T)$*-model on a* d*-graph* $G = (V, E)$*.*

- *If*

$$\limsup_{t\to\infty} T(t)\ln t < 4, \tag{18}$$

then the temperature profile T *is fast decreasing to zero.*

- *If, for a given* $x \in V$*,*

$$\liminf_{t\to\infty} T(t)\ln t > 4d_x, \tag{19}$$

then $\mathcal{N}_x$ *is infinite almost surely.*

Proof. We start by proving the first item of the theorem.

By (18) one has

$$r = \frac{1}{4}\cdot\limsup_{t\to\infty} T(t)\ln t < 1. \tag{20}$$

We also consider a constant $a \in (1, \frac{1}{r})$. Given $\mathcal{N}_x^-$ (resp. $\mathcal{N}_x$), for $x \in V$ and $n \in \mathbb{N}$, we define $\mathcal{N}_{x,n}^- = \mathcal{N}_x^- \cap [n-1, n)$ (resp. $\mathcal{N}_{x,n} = \mathcal{N}_x \cap [n-1,n)$).

Thus, there exists $\bar{n} \in \mathbb{N}$ large enough such that, for any $n \geq \bar{n}$, $t \in [n-1,n)$, $\mathcal{J} \in \{-1,+1\}^E$, $\sigma \in \{-1,+1\}^V$ one has

$$\frac{1}{(n-1)^a} \geq \sup\bigg\{c_t^{\mathcal{J},T,(A)}(\sigma) : c_t^{\mathcal{J},T,(A)}(\sigma) < \frac{1}{2}\bigg\}, \quad \text{for } A \ni x. \tag{21}$$

In fact, for any $t \in [n-1,n)$ using (4) and the fact that $\frac{1}{1+e^x}$ decreases in x, one has

$$c_t^{\mathcal{J},T,(A)}(\sigma) < \frac{1}{2} \quad\Rightarrow\quad c_t^{\mathcal{J},T,(A)}(\sigma) < \frac{1}{1+e^{4/T(t)}}. \tag{22}$$

By (20) and for a given $r' \in (r, 1)$, there exists t' such that for any $t > t'$ then $4/T(t) > \ln t/r'$. Now, it is sufficient to take $\bar{n} = \lceil t' \rceil + 1$.

The second inequality in (22) gives that, for any $n \geq \bar{n}$ and for any $t \in [n-1, n)$

$$c_t^{\mathcal{J},T,(A)}(\sigma) < \frac{1}{1 + e^{\ln t/r'}} = \frac{1}{1 + t^{1/r'}} \leq \frac{1}{(n-1)^{1/r'}}.$$

By setting $r' = 1/a$, we obtain (21). We set $p_n = \frac{1}{(n-1)^a}$, for $n \geq \bar{n}$.

Then, for $t \geq \bar{n}$,

$$\begin{aligned}\sup_{\hat{\sigma} \in \{-1,+1\}^V} \mathbb{E}\big(\big|\mathcal{N}_x^- \cap [t, \infty)\big| \big| \sigma(t) = \hat{\sigma}\big) &\leq \sup_{\hat{\sigma} \in \{-1,+1\}^V} \sum_{n=\lfloor t \rfloor}^{\infty} \mathbb{E}\big(\big|\mathcal{N}_{x,n}^-\big|\big) \\ &\leq \sum_{n=\lfloor t \rfloor}^{\infty} \mathbb{E}\left(\sum_{\ell=1}^{|\mathcal{Q}_{x,n}|} Y_{x,n,\ell}\right),\end{aligned} \tag{23}$$

where $\mathcal{Q}_{x,n} = \mathcal{Q}_x \cap [n-1, n)$ and $(Y_{x,n,\ell} : \ell \in \mathbb{N})$ are i.i.d. Bernoulli random variables with parameter p_n independent from $\mathcal{Q}_{x,n}$. Notice that we are implicitly considering a common probability space for all the random variables. The last inequality in (23) comes from the graphical representation.

Using the definition of K given in (17), the proof of the first item of this theorem ends by noticing that the last term in (23) is smaller or equal than

$$\sum_{n=\lfloor t \rfloor}^{\infty} \mathbb{E}\big(|\mathcal{Q}_{x,n}|\big)\mathbb{E}(Y_{x,n,1}) \leq K \sum_{n=\lfloor t \rfloor}^{\infty} \mathbb{E}(Y_{x,n,1}) = K \sum_{n=\lfloor t \rfloor}^{\infty} \frac{1}{(n-1)^a},$$

the last term tends to zero for $t \to \infty$.

We give now the proof of the second item of the theorem.

Condition (19) gives

$$\eta = \frac{1}{4d_x} \cdot \liminf_{t \to \infty} T(t) \ln t > 1. \tag{24}$$

For $x \in V$ and $n \in \mathbb{N}$, we define the event

$$F_{x,n} = \big\{\exists \ell \in \mathbb{N} : \tau_{\{x\},\ell} \in [n-1, n)\big\},$$

i.e. there is at least an arrive for the Poisson $\mathcal{P}_{\{x\}}$ in the interval $[n-1, n)$. We consider the collection of independent and equiprobable events $(F_{x,n} : x \in V, n \in \mathbb{N})$. In particular, $\mathbb{P}(F_{x,1}) = 1 - e^{-1}$.

There exists $\bar{n} \in \mathbb{N}$ large enough such that, for any $n \geq \bar{n}$, it results

$$\frac{1}{n} \leq \inf\big\{c_t^{\mathcal{J},T,(A)}(\sigma) : t \in [n-1, n), \mathcal{J} \in \{-1,+1\}^E, \sigma \in \{-1,+1\}^V\big\} \quad A \ni x.$$

In fact, by (4), one has

$$c_t^{\mathcal{J},T,(A)}(\sigma) \geq \frac{1}{1 + e^{4d_x/T(t)}}.$$

Therefore, by (24), there exists t' such that for $t > t'$ one has $4d_x/T(t) < \ln t$. Analogously to the previous item, define $\bar{n} = \lceil t' \rceil + 1$.

Hence, for any $n \geq \bar{n}$ and for any $t \in [n-1, n)$

$$c_t^{\mathcal{J},T,(A)}(\sigma) > \frac{1}{1 + e^{\ln t}} = \frac{1}{1+t} \geq \frac{1}{1+n}.$$

Then by Lévy's conditional form of the Borel–Cantelli lemma (see [31]), one obtains the proof of the statement.

In more details, let us consider $n \in \mathbb{N}$ and define $\mathcal{G}_n$ as the σ-algebra generated by all the Poisson processes until time n and the related U's. Hence, the process $\sigma(\cdot)$ is seen as a function of the Harris' graphical representation. Let us define the event

$$E_n = F_{x,n} \cap \left\{U_{\{x\},\bar{\ell}(n)} < c^{\mathcal{J},T,(\{x\})}_{\tau_{\{x\},\bar{\ell}(n)}}\big(\sigma(\tau_{\{x\},\bar{\ell}(n)})\big)\right\} \supset F_{x,n} \cap \left\{U_{\{x\},\bar{\ell}(n)} < \frac{1}{1+n}\right\},$$

where $\bar{\ell}(n) = \inf\{\ell : \tau_{\{x\},\ell} \in [n-1, n)\}$.

Therefore, $E_n \in \mathcal{G}_n$ and $\mathbb{P}(E_n|\mathcal{G}_{n-1}) \geq \frac{1-e^{-1}}{1+n}$ and this implies that

$$\sum_{n=1}^{\infty} \mathbf{1}_{E_n} = \infty \quad \text{a.s.} \tag{25}$$

Formula (25) guarantees that $S^-_{\{x\},\infty} \cup S^0_{\{x\},\infty} \cup S^+_{\{x\},\infty}$ is unbounded a.s. Hence, $|\mathcal{N}_x| = \infty$ a.s. □

Remark 2. Consider the cubic d-graph $\mathbb{L}_d = (\mathbb{Z}^d, \mathbb{E}_d)$. In this case $d_x = 2d$, for any $x \in \mathbb{Z}^d$. Moreover, given a connected finite subset $A \subset \mathbb{Z}^d$, then $|\{e = \{x, y\} \in \mathbb{E}_d : x \in \partial A, y \in \partial^{\text{ext}} A\}|$ is even. In this case, condition (18) can be replaced with a less restrictive one as follows

$$\limsup_{t\to\infty} T(t) \ln t < 8. \tag{26}$$

In fact, for each $\mathcal{J} \in \{-1, +1\}^E$ and $\sigma \in \{-1, +1\}^V$, the flips in opposition of the Hamiltonian have $\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) \geq 4$. This statement can be generalized to the case of a site x with even degree d_x.

In particular, $d = 1$ is associated to $d_x = 2$, for each $x \in \mathbb{Z}$, and also $|\{e = \{x, y\} \in \mathbb{E}_1 : x \in \partial A, y \in \partial^{\text{ext}} A\}| = 2$, where $A \subset \mathbb{Z}$ is connected. If there exists $\lim_{t\to\infty} T(t) \ln t \neq 8$, then one among hypotheses (26) and (19) is verified.

As an immediate consequence of Theorem 1 we obtain

Corollary 1. *Let us consider a d-graph $G = (V, E)$ and a temperature profile T such that* $\liminf_{t\to\infty} T(t) \ln t > 4d_G$. *Then the $(k, \langle\mathcal{J}\rangle, \langle\sigma\rangle; T)$-model on G is of type $\mathcal{I}$, for any $k \in \mathbb{N}$, $\mathcal{J} \in \{-1, +1\}^E$ and $\sigma \in \{-1, +1\}^V$.*

To the benefit of the reader, we now adapt Lemma 5 in [14] and related definitions to our context of nonhomogeneous Markov process.

Definition 2. Let us consider a continuous-time Markov process $(Z_s : s \geq 0)$ with state space $\mathcal{Z}$. Moreover, let us take a measurable set $A \subset \mathcal{Z}$. The set A *recurs with probability* $p \in [0, 1]$, if

$$P\big(\{s > 0 : Z_s \in A\} \text{ unbounded}\big) = p.$$

We also say that a measurable set $B \subset \mathcal{Z}^{[0,1]}$ *recurs with probability* $p \in [0, 1]$, if

$$P\Big(\big\{t > 0 : \big(Z_{t+s} : s \in [0, 1]\big) \in B\big\} \text{ unbounded}\Big) = p.$$

Lemma 2. *Let us consider the process $Z = (Z_t : t \geq 0)$ and the events A and B as in Definition 2. If A recurs with probability $p \in (0, 1]$ and*

$$\inf_{z\in A} \inf_{t\geq 0} P\big(\big(Z_{t+s} : s \in [0, 1]\big) \in B | Z_t = z\big) \geq \varsigma > 0, \tag{27}$$

then B recurs with probability $p' \geq p$.

Proof. Let us define $W = \{\{s > 0 : Z_s \in A\}$ unbounded$\}$. If W occurs we can define recursively an infinite sequence of stopping times $(T_j : j \geq 0)$ such that $T_0 = 0$ and

$$T_{j+1} = \inf\{t \geq T_j + 1 : Z_t \in A\}, \quad \text{for } j \geq 0, \tag{28}$$

with the convention that $\inf \varnothing = \infty$.

Let

$$\mathcal{G}_n = \sigma\text{-algebra}(Z_t : t \leq T_{n+1}), \tag{29}$$

for $n \geq 0$.

By the strong Markov property and formula (27), one obtains

$$P\big((Z_{T_j+s} : s \in [0,1]) \in B | \mathcal{G}_{j-1}\big) \geq \varsigma, \tag{30}$$

on W, for $j \in \mathbb{N}$. By (28) and (29) one has $\{(Z_{T_j+s} : s \in [0,1]) \in B\} \in \mathcal{G}_j$.

Formula (30) gives that

$$\sum_{j=1}^{\infty} P\big((Z_{T_j+s} : s \in [0,1]) \in B | \mathcal{G}_{j-1}\big) = \infty \quad \text{on } W.$$

The Lévy's extensions of the Borel–Cantelli lemma implies that

$$\sum_{j=1}^{\infty} \mathbf{1}_{\{(Z_{T_j+s}:s\in[0,1])\in B\}} = \infty \quad P\text{-a.s.}$$

on W. Therefore B recurs at least with probability p. □

Now we introduce a new condition on the d-$\mathcal{E}$ graphs.

Definition 3. Let us consider $k \in \mathbb{N}$. We say that a d-$\mathcal{E}$ graph $G = (V, E)$ is k-*stable* if there exists a vertex $v \in V$ having d_v even such that both the following conditions are satisfied

- if $x \in \partial^{\text{ext}}\{v\}$, then $d_x \geq 3$;
- if $A \in \mathcal{A}_k$ is such that $v \in \partial A \cup \partial^{\text{ext}} A$ and $|A| \geq 2$ then

$$\big|\big\{\{x, y\} \in E : x \in \partial A, y \in \partial^{\text{ext}} A\big\}\big| - d_v > 0.$$

Notice that if a d-$\mathcal{E}$ graph is k-stable, then it is k'-stable, for any $k' < k$, being $\mathcal{A}_{k'} \subset \mathcal{A}_k$. We also notice that for $k = 1$ the second condition of Definition 3 is automatically satisfied.

An example of a k-stable d-$\mathcal{E}$ graph is the cubic lattice $\mathbb{L}_d = (\mathbb{Z}^d, \mathbb{E}_d)$, for each $k \in \mathbb{N}$ and $d \geq 2$.

The case $\mathbb{L}_1 = (\mathbb{Z}, \mathbb{E}_1)$ is an example of a d-$\mathcal{E}$ graph that is not 1-stable, and thus it is not k-stable for each $k \in \mathbb{N}$. In fact, the first condition in Definition 3 is not satisfied.

Lemma 3. *Let $k \in \mathbb{N}$ and $G = (V, E)$ be a k-stable d-$\mathcal{E}$ graph. The vertex $v \in V$ is as in Definition 3. Consider $F_v = \{f_1, \dots, f_{d_v}\}$ as the set of incident edges to v, i.e. $f_i \cap \{v\} = \{v\}$, for $i = 1, \dots, d_v$. The interactions $\mathcal{J}$ are taken as follows*

$$J_e = \begin{cases} +1, & \text{if } e \notin F_v, \\ -1, & \text{if } e = f_i \text{ for } i = 1, \dots, d_v/2, \\ +1, & \text{if } e = f_i \text{ for } i = d_v/2 + 1, \dots, d_v, \end{cases}$$

and the configuration $\sigma = (\sigma_u : u \in V) \in \{-1, +1\}^V$ with $\sigma_u = +1$ for $u \in V \setminus \{v\}$. Then, for any $A \in \mathcal{A}_k$ with $A \neq \{v\}$ one has $\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) \geq 2$.

Proof. If $A \in \mathcal{A}_k$ and $v \notin \partial A \cup \partial^{\text{ext}} A$, then there is nothing to prove because all the interactions and spins involved in the computation of $\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma)$ are equal to $+1$.

If $A = \{x\}$ with $v \in \partial^{\text{ext}} A$ then

$$\Delta_{\{x\}} \mathcal{H}_{\mathcal{J}}(\sigma) = 2 \sum_{y \in V : \{x,y\} \in E} J_{\{x,y\}} \sigma_x \sigma_y = 2 J_{\{x,v\}} \sigma_x \sigma_v + 2 \sum_{y \in V \setminus \{v\} : \{x,y\} \in E} J_{\{x,y\}} \sigma_x \sigma_y$$
$$= 2 J_{\{x,v\}} \sigma_x \sigma_v + 2(d_x - 1) \geq 2(d_x - 2) \geq 2.$$

If $A \in \mathcal{A}_k$ is such that $v \in \partial A \cup \partial^{\text{ext}} A$ and $|A| \geq 2$ then

$$\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) = 2 \sum_{\{x,y\} \in E : x \in \partial A, y \in \partial^{\text{ext}} A} J_{\{x,y\}} \sigma_x \sigma_y.$$

The previous sum is done on $|\{\{x, y\} \in E : x \in \partial A, y \in \partial^{\text{ext}} A\}|$ terms. At most $d_v/2$ of theses terms are equal to -1. Therefore one has

$$\Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) \geq 2\left(\left|\left\{\{x, y\} \in E : x \in \partial A, y \in \partial^{\text{ext}} A\right\}\right| - d_v\right) \geq 2. \qquad \square$$

Theorem 2. *Let us consider a $(k, \alpha, \gamma; T)$-model on a k-stable d-$\mathcal{E}$ graph $G = (V, E)$, with $k \in \mathbb{N}$, $\alpha \in (0, 1)$, and $\gamma \in [0, 1]$. If T is fast decreasing to zero and positive, then $\rho_{\mathcal{I}} > 0$.*

Proof. By contradiction, we suppose that $\rho_{\mathcal{I}} = 0$.

In this proof we will use several stochastic Ising models sharing the same realized random configuration σ but with different interactions. Among them, we denote by *original system* the model on the k-stable d-$\mathcal{E}$ graph $G = (V, E)$ with realized random interactions $\mathcal{J}$. In the graphical representation we denote the Poisson processes associated to the original model by $(\mathcal{P}_A : A \in \mathcal{A}_k)$, and the related uniform random variables by $(U_{A,n} : A \in \mathcal{A}_k, n \in \mathbb{N})$.

By the assumption that the d-$\mathcal{E}$ graph is k-stable, there exists a vertex $v \in V$ with even degree satisfying the properties given in Definition 3. We select such a vertex. Let us consider the sets $B_{4k}(v)$ and $B_{8k}(v)$, where we are using the constant k given in the statement of the theorem.

Define

$$T_{V_0} = \inf\left\{t \geq 0 : \forall t' \geq t, \forall x \in V_0, \sigma_x\left(t'\right) = \sigma_x(t)\right\},$$

for a finite subset $V_0 \subset V$. We notice that T_{V_0} is not a stopping time.

For $M \in \mathbb{R}^+$, let us introduce the event $F_M^{(1)} = \{T_{B_{8k}(v) \setminus B_{4k}(v)} < M\}$. It results that $F_{M_1}^{(1)} \subset F_{M_2}^{(1)}$ when $M_1 < M_2$.

By hypothesis that $\rho_{\mathcal{I}} = 0$,

$$\lim_{M \to \infty} \mathbb{P}\left(F_M^{(1)}\right) = 1. \tag{31}$$

Now, recall the concept of $F_v = \{f_1, \ldots, f_{d_v}\}$ introduced in Lemma 3 and consider all the finite possible interactions $\mathcal{J}^{(1)}, \ldots, \mathcal{J}^{(q)}, \ldots, \mathcal{J}^{(Q)} \in \{-1, +1\}^E$ obtained by taking:

(A) all the interactions between two vertices $x, y \in B_{4k}(v)$ such that $\{x, y\} \in E \setminus \{f_1, \ldots, f_{d_v/2}\}$ are equal to $+1$, while the interactions J_{f_i}, where $f_i \in F_v$ with $i = 1, \ldots, d_v/2$, that are equal to -1;
(B) if $e = \{x, y\} \in E$ with $x \in \partial^{\text{ext}} B_{4k}(v)$ and $y \in \partial B_{4k}(v)$, the interaction J_e is arbitrary;
(C) all the remaining interactions coincide with those of the realized interactions $\mathcal{J}$ of the original system.

The value Q is finite because we are dealing with d-graphs, and it depends on G, k and v.

We will construct a coupling among all the different systems with interactions $\mathcal{J}^{(1)}, \ldots, \mathcal{J}^{(q)}, \ldots, \mathcal{J}^{(Q)}$.

All the quantities related to the qth system are denoted, in a natural way, by adding, when needed, to the original quantities the superscript (q).

Recall that all the systems described above share the initial configuration σ of the original one. Moreover, the collection of Poisson processes $(\mathcal{P}_A : A \in \mathcal{A}_k)$ are assumed to be common for all the Q systems until time M and, accordingly, the systems share also the same variables U's defined in (5).

Given the qth system, we define the set

$$\mathcal{D}^{(q)} = \{x \in V : \sigma_x(M) \neq \sigma_x^{(q)}(M)\}.$$

It is known that $|\mathcal{D}^{(q)}| < \infty$ a.s. (see e.g. [25]).

We introduce the random set

$$\mathcal{D} = \left[\bigcup_{q=1}^{Q} \mathcal{D}^{(q)}\right] \cup B_{4k}(v),$$

so that $|\mathcal{D}| < \infty$ a.s.

For any $M \in \mathbb{R}^+$ and $M' > M$, we define

$$F^{(2)}_{M,M'} = \bigcap_{A \in \mathcal{A}_k : (A \cup \partial^{\text{ext}} A) \cap \mathcal{D} \neq \varnothing} \{\mathcal{T}_A \cap [M, M'] = \varnothing\},$$

i.e. there are no arrivals in the time interval $[M, M']$ for the Poisson processes $\mathcal{P}_A$ labeled by $A \in \mathcal{A}_k$ with $(A \cup \partial^{\text{ext}} A) \cap \mathcal{D} \neq \varnothing$. In this case one has $F^{(2)}_{M,M'_1} \supset F^{(2)}_{M,M'_2}$ when $M < M'_1 < M'_2$. Furthermore,

$$\lim_{M' \to M^+} \mathbb{P}(F^{(2)}_{M,M'}) = 1. \tag{32}$$

Notice that the probability in (32) depends only on the difference $(M' - M)$.

Among all the different systems – described in (A), (B) and (C) – we select the only one having interactions such that $J_e = \sigma_x(M)$ when $e = \{x, y\} \in E$ with $x \in \partial^{\text{ext}} B_{4k}(v)$ and $y \in \partial B_{4k}(v)$. We denote such random interactions as $\bar{\mathcal{J}}$. We call the corresponding system as *capital system*, this process is denoted by $(\Sigma(t) : t \geq 0)$. We assume that it is a stochastic Ising model, and we will check it below.

We denote by $\bar{\mathcal{S}}^-_{A,t}$ (resp. $\bar{\mathcal{N}}^-_x$) the random set in (6) (resp. in (9)), by replacing σ with Σ.

For $M \in \mathbb{R}^+$, we define

$$F^{(3)}_M = \bigcap_{A \in \mathcal{A}_k : A \subset B_{4k}(v)} \{\bar{\mathcal{S}}^-_{A,\infty} \cap [M, \infty) = \varnothing\}.$$

One has that $F^{(3)}_{M_1} \subset F^{(3)}_{M_2}$ when $M_1 < M_2$.

Observe that

$$\bigcap_{x \in B_{4k}(v)} \{\bar{\mathcal{N}}^-_x \cap [M, \infty) = \varnothing\} \subset F^{(3)}_M.$$

Since $(\Sigma(t) : t \geq 0)$ will be proved to be a stochastic Ising model, we can use (15) in Lemma 1 in order to obtain

$$\lim_{M \to \infty} \mathbb{P}(F^{(3)}_M) = 1. \tag{33}$$

By (31) and (33) one can select M large enough to have that $\mathbb{P}(F^{(i)}_M) > 3/4$, for $i = 1, 3$. Now, by (32), one can take M' close enough to M so that $\mathbb{P}(F^{(2)}_{M,M'}) > 3/4$. Therefore, under such choices of M and M', one has

$$\mathbb{P}(F^{(1)}_M \cap F^{(2)}_{M,M'} \cap F^{(3)}_{M'}) \geq \mathbb{P}(F^{(1)}_M \cap F^{(2)}_{M,M'} \cap F^{(3)}_M) > \frac{1}{4}.$$

We define $(\bar{\mathcal{P}}_A : A \in \mathcal{A}_k)$ a set of Poisson processes with rate 1 that are active in the time interval $[M, M']$ and independent from all the other Poisson processes and also from the U's of the original system.

We call $\bar{\mathcal{T}}_A = (\bar{\tau}_{A,n} \in [M, M'] : n \in \mathbb{N})$ the set of arrivals of the Poisson process $\bar{\mathcal{P}}_A$. In order to use Harris' graphical representation, we define a set of uniform random variables $(\bar{U}_{A,n} : n \in \mathbb{N})$ associated to $\bar{\mathcal{T}}_A$, for any $A \in \mathcal{A}_k$. The $\bar{U}$'s are independent from all the Poisson processes and from the U's of the original system.

Given $A \in \mathcal{A}_k$, we consider a new process $\mathcal{P}^\star_A$ whose set of arrivals $\mathcal{T}^\star_A$ is defined as follows:

$$\mathcal{T}^\star_A = \begin{cases} \{\mathcal{T}_A \cap [0, M]\} \cup \bar{\mathcal{T}}_A \cup \{\mathcal{T}_A \cap [M', \infty)\}, & \text{if } A = \{x\} \text{ and } x \in \mathcal{D}; \\ \mathcal{T}_A, & \text{otherwise.} \end{cases} \tag{34}$$

For each $A \in \mathcal{A}_k$, the set $\mathcal{T}^\star_A$ is distributed as a Poisson process of rate 1, and $(\mathcal{P}^\star_A : A \in \mathcal{A}_k)$ is a set of independent Poisson processes with rate 1, which are also independent from the U's and from the $\bar{U}$'s.

The U's (resp. $\bar{U}$'s) are used when the arrivals of the Poisson processes are taken from $\mathcal{T}_A$ (resp. from $\bar{\mathcal{T}}_A$).

All these random quantities are employed to construct the capital system $(\Sigma(t) : t \geq 0)$ via the graphical representation. Since $(\mathcal{P}^\star_A : A \in \mathcal{A}_k)$ is a collection of independent Poisson processes and by definition of the U's and the $\bar{U}$'s, the capital system is a stochastic Ising model.

We set the constant

$$\eta_{M,M'} = \inf_{t \in [M,M']} \inf_{x \in V} \inf_{\sigma \in \{-1,+1\}^V} \inf_{\mathcal{J} \in \{-1,+1\}^E} \bigl\{ c_t^{\mathcal{J},T,(\{x\})}(\sigma), 1 - c_t^{\mathcal{J},T,(\{x\})}(\sigma) \bigr\}.$$

We notice that $\eta_{M,M'} > 0$. In fact, the temperature profile T is positive in $[M, M']$ and t ranges in a compact interval. Moreover, x could be taken only in Cell for the periodicity of the d-graph, and this implies also that $\mathcal{J}$ and σ could be considered only in a finite region.

We define the events

$$F^{(4)}_{M,M'} = \bigcap_{x \in \mathcal{D} \setminus B_{4k}(v)} \bigl[\bigl\{ \bar{\mathcal{T}}_{\{x\}} = \{\bar{\tau}_{\{x\},1}\} \bigr\} \cap \{ \bar{U}_{\{x\},1} \mathbf{1}_{\{\Sigma_x(M) \neq \sigma_x(M)\}} < \eta_{M,M'} \} \cap \bigl\{ \bar{U}_{\{x\},1} \geq \mathbf{1}_{\{\Sigma_x(M) = \sigma_x(M)\}} (1 - \eta_{M,M'}) \bigr\} \bigr]$$

and

$$F^{(5)}_{M,M'} = \bigcap_{x \in B_{4k}(v)} \bigl[\bigl\{ \bar{\mathcal{T}}_{\{x\}} = \{\bar{\tau}_{\{x\},1}\} \bigr\} \cap \{ \bar{U}_{\{x\},1} \mathbf{1}_{\{\Sigma_x(M) = -1\}} < \eta_{M,M'} \} \cap \bigl\{ \bar{U}_{\{x\},1} \geq \mathbf{1}_{\{\Sigma_x(M) = +1\}} (1 - \eta_{M,M'}) \bigr\} \bigr].$$

By the independence properties of the considered random quantities, it results

$$\mathbb{P}\bigl(F^{(4)}_{M,M'} \cap F^{(5)}_{M,M'} | F^{(1)}_M \cap F^{(2)}_{M,M'} \cap F^{(3)}_{M'} \bigr) \geq \sum_{\ell=0}^{\infty} \bigl[\bigl(M' - M \bigr) e^{-(M'-M)} \eta_{M,M'} \bigr]^\ell \cdot \mathbb{P}\bigl(|\mathcal{D}| = \ell | F^{(1)}_M \cap F^{(2)}_{M,M'} \cap F^{(3)}_{M'} \bigr) > 0,$$

and this gives $\mathbb{P}(F^{(1)}_M \cap F^{(2)}_{M,M'} \cap F^{(3)}_{M'} \cap F^{(4)}_{M,M'} \cap F^{(5)}_{M,M'}) > 0$.

Assume from now on that $F^{(1)}_M \cap F^{(2)}_{M,M'} \cap F^{(3)}_{M'} \cap F^{(4)}_{M,M'} \cap F^{(5)}_{M,M'}$ occurs.

We claim that $\Sigma_u(M') = +1$, for each $u \in B_{4k}(v)$. In particular, on $F^{(2)}_{M,M'} \cap F^{(5)}_{M,M'}$, for $u \in B_{4k}(v)$ one has the occurrence of only one arrival in $[M, M']$ of the Poisson processes involving u, and such an arrival is $\bar{\tau}_{\{u\},1}$. On $F^{(5)}_{M,M'}$ this arrival generates a flip at u for the capital system if and only if $\Sigma_u(M) = -1$.

We also claim that $\Sigma_u(M') = \sigma_u(M')$ for each $u \notin B_{4k}(v)$. We prove it by separating two subcases: (i) $u \in \mathcal{D} \setminus B_{4k}(v)$ and (ii) $u \notin \mathcal{D}$.

(i) On $F^{(2)}_{M,M'} \cap F^{(4)}_{M,M'}$ for $u \in \mathcal{D} \setminus B_{4k}(v)$ there is only one arrival in $[M, M']$ of the Poisson processes involving u, and the arrival is $\bar{\tau}_{\{u\},1}$. On $F^{(4)}_{M,M'}$ this arrival generates a flip at u for the capital system if and only if $\Sigma_u(M) \neq \sigma_u(M)$.

(ii) On $F^{(2)}_{M,M'}$ one knows that the arrivals in the interval $[M, M']$ involving the sites outside $\mathcal{D}$ are the same for the original and the capital system. Moreover, the original and the capital system share the same configuration outside $\mathcal{D}$ and the same U's. Therefore, such systems have identical flips outside $\mathcal{D}$. Since $\Sigma_u(M) = \sigma_u(M)$ for $u \notin \mathcal{D}$ then $\Sigma_u(t) = \sigma_u(t)$ for $u \notin \mathcal{D}$ and $t \in [M, M']$.

We now need to check the following:

(H) for any $t \geq M'$ we have $\Sigma_u(t) = \sigma_u(t)$ when $u \notin B_{4k}(v)$ and $\Sigma_u(t) = +1$ when $u \in B_{4k}(v) \setminus \{v\}$.

We can define recursively the sequence of random times $(\psi_\ell : \ell \in \mathbb{N})$. Let

$$\psi_\ell = \inf\bigl\{\tau_{A,n} > \psi_{\ell-1} : A \in \mathcal{A}_k \text{ with } \bigl(A \cup \partial^{\text{ext}} A\bigr) \cap B_{4k}(v) \neq \varnothing, n \in \mathbb{N}\bigr\},$$

for $\ell \geq 1$ and the conventional agreement that $\psi_0 = M'$.

For each $\ell \geq 1$ one has that the random set

$$\Upsilon_\ell = \bigl\{\tau_{A,n} \in [\psi_{\ell-1}, \psi_\ell] : A \in \mathcal{A}_k \text{ with } \bigl(A \cup \partial^{\text{ext}} A\bigr) \cap B_{4k}(v) = \varnothing, n \in \mathbb{N}\bigr\}$$

has infinite cardinality a.s.

In the interval $[M', \psi_1)$ we are dealing only with arrivals having $A \in \mathcal{A}_k$ such that $(A \cup \partial^{\text{ext}} A) \cap B_{4k}(v) = \varnothing$, hence belonging to Υ_1. Therefore, by construction and from the fact that $\Sigma_u(M') = \sigma_u(M')$ for each $u \notin B_{4k}(v)$, the original and the capital system share the same flips in Υ_1. This leads to $\Sigma_u(t) = \sigma_u(t)$ when $u \notin B_{4k}(v)$ and $\Sigma_u(t) = +1$ when $u \in B_{4k}(v) \setminus \{v\}$, for $t \in [M', \psi_1]$.

Now we analyze the arrival in ψ_1. There exist unique (a.s.) $A \in \mathcal{A}_k$ and $n \in \mathbb{N}$ such that the arrival $\tau_{A,n} = \psi_1$. Four cases can be considered for the set A. They are presented and discussed below.

- $A = \{v\}$.
  In this case a flip at A influences only $\Sigma_v(\psi_1^+)$. Therefore $\Sigma_u(\psi_1^+) = \sigma_u(\psi_1^+)$ when $u \notin B_{4k}(v)$ and $\Sigma_u(\psi_1^+) = +1$ when $u \in B_{4k}(v) \setminus \{v\}$.
- $(A \cup \partial^{\text{ext}} A) \subset B_{4k}(v)$ and $A \neq \{v\}$.
  By Lemma 3 we know that $\Delta_A \mathcal{H}_{\bar{\mathcal{J}}}(\Sigma(\psi_1)) \geq 2$. However, the occurrence of $F^{(3)}_{M'}$ guarantees that this arrival does not correspond to a flip.
- $A \subset B_{4k}(v)$ and $\partial^{\text{ext}} A \not\subset B_{4k}(v)$.
  By the selected interactions $\bar{\mathcal{J}}$ for the capital system (in particular: if $e = \{x, y\} \in E$ with $x \in \partial^{\text{ext}} B_{4k}(v)$ and $y \in \partial B_{4k}(v)$, then $J_e = \sigma_x(M) = \sigma_x(M') = \Sigma_x(M')$) we are in the position of applying Lemma 3. Analogously to the previous case, $F^{(3)}_{M'}$ assures that there is not a flip at A in ψ_1.
- $A \not\subset B_{4k}(v)$ and $(A \cup \partial^{\text{ext}} A) \cap B_{4k}(v) \neq \varnothing$.
  By the selected interactions $\bar{\mathcal{J}}$ one has

  $$\Delta_A \mathcal{H}_{\bar{\mathcal{J}}}\bigl(\Sigma(\psi_1)\bigr) \geq \Delta_A \mathcal{H}_{\mathcal{J}}\bigl(\sigma(\psi_1)\bigr).$$

  Formula (4) says that

  $$c^{\bar{\mathcal{J}},T,(A)}_{\psi_1}\bigl(\Sigma(\psi_1)\bigr) \leq c^{\mathcal{J},T,(A)}_{\psi_1}\bigl(\sigma(\psi_1)\bigr). \tag{35}$$

  The occurrence of $F^{(1)}_M$ assures that there are no flips at A after time M for the original system. Hence, since the original and the capital system share the same U's, (35) states that there is not a flip at A in ψ_1 for the capital system.

By iterating the arguments above, we obtain (**H**). In particular, the event

$$\bar{A} = \bigl\{\text{all the spins in } B_{4k}(v) \setminus \{v\} \text{ are equal to } +1\bigr\}$$

recurs with probability $p \geq \mathbb{P}(F^{(1)}_M \cap F^{(2)}_{M,M'} \cap F^{(3)}_{M'} \cap F^{(4)}_{M,M'} \cap F^{(5)}_{M,M'}) > 0$.

When $\Sigma(t) \in \bar{A}$ there is a positive probability that there is an arrival for the Poisson $\mathcal{P}_{\{v\}}$ in the interval $[t, t+1]$. Such an arrival has probability $1/2$ to generate a flip at $\{v\}$, being $\Delta_{\{v\}}\mathcal{H}_{\bar{\mathcal{J}}}(\Sigma(t)) = 0$ for each $t \geq M'$. Thus, by Lemma 2, the event

$$\{\Sigma_v(\cdot) \text{ has a flip in } [0,1]\}$$

recurs with positive probability greater than or equal to p.

Re-sampling the interactions of the original system only over the finite set

$$\{e = \{u, w\} \in E : \{u, w\} \cap B_{4k}(v) \neq \varnothing\}$$

there is a positive probability that it coincides with the interactions of the capital system, since $\alpha \in (0,1)$. Therefore, also in the original one, the probability that $|\mathcal{N}_v| = \infty$ is positive. Hence, by ergodicity, $\rho_{\mathcal{I}}$ must be larger than zero. This concludes the proof. □

Remark 3. We notice that, the previous theorem can be generalized by considering the parameter $\alpha \in (0,1)$ but giving the initial configuration $\sigma \in \{-1,+1\}^V$. In this case we can only prove, following the same line of the previous proof, that there exists a positive frequency of vertices that flip infinitely often. Hence, in general, we only know that

$$\liminf_{\ell\to\infty} \frac{|\{x \in B_\ell(O) : x \text{ flips finitely many times}\}|}{|B_\ell(O)|} > 0,$$

but the limsup could not coincide with the liminf and they could depend on the initial configuration σ. We can conclude that for any initial configuration σ the model is of type $\mathcal{M}$ or $\mathcal{I}$. This means that there is not convergence of the process almost surely.

We now present a theorem in which we explore the possibility that $\rho_{\mathcal{I}}$ is larger than zero without requiring the positivity of the temperature profile. The basic assumption is that some specific sites fixate. We address the reader to the next section for sufficient conditions to obtain $\rho_{\mathcal{F}} > 0$.

Theorem 3. *Let us consider a $(k, \alpha, \gamma; T)$-model on a d-graph $G = (V, E)$ with $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\gamma \in [0,1]$ and T fast decreasing to zero. Suppose that there exist finite sets $R, R' \subset V$, a finite set $S \subset E$, $\hat{\sigma}_R = (\hat{\sigma}_u : u \in R) \in \{-1,+1\}^R$, and $\hat{\mathcal{J}}_S = (\hat{J}_e : e \in S) \in \{-1,+1\}^S$ with the following properties*

(i)

$$\mathbb{P}\left(\bigcap_{u\in R}\left\{\lim_{t\to\infty} \sigma_u(t) = \hat{\sigma}_u\right\} \Big| \bigcap_{e\in S}\{J_e = \hat{J}_e\}\right) > 0. \tag{36}$$

(ii) *For any $\mathcal{J}$ coinciding with $\hat{\mathcal{J}}_S$ on S and for any $\sigma \in \{-1,+1\}^V$ coinciding with $\hat{\sigma}_R$ on R there exists $D \in \mathcal{A}_k$ such that $D \subset R'$, with $\Delta_D\mathcal{H}_{\mathcal{J}}(\sigma) \leq 0$.*

Then, the model is of type $\mathcal{M}$.

Proof. First of all, we notice that there exists $\mathcal{J} \in \{-1,+1\}^E$ coinciding with $\hat{\mathcal{J}}_S$ on S with positive probability, since S is finite and $\alpha \in (0,1)$.

By (36) and by ergodicity one has that $\rho_{\mathcal{F}} > 0$. So, it is sufficient to prove that $\rho_{\mathcal{I}} > 0$.

In the following we use the probability measure $P(\cdot) = \mathbb{P}(\cdot | \bigcap_{e\in S}\{J_e = \hat{J}_e\})$.

We define the set

$$\hat{A} = \{\sigma \in \{-1,+1\}^V : \sigma_u = \hat{\sigma}_u, \forall u \in R\}.$$

By definition of $\hat{\sigma}_R$, we know that $\hat{A}$ recurs with positive probability. The set D in the statement of the theorem depends on σ. We write

$$A_D = \{\sigma \in \{-1,+1\}^V : \sigma_u = \hat{\sigma}_u, \forall u \in R, \Delta_D\mathcal{H}_{\mathcal{J}}(\sigma) \leq 0\},$$

for $D \in \mathcal{A}_k$ such that $D \subset R'$. Thus, $\hat{A} = \bigcup_{D \in \mathcal{A}_k : D \subset R'} A_D$ is a finite union. Therefore, one can select a $\bar{D} \in \mathcal{A}_k$ with $\bar{D} \subset R'$ such that $A_{\bar{D}}$ recurs with positive probability.

Now, the set B, in the frame of Lemma 2, is taken as

$$B = \{\mathcal{S}^{+}_{\bar{D},1} \cup \mathcal{S}^{0}_{\bar{D},1} \neq \varnothing\}.$$

Then

$$\inf_{\sigma \in A_{\bar{D}}} \inf_{t \geq 0} P\big(\big(\sigma(t+s) : s \in [0,1]\big) \in B \,|\, \sigma(t) = \sigma\big)$$

$$\geq \frac{1}{2} P\big(\{\mathcal{T}_{\bar{D}} \cap [t, t+1] \neq \varnothing\} \cap \{\mathcal{T}_A \cap [t, t+1] = \varnothing : \forall A \in \mathcal{A}_k \text{ with } A \neq \bar{D}, A \cap \big(R' \cup \partial^{\text{ext}} R'\big) \neq \varnothing\}\big), \quad (37)$$

where $1/2$, in the previous formula, is due to $\Delta_{\bar{D}} \mathcal{H}_{\mathcal{J}}(\sigma) \leq 0$.

By the independence of the Poisson processes, (37) is larger than or equal to

$$\frac{1}{2}\big(1 - e^{-1}\big) e^{-K|R' \cup \partial^{\text{ext}} R'|} > 0,$$

where K is given in (17).

Lemma 2 guarantees that B recurs with positive probability. Thus, the ergodic theorem assures that $\rho_{\mathcal{I}} > 0$. □

Example 1. In Figure 1 we provide an example. It is here represented a region of the d-graph G, with $d = 2$. For the notation, see Theorem 3. Consider $k = 1$, $\alpha \in (0, 1)$, $\gamma \in (0, 1]$ (resp. $\gamma \in [0, 1)$) and T fast decreasing to zero. Set the black bullets as sites with spin $+1$ (resp. -1) and the other spins can be arbitrarily selected. The set R is formed by the black bullets. The set S is given by all the edges connecting a black bullet with another black bullet or with the white one. The continuous line segment represents an edge e with $J_e = +1$, while the dotted line segment is associated to $J_e = -1$. With positive probability the spins over R do not flip, and remain constant at $+1$ (resp. -1) if initially taken positive (resp. negative) (see (16) in Lemma 1). Assume that this is the case. R' is formed by the cental white bullet. So $D = R'$ and $\Delta_D \mathcal{H}_{\mathcal{J}}(\sigma) = 0$. Hence, $\rho_{\mathcal{I}} > 0$. Notice that the only interactions which really matter are those in S, while the other ones can be arbitrarily selected.

We finish the section with a particular case in which we show that the parameter k plays a role to establish the class of a $(k, \alpha, \gamma; T)$-model. It is known that the $(1, \alpha, \gamma; T)$-model on the hexagonal lattice is of type $\mathcal{F}$ when $T \equiv 0$ (see [23]) and in the next section this result is proven when T is fast decreasing to zero (see Theorem 5). It is important to notice that the hexagonal lattice in our framework is a 2-graph.

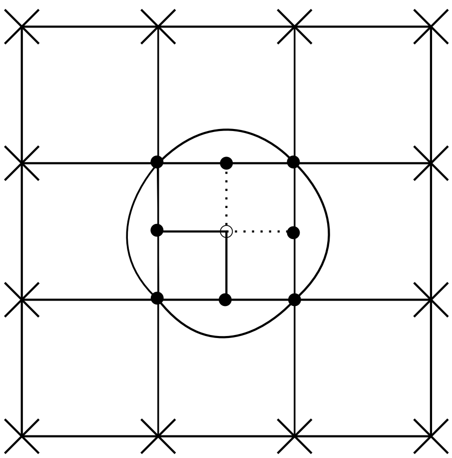

Fig. 1. Example of 2-graph in which Theorem 3 can be applied. Such a graph can be also used as an example of Theorem 6.

In the following result we show that a $(k, 1, 1/2; T)$-model with $k \geq 2$ cannot belong to $\mathcal{F}$. The question if it is of kind $\mathcal{M}$ or $\mathcal{I}$ remains open.

Theorem 4. *The $(k, 1, 1/2; T)$-model on the hexagonal 2-graph $G_H = (V_H, E_H)$ with $k \geq 2$ is not of type $\mathcal{F}$, for any temperature profile T.*

Proof. By contradiction suppose that the model is of type $\mathcal{F}$. Therefore there exists, for any $x \in V_H$,

$$\sigma_x(\infty) = \lim_{t \to \infty} \sigma_x(t) \quad \text{a.s.}$$

We denote with $\sigma(\infty) = (\sigma_x(\infty) : x \in V_H)$ the random limit configuration. By ergodicity and since $\gamma = 1/2$ we obtain that

$$\lim_{\ell \to \infty} \frac{|\{x \in B_\ell(v) : \sigma_x(\infty) = +1\}|}{|B_\ell(v)|} = \frac{1}{2} \quad \text{a.s.} \tag{38}$$

for any $v \in V_H$.

As in [23] we define the domain wall $\mathcal{D}^H$ as a subset of the hexagonal dual lattice, which separates the positive spins from the negative ones by joining the center of the hexagons (see Figure 2). By (38) $\mathcal{D}^H \neq \varnothing$ almost surely. The dual lattice is formed by three classes of edges: the vertical edges, the ascendent ones and the descendent ones (see Figure 2). As in (38), there exists the density ρ_{ver} (ρ_{asc}, ρ_{des}, resp.) of the vertical (ascendent, descendent, resp.) edges of the domain wall. By symmetry and being $\mathcal{D}^H \neq \varnothing$, we have that $\rho_{\text{ver}} = \rho_{\text{asc}} = \rho_{\text{des}} > 0$ almost surely.

On each site $v \in V_H$ with $\sigma_v = -1$ we consider the triangle $\Phi_{v,H}$ connecting the centers of the three hexagons containing v. Such triangle is a closed convex set, in that it is given by its internal part and its boundary. We introduce the set containing all such triangles

$$\Phi_H = \bigcup_{v \in V_H : \sigma_v = -1} \Phi_{v,H}.$$

The set $\mathcal{D}^H$ is the boundary of the set Φ_H. Simplicial homology arguments (see [27]) gives that the domain wall $\mathcal{D}^H$ can be decomposed in connected components, each of them belonging to one of the following families:

- *Closed component*: the component is given by a sequence of edges forming a cycle.
- *Bidirectional infinite component*: the component is not closed and each edge belonging to the component has two adjacent edges; moreover, such adjacent edges are disjoint.

Two adjacent edges of the set $\mathcal{D}^H$ may form an angle of π, $2\pi/3$ and $\pi/3$.

If there exists a closed component, then there exists at least one couple of adjacent edges forming an angle different from π. In this case, by ergodicity, the sum of the densities of angles $2\pi/3$ and $\pi/3$ is different from zero.

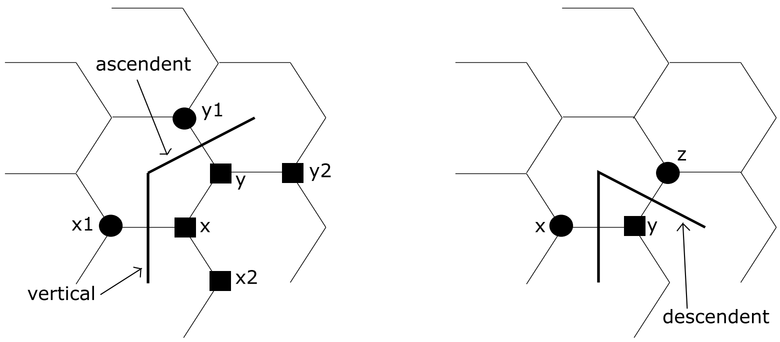

Fig. 2. The hexagonal lattice with the domain wall $\mathcal{D}^H$ in bold. The squares and the bullets represents spins with opposite signs in the two situations of $2\pi/3$ (left panel) and $\pi/3$ (right panel) angles.

If all the components are of type bidirectional infinite, then we have two subcases. One of the components has an angle different from π or, otherwise, the components are straight lines (for example, with all vertical edges). In the case of components with an angle different from π, ergodicity leads that, as in the case of existence of a closed component, the sum of the densities of angles $2\pi/3$ and $\pi/3$ is different from zero. The straight line case has zero probability to occur. In fact, ergodicity and symmetry guarantee the existence of another component of $\mathcal{D}^H$ which is a straight line as well and it is not parallel to the vertical one. The intersection of the two components forms an angle different from π, and we fall in the previous case. However, we notice that the coexistence of two nonparallel straight lines is absurd, in that one of the spins at the four corners formed by the intersection of the components should have simultaneously value $+1$ and -1.

If the density of the angles $\pi/3$ is different from zero (see Figure 2, right panel), then we have two edges $\{x, y\}, \{y, z\} \in E_H$ such that the random time

$$T_{\{x,y,z\}} = \inf\{t \in \mathbb{R}_+ : \forall t' \geq t, \sigma_x(t') = \sigma_z(t') = -\sigma_y(t')\}$$

is finite a.s. However, there are infinite arrivals of the Poisson $\mathcal{P}_{\{y\}}$ in the random interval $(T_{\{x,y,z\}}, \infty)$ and at each arrival the site y flips with a probability at least $1/2$ (the value of such a probability depends on the temperature at the arrival time, and it is one in the case of zero-temperature). By Lemma 2, we have a contradiction and the $(k, 1, 1/2; T)$-model is of type $\mathcal{M}$ or $\mathcal{I}$.

If the density of the angles $\pi/3$ is zero, then the density of the angles of $2\pi/3$ is different from zero (see Figure 2, left panel). In this case there exists an edge $\{x, y\} \in E_H$ and its four adjacent edges $\{x_1, x\}, \{x_2, x\}, \{y_1, y\}, \{y_2, y\} \in E_H$ such that the random time

$$\begin{aligned} &T_{\{x,y,x_1,x_2,y_1,y_2\}} \\ &\quad = \inf\{t \in \mathbb{R}_+ : \forall t' \geq t, \sigma_{x_1}(t') = \sigma_{y_1}(t') = -\sigma_{x_2}(t') = -\sigma_{y_2}(t') \text{ and } \sigma_x(t') = \sigma_y(t')\} \end{aligned}$$

is finite a.s. Analogously to the previous case, there are infinite arrivals of the Poisson $\mathcal{P}_{\{x,y\}}$ in $(T_{\{x,y,x_1,x_2,y_1,y_2\}}, \infty)$ and at each arrival the set $\{x, y\}$ flips with a probability equals to $1/2$. Lemma 2 leads to a contradiction, therefore the $(k, 1, 1/2; T)$-model cannot be of type $\mathcal{F}$. □

4. Conditions for $\rho_{\mathcal{F}} > 0$

In this section we give some sufficient conditions to obtain $\rho_{\mathcal{F}} > 0$. Let us consider a d-graph $G = (V, E)$. For a finite set $\Lambda \subset V$, we define

$$E(\Lambda) = \{\{u, v\} \in E : u, v \in \Lambda\}.$$

For $\mathcal{J} \in \{-1, +1\}^E$, $\sigma \in \{-1, +1\}^V$, we also set

$$\mathcal{H}_{\Lambda,\mathcal{J}}(\sigma) = -\sum_{e=\{x,y\} \in E(\Lambda)} J_e \sigma_x \sigma_y \tag{39}$$

and

$$D_{\Lambda,\mathcal{J}}(\sigma) = \frac{\mathcal{H}_{\Lambda,\mathcal{J}}(\sigma)}{|\Lambda|}. \tag{40}$$

We are now ready to present the following

Lemma 4. *Let us consider a d-graph $G = (V, E)$. For any $\mathcal{J} \in \{-1, +1\}^E$, $\sigma \in \{-1, +1\}^V$ and any finite set $\Lambda \subset V$ one has*

$$D_{\Lambda,\mathcal{J}}(\sigma) \in [-d_G/2, d_G/2],$$

where d_G is the maximal degree of the graph G.

Proof. We rewrite the expression in (39) as

$$\mathcal{H}_{\Lambda,\mathcal{J}}(\sigma) = -\frac{1}{2}\sum_{x\in\Lambda}\biggl[\sum_{\{x,y\}\in E(\Lambda)} J_{\{x,y\}}\sigma_x\sigma_y\biggr].$$

The term inside the square brackets belongs to $[-d_G, d_G]$. In fact, for any $x \in \Lambda$, by definition we have $d_x \le d_G$ and $J_{\{x,y\}}, \sigma_x, \sigma_y \in \{-1, +1\}$. Thus

$$\mathcal{H}_{\Lambda,\mathcal{J}}(\sigma) \in \biggl[-\frac{d_G|\Lambda|}{2}, \frac{d_G|\Lambda|}{2}\biggr].$$

By (40), we have the thesis. □

We give two more definitions that will be used in the proof of the next lemma. They are based on the embeddedness of the d-graph and on the concept of cell. For a d-graph $G = (V, E)$ and $\ell \in \mathbb{N}$ we define

$$\mathcal{Z}_\ell = \bigl\{A \in \mathcal{A}_k : A \cap [-\ell, \ell)^d \neq \varnothing \text{ and } \bigl(A \cup \partial^{\text{ext}} A\bigr) \not\subset [-\ell, \ell)^d\bigr\}$$

and

$$\mathcal{Z}_\ell^+ = \bigl\{A \in \mathcal{A}_k : \bigl(A \cup \partial^{\text{ext}} A\bigr) \subset [-\ell, \ell)^d\bigr\}.$$

Thus $\mathcal{Z}_\ell \cap \mathcal{Z}_\ell^+ = \varnothing$. By definition, there exists a constant $M_{k,G} > 0$, depending on k and the graph G, such that $|\mathcal{Z}_\ell| \le M_{k,G}\ell^{d-1}$ for any $\ell \in \mathbb{N}$. Analogously, there exists a constant $M_{k,G}^+ > 0$ such that $|\mathcal{Z}_\ell^+| \le M_{k,G}^+\ell^d$ for any $\ell \in \mathbb{N}$.

Lemma 5. *Let us consider a $(k, \alpha, \gamma; T)$-model on a d-graph $G = (V, E)$ with $k \in \mathbb{N}$, $\alpha \in [0, 1]$, $\gamma \in [0, 1]$ and temperature profile T fast decreasing to zero. Then $\mathbb{E}[|\mathcal{S}_{A,\infty}^+|] < \infty$ for any $A \in \mathcal{A}_k$.*

Proof. Let $\ell \in \mathbb{N}$. We denote the quantity $\mathcal{H}_{V\cap[-\ell,\ell)^d,\mathcal{J}}(\cdot)$ by $\mathcal{H}_{\ell,\mathcal{J}}(\cdot)$.

We claim that there exists the following bound for $\mathcal{H}_{\ell,\mathcal{J}}(\sigma(t))$

$$\begin{aligned}&\mathcal{H}_{\ell,\mathcal{J}}\bigl(\sigma(t)\bigr) - \mathcal{H}_{\ell,\mathcal{J}}\bigl(\sigma(0)\bigr)\\ &\quad\le \sum_{A\in\mathcal{A}_k:A\cap[-\ell,\ell)^d\neq\varnothing} kd_G\bigl|\mathcal{S}_{A,t}^-\bigr| + \sum_{A\in\mathcal{Z}_\ell} kd_G\bigl|\mathcal{S}_{A,t}^- \cup \mathcal{S}_{A,t}^0 \cup \mathcal{S}_{A,t}^+\bigr| - 2\sum_{A\in\mathcal{Z}_\ell^+}\bigl|\mathcal{S}_{A,t}^+\bigr|,\end{aligned} \tag{41}$$

where kd_G represents an upper bound for $\Delta_A\mathcal{H}_{\mathcal{J}}(\sigma)$, for each $\sigma \in \{-1, +1\}^V$ and $A \in \mathcal{A}_k$ while -2 is a lower bound for the case in favour to the Hamiltonian, and comes out from (2).

Notice that the sets $A \in \mathcal{A}_k$ such that $A \cap [-\ell, \ell)^d = \varnothing$ do not appear in (41) because, in this case, $\mathcal{H}_{\ell,\mathcal{J}}(\sigma) = \mathcal{H}_{\ell,\mathcal{J}}(\sigma^{(A)})$.

We check (41) by recursion.

Now, define

$$\bar{s} = \inf\bigl\{s \in \mathcal{S}_{A,t}^- \cup \mathcal{S}_{A,t}^0 \cup \mathcal{S}_{A,t}^+ : A \in \mathcal{A}_k \text{ and } A \cap [-\ell, \ell)^d \neq \varnothing\bigr\}.$$

By the finiteness of the set of all the $A \in \mathcal{A}_k$ intersecting $[-\ell, \ell)^d$, then there exists a unique (a.s.) $A_1 \in \mathcal{A}_k$ with $A_1 \cap [-\ell, \ell)^d \neq \varnothing$ such that

$$\bar{s} \in \mathcal{S}_{A_1,t}^- \cup \mathcal{S}_{A_1,t}^0 \cup \mathcal{S}_{A_1,t}^+.$$

Then

$$\begin{aligned}&\mathcal{H}_{\ell,\mathcal{J}}\bigl(\sigma^{(A_1)}(0)\bigr) - \mathcal{H}_{\ell,\mathcal{J}}\bigl(\sigma(0)\bigr)\\ &\quad\le kd_G \cdot [\mathbf{1}_{\{A_1\in\mathcal{A}_k:A_1\cap[-\ell,\ell)^d\neq\varnothing\}} \cdot \mathbf{1}_{\{\bar{s}\in\mathcal{S}_{A_1,t}^-\}} + \mathbf{1}_{\{A_1\in\mathcal{Z}_\ell\}}] - 2 \cdot \mathbf{1}_{\{A_1\in\mathcal{Z}_\ell^+\}} \cdot \mathbf{1}_{\{\bar{s}\in\mathcal{S}_{A_1,t}^+\}}.\end{aligned}$$

Thus, by induction one obtains (41).

By Lemma 4 one has

$$\mathbb{E}\big[\mathcal{D}_{V\cap[-\ell,\ell)^d,\mathcal{J}}\big(\sigma(t)\big)\big]\in[-d_G/2,d_G/2]. \tag{42}$$

By applying the expected value operator to the terms in inequality (41) and by (42), one has

$$\mathbb{E}\big[\mathcal{H}_{\ell,\mathcal{J}}\big(\sigma(t)\big)\big]\le\frac{1}{2}d_G|V_{\text{Cell}}|(2\ell)^d+\mathbb{E}\bigg[\sum_{A\in\mathcal{A}_k:A\cap[-\ell,\ell)^d\neq\varnothing}d_Gk\big|\mathcal{S}^-_{A,t}\big|\bigg]$$
$$+\mathbb{E}\bigg[\sum_{A\in\mathcal{Z}_\ell}d_Gk\big|\mathcal{T}_A\cap[0,t]\big|\bigg]-2\cdot\mathbb{E}\bigg[\sum_{A\in\mathcal{Z}^+_\ell}\big|\mathcal{S}^+_{A,t}\big|\bigg], \tag{43}$$

since $\mathcal{S}^-_{A,t}\cup\mathcal{S}^0_{A,t}\cup\mathcal{S}^+_{A,t}\subset\mathcal{T}_A\cap[0,t]$.

Now, we bound the single terms of inequality (43).

By (14) in Lemma 1, one has

$$\mathbb{E}\bigg[\sum_{A\in\mathcal{A}_k:A\cap[-\ell,\ell)^d\neq\varnothing}d_Gk\big|\mathcal{S}^-_{A,t}\big|\bigg]\le\mathbb{E}\bigg[\sum_{A\in\mathcal{A}_k:A\cap[-\ell,\ell)^d\neq\varnothing}d_Gk\big|\mathcal{S}^-_{A,\infty}\big|\bigg]\le C_{1,k}\ell^d, \tag{44}$$

where the constant $C_{1,k}$ can be chosen in such a way that it does not depend on ℓ. In fact, by translation invariance, the expected value over a cell is equal to the expected value on any other cell.

By the bound on the cardinality of $\mathcal{Z}_\ell$ we obtain

$$\mathbb{E}\bigg[\sum_{A\in\mathcal{Z}_\ell}d_Gk\big|\mathcal{T}_A\cap[0,t]\big|\bigg]\le d_Gk|\mathcal{Z}_\ell|\mathbb{E}\big[\big|\mathcal{T}_A\cap[0,t]\big|\big]=d_Gkt|\mathcal{Z}_\ell|\le d_GktM_{k,G}\ell^{d-1}. \tag{45}$$

Now, by contradiction, suppose that there exists $\bar{A}\in\mathcal{A}_k$ such that $\mathbb{E}[|S^+_{\bar{A},\infty}|]=\infty$. For ℓ large enough, one has

$$2\cdot\mathbb{E}\bigg[\sum_{A\in\mathcal{Z}^+_\ell}\big|\mathcal{S}^+_{A,t}\big|\bigg]\ge\mathbb{E}\big[\big|S^+_{\bar{A},t}\big|\big]\ell^d. \tag{46}$$

By (43)–(46) and for ℓ large enough, one gets

$$\mathbb{E}\big[\mathcal{H}_{\ell,\mathcal{J}}\big(\sigma(t)\big)\big]\le\frac{1}{2}d_G|V_{\text{Cell}}|(2\ell)^d+C_{1,k}\ell^d+d_GktM_{k,G}\ell^{d-1}-\mathbb{E}\big[\big|S^+_{\bar{A},t}\big|\big]\ell^d. \tag{47}$$

The monotone convergence theorem leads to $\lim_{t\to\infty}\mathbb{E}[|S^+_{\bar{A},t}|]=\infty$. Let us take $t=\ell^{1/2}$. Therefore

$$\mathbb{E}\big[D_{\Lambda,\mathcal{J}}\big(\sigma\big(\ell^{1/2}\big)\big)\big]\le\frac{1}{2}d_G+\frac{C_{1,k}}{2^d|V_{\text{Cell}}|}+\frac{d_GkM_{k,G}}{2^d|V_{\text{Cell}}|}\ell^{-1/2}-\frac{1}{2^d|V_{\text{Cell}}|}\mathbb{E}\big[\big|S^+_{\bar{A},\ell^{1/2}}\big|\big].$$

If $\mathbb{E}[|S^+_{\bar{A},\infty}|]=\infty$ the r.h.s. of the previous inequality should become smaller than $-\frac{1}{2}d_G$, for ℓ large. This is in contradiction with Lemma 4, therefore $\mathbb{E}[|S^+_{\bar{A},\infty}|]<\infty$. □

We notice that only in the case of a temperature profile T that does not satisfy Corollary 1 one can hope to obtain $\rho_{\mathcal{F}}>0$. In the following we work under the assumption that the temperature profile T is fast decreasing to zero and we give, jointly to it, some sufficient conditions to obtain $\rho_{\mathcal{F}}>0$.

Next two theorems provide conditions for $\rho_{\mathcal{F}}>0$. Analogous results are in [14,23] for the case $k=1$.

Theorem 5. *Let us consider a $(k, \alpha, \gamma; T)$-model on a d-graph $G = (V, E)$ with $k \in \mathbb{N}$, $\alpha \in [0, 1]$, $\gamma \in [0, 1]$ and T fast decreasing to zero. Suppose that there exists $v \in V$ such that, for any $A \in \mathcal{A}_k$ with $A \ni v$, the cardinality of the set $\Gamma_A = \{\{u, w\} \in E : u \in \partial A, w \in \partial^{\text{ext}} A\}$ is odd. Then $\rho_{\mathcal{F}} > 0$. Specifically, $\lim_{t\to\infty} \sigma_v(t)$ exists a.s.*

Proof. Let us consider $A \in \mathcal{A}_k$ with $A \ni v$. As in [14,23], we notice that if $t \in \mathcal{T}_A$ corresponds to a flip at A, then $t \in \mathcal{S}^-_{A,\infty} \cup \mathcal{S}^+_{A,\infty}$. Formula (14) in Lemma 1 implies that $|\mathcal{S}^-_{A,t}| < \infty$ and Lemma 5 states that $|\mathcal{S}^+_{A,t}| < \infty$, almost surely. Thus, we have $|\mathcal{N}_v| < \infty$ almost surely. By ergodicity, $\rho_{\mathcal{F}} > 0$. □

Theorem 6. *Let us consider a $(k, \alpha, \gamma; T)$-model on a d-graph $G = (V, E)$ with $k \in \mathbb{N}$, $\alpha \in (0, 1]$, $\gamma \in (0, 1]$ and T fast decreasing to zero.*

Assume that there exists a finite set $B \subset V$ such that $|B| > k$ and for any $A \in \mathcal{A}_k$ with $A \cap B \neq \varnothing$ one has

$$\begin{aligned} &\big|\big\{\{u, v\} \in E : u \in A \cap B, v \in A^c \cap B\big\}\big| \\ &\quad > \big|\big\{\{u, v\} \in E : u \in A, v \in A^c \cap B^c\big\}\big| + \big|\big\{\{u, v\} \in E : u \in A \cap B^c, v \in A^c \cap B\big\}\big|. \end{aligned} \tag{48}$$

Then $\rho_{\mathcal{F}} > 0$. In particular, $\mathbb{P}(\bigcap_{u\in B}\{\lim_{t\to\infty} \sigma_u(t) = +1\}) > 0$.

Proof. Let us consider B as in the statement of the theorem. Since $\alpha > 0$ and $E(B)$ is finite, then with positive probability one has that $J_e = +1$ for any $e \in E(B)$. In fact, the independence of the interaction leads to $\mathbb{P}(\bigcap_{e\in E(B)}\{J_e = +1\}) = \alpha^{|E(B)|} > 0$. Analogously, it is $\mathbb{P}(\bigcap_{v\in B}\{\sigma_v(0) = +1\}) = \gamma^{|B|} > 0$.

Notice that $\{\{u, v\} \in E : u \in A \cap B, v \in A^c \cap B\}$, $\{\{u, v\} \in E : u \in A, v \in A^c \cap B^c\}$ and $\{\{u, v\} \in E : u \in A \cap B^c, v \in A^c \cap B\}$ is a partition of $\{\{u, v\} \in E : u \in A, v \in A^c\}$.

Hence, inequality (48) and conditions $\sigma_v = +1$ for each $v \in B$ and $J_e = +1$ for each $e \in E(B)$ imply that for each $A \in \mathcal{A}_k$ such that $A \cap B \neq \varnothing$ one has

$$\begin{aligned} \Delta_A \mathcal{H}_{\mathcal{J}}(\sigma) &= 2 \sum_{\{u,v\}\in E: u\in A, v\in A^c} J_{\{u,v\}}\sigma_u\sigma_v \\ &\geq 2\big|\big\{\{u, v\} \in E : u \in A \cap B, v \in A^c \cap B\big\}\big| \\ &\quad - 2\big[\big|\big\{\{u, v\} \in E : u \in A, v \in A^c \cap B^c\big\}\big| + \big|\big\{\{u, v\} \in E : u \in A \cap B^c, v \in A^c \cap B\big\}\big|\big] \geq 2. \end{aligned}$$

By formula (16) in Lemma 1, there is a positive probability that all the spins maintain the initial value $+1$ in the region B. By ergodicity, we obtain $\rho_{\mathcal{F}} > 0$. □

Remark 4. Let us consider the case $k = 1$ (see [23]). The hypothesis of Theorem 5 can be rephrased by stating that there exists $v \in V$ such that d_v is odd. Furthermore, condition in Theorem 6 becomes

$$\big|\big\{\{u, v\} \in E : u \in A \cap B, v \in A^c \cap B\big\}\big| > \big|\big\{\{u, v\} \in E : u \in A, v \in A^c \cap B^c\big\}\big|.$$

Example 2. Let us consider $k = 1$ and the hexagonal planar lattice. In this situation $\rho_{\mathcal{F}} = 1$, for any $\alpha, \gamma \in [0, 1]$ and T fast decreasing to zero (see Theorem 5 and Remark 4). However, for $k \geq 2$, it is easy to check that the hypothesis of Theorem 5 is not verified for such a graph. In this case, Theorem 4 guarantees that $\rho_{\mathcal{F}} < 1$ when $\alpha = 1$ and $\gamma = 1/2$.

We present here a 2-graph on which Theorem 5 can be applied when $k = 2$ (see Figure 3). All the intersections among line segments represent sites of the graph. The site denoted with the black bullet is v. Each connected set $A \ni v$ of cardinality 2 is such that $|\Gamma_A| = 5$. The case $A = \{v\}$ is such that $|\Gamma_A| = 3$.

For what concerns Theorem 6, we address the reader to Figure 1. In this case we consider $k = 1$. Take B as the set of the black bullets. A is a set containing only a vertex of B. It is clear that Theorem 6 can be applied.

In the following results we adapt the definition of *e-absent* set in [2,14].

Definition 4. Let us consider a d-graph $G = (V, E)$, interactions $\mathcal{J} \in \{-1, +1\}^E$. Take the $(k, \langle\mathcal{J}\rangle, \langle\sigma\rangle; T)$-model on G, with generic initial configuration $\sigma \in \{-1, +1\}^V$, and a finite $C \subset V$. We say that $\hat{\sigma}_C = (\hat{\sigma}_x : x \in C)$ is *k-absent*

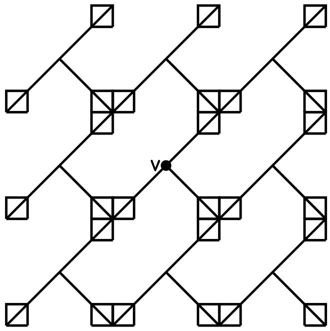

Fig. 3. Example of a 2-graph on which Theorem 5 can be applied when $k = 2$.

on $\mathcal{J}$ if there exists a sequence of m elements

$$\left(A^{(j)} \in \mathcal{A}_k : A^{(j)} \subset C, \text{ for } j = 1, \dots, m\right),$$

such that $\Delta_{A^{(m)}} \mathcal{H}_{\mathcal{J}}(\sigma^{(m-1)}) < 0$, while $\Delta_{A^{(j)}} \mathcal{H}_{\mathcal{J}}(\sigma^{(j-1)}) = 0$, for any $j = 1, \dots, m-1$, where $\sigma^{(0)}$ coincides with $\hat{\sigma}_C$ on C. Recursively, $\sigma^{(j)} = (\sigma^{(j-1)})^{(A^{(j)})}$, for $j = 1, \dots, m-1$.

Remark 5. (i) For any given $\mathcal{J}$, if $\hat{\sigma}_C$ is k-absent on $\mathcal{J}$, then it is also k'-absent on $\mathcal{J}$, for each $k' > k$, being $\mathcal{A}_k \subset \mathcal{A}_{k'}$.

(ii) Let us consider a finite set $C \subset V$. We notice that $\hat{\sigma}_C$ is k-absent on $\mathcal{J}$ only on the basis of the restriction of $\mathcal{J}$ to the finite set of edges $\hat{E}(C) = \{\{x, y\} \in E : \{x, y\} \cap C \neq \varnothing\}$.

(iii) Let us consider $\hat{\sigma}_C$ k-absent on $\mathcal{J}$ and a configuration $\bar{\sigma} \in \{-1, +1\}^V$.

If $\bar{\sigma}_C$ is such that there exists a finite sequence of $\bar{m}$ flips $A^{(j)} \subset C$, $A^{(j)} \in \mathcal{A}_k$, with $\Delta_{A^{(j)}} \mathcal{H}_{\mathcal{J}}(\bar{\sigma}^{(j-1)}) \leq 0$, where $\bar{\sigma}^{(0)}$ coincides with $\bar{\sigma}_C$ on C, $\bar{\sigma}^{(j)} = (\bar{\sigma}^{(j-1)})^{(A^{(j)})}$, for each $j = 1, \dots, \bar{m}$, and $\bar{\sigma}_C^{(\bar{m})} = \hat{\sigma}_C$, then $\bar{\sigma}_C$ is k-absent on $\mathcal{J}$ as well.

Theorem 7. *Let us consider a* $(k, \alpha, \gamma; T)$*-model on a* d*-graph* $G = (V, E)$ *with* $k \in \mathbb{N}$, $\alpha \in (0, 1)$, $\gamma \in [0, 1]$ *and temperature profile* T *fast decreasing to zero. Moreover, let us take a finite subset* $C \subset V$, *interactions* $\hat{\mathcal{J}} = (\hat{J}_e : e \in E) \in \{-1, +1\}^E$ *and a configuration* $\hat{\sigma}_C = (\hat{\sigma}_v : v \in C) \in \{-1, +1\}^C$ k*-absent on* $\hat{\mathcal{J}}$. *Then the set*

$$A(\hat{\sigma}_C) = \left\{\sigma \in \{-1, +1\}^V : \sigma_v = \hat{\sigma}_v, \forall v \in C\right\} \tag{49}$$

recurs with null probability by using the probability measure

$$P(\cdot) = \mathbb{P}\left(\cdot \,\middle|\, \bigcap_{e \in \hat{E}(C)} \{J_e = \hat{J}_e\}\right), \tag{50}$$

where $\hat{E}(C) = \{\{x, y\} \in E : \{x, y\} \cap C \neq \varnothing\}$.

Proof. We consider the Harris' graphical model and denote it by $Z = (Z_t : t \geq 0)$. By hypothesis, $\mathcal{J}$ of the model coincides with $\hat{\mathcal{J}}$ over the set $\hat{E}(C)$.

By contradiction, assume that $A(\hat{\sigma}_C)$ defined in (49) recurs with positive probability.

Define the random set

$$\mathcal{K} = \bigcup_{A \in \mathcal{A}_k : A \cap C \neq \varnothing} \mathcal{T}_A \cap [0, 1].$$

Therefore $\mathcal{K}$ is a function of the process Z. Define the set

$$B = \Big\{\big(Z_t : t \in [0, 1]\big) : \mathcal{K} = \{t_1, \dots, t_m\} \text{ with } 0 < t_1 < \dots < t_m < 1 \text{ and }$$
$$t_1 \in \mathcal{S}^0_{A^{(1)},1}, \dots, t_{m-1} \in \mathcal{S}^0_{A^{(m-1)},1}, t_m \in \mathcal{S}^+_{A^{(m)},1}\Big\},$$

where m and the $A^{(j)}$'s are given as in Definition 4.

We notice that we are under the conditions of Lemma 2. In fact,

$$\inf_{z\in A(\hat{\sigma}_C)} \inf_{t\geq 0} P\big(\{Z_{t+s} : s\in[0,1]\}\in B|Z_t = z\big)\geq \varsigma > 0,$$

because the probability that the U's are smaller than $1/2$ and that the only finite sequence of arrivals in $[0,1]$ of the Poisson processes $\mathcal{P}_A$, with $A\in\mathcal{A}_k$ and $A\cap C\neq\varnothing$, is ordered as in $\mathcal{K}$, is larger than zero (see the proof of Theorem 3 for a similar argument).

Since $A(\hat{\sigma}_C)$ recurs with positive probability, then Lemma 2 gives that B recurs with positive probability. In particular, one also has that $P(|S^+_{A^{(m)},\infty}|=\infty)>0$ but $\mathbb{P}(\bigcap_{e\in\hat{E}(C)}\{J_e=\hat{J}_e\})>0$. Therefore $\mathbb{P}(|S^+_{A^{(m)},\infty}|=\infty)>0$, and this contradicts Lemma 5. □

Remark 6. We can extend Theorem 7 also to $\alpha=0$ (resp. $\alpha=1$) by selecting $\hat{\mathcal{J}}\equiv -1$ (resp. $\hat{\mathcal{J}}\equiv +1$).

Theorem 8. *Let us consider a $(k,\alpha,\gamma;T)$-model on a d-graph $G=(V,E)$ with $k\in\mathbb{N}$, $\alpha\in(0,1)$, $\gamma\in[0,1]$ and temperature profile T fast decreasing to zero. If there exist interactions $\hat{\mathcal{J}}\in\{-1,+1\}^E$, a finite set $C\subset V$ and $u,v\in C$ with $\nu_G(u,v)\geq k$ such that any $\sigma_C\in\{-1,+1\}^C$ with $\sigma_u=\sigma_v$ (resp. $\sigma_u=-\sigma_v$) is k-absent on $\hat{\mathcal{J}}$. Then*

$$P\Big(\lim_{t\to\infty}\sigma_u(t)=-\lim_{t\to\infty}\sigma_v(t)\Big)=1,\quad \Big(\textit{resp. } P\Big(\lim_{t\to\infty}\sigma_u(t)=\lim_{t\to\infty}\sigma_v(t)\Big)=1\Big),$$

where P is the conditioned probability as in (50), and $\rho_{\mathcal{F}}>0$.

Proof. We only prove the case when any $\sigma_C\in\{-1,+1\}^C$, with $\sigma_u=\sigma_v$, is k-absent on $\hat{\mathcal{J}}$. The other case is analogous.

The interactions $\mathcal{J}$ coincide with $\hat{\mathcal{J}}$ on $\hat{E}(C)$ with positive probability, since $\alpha\in(0,1)$ and the set $\hat{E}(C)$ is finite.

Let us suppose that $\mathcal{J}$ coincides with $\hat{\mathcal{J}}$ on $\hat{E}(C)$. The vertices u and v, as defined in the theorem, flip simultaneously with probability zero because $\nu_G(u,v)\geq k$. In fact, for any $A\in\mathcal{A}_k$, the set $\{u,v\}$ is not a subset of A. By Theorem 7, we know that the set $A(u,v)=\{\sigma\in\{-1,+1\}^V:\sigma_u=\sigma_v\}$ recurs with zero probability. Therefore there exists $\lim_{t\to\infty}\sigma_u(t)$, $\lim_{t\to\infty}\sigma_v(t)$ P-a.s., and they have opposite sign P-a.s.

So, if we remove the conditioning assumption that $\mathcal{J}$ coincides with $\hat{\mathcal{J}}$ on $\hat{E}(C)$, we have that the spins on the vertices u,v fixate with positive probability.

By ergodicity we obtain that $\rho_{\mathcal{F}}>0$. □

We present an example in which Theorem 8 can be applied.

Example 3. We consider $k\leq 3$ and a 1-graph $G=(V,E)$, as in Figure 4, whose cell has vertices (the cell will be scaled by $\frac{1}{5}$ and translated by $-\frac{1}{10}$) given by

$$V_{\text{Cell}}=\{1,2,3,4,5\}$$

and edges

$$E_{\text{Cell}}=\big\{\{-1,1\},\{0,1\},\{1,2\},\{1,3\},\{2,4\},\{3,5\},\{4,6\},\{5,6\}\big\}.$$

We consider as C of Definition 4 the vertices $\{1,2,\dots,11\}$. The interactions $\mathcal{J}$, when restricted to the elements of C, are taken all positive with the exception of $J_{\{2,4\}}=-1$. We show that

$$A=\big\{\sigma\in\{-1,+1\}^V:\sigma_6=-\sigma_{11}\big\}$$

recurs with null probability. In particular, all the configurations $\sigma_C\in\{-1,+1\}^C$ having $\sigma_6=-\sigma_{11}$ are 1-absent on $\mathcal{J}$.

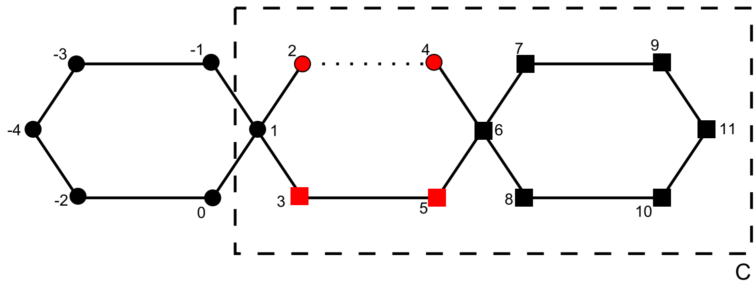

Fig. 4. These sites constitute a region of a graph $G=(V,E)$ with $V=\mathbb{L}_1$. The interactions $\mathcal{J}$ are represented in the figure. In particular, continuous line stands for an edge e such that $J_e=+1$, while the dotted line describes $J_e=-1$. The black bullets and squares are spins fixating definitively at a constant configuration ($+1$ or -1) and, in general, bullets can be different from squares. When bullets and squares share the same constant configuration, then sites 2 and 4 flip infinitely often, i.e. it does not exist $\lim_{t\to\infty}\sigma_j(t)$, with $j=2,4$. Differently, when bullets and squares have opposite signs, then it does not exist $\lim_{t\to\infty}\sigma_j(t)$, with $j=3,5$. All the configurations σ_C having $\sigma_6=-\sigma_{11}$ are 1-absent on $\mathcal{J}$.

Since $\nu_G(6,11)=3$, then Theorem 8 gives that $\rho_{\mathcal{F}}>0$. By symmetry of the dynamics with respect to global flips, i.e. $\sigma\to-\sigma$, we can take $\sigma_6=+1$ and $\sigma_{11}=-1$. In principle one should check 2^9 configurations σ_C to be 1-absent on $\mathcal{J}$.

By item (iii) of Remark 5, there exists a finite sequence of flips in favour of or indifferent to the Hamiltonian leading to $\sigma_7=\sigma_8=-1$. We assume that $\sigma_7=\sigma_8=-1$ for showing the 1-absence on $\mathcal{J}$ for all the configurations σ_C having $\sigma_6=+1$ and $\sigma_{11}=-1$.

We now distinguish the cases of $\sigma_1=+1$ and $\sigma_1=-1$.

Consider $\sigma_1=+1$.

(1) If $\sigma_4=-1$ then, by flipping the site 6, we obtain $\Delta_{\{6\}}\mathcal{H}_{\mathcal{J}}(\sigma)\le-4$. Thus the 2^7 configurations $\sigma_C\in\{-1,+1\}^C$ with $\sigma_4=\sigma_{11}=-1$ and $\sigma_1=\sigma_6=+1$ are 1-absent on the considered $\mathcal{J}$.
(2) If $\sigma_4=+1$ and $\sigma_2=+1$ then, by flipping the site 4, we have $\Delta_{\{4\}}\mathcal{H}_{\mathcal{J}}(\sigma)=0$. By flipping the site 6, we obtain $\Delta_{\{6\}}\mathcal{H}_{\mathcal{J}}(\sigma)\le-4$. The 2^6 configurations $\sigma_C\in\{-1,+1\}^C$ with $\sigma_{11}=-1$ and $\sigma_1=\sigma_2=\sigma_4=\sigma_6=+1$ are 1-absent on the considered $\mathcal{J}$.
(3) If $\sigma_4=+1$ and $\sigma_2=-1$ then, by flipping the site 2, we have $\Delta_{\{2\}}\mathcal{H}_{\mathcal{J}}(\sigma)=0$ and we are in case (2). The 2^6 configurations $\sigma_C\in\{-1,+1\}^C$ with $\sigma_2=\sigma_{11}=-1$ and $\sigma_1=\sigma_4=\sigma_6=+1$ are 1-absent on the considered $\mathcal{J}$.

The case $\sigma_1=-1$ is analogous to the previous one by replacing site 2 with 3 and site 4 with 5. Therefore $\lim_{t\to\infty}\sigma_6(t)=\lim_{t\to\infty}\sigma_{11}(t)$ and this also implies that $\lim_{t\to\infty}\sigma_i(t)=\lim_{t\to\infty}\sigma_6(t)$, for $i=7,8,9,10,11$.

We notice that all the interactions on $\hat{E}(C)$ are positive with the exception of $J_{\{2,4\}}=-1$ with positive probability. In this case we can apply the arguments above and have that $\lim_{t\to\infty}\sigma_i(t)=\lim_{t\to\infty}\sigma_1(t)$, for $i=-4,-3,-2,-1,0$.

If $\lim_{t\to\infty}\sigma_1(t)=\lim_{t\to\infty}\sigma_6(t)$ then the sites 2 and 4 flip infinitely many times. Differently if $\lim_{t\to\infty}\sigma_1(t)=-\lim_{t\to\infty}\sigma_6(t)$ then the sites 3 and 5 flip infinitely many times.

This framework can be though as an illustrative example also of Theorem 3. In fact, the set R can be taken as coinciding with $\{-4,-3,-2,-1,0,1,6,7,8,9,10,11\}$, S is the set of the edges represented in Figure 4, at least one among the four combinations of black bullets ±1 and black squares ±1 fixates with positive probability and, accordingly, D is given by $\{2,4\}$ (bullets and squares with the same sign) or $\{3,5\}$ (otherwise).

Two important remarks are in order: first, the model described in this example is of type $\mathcal{M}$; second, this model can be extended to d-dimensional graphs (see Figure 5).

Remark 7. In [23] the Ising stochastic model at zero temperature on $\mathbb{Z}^2$ is studied. Their model can be rewritten in our language as $(1,1,\gamma;T)$-model on the 2-graph $\mathbb{L}_2=(\mathbb{Z}^2,\mathbb{E}_2)$ with $\gamma=1/2$ and $T\equiv0$. The authors show that the model is of type $\mathcal{I}$. With a coupling argument one can prove that also for $\mathcal{J}\equiv-1$, i.e. $\alpha=0$, the system is of type $\mathcal{I}$ (see [14]). In fact, consider two systems on the d-graph $\mathbb{L}_d=(\mathbb{Z}^d,\mathbb{E}_d)$, $\gamma=1/2$ and $\mathcal{J}\equiv+1$ for one of the systems and $\tilde{\mathcal{J}}\equiv-1$ for the other one. We say that a vertex $v=(v_1,\dots,v_d)\in\mathbb{Z}^d$ is *even* (resp. *odd*) if $|\sum_{i=1}^d v_i|$ is even

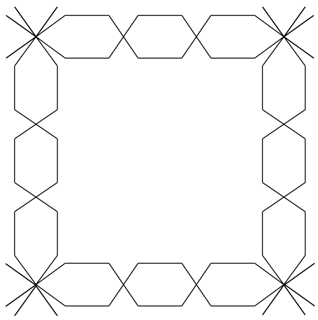

Fig. 5. Representation of $\Gamma_{3,3}(\mathbb{L}_2)$.

(resp. odd). We take a initial configuration σ for the system with $\mathcal{J}$ and initial configuration $\tilde{\sigma}$ for the system with $\tilde{\mathcal{J}}$ as follows

$$\tilde{\sigma}_v = \begin{cases} \sigma_v & \text{if } v \text{ is even;} \\ -\sigma_v & \text{if } v \text{ is odd.} \end{cases}$$

Therefore, also the random vector $(\tilde{\sigma}_v : v \in \mathbb{Z}^d)$ is a collection of i.i.d. Bernoulli random variables with $\gamma = 1/2$. Taking the same Poisson processes and the same random variables U's for both the systems at any time t, one obtains

$$\tilde{\sigma}_v(t) = \begin{cases} \sigma_v(t) & \text{if } v \text{ is even;} \\ -\sigma_v(t) & \text{if } v \text{ is odd.} \end{cases}$$

Therefore the two systems are of the same type ($\mathcal{I}$, $\mathcal{F}$ or $\mathcal{M}$). The same construction works for any bipartite graph.

Example 3 suggests a general result on a particular class of graphs. We firstly define such a class, and then present a simple and meaningful theorem. The proof of such a result can be viewed as a generalization of the arguments carried out in Example 3.

Definition 5. For $\ell \geq 2$ and $m \geq 1$, let us consider a d-graph $G = (V, E)$, with $d \in \mathbb{N}$. We denote by $\Gamma_{\ell,m}(G) = (V_\Gamma, E_\Gamma)$ the d-graph such that each edge of G is replaced with m identical cycles whose number of vertices is 2ℓ with the following property:

(i) two adjacent cycles have only one vertex in common. We denote such vertices as *common vertices*, and collect them in the set $\mathcal{V}_I$.
(ii) $\min_{u,v \in \mathcal{V}_I} \nu_{\Gamma_{\ell,m}(G)}(u, v) = \ell$.

The vertices $V \subset V_\Gamma$ will be called *original vertices*.

Notice that $V \subset \mathcal{V}_I$. In Figure 5 we have the representation of $\Gamma_{3,3}(\mathbb{L}_2)$.

Theorem 9. *Let $G = (V, E)$ be a d-graph, with $d \in \mathbb{N}$. Consider a $(k, \alpha, \gamma; T)$-model on $\Gamma_{\ell,m}(G)$ with $\alpha \in (0, 1)$, $\gamma \in [0, 1]$ and temperature profile T fast decreasing to zero. If $\ell \geq k \wedge 2$ and $m \geq 3$, the model is of type $\mathcal{M}$.*

Proof. We briefly denote $\nu_{\Gamma_{\ell,m}(G)}$ by ν. Consider two original vertices $u, v \in V$ such that $\nu(u, v) = \ell m$. The finite subgraph composed by the m cycles between u and v will be denoted by $\Gamma^{u,v} = (V_\Gamma^{u,v}, E_\Gamma^{u,v})$. The cycles composing $\Gamma^{u,v}$ are subgraphs as well. They will be denoted by $Y_1 = (V_1, E_1), \ldots, Y_m = (V_m, E_m)$, so that if one takes $h, \ell = 1, \ldots, m$ such that $h < \ell$, then for each $x \in V_h$ and $y \in V_\ell$ it results $\nu(x, u) < \nu(y, u)$. We consider the first three cycles Y_1, Y_2 and Y_3 and we label the vertices of these cycles in the following way: $v_1 = u$, $v_2 \in V_1 \cap V_2$, $v_3 \in V_2 \cap V_3$ and v_4 is different from v_3 and belong to $\mathcal{V}_I \cap V_3$. The other vertices are denoted by $v_{a,b,s}$, with $a = 1, 2, 3$, $b = 1, 2$, $s = 1, \ldots, \ell - 1$. Specifically, the index a means that $v_{a,b,s}$ belongs to V_a; the index b identifies one of the two paths of length ℓ connecting the common vertices of the cycle Y_a; the index s is such that $\nu(v_{a,b,s}, u) < \nu(v_{a,b,s+1}, u)$, for each $s = 1, \dots, \ell - 1$.

Let us consider the following condition

(H1) $J_{\{v_2, v_{2,1,1}\}} = -1$; $J_e = +1$, for each $e \in (E_1 \cup E_2 \cup E_3) \setminus \{v_2, v_{2,1,1}\}$.

With positive probability, there exist interactions $\mathcal{J}$ that satisfies the previous condition (H1). Define $C = V_1 \cup V_2$.

We prove that the configurations belonging to

$$\big\{\big(\sigma_v \in \{-1, +1\} : v \in C\big) : \sigma_{v_1} \neq \sigma_{v_2}\big\} \subset \{-1, +1\}^C \tag{51}$$

are 1-absent on the interactions $\mathcal{J}$ that satisfy condition (H1). Without loss of generality in (51) one can take $\sigma_{v_1} = +1$ and $\sigma_{v_2} = -1$. Now we consider two cases

(i) $\sigma_{v_1} = \sigma_{v_3} = +1$ and $\sigma_{v_2} = -1$;
(ii) $\sigma_{v_1} = +1$ and $\sigma_{v_2} = \sigma_{v_3} = -1$.

In case (i) let us consider the following elements

$$\hat{A}^{(i)} = \begin{cases} \{v_{1,1,i}\}, & \text{for } i = 1, \dots, \ell - 1; \\ \{v_{1,2,i-\ell+1}\}, & \text{for } i = \ell, \dots, 2\ell - 2; \\ \{v_{2,2,3\ell-2-i}\}, & \text{for } i = 2\ell - 1, \dots, 3\ell - 3; \\ \{v_2\}, & \text{for } i = 3\ell - 2. \end{cases}$$

For a given $\sigma_C = (\sigma_v : v \in C)$ satisfying item (i) we eliminate from the sequence $(\hat{A}^{(i)} : i = 1, \dots, 3\ell - 2)$ the elements of the form $\{v\}$ with $\sigma_v = +1$ to obtain the new sequence $(A^{(i)} : i = 1, \dots, m)$ (thus $m \leq 3\ell - 2$). Using these elements and notation in Definition 4, one can check that all the $\Delta_{A^{(i)}} \mathcal{H}_{\mathcal{J}}(\sigma^{(i-1)}) \leq 0$ for $i = 1, \dots, m - 1$, and $\Delta_{A^{(m)}} \mathcal{H}_{\mathcal{J}}(\sigma^{(m-1)}) < 0$. Notice that $m \geq 1$, and $A^{(m)} = \{v_2\}$. Therefore, by using item (iii) of Remark 5, one has that all the configurations in which item (i) holds true are 1-absent on $\mathcal{J}$.

In case (ii), we define

$$\tilde{A}^{(i)} = \begin{cases} \{v_{1,1,i}\}, & \text{for } i = 1, \dots, \ell - 1; \\ \{v_{1,2,i-\ell+1}\}, & \text{for } i = \ell, \dots, 2\ell - 2; \\ \{v_{2,1,3\ell-2-i}\}, & \text{for } i = 2\ell - 1, \dots, 3\ell - 3; \\ \{v_2\}, & \text{for } i = 3\ell - 2. \end{cases}$$

For $\sigma_C = (\sigma_v : v \in C)$ which satisfies item (ii) we eliminate from the sequence $(\hat{A}^{(i)} : i = 1, \dots, 3\ell - 2)$ the elements of the form $\{v_{1,b,s}\}$ with $\sigma_{v_{1,b,s}} = +1$ and $\{v_{2,1,s}\}$ with $\sigma_{v_{2,1,s}} = -1$. In so doing, we have the new sequence $(A^{(i)} : i = 1, \dots, m)$. By following the argument developed in case (i), all the configurations in which item (ii) is satisfied are 1-absent on $\mathcal{J}$.

Since $\nu(v_1, v_2) = \ell \geq k$, then Theorem 8 guarantees that, for large t, $\sigma_{v_1}(t)$ and $\sigma_{v_2}(t)$ fixate and are equal. Therefore, $\rho_{\mathcal{F}} > 0$.

With an analogous argument one can prove the same result for v_3 and v_4, i.e. there exist the limits $\lim_{t\to\infty} \sigma_{v_3}(t)$ and $\lim_{t\to\infty} \sigma_{v_4}(t)$, and they coincide.

As in Theorem 3 we take $R = \{v_2, v_3\}$. In fact, condition (36) is verified, i.e.: with positive probability, at least one of the following four alternatives is verified: $\lim_{t\to\infty} \sigma_{v_2}(t) = \pm 1$ and $\lim_{t\to\infty} \sigma_{v_3}(t) = \pm 1$. We analyze separately the four cases.

If $\lim_{t\to\infty} \sigma_{v_2}(t) = +1$ and $\lim_{t\to\infty} \sigma_{v_3}(t) = +1$ with positive probability, then we select $R' = \{v_{2,1,s} : s = 1, \dots, \ell - 1\}$. Let us consider

$$\bar{s}_1 = \sup\big\{s \in \{1, \dots, \ell - 1\} : \sigma_{v_{2,1,s}} = -1\big\},$$

with $\sup \varnothing = 1$. We define $D = \{v_{2,1,\bar{s}_1}\}$. Notice that $\sigma_{v_{2,1,\bar{s}_1+1}} = +1$, with the conventional agreement that $v_{2,1,\ell} = v_3$. Then, condition (ii) of Theorem 3 is satisfied and the model is of type $\mathcal{M}$.

An analogous argument applies to the case of $\lim_{t\to\infty}\sigma_{v_2}(t) = \lim_{t\to\infty}\sigma_{v_3}(t) = -1$.

If $\lim_{t\to\infty}\sigma_{v_2}(t) = +1$ and $\lim_{t\to\infty}\sigma_{v_3}(t) = -1$ with positive probability, then $R' = \{v_{2,2,s} : s = 1, \dots, \ell - 1\}$. In this case

$$\bar{s}_2 = \sup\big\{s \in \{1, \dots, \ell - 1\} : \sigma_{v_{2,2,s}} = +1\big\},$$

with $\sup\varnothing = 1$. By taking $D = \{v_{2,2,\bar{s}_2}\}$ and noticing that $\sigma_{v_{2,2,\bar{s}_2+1}} = -1$, with $v_{2,2,\ell} = v_3$, we obtain condition (ii) of Theorem 3 and the model is of type $\mathcal{M}$.

In a similar way we can treat the case of $\lim_{t\to\infty}\sigma_{v_2}(t) = -1$ and $\lim_{t\to\infty}\sigma_{v_3}(t) = +1$. □

We recall the definition of cluster that will be used in the next Theorem. Let us consider a d-graph $G = (V, E)$ and a stochastic Ising model $\sigma(\cdot) = (\sigma_u(t) : u \in V, t \geq 0)$. *The cluster $C_v(t)$ of the site v at time t* is the maximal connected component of the set $\{u \in V : \sigma_u(t) = \sigma_v(t)\}$ which contains v.

Theorem 10. *Let us consider a $(1, \alpha, 1/2; T)$-model on $\Gamma_{\ell,m}(\mathbb{L}_2)$, with $\alpha \in \{0, 1\}$, $\ell \geq 2$, $m \geq 1$ and temperature profile T fast decreasing to zero. Then the $(1, \alpha, 1/2; T)$-model is of type $\mathcal{I}$.*

Proof. First of all, we notice that $\Gamma_{\ell,m}(\mathbb{L}_2)$ is a bipartite graph. Then, by Remark 7, it is sufficient to prove the result for $\alpha = 1$.

The origin O of the graph is placed in $(0, 0) \in \mathbb{Z}^2$. Also in this case, for the sake of simplicity, we will write ν instead of $\nu_{\Gamma_{\ell,m}(\mathbb{L}_2)}$.

We first show that if

$$p = \mathbb{P}\Big(\text{There exists } \lim_{t\to\infty}\sigma_O(t) = +1\Big) = \mathbb{P}\Big(\text{There exists } \lim_{t\to\infty}\sigma_O(t) = -1\Big) = 0$$

then the model is of type $\mathcal{I}$. By translation invariance of the graph, each vertex belonging to $\mathbb{Z}^2$ reaches the fixation with null probability.

For a vertex $x \notin \mathbb{Z}^2$ let us take the two vertices $u, v \in \mathbb{Z}^2$ having Euclidean distance equal to one and such that $\nu(x, u), \nu(x, v) < \nu(u, v)$. The fact that $p = 0$ means that the event

$$A = \big\{\sigma \in \{-1, +1\}^{V_\Gamma} : \sigma_u \neq \sigma_v\big\}$$

recurs with probability one.

Now, define the sets

$$B^+ = \big\{\big(\sigma(s) : s \in [0, 1]\big) : \exists t \in [0, 1] \text{ such that } \sigma_x(t) = +1\big\}$$

and

$$B^- = \big\{\big(\sigma(s) : s \in [0, 1]\big) : \exists t \in [0, 1] \text{ such that } \sigma_x(t) = -1\big\}.$$

One can check that

$$\inf_{\sigma\in A}\inf_{t\geq 0}\mathbb{P}\big(\big(\sigma(t+s) : s \in [0, 1]\big) \in B^+ | \sigma(t) = \sigma\big) > 0,$$

the same for B^-. In fact, the definition of u, v and x, along with the opposite signs of σ_u and σ_v, allow to have the propagation of the sign of u or v over the vertex x with positive probability. Since A recurs with probability one, by Lemma 2, we get that also B^+ and B^- recur with probability one. This means that the model is of type $\mathcal{I}$.

Now the proof proceeds by contradiction. Suppose that

$$p = \mathbb{P}\Big(\text{There exists } \lim_{t\to\infty}\sigma_O(t) = +1\Big) = \mathbb{P}\Big(\text{There exists } \lim_{t\to\infty}\sigma_O(t) = -1\Big) > 0. \tag{52}$$

The second equality in (52) follows by $\gamma = 1/2$. Then, by the ergodic theorem, one should have that $\rho_{\mathcal{F}} > 0$.

By the ergodic theorem, for any $\varepsilon > 0$ there exists an $L \in \mathbb{N}$ such that

$$\mathbb{P}\Big(\limsup_{t\to\infty}\big|\{v \in [-L, L]^2 : \sigma_v(t) = -1\}\big| = 0\Big) < \varepsilon. \tag{53}$$

The previous inequality means that the event

$$\hat{A}_L = \{\sigma_v = +1, \forall v \in [-L, L]^2\}$$

recurs with probability smaller than ε.

Now we notice that all the finite clusters are 1-absent on $\mathcal{J} \equiv +1$. In fact, given a cluster $C_v(t)$, there exists a sequence of flips, which are indifferent for or in favour of the Hamiltonian, such that all the spins associated to the vertices of $C_v(t)$ change their sign (see Theorem 7). This means that, for $v \in V_\Gamma$ and $M \in \mathbb{N}$,

$$\lim_{t\to\infty} \mathbb{P}\big(\big|C_v(t)\big| > M\big) = 1.$$

Therefore,

$$\lim_{t\to\infty} \mathbb{P}\big(C_O(t) \cap \partial[-L, L]^2 \neq \varnothing\big) = 1.$$

Let us define the four events associated to the four sides of $\partial[-L, L]^2$

$$E_1(t) = \{\sigma_O(t) = +1 \text{ and } C_O(t) \cap ([-L, L] \times \{L\}) \neq \varnothing\},$$
$$E_2(t) = \{\sigma_O(t) = +1 \text{ and } C_O(t) \cap ([-L, L] \times \{-L\}) \neq \varnothing\},$$
$$E_3(t) = \{\sigma_O(t) = +1 \text{ and } C_O(t) \cap (\{L\} \times [-L, L]) \neq \varnothing\},$$

and

$$E_4(t) = \{\sigma_O(t) = +1 \text{ and } C_O(t) \cap (\{-L\} \times [-L, L]) \neq \varnothing\}.$$

By symmetry of the graph, the events $E_1(t), E_2(t), E_3(t), E_4(t)$ have the same probability. Therefore, using also that $\gamma = 1/2$, one obtain

$$\liminf_{t\to\infty}\big(E_i(t)\big) \geq \frac{1}{8}, \quad \forall i = 1, 2, 3, 4.$$

We notice that $E_i(t)$ is an increasing event, for each $i = 1, 2, 3, 4$.

Let us define

$$E(t) = \bigcap_{i=1}^{4} E_i(t).$$

By the fact that the events E_i's are increasing and $k = 1$ one can use the FKG inequality to bound the probability of $E(t)$ (see e.g. [18]). Therefore

$$\liminf_{t\to\infty} \mathbb{P}\big(E(t)\big) \geq \liminf_{t\to\infty}\big(\mathbb{P}\big(E_N(t)\big)\mathbb{P}\big(E_E(t)\big)\mathbb{P}\big(E_S(t)\big)\mathbb{P}\big(E_O(t)\big)\big) \geq \left(\frac{1}{8}\right)^4.$$

Consider the sequence of events $(E(n) : n \in \mathbb{N})$; by Fatou's lemma,

$$\mathbb{P}\Big(\limsup_{n\to\infty} E(n)\Big) \geq \limsup_{n\to\infty} \mathbb{P}\big(E(n)\big) \geq \left(\frac{1}{8}\right)^4.$$

In the frame of Lemma 2 this means that the event

$$A_L = \{O \text{ is connected to the four sides of } [-L, L]^2 \text{ with } +1 \text{ spins}\}$$

recurs with a positive probability larger or equal to $(\frac{1}{8})^4$.

Let us define the event

$$B_L = \{\exists s \in [0,1] : \sigma_u(s) = +1, \forall u \in [-L, L]^2\}.$$

One has

$$\inf_{\sigma \in A_L} \inf_{t \geq 0} \mathbb{P}\big((\sigma(t+s) : s \in [0,1]) \in B_L | \sigma(t) = \sigma\big) > 0.$$

In fact, if the set of vertices with negative spins is not empty, then there exists at least one vertex that can change its spin from -1 to $+1$ with a flip that is indifferent for or in favour of the Hamiltonian. Then, recursively, all the spins of the vertices belonging to $[-L, L]^2$ can become positive at a same time. By Lemma 2, one obtains that B_L recurs with probability larger or equal than $(\frac{1}{8})^4$. Therefore, for each $L \in \mathbb{N}$, the probability that there are not vertices belonging to $[-L, L]^2$ whose spins fixate to -1 is at least $(\frac{1}{8})^4$. This last assertion contradicts formula (53) when $\varepsilon < (\frac{1}{8})^4$. □

5. Conclusions

In this paper we have presented a generalization of the Glauber dynamics of the Ising model. The temperature is assumed to be time-dependent and fast decreasing to zero, hence including the case of $T \equiv 0$. Moreover, it is allowed that spins flip simultaneously when belonging to some connected regions. The dynamics is taken over general periodic graphs embedded in $\mathbb{R}^d$. The obtained results can be compared with the standard case of zero-temperature, cubic lattice and $k = 1$.

For the cubic lattice $\mathbb{L}_2$ the paper [23] says that for $\alpha = 1$ or $\alpha = 0$ and $\gamma = \frac{1}{2}$ the model is of type $\mathcal{I}$. On the same graph, [14] proves that the model is of type $\mathcal{M}$ when $\alpha \in (0,1)$ and $\gamma = \frac{1}{2}$ (actually, their arguments hold true for $\gamma \in (0,1)$ as well). In [13,22] it is shown that, for $\mathbb{L}_d$ with $d \geq 2$, the model is of type $\mathcal{F}$ when $\alpha = 1$ and γ sufficiently close to one (or, by symmetry, sufficiently close to zero). In particular, it is known that the limit configuration is given by spins whose values are $+1$ (or, by symmetry, -1). For a better visualization of the results, see Figure 6 (left panel).

For what concerns the graphs $\Gamma_{\ell,m}(G)$ of Definition 5, see Figure 6 (central panel). In this case, when $\ell \geq k \wedge 2$ and $m \geq 3$, we have shown that the $(k, \alpha, \gamma; T)$-model is of type $\mathcal{M}$ for $\alpha \in (0,1)$ and $\gamma \in [0,1]$. The particular case of $\Gamma_{\ell,m}(\mathbb{L}_2)$ gives that the $(1, \alpha, 1/2; T)$-model is of type $\mathcal{I}$ for $\alpha = 0, 1$.

When T is fast decreasing to zero and positive, and by considering $\mathbb{L}_d$ with $d \geq 2$, then it is possible to exclude that the $(k, \alpha, \gamma; T)$-model is of type $\mathcal{F}$ (see Theorem 2) and refer to Figure 6 (right panel).

We feel that Figure 6 might also contribute to highlight some problems left open by this paper.

To conclude, we think that our results may represent a first move towards the following three conjectures.

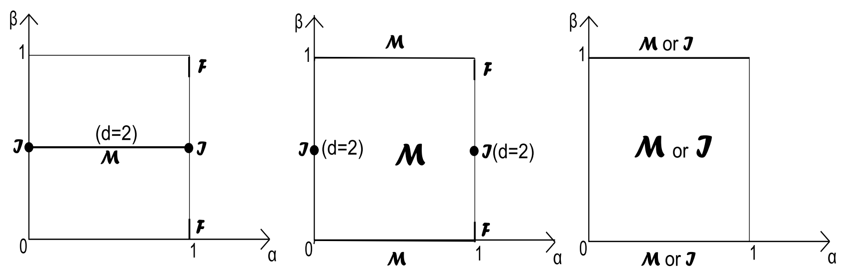

Fig. 6. Graphical visualization of the Conclusions. The panel on the left describes the main results in the literature for the stochastic Ising models at zero-temperature on the graph $\mathbb{L}_d$. Notice that the case of $\gamma = 1/2$ holds only for $d = 2$. The central panel represents our results for the graphs $\Gamma_{\ell,m}(\mathbb{L}_d)$. The panel on the right depicts Theorem 2.

(i) If $\alpha, \gamma \in (0,1)$ and T is fast decreasing to zero, then the type of the $(k, \alpha, \gamma; T)$-model over a d-graph $G = (V, E)$ is identified by the value of $k \in \mathbb{N}$ and by the graph G.
(ii) Fix $\alpha, \gamma \in [0,1]$, a temperature profile fast decreasing to zero T and a d-graph G. Set $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{I}$ to 1, 2, 3, respectively. Define the function $\xi_{\alpha,\gamma,T,G} : \mathbb{N} \to \{1, 2, 3\}$ assigning to any $k \in \mathbb{N}$ the type of the $(k, \alpha, \gamma; T)$-model over G. The function $\xi_{\alpha,\gamma,T,G}$ is nondecreasing.
(iii) Let $G = \mathbb{L}_d$, for $d \geq 2$. Consider a $(k, 1, \gamma; T)$-model on $\Gamma_{\ell,m}(G)$. Then there exists $\varepsilon \in (0, 1/2)$ and a temperature profile $T^\star$ fast decreasing to zero such that: for each $\gamma \in [0, \varepsilon) \cup (\varepsilon, 1]$ and for each temperature profile T such that $T(t) \leq T^\star(t)$, for each $t \geq 0$, then the $(k, 1, \gamma; T)$-model is of type $\mathcal{F}$.

Acknowledgement

We thank two anonymous referees for their many insightful comments and suggestions. We thank also Lorenzo Bertini, Emanuele Caglioti and Mauro Piccioni for helpful discussions. All the remaining errors are our own responsibility.

References

[1] R. Arratia. Site recurrence for annihilating random walks on $\mathbf{Z}_d$. *Ann. Probab.* **11** (3) (1983) 706–713. MR0704557
[2] F. Camia, E. De Santis and C. M. Newman. Clusters and recurrence in the two-dimensional zero-temperature stochastic Ising model. *Ann. Appl. Probab.* **12** (2) (2002) 565–580. MR1910640
[3] P. Caputo and F. Martinelli. Phase ordering after a deep quench: The stochastic Ising and hard core gas models on a tree. *Probab. Theory Related Fields* **136** (1) (2006) 37–80. MR2240782
[4] M. Damron, S. M. Eckner, H. Kogan, C. M. Newman and V. Sidoravicius. Coarsening dynamics on $\mathbf{Z}^d$ with frozen vertices. *J. Stat. Phys.* **160** (1) (2015) 60–72. MR3357967
[5] M. Damron, H. Kogan, C. M. Newman and V. Sidoravicius. Fixation for coarsening dynamics in 2D slabs. *Electron. J. Probab.* **18** (105) (2013) 1–20. MR3145052
[6] E. De Santis. Glauber dynamics of spin glasses at low and high temperature. *Ann. Inst. Henri Poincaré B, Probab. Stat.* **38** (5) (2002) 681–710. MR1931583
[7] E. De Santis. Torpid mixing of Swendsen-Wang dynamics on $\mathbf{Z}^d$ for low-temperature non-ferromagnetic disordered systems. *Markov Process. Related Fields* **9** (1) (2003) 35–57. MR1973320
[8] E. De Santis and A. Lissandrelli. Developments in perfect simulation of Gibbs measures through a new result for the extinction of Galton–Watson-like processes. *J. Stat. Phys.* **147** (2) (2012) 231–251. MR2922768
[9] E. De Santis and A. Maffei. Perfect simulation for the infinite random cluster model, Ising and Potts models at low or high temperature. *Probab. Theory Related Fields* **164** (1) (2016) 109–131. MR3449387
[10] E. De Santis and C. Marinelli. A class of stochastic games with infinitely many interacting agents related to Glauber dynamics on random graphs. *J. Phys. A* **40** (39) (2007) 11777–11790. MR2374050
[11] E. De Santis and C. M. Newman. Convergence in energy-lowering (disordered) stochastic spin systems. *J. Stat. Phys.* **110** (1–2) (2003) 431–442. MR1966336
[12] S. M. Eckner and C. M. Newman. Fixation to consensus on tree-related graphs. *ALEA Lat. Am. J. Probab. Math. Stat.* **12** (1) (2015) 357–374. MR3357131
[13] L. R. Fontes, R. H. Schonmann and V. Sidoravicius. Stretched exponential fixation in stochastic Ising models at zero temperature. *Comm. Math. Phys.* **228** (3) (2002) 495–518. MR1918786
[14] A. Gandolfi, C. M. Newman and D. L. Stein. Zero-temperature dynamics of $\pm J$ spin glasses and related models. *Comm. Math. Phys.* **214** (2) (2002) 373–387. MR1796026
[15] T. E. Harris. Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** (3) (1978) 355–378. MR0488377
[16] C. D. Howard and C. M. Newman. The percolation transition for the zero-temperature stochastic Ising model on the hexagonal lattice. *J. Stat. Phys.* **111** (1–2) (2003) 57–72. MR1964265
[17] U. Krengel. *Ergodic Theorems*. *De Gruyter Studies in Mathematics* **6**. Walter de Gruyter & Co., Berlin, 1985. MR0797411
[18] T. M. Liggett. *Interacting Particle Systems*. *Classics in Mathematics*. Springer, Berlin, 2005. MR2108619
[19] F. Martinelli, E. Olivieri and E. Scoppola. On the Swendsen–Wang dynamics. I. Exponential convergence to equilibrium. *J. Stat. Phys.* **62** (1–2) (1991) 117–133. MR1105260
[20] F. Martinelli and F. L. Toninelli. On the mixing time of the 2D stochastic Ising model with "plus" boundary conditions at low temperature. *Comm. Math. Phys.* **296** (1) (2010) 175–213. MR2606632
[21] P. Mathieu. Convergence to equilibrium for spin glasses. *Comm. Math. Phys.* **215** (1) (2000) 57–68. MR1799876
[22] R. Morris. Zero-temperature Glauber dynamics on $\mathbb{Z}^d$. *Probab. Theory Related Fields* **149** (3–4) (2011) 417–434. MR2776621
[23] S. Nanda, C. M. Newman and D. L. Stein. Dynamics of Ising spin systems at zero temperature. In *On Dobrushin's Way. From Probability Theory to Statistical Physics* 183–194. *Amer. Math. Soc. Transl. Ser.* **198**. Amer. Math. Soc., Providence, RI, 2000. MR1766351

[24] M. Ostilli and J. F. F. Mendes. Communication and correlation among communities. *Phys. Rev. E (3)* **80** (011142) (2009). 515–528. MR2552037

[25] D. Richardson. Random growth in a tessellation. *Math. Proc. Cambridge Philos. Soc.* **74** (1973) 515–528. MR0329079

[26] T. Seppäläinen. Translation Invariant Exclusion Processes, 2008. Book in progress. Available at http://www.math.wisc.edu/~seppalai/excl-book/ajo.pdf.

[27] E. H. Spanier. *Algebraic Topology*. McGraw-Hill Book Co., New York–Toronto, Ont.–London, 1966. MR0210112

[28] D. Stauffer. A biased review of sociophysics. *J. Stat. Phys.* **151** (1–2) (2013) 9–20. MR3045804

[29] O. Tamuz and R. J. Tessler. Majority dynamics and the retention of information. *Israel J. Math.* **206** (1) (2015) 483–507. MR3319649

[30] R. J. Tessler Geometry and dynamics in zero temperature statistical mechanics models. Preprint, 2010. Available at arXiv:1008.5279.

[31] D. Williams. *Probability with Martingales. Cambridge Mathematical Textbooks*. Cambridge University Press, Cambridge, 1991. MR1155402