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Stochastic integral equations for Walsh semimartingales

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Abstract. We construct planar semimartingales that include the Walsh Brownian motion as a special case, and derive Harrison–Shepp-type equations and a change-of-variable formula in the spirit of Freidlin–Sheu for these so-called "Walsh semimartingales". We examine the solvability of the resulting system of stochastic integral equations. In appropriate Markovian settings we study two types of connections to martingale problems, questions of uniqueness in distribution for such processes, and a few examples.

Résumé. Nous construisons des semimartingales planaires qui incluent le mouvement brownien de Walsh comme cas particulier, et nous établissons pour ces « semimartingales de Walsh » des équations de type Harrison–Shepp, et une formule de changement de variable dans l'esprit de Freidlin–Sheu. Dans des cadres markoviens appropriés, nous étudions deux types de relations aux problèmes de martingale, des questions d'unicité en loi pour de tels processus, et quelques exemples.

MSC: Primary 60G42; secondary 60H10

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1. Introduction and summary

We consider the following questions: *What is a two-dimensional analogue of the scalar skew Brownian motion*? *If such a process exists, what stochastic integral equation realizes its construction and describes its dynamics*? *Are there more general planar semimartingales with similar skew-unfolding-type structure*?

In order to answer the first question, WALSH [27] introduced a singular planar diffusion with these properties. This diffusion is known now as the WALSH Brownian motion. In its description by BARLOW, PITMAN & YOR [1], "started at a point in the plane away from the origin, this process moves like a standard Brownian motion along the ray joining the starting point and the origin $\mathbf{0}$, until it reaches $\mathbf{0}$. Then it is kicked away from $\mathbf{0}$ by an entrance law that makes the radial part of the diffusion a reflecting Brownian motion, while randomizing the angular part". The WALSH Brownian motion has been generalized to the so-called spider martingales, and has been studied by several researchers (among them [1,4,6–10,18,19,26,28]). In this paper we construct a family of planar semimartingales which includes the spider martingales and the WALSH Brownian motion as special cases.

There are several constructions of WALSH's Brownian motions in terms of resolvents, infinitesimal generators, semigroups, and excursion theory. Our approach can be thought of as a bridge between excursion theory and stochastic integral equations, via the folding and unfolding of semimartingales. It represents also an attempt to study higher-dimensional analogues of the skew-TANAKA equation, and the semimartingale properties of planar processes that hit points.

Preview: We provide in Section 2 a system of stochastic equations (2.8) that these planar semimartingales satisfy. This system is a two-dimensional analogue of the equation introduced by HARRISON & SHEPP [11] for the skew Brownian motion, and answers the second and third questions stated above. Based on this integral-equation description, we develop in Sections 3 and 4 a stochastic calculus, and establish a FREIDLIN–SHEU type change-of-variable formula (4.2), for such WALSH semimartingales. We also develop a condition (3.6) closely analogous to that of [11], for the solvability of our system of equations (2.8). In Section 5 we examine by the method of [20] this two-dimensional HARRISON–SHEPP equation driven by a continuous semimartingale, as in [12].

Pathwise uniqueness fails for the system of (2.8). For the ITÔ diffusion case, we recast this system in the form (6.4), and study its connections to an appropriate martingale problem in Sections 6 and 8. The well-posedness of this martingale problem, in the form of conditions under which a weak solution exists for the system of (6.4) and is unique in distribution, is based on the stochastic calculus of Section 4.

The WALSH Brownian motion constructed via the FELLER semigroup, is then shown in Section 7 to be a special case of our "WALSH diffusion" framework. Another type of connection to martingale problems is established in Section 9, allowing us to show that WALSH diffusions are the only time-homogeneous and strongly Markovian solutions of the system (6.4). A notable difference from the HARRISON–SHEPP equation is also given there; whereas in Section 10 we study additional examples. Some auxiliary results and proofs are provided in the appendices, Sections A and B.

2. The setting and results

On a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}(t)\}_{0\le t<\infty}$ that satisfies the "usual conditions" of right-continuity and augmentation by null sets, we consider a real-valued, continuous semimartingale

$$U(t) = M(t) + V(t), \quad 0 \le t < \infty. \tag{2.1}$$

Here $M(\cdot)$ is a continuous local martingale and $V(\cdot)$ has finite variation on compact intervals; we assume that the initial position $U(0) \ge 0$ is a given real number. We denote by

$$S(t) := U(t) + \Lambda(t), \quad \text{where } \Lambda(t) = \max_{0\le s\le t} \big(-U(s)\big)^+, 0 \le t < \infty, \tag{2.2}$$

the SKOROKHOD reflection (or "folding") of $U(\cdot)$; see, for instance, Section 3.6 in [17] for relevant theory. In particular, the continuous, increasing process $\Lambda(\cdot)$ is flat off the zero set

$$\mathfrak{Z} := \big\{0 \le t < \infty : S(t) = 0\big\}. \tag{2.3}$$

We shall impose the "non-stickiness" condition

$$\mathrm{Leb}(\mathfrak{Z}) \equiv \int_0^\infty \mathbf{1}_{\{S(t)=0\}}\, \mathrm{d}t = 0. \tag{2.4}$$

Finally, we recall the (*right*) *local time* $L^{\Xi}(\cdot)$ accumulated at the origin during the time-interval $[0, T]$ by a generic one-dimensional continuous semimartingale $\Xi(\cdot)$, namely

$$L^{\Xi}(T) := \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^T \mathbf{1}_{\{0\le \Xi(t)<\varepsilon\}}\, \mathrm{d}\langle \Xi\rangle(t), \quad 0 \le T < \infty. \tag{2.5}$$

2.1. *The main result*

Theorem 2.1 below, is our first key result. It produces a planar "skew-unfolding" $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ for the folding $S(\cdot)$ of the given continuous semimartingale $U(\cdot)$. This planar "skew-unfolded" process has radial part $\|X(\cdot)\| = S(\cdot)$, and its motion away from the origin follows the one-dimensional dynamics of $S(\cdot)$ along rays emanating from the origin. Once at the origin, the process $X(\cdot)$ chooses the next ray for its voyage (according to the dynamics of $S(\cdot)$)

independently of its past history, according to a given probability measure on the collection of angles in $[0, 2\pi)$. Whenever $S(\cdot)$ is a reflecting Brownian motion or, more generally, a reflecting diffusion, these one-dimensional dynamics away from the origin are diffusive.

In order to describe this skew-unfolding with some detail and rigor, we shall need appropriate notation. Let us consider the unit circumference

$$\mathfrak{S} := \big\{(z_1, z_2)' : z_1^2 + z_2^2 = 1\big\}.$$

Here and throughout the paper, vectors are columns and the superscript $'$ denotes transposition. For every point $x := (x_1, x_2)' \in \mathbb{R}^2$ we introduce the mapping $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)' : \mathbb{R}^2 \to \mathfrak{S} \cup \{\mathbf{0}\}$ via $\mathfrak{f}(\mathbf{0}) := \mathbf{0}$ and

$$\mathfrak{f}(x) := \frac{x}{\|x\|} = \big(\cos\big(\arg(x)\big), \sin\big(\arg(x)\big)\big)'; \quad x \in E := \mathbb{R}^2 \setminus \{\mathbf{0}\} \tag{2.6}$$

with the notation $\mathbf{0} := (0, 0)'$ and with $\arg(x) \in [0, 2\pi)$ denoting the *argument of the vector* $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ in its polar coordinates. We fix a probability measure $\boldsymbol{\mu}$ on the collection $\mathcal{B}(\mathfrak{S})$ of Borel subsets of the unit circumference $\mathfrak{S}$, and introduce the unit vector $\boldsymbol{\gamma} := (\gamma_1, \gamma_2)'$ with the following real constants

$$\alpha_i^{(\pm)} := \int_{\mathfrak{S}} z_i^{\pm} \boldsymbol{\mu}(\mathrm{d}z), \qquad \gamma_i := \alpha_i^{(+)} - \alpha_i^{(-)} = \int_{\mathfrak{S}} z_i \boldsymbol{\mu}(\mathrm{d}z), \quad i = 1, 2. \tag{2.7}$$

Theorem 2.1 (Construction of WALSH semimartingales). *Consider the* SKOROKHOD *reflection $S(\cdot)$ of the continuous semimartingale $U(\cdot)$ as in* (2.1)–(2.4), *and fix a probability measure $\boldsymbol{\mu}$ as above, as well as a vector $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)' \in \mathbb{R}^2$ with $\|\mathrm{x}\| = S(0)$.*

On a suitable enlargement $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}(t)\}_{0 \le t < \infty}$ of the filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\widetilde{\mathbb{F}}$ with a measure-preserving map $\boldsymbol{\pi} : \Omega \to \widetilde{\Omega}$, there exists a planar continuous semimartingale $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$ which satisfies the system of stochastic integral equations

$$X_i(T) = \mathrm{x}_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}S(t) + \gamma_i L^S(T), \quad 0 \le T < \infty \tag{2.8}$$

for $i = 1, 2$ in the notation of (2.6)–(2.7), *and whose radial part is*

$$\big\|X(\cdot)\big\| := \sqrt{X_1^2(\cdot) + X_2^2(\cdot)} = S(\cdot). \tag{2.9}$$

This continuous semimartingale $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$ has the following properties:

(i) *With $\mathrm{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and with $\tau(s) := \inf\{t > s : X(t) = \mathbf{0}\}$ the first visit to the origin after time $s \ge 0$, this process $X(\cdot)$ satisfies, for every $s \in (0, \infty)$, $B \in \mathcal{B}(\mathfrak{S})$ and for* LEBESGUE-*almost every $t \in (0, \infty)$, the properties*

$$\mathfrak{f}\big(X(s)\big) = \mathfrak{f}(\mathrm{x}), \quad \mathbb{P}\text{-a.e. on } \big\{\tau(0) > s\big\}, \tag{2.10}$$

$$\mathbb{P}\big(\mathfrak{f}\big(X\big(\tau(s) + t\big)\big) \in B \,|\, \mathcal{F}\big(\tau(s)\big)\big) = \boldsymbol{\mu}(B), \quad \mathbb{P}\text{-a.e. on } \big\{\tau(s) < \infty\big\}. \tag{2.11}$$

(ii) *The local times at the origin of the component processes $X_i(\cdot)$, are given as*

$$L^{X_i}(\cdot) \equiv \alpha_i^{(+)} L^{\|X\|}(\cdot), \quad i = 1, 2 \tag{2.12}$$

and are thus flat off the random set $\mathfrak{Z}$ in (2.3); *in particular,*

$$\int_0^\infty \mathbf{1}_{\{X(t) = \mathbf{0}\}}\,\mathrm{d}t \equiv 0. \tag{2.13}$$

(iii) *Finally, for every $A \in \mathcal{B}(\mathfrak{S})$, the semimartingale local time at the origin of the "thinned" process $R^A(\cdot) := \|X(\cdot)\| \cdot \mathbf{1}_A(\mathfrak{f}(X(\cdot)))$ is given by*

$$L^{R^A}(\cdot) \equiv \boldsymbol{\mu}(A) L^{\|X\|}(\cdot). \tag{2.14}$$

Terminology 2.1. The process $X(\cdot)$ constructed in the above Theorem can be thought of as a planar skew-unfolding of the SKOROKHOD reflection $S(\cdot)$ of the driving continuous semimartingale $U(\cdot)$; we shall call it a WALSH *semimartingale with* "*driver*" $U(\cdot)$ (and "*folded driver*" $S(\cdot)$). We shall call $X(\cdot)$ a WALSH diffusion, whenever the folded driver $S(\cdot)$ is an ITÔ diffusion with reflection at the origin.

As the referee observes, in all this the unit circle can be replaced by an n-dimensional sphere, with the process $X(\cdot)$ then evolving in $\mathbb{R}^{n+1}$ for some $n \geq 1$. The proofs presented here go then through as well.

3. Discussion and ramifications

An intuitive interpretation of the stochastic integral equations (2.8) with the property (2.9) is as follows: We first "fold" the driving semimartingale $U(\cdot)$ to get its SKOROKHOD reflection $S(\cdot)$ as in (2.2) and then, starting from the point $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)' \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ with $\mathrm{x}_i = \mathfrak{f}_i(\mathrm{x})S(0)$, $i = 1, 2$ and up until the time $\tau(0)$ of Theorem 2.1(i), we run the planar process $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ according to the integral equation

$$X_i(\cdot) = \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\, \mathrm{d}S(t), \quad \text{for } i = 1, 2 \tag{3.1}$$

on $[0, \tau(0))$. This is the equation to which (2.8) reduces on the interval $[0, \tau(0))$. Applying ITÔ's rule to the function $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)'$ of (2.6), we verify that *the motion of the planar process* $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ *during the time-interval* $[0, \tau(0))$ *is along the ray that connects the origin* $\mathbf{0}$ *to the starting point* x.

Now, every time the planar process $X(\cdot)$ visits the origin, the direction of the next ray for its $S(\cdot)$-governed motion is instantaneously chosen, at random, according to the probability distribution $\boldsymbol{\mu}$, the "spinning measure" of the process $X(\cdot)$, in a manner described in more detail later. If the origin is visited infinitely often during a time-interval of finite length, it is not surprising that this random choice should lead to the accumulation of local time at the origin, as indicated in the equations (2.8). It follows from (2.13) that the set of times $X(\cdot)$ spends at the origin, has zero LEBESGUE measure. The process continues to move then along the newly chosen ray, its motion governed by the stochastic integral equations of (3.1) just described, as long as it stays away from the origin. The path $t \mapsto X(t)$ is, with probability one, continuous in the usual topology of $\mathbb{R}^2$, as well as in the topology induced by the "tree-metric" (French railway metric)

$$\varrho(x, y) := (r_1 + r_2)\mathbf{1}_{\{\theta_1 \neq \theta_2\}} + |r_1 - r_2|\mathbf{1}_{\{\theta_1 = \theta_2\}}, \quad x = (r_1, \theta_1),\ y = (r_2, \theta_2). \tag{3.2}$$

The reader may find it useful at this juncture to think of a roundhouse at the origin, of the spokes of a bicycle wheel – or of the Aeolian winds of Homeric lore, that blow the raft of Odysseus in all directions at once.

3.1. *Spider semimartingales*

Suppose that the measure $\boldsymbol{\mu}$ charges only a finite number m of points on the unit circumference. We can think then of the planar process $X(\cdot)$ in Theorem 2.1 as a *Spider Semimartingale*, whose radial part $\|X(\cdot)\| = S(\cdot)$ is the SKOROKHOD reflection of the driver $U(\cdot)$ according to (2.9).

When the driving semimartingale $U(\cdot)$ is Brownian motion, the process $X(\cdot)$ of Theorem 2.1 becomes the original WALSH *Brownian Motion* (as constructed, for instance, in BARLOW, PITMAN & YOR [1]) with roundhouse singularity in a multipole field; this will be shown in Proposition 7.2 below. When $m = 2$ and $\boldsymbol{\mu}(\{(1, 0)\}) = \alpha \in (0, 1)$, $\boldsymbol{\mu}(\{(-1, 0)\}) = 1 - \alpha$, this construction recovers the familiar *Skew Brownian Motion*, introduced in [15] and studied by WALSH [27] and by HARRISON & SHEPP [11]. Using MARKOV semigroups and excursions, BARLOW, PITMAN & YOR [1] study WALSH's Brownian motion on a finite collection of rays. Their approach has been generalized to "multiple spider martingales" by YOR [29], and has been studied by TSIREL'SON [26], BARLOW, ÉMERY, KNIGHT, SONG & YOR [2], WATANABE [28] and MANSUY & YOR [18].

3.2. *Generalized skew-*TANAKA *and* HARRISON–SHEPP *equations*

In the context of Theorem 2.1 (in particular, with the property (2.9)), the equations of (2.8) can be cast in equivalent forms, now driven by the original semimartingale $U(\cdot)$, as follows (the latter when $\alpha_i^{(+)} > 0$):

$$X_i(\cdot) = \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}U(t) + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2, \tag{3.3}$$

$$X_i(\cdot) = \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}U(t) + \bigg(1 - \frac{\alpha_i^{(-)}}{\alpha_i^{(+)}}\bigg) L^{X_i}(\cdot), \quad i = 1, 2. \tag{3.4}$$

This last system (3.4) is a two-dimensional analogue of the *skew-*TANAKA *equation* studied by [12].

The system of equations (3.3), on the other hand, can be thought of as a planar semimartingale version of the HARRISON & SHEPP equation [11] for the skew Brownian motion. For two fixed real constants γ_1, γ_2, and a folded driver $S(\cdot)$ that satisfies the condition

$$\mathbb{P}\big(L^S(\infty) > 0\big) > 0 \tag{3.5}$$

(e.g., reflecting Brownian motion), we have the following necessary and sufficient condition (3.6), for the solvability of the system (3.3) subject to the requirement (2.9). The condition (3.6) is a two-dimensional analogue of the condition in [11] for the solvability of the stochastic equation that characterizes the skew Brownian motion. The proof of Theorem 3.1 right below, is given in Section 5.

Theorem 3.1 (A generalized HARRISON–SHEPP system of equations). *Consider a real-valued, continuous semimartingale $U(\cdot)$ along with its* SKOROKHOD *reflection $S(\cdot)$ as in Section 2.1, two real numbers γ_1, γ_2, and a vector $\mathrm{x} := (\mathrm{x}_1, \mathrm{x}_2)' \in \mathbb{R}^2$ with $\|\mathrm{x}\| = S(0)$.*

(i) *Suppose that γ_1, γ_2 satisfy the condition*

$$\gamma_1^2 + \gamma_2^2 \le 1. \tag{3.6}$$

There exists then a continuous planar semimartingale $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ that satisfies the system (3.3) and the condition (2.9).

(ii) *Conversely, suppose that (3.5) holds, and that there exists a continuous planar semimartingale $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ that satisfies the system (3.3) and the condition (2.9). Then (3.6) is satisfied by γ_1, γ_2.*

3.3. *An open question*

It would be of considerable interest to extend the methodology of this paper to a situation with an entire family $U(\cdot; z), z \in \mathfrak{S}$ of semimartigales so that, when the point z is selected on the unit circumference by the spinning measure $\boldsymbol{\mu}$, the motion along the corresponding ray is according to the SKOROKHOD reflection $S(\cdot; z)$ of this semimartingale $U(\cdot; z)$. Some results in this vein are obtained in Section 8, in the context of the diffusion case and by the method of scale function and time-change.

4. A FREIDLIN–SHEU-type formula

Definition 4.1. We consider the class $\mathfrak{D}$ of BOREL-measurable functions $g : \mathbb{R}^2 \to \mathbb{R}$ with the properties:

(i) for every $z \in \mathfrak{S}$, the function $r \longmapsto g_z(r) := g(rz)$ is twice continuously differentiable on $(0, \infty)$ and has finite first and second right-derivatives at the origin; and
(ii) $\sup_{\substack{z \in \mathfrak{S} \\ 0<r<K}} (|g_z'(r)| + |g_z''(r)|) < \infty$ holds for all $K \in (0, \infty)$.

Here we consider BOREL sets with respect to the Euclidean topology. We introduce also the subclasses

$$\mathfrak{D}^{\boldsymbol{\mu}} := \left\{ g \in \mathfrak{D} : \int_{\mathfrak{S}} g_z'(0+)\boldsymbol{\mu}(\mathrm{d}z) = 0 \right\}, \qquad \mathfrak{D}_+^{\boldsymbol{\mu}} := \left\{ g \in \mathfrak{D} : \int_{\mathfrak{S}} g_z'(0+)\boldsymbol{\mu}(\mathrm{d}z) \geq 0 \right\}. \tag{4.1}$$

Definition 4.2. For every given function $g : \mathbb{R}^2 \to \mathbb{R}$ in $\mathfrak{D}$, we set $\partial_r g(x) := g_z'(r)$, $\partial_{rr}^2 g(x) := g_z''(r)$ with $z = \mathfrak{f}(x)$, $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and introduce the notation $D_{\boldsymbol{\mu}} g(0) := \int_{\mathfrak{S}} g_z'(0+)\boldsymbol{\mu}(\mathrm{d}z)$.

With this notation in place, we can formulate our second major result. Its proof appears in Section 5.

Theorem 4.1 (A generalized FREIDLIN–SHEU formula). *With the above notation, every continuous semimartingale $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ which satisfies the system of equations (2.8) and the properties (2.9), (2.14), also satisfies for every $g \in \mathfrak{D}$ the generalized* FREIDLIN–SHEU *identity*

$$g\big(X(\cdot)\big) = g(\mathrm{x}) + \int_0^{\cdot} \mathbf{1}_{\{X(t)\neq\mathbf{0}\}} \left(\partial_r g\big(X(t)\big)\,\mathrm{d}S(t) + \frac{1}{2}\partial_{rr}^2 g\big(X(t)\big)\,\mathrm{d}\langle S\rangle(t) \right) + D_{\boldsymbol{\mu}} g(0) L^S(\cdot). \tag{4.2}$$

When the function g belongs to $C^2(\mathbb{R}^2)$, the identity (4.2) can be proved by the usual ITÔ formula.

4.1. *Slope-averaging martingales*

For any given bounded, measurable $\phi : \mathfrak{S} \to \mathbb{R}$, let us define the functions

$$h_{(\phi)}(x) := \big(\phi\big(\mathfrak{f}(x)\big) - \mathbb{E}\big[\phi(\boldsymbol{\xi}_1)\big]\big) \cdot \mathbf{1}_{\{x\neq\mathbf{0}\}}, \qquad g_{(\phi)}(x) := \|x\| \cdot h_{\phi}(x) \tag{4.3}$$

for $x \in \mathbb{R}^2$, where $\boldsymbol{\xi}_1$ is an $\mathfrak{S}$-valued random variable with distribution $\boldsymbol{\mu}$ as in (5.1). Such functions were first introduced by BARLOW, PITMAN & YOR [1], in their study of the WALSH Brownian motion. We observe that $g_{(\phi)}(\cdot)$ belongs to the class $\mathfrak{D}$ and satisfies $\partial_r g_{(\phi)}(\cdot) = h_{(\phi)}(\cdot)$, $\partial_{rr}^2 g_{(\phi)}(\cdot) = 0$ and $D_{\boldsymbol{\mu}} g_{(\phi)}(0) = 0$. The following result is now a corollary of Theorem 4.1.

Proposition 4.1. *Assume that $U(\cdot)$ in (2.1) is a real-valued, continuous local martingale, and construct its* SKOROKHOD *reflection $S(\cdot)$ as in (2.2). Consider any planar continuous semimartingale $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$ which satisfies the system of equations (2.8), along with the properties (2.9) and (2.14).*

Then for any given bounded, measurable function $\phi : \mathfrak{S} \to \mathbb{R}$ and with the notation of (4.3), the process below is a continuous, real-valued local martingale:

$$g_{(\phi)}\big(X(\cdot)\big) = \big\|X(\cdot)\big\| h_{(\phi)}\big(X(\cdot)\big) = g_{(\phi)}(\mathrm{x}) + \int_0^{\cdot} h_{(\phi)}\big(X(t)\big)\,\mathrm{d}U(t).$$

5. The proofs of Theorems 2.1, 4.1 and 3.1

We construct a planar process $X(\cdot)$ which satisfies the equation (2.8) via "folding and unfolding of semimartingales", with additional randomness coming from a sequence $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ of $\mathfrak{S}$-valued, I.I.D. random variables. These have common probability distribution $\boldsymbol{\mu}$ on $\mathfrak{S}$, such that the components of the random vector $\boldsymbol{\xi}_1 := (\xi_{1,1}, \xi_{1,2})'$ have expectations that are matched with the vector $(\alpha_1^{(+)}, \alpha_1^{(-)}, \alpha_2^{(+)}, \alpha_2^{(-)}) \in [0, 1]^4$ in (2.7), (2.8) as

$$\mathbb{E}\big(\xi_{1,i}^{\pm}\big) = \alpha_i^{(\pm)}, \qquad \mathbb{E}(\xi_{1,i}) = \alpha_i^{(+)} - \alpha_i^{(-)} = \gamma_i, \qquad \mathbb{E}\big(|\xi_{1,i}|\big) = \alpha_i^{(+)} + \alpha_i^{(-)}; \quad i = 1, 2. \tag{5.1}$$

Proof of Theorem 2.1. Following [20] and [12], we enlarge the probability space by means of the above sequence $\{\boldsymbol{\xi}_k\}_{k\in\mathbb{N}}$ of $\mathfrak{S}$-valued, I.I.D. random variables. These are independent of the σ-algebra $\widetilde{\mathcal{F}}(\infty) := \bigvee_{0\leq t<\infty} \widetilde{\mathcal{F}}(t)$ and have expectation $\mathbb{E}(\boldsymbol{\xi}_1) = \boldsymbol{\gamma}$ as in (5.1).

• Let us decompose the nonnegative half-line into the zero set $\mathfrak{Z}$ of $S(\cdot)$ as in (2.3) on the one hand, and the countable collection $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$ of open disjoint components of $[0,\infty)\setminus\mathfrak{Z}$ on the other. Each of them represents an excursion interval away from the origin for the process $S(\cdot)$ in (2.2). We enumerate these countably-many excursion intervals $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$ in a measurable manner, so that $\{t\in\mathcal{C}_k\}\in\widetilde{\mathcal{F}}(\infty)$ holds for all $t\geq 0$, $k\in\mathbb{N}$. For notational simplicity, we declare also $\mathcal{C}_0:=\mathfrak{Z}$, $\boldsymbol{\xi}_0:=\mathbf{0}$. We denote now

$$Z(t):=\mathfrak{f}(\mathrm{x})\cdot\mathbf{1}_{[0,\tau_0)}(t)+\sum_{k\in\mathbb{N}_0}\boldsymbol{\xi}_k\cdot\mathbf{1}_{\mathcal{C}_k\cap[\tau_0,\infty)}(t),\qquad X(t):=Z(t)S(t),\tag{5.2}$$

$\mathcal{F}^Z(t):=\sigma(Z(s),0\leq s\leq t)$ for $0\leq t<\infty$ with $\tau_0:=\inf\{t:S(t)=0\}$, and introduce the enlarged filtration $\mathbb{F}:=\{\mathcal{F}(t),0\leq t<\infty\}$ via $\mathcal{F}(t):=\widetilde{\mathcal{F}}(t)\vee\mathcal{F}^Z(t)$. Note the zero set of (2.3) is

$$\mathfrak{Z}=\big\{t\geq 0:S(t)=0\big\}=\big\{t\geq 0:Z(t)=\mathbf{0}\big\}=\big\{t\geq 0:X(t)=\mathbf{0}\big\}.\tag{5.3}$$

This procedure corresponds exactly to the program outlined by J. B. WALSH in the appendix to his 1978 paper: "*The idea is to take each excursion of* (*reflecting Brownian motion*) *and, instead of giving it a random sign, to assign it a random variable with a given distribution in* $[0,2\pi)$*, and to do so independently for each excursion*". We shall see presently the so-constructed process $X(\cdot)$ satisfies the system of equations (2.8).

• We claim that, because of independence and of the way the probability space was enlarged, *both processes* $U(\cdot)$ *and* $S(\cdot)$ *are continuous* $\mathbb{F}$*-semimartingales*. This claim can be established as in the proof of Proposition 2 in [20]; see also Proposition 3.1 in [12].

• In order to describe the dynamics of the process $X(\cdot)$ defined in (5.2), we approximate the process $Z(\cdot)$ also defined there by a family of processes $Z^\varepsilon(\cdot)$ with finite first variation over compact intervals, indexed by $\varepsilon\in(0,1)$, as follows. We define the sequence of stopping times $\tau_0^\varepsilon:=\inf\{t\geq 0:\|X(t)\|=0\}$ and

$$\tau_{2\ell+1}^\varepsilon:=\inf\big\{t>\tau_{2\ell}^\varepsilon:\big\|X(t)\big\|\geq\varepsilon\big\},\qquad\tau_{2\ell+2}^\varepsilon:=\inf\big\{t>\tau_{2\ell+1}^\varepsilon:\big\|X(t)\big\|=0\big\};\quad\ell\in\mathbb{N}_0\tag{5.4}$$

recursively. We also introduce a piecewise-constant process $Z^\varepsilon(\cdot):=(Z_1^\varepsilon(\cdot),Z_2^\varepsilon(\cdot))'$ with

$$Z^\varepsilon(t):=\sum_{\ell\in\mathbb{N}_0}Z(t)\mathbf{1}_{[\tau_{2\ell+1}^\varepsilon,\tau_{2\ell+2}^\varepsilon)}(t)=\sum_{(k,\ell)\in\mathbb{N}_0^2}\boldsymbol{\xi}_k\mathbf{1}_{\mathcal{C}_k\cap[\tau_{2\ell+1}^\varepsilon,\tau_{2\ell+2}^\varepsilon)}(t),\quad 0\leq t<\infty,\tag{5.5}$$

i.e., constant on each of the "downcrossing intervals" $[\tau_{2\ell+1}^\varepsilon,\tau_{2\ell+2}^\varepsilon)$. For this process, the product rule gives

$$X^\varepsilon(T):=Z^\varepsilon(T)S(T)=\int_0^T Z^\varepsilon(t)\,\mathrm{d}S(t)+\int_0^T S(t)\,\mathrm{d}Z^\varepsilon(t),\quad 0\leq T<\infty.\tag{5.6}$$

Passing to the limit as $\varepsilon\downarrow 0$ and using (5.1)–(5.2), as well as the characterization of the local time $L^S(\cdot)$ of the semimartingale $S(\cdot)$ in terms of the number of its downcrossings, we obtain in the notation of (2.7):

$$X(T)=Z(T)S(T)=\int_0^T Z(t)\,\mathrm{d}S(t)+\mathbb{E}[\boldsymbol{\xi}_1]L^S(T)=\int_0^T\mathfrak{f}\big(X(t)\big)\,\mathrm{d}S(t)+\boldsymbol{\gamma}L^S(T).\tag{5.7}$$

Indeed, let $\{\boldsymbol{\xi}_{\ell_j}\}_{j=1}^{N(T,\varepsilon)}$ denote an enumeration of $Z^\varepsilon(\tau_{2\ell+1}^\varepsilon)$, and $N(T,\varepsilon):=\sharp\{\ell\in\mathbb{N}:\tau_{2\ell}^\varepsilon<T\}$ the number of downcrossings of the interval $(0,\varepsilon)$ that the process $S(\cdot)$ has completed during $[0,T)$. Then the second term on the right-hand side of (5.6) can be estimated by the strong law of large numbers, using Theorem VI.1.10 in [21]; namely, we have in probability

$$\int_0^T S(t)\,\mathrm{d}Z^\varepsilon(t)=\sum_{\{\ell:\tau_{2\ell+1}^\varepsilon<T\}}S\big(\tau_{2\ell+1}^\varepsilon\big)Z^\varepsilon\big(\tau_{2\ell+1}^\varepsilon\big)=\varepsilon\sum_{j=1}^{N(T,\varepsilon)}\boldsymbol{\xi}_{\ell_j}+O(\varepsilon)\xrightarrow[\varepsilon\downarrow 0]{}L^S(T)\cdot\mathbb{E}[\boldsymbol{\xi}_1].\tag{5.8}$$

We deduce from (5.7) that the process $X(\cdot)$ is a continuous planar $\mathbb{F}$-semimartingale. And by analogy with (5.7), we can approximate the process $|Z_i(\cdot)|$ by $|Z_i^\varepsilon(\cdot)|$, the absolute value of each of the components $Z_i^\varepsilon(\cdot)$ of the piecewise-constant process in (5.5); passing to the limit as $\varepsilon \downarrow 0$, we obtain

$$\big|X_i(T)\big| = \big|Z_i(T)\big|S(T) = \int_0^T \big|Z_i(t)\big|\,\mathrm{d}S(t) + \mathbb{E}\big(|\xi_{1,i}|\big)L^S(T), \quad 0 \le T < \infty \tag{5.9}$$

for $i = 1, 2$. We appeal now to Exercise VI (1.16) 3^o) of [21] recalling the form of $S(\cdot)$ in (2.2) along with (5.3) we deduce that, with the normalization of (2.5), the continuous, nonnegative semimartingale $|X_i(\cdot)|, i = 1, 2$ with the decomposition (5.9) has local time at the origin

$$L^{|X_i|}(\cdot) = \int_0^\cdot \mathbf{1}_{\{X_i(t)=0\}}\big[\big|Z_i(t)\big|\,\mathrm{d}S(t) + \big(\alpha_i^{(+)} + \alpha_i^{(-)}\big)\,\mathrm{d}L^S(t)\big] = \big(\alpha_i^{(+)} + \alpha_i^{(-)}\big)L^S(\cdot). \tag{5.10}$$

• At this point, we need to identify the local times $L^{X_i}(\cdot)$ of each component $X_i(\cdot)$ in terms of the local time $L^S(\cdot)$. Since $X_i(\cdot) = Z_i(\cdot)S(\cdot)$ is a continuous semimartingale for $i = 1, 2$, we recall the decomposition (5.7) and properties of semimartingale local time and obtain the string of identities

$$2L^{X_i}(\cdot) - L^{|X_i|}(\cdot) = \int_0^\cdot \mathbf{1}_{\{X_i(t)=0\}}\,\mathrm{d}X_i(t) = \gamma_i L^S(\cdot) = \big(\alpha_i^{(+)} - \alpha_i^{(-)}\big)L^S(\cdot). \tag{5.11}$$

Thus, combining with (5.10), we deduce $2L^{X_i}(\cdot) = 2\alpha_i^{(+)}L^S(\cdot), i = 1, 2$, i.e., property (2.12). The equations (2.8) and (2.9), (2.13) follow now from (2.4), (2.7) and (5.7). The property (2.14) can be shown by taking the “thinned” sequence $\boldsymbol{\xi}_k^A := \mathbf{1}_A(\arg(\boldsymbol{\xi}_k))$ in place of $\boldsymbol{\xi}_k$, $k \in \mathbb{N}_0$ in the proof of (2.12).

• We note now $\{\mathfrak{f}(X(s)) = \mathfrak{f}(\mathrm{x}), s < \tau(0)\} = \{s < \tau(0)\}$, mod. $\mathbb{P}$, which verifies (2.10). Moreover, for every $(s,t) \in (0,\infty)^2$, there exists by construction an $\widetilde{\mathcal{F}}(\infty)$-measurable random index $\kappa_0(s,t) : \Omega \to \mathbb{N}$ such that we have, either $\tau(s) + t \in \mathcal{C}_{\kappa_0(s,t)}$, or $\tau(s) + t \in \mathfrak{Z}$ on $\{\tau(s) < \infty\}$. If $\tau(s) + t \in \mathfrak{Z}$ and $\tau(s) < \infty$, then $\mathfrak{f}(X(\tau(s) + t)) = \mathbf{0}$. By (2.4) we obtain $\mathbb{P}(\mathfrak{f}(X(\tau(s) + t)) = \mathbf{0}) = \mathbb{P}(S(\tau(s) + t) = 0) = 0$ for a.e. $t \in (0,\infty)$. Therefore,

$$\big\{\mathfrak{f}\big(X\big(\tau(s) + t\big)\big) \in B, \tau(s) < \infty\big\} = \big\{\boldsymbol{\xi}_{\kappa_0(s,t)} \in B, \tau(s) < \infty\big\}$$

holds mod. $\mathbb{P}$ for every $B \in \mathcal{B}(\mathfrak{S})$ and almost every $t \in (0,\infty)$. We conclude that (2.11) holds, namely

$$\mathbb{P}\big(\mathfrak{f}\big(X\big(\tau(s) + t\big)\big) \in B|\mathcal{F}\big(\tau(s)\big)\big) = \mathbb{P}\big(\boldsymbol{\xi}_{\kappa_0(s,t)} \in B|\mathcal{F}\big(\tau(s)\big)\big) = \mathbb{E}\big[\mathbb{P}(\boldsymbol{\xi}_1 \in B)|\mathcal{F}\big(\tau(s)\big)\big] = \boldsymbol{\mu}(B),$$

for every $s \in (0,\infty)$, $B \in \mathcal{B}(\mathfrak{S})$ and almost every $t \in (0,\infty)$. We have used here the definitions of $\mathcal{F}(\cdot) = \widetilde{\mathcal{F}}(\cdot) \vee \mathcal{F}^Z(\cdot)$ and $Z(\cdot)$ in (5.2), and the independence between $\widetilde{\mathcal{F}}(\infty)$ and the sequence of I.I.D. random variables $\{\boldsymbol{\xi}_k\}_{k\in\mathbb{N}}$. This completes the proof of Theorem 2.1. □

Proof of Theorem 4.1. We fix a function $g : \mathbb{R}^2 \to \mathbb{R}$ in the class $\mathfrak{D}$ as in the statement of the theorem, and recall the notation established in Definitions 4.1, 4.2. We consider also a continuous planar semimartingale $X(\cdot)$ satisfying the equations of (2.8) along with the properties (2.9) and (2.14). With $\{\tau_k^\varepsilon\}_{k\in\mathbb{N}_0}$ defined as in (5.4), and with $\tau_{-1}^\varepsilon \equiv 0$ and $\mathbb{N}_{-1} := \mathbb{N}_0 \cup \{-1\}$, the value $g(X(T))$ is decomposed into

$$\begin{aligned} g\big(X(T)\big) = g(\mathrm{x}) &+ \sum_{\ell\in\mathbb{N}_{-1}} \big(g\big(X\big(T \wedge \tau_{2\ell+2}^\varepsilon\big)\big) - g\big(X\big(T \wedge \tau_{2\ell+1}^\varepsilon\big)\big)\big) \\ &+ \sum_{\ell\in\mathbb{N}_0} \big(g\big(X\big(T \wedge \tau_{2\ell+1}^\varepsilon\big)\big) - g\big(X\big(T \wedge \tau_{2\ell}^\varepsilon\big)\big)\big). \end{aligned} \tag{5.12}$$

• We recall from the beginning of Section 3, that the process $X(\cdot)$ moves along the ray that connects $\mathbf{0}$ to the starting point $\mathrm{x} \neq \mathbf{0}$, during the time-interval $[0,\tau(0)) = [\tau_{-1}^\varepsilon, \tau_0^\varepsilon)$. In a similar manner, the processes $\mathfrak{f}_i(X(\cdot))$ are constant on every interval $[\tau^\varepsilon_{2\ell+1},\tau^\varepsilon_{2\ell+2})$ for $\ell\in\mathbb{N}_0$, $i=1,2$. The first summation in (5.12) is thus rewritten as $\sum_{\ell\in\mathbb{N}_{-1}}(g_z(S(T\wedge\tau^\varepsilon_{2\ell+2}))-g_z(S(T\wedge\tau^\varepsilon_{2\ell+1})))|_{z=Z(T\wedge\tau^\varepsilon_{2\ell+1})}$, or equivalently as

$$\sum_{\ell\in\mathbb{N}_{-1}}\int_{T\wedge\tau^\varepsilon_{2\ell+1}}^{T\wedge\tau^\varepsilon_{2\ell+2}}\Big(g_z'\big(S(t)\big)\,\mathrm{d}S(t)+\frac{1}{2}g_z''\big(S(t)\big)\,\mathrm{d}\langle S\rangle(t)\Big)\Big|_{z=Z(t)}$$
$$=\int_0^T\Big(\sum_{\ell\in\mathbb{N}_{-1}}\mathbf{1}_{(\tau^\varepsilon_{2\ell+1},\tau^\varepsilon_{2\ell+2})}(t)\Big)\Big(\partial_r g\big(X(t)\big)\,\mathrm{d}S(t)+\frac{1}{2}\partial^2_{rr}g\big(X(t)\big)\,\mathrm{d}\langle S\rangle(t)\Big).$$

We have set here $Z(\cdot)=\mathfrak{f}(X(\cdot))$, and applied Itô's rule (Problem 3.7.3 in [17]) to the process $g_z(S(\cdot))$. Letting $\varepsilon\downarrow 0$ in the above expression, we obtain in the limit (in probability)

$$\int_0^T\mathbf{1}_{\{X(t)\neq\mathbf{0}\}}\Big(\partial_r g\big(X(t)\big)\,\mathrm{d}S(t)+\frac{1}{2}\partial^2_{rr}g\big(X(t)\big)\,\mathrm{d}\langle S\rangle(t)\Big). \tag{5.13}$$

- Now we observe that $g(\mathbf{0})=g_z(0)$ holds by definition, so the second summation in (5.12) is cast as

$$\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}\big(g_z\big(S\big(\tau^\varepsilon_{2\ell+1}\big)\big)-g_z(0)\big)\Big|_{z=Z(\tau^\varepsilon_{2\ell+1})}+O(\varepsilon)$$
$$=\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}\big(g_z(\varepsilon)-g_z(0)\big)\Big|_{z=Z(\tau^\varepsilon_{2\ell+1})}+O(\varepsilon)$$
$$=\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}\Big(\varepsilon g_z'(0+)+\int_0^\varepsilon(\varepsilon-u)g_z''(u)\,\mathrm{d}u\Big)\Big|_{z=Z(\tau^\varepsilon_{2\ell+1})}+O(\varepsilon)\xrightarrow[\varepsilon\downarrow 0]{}L^S(T)\int_{\mathfrak{S}}g_z'(0+)\boldsymbol{\mu}(\mathrm{d}z) \tag{5.14}$$

in probability. Indeed, by analogy with (5.8) we can verify

$$\Big|\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}\Big(\int_0^\varepsilon(\varepsilon-u)g_z''(u)\,\mathrm{d}u\Big)\Big|_{z=Z(\tau^\varepsilon_{2\ell+1})}\Big|\le c\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}\varepsilon^2=c\varepsilon\cdot\big(\varepsilon N(T,\varepsilon)+O(\varepsilon)\big)\xrightarrow[\varepsilon\downarrow 0]{}0$$

in probability, where $c:=\sup_{z\in\mathfrak{S}}\max_{0\le u\le 1}(g_z''(u)/2)<+\infty$ by assumption.

We also check for every set $A\in\mathcal{B}(\mathfrak{S})$, on account of the property (2.14) for the process $R^A(\cdot)=\|X(\cdot)\|\mathbf{1}_A(Z(\cdot))$, the convergence in probability

$$\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}\varepsilon\mathbf{1}_{\{Z(\tau^\varepsilon_{2\ell+1})\in A\}}=\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}S\big(\tau^\varepsilon_{2\ell+1}\big)\mathbf{1}_{\{Z(\tau^\varepsilon_{2\ell+1})\in A\}}=\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}\big\|X\big(\tau^\varepsilon_{2\ell+1}\big)\big\|\mathbf{1}_{\{Z(\tau^\varepsilon_{2\ell+1})\in A\}}$$
$$=\sum_{\{\ell:\widetilde{\tau}^\varepsilon_{2\ell+1}<T\}}R^A\big(\widetilde{\tau}^\varepsilon_{2\ell+1}\big)=\varepsilon\widetilde{N}(T,\varepsilon)+O(\varepsilon)\xrightarrow[\varepsilon\downarrow 0]{}L^{R^A}(T)=\boldsymbol{\mu}(A)L^S(T).$$

Here we set $\widetilde{\tau}^\varepsilon_0:=\inf\{t\ge 0:R^A(t)=0\}$, and recursively $\widetilde{\tau}^\varepsilon_{2\ell+1}:=\inf\{t>\widetilde{\tau}^\varepsilon_{2\ell}:R^A(t)\ge\varepsilon\}$, $\widetilde{\tau}^\varepsilon_{2\ell+2}:=\inf\{t>\widetilde{\tau}^\varepsilon_{2\ell+1}:R^A(t)=0\}$ for $\ell\in\mathbb{N}_0$; and also denote by $\widetilde{N}(\varepsilon,T)$ the number of downcrossings of the interval $(0,\varepsilon)$ that the process $R^A(\cdot)$ has completed during the interval $[0,T)$ (we count the number of downcrossings corresponding to the rays in the directions in the subset A of $[0,2\pi)$). Approximating the function $z\mapsto g_z'(0+)$ by indicators $z\mapsto\mathbf{1}_A(z)$, $A\in\mathcal{B}(\mathfrak{S})$, we verify the convergence

$$\sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}}\varepsilon g'_{Z(\tau^\varepsilon_{2\ell+1})}(0+)\xrightarrow[\varepsilon\downarrow 0]{}L^S(T)\int_{\mathfrak{S}}g_z'(0+)\boldsymbol{\mu}(\mathrm{d}z),\qquad\text{in probability.} \tag{5.15}$$

• Therefore, the limit of the expression in (5.12) is the sum of the limits of the expressions in (5.13) and (5.14). Thus we obtain (4.2), since both its sides are continuous processes. This fact is given by the continuity of g in the topology induced by the tree-metric, an easy consequence of Definition 4.1. □

Proof of Theorem 3.1. (i) Assume $\gamma_1^2 + \gamma_2^2 \le 1$, and consider the vector $\boldsymbol{\gamma} := (\gamma_1, \gamma_2)' \in \mathbb{R}^2$. Then we define the probability measure $\boldsymbol{\mu} := ((1+\beta)/2)\delta_{z_0} + ((1-\beta)/2)\delta_{-z_0}$ on $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ with $\beta := \|\boldsymbol{\gamma}\| \le 1$ and $z_0 := \boldsymbol{\gamma}/\beta \in \mathfrak{S}$ provided that $\beta \neq 0$ (if $\beta = 0$, we simply pick an arbitrary $z_0 \in \mathfrak{S}$), and note

$$\int_{\mathfrak{S}} z\boldsymbol{\mu}(\mathrm{d}z) = \frac{1+\beta}{2} z_0 + \frac{1-\beta}{2}(-z_0) = \beta z_0 = \boldsymbol{\gamma}.$$

Thus, if we take the process $S(\cdot)$ in Section 2 as the "folded driver", and the above $\boldsymbol{\mu}$ as the "spinning measure", Theorem 2.1 constructs a continuous planar semimartingale $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ that satisfies the condition (2.9) as well as the system of equations (2.8) – thus also the system (3.3).

(ii) Suppose now that (3.5) holds, and that there exists a continuous semimartingale $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ which satisfies (2.9) and the system of equations (3.3), thus also of (2.8). For every $\varepsilon > 0$, we define $\tau_{-1}^{\varepsilon} \equiv 0$ and the sequence $\{\tau_m^{\varepsilon}\}_{m \in \mathbb{N}_0}$ as in (5.4).

Following the proof for Theorem 4.1, we write the equation (5.12) for the identity function $g(x) \equiv x$. Then, as $\varepsilon \downarrow 0$ and on account of (2.8), the first summation $\sum_{\ell \in \mathbb{N}_{-1}} (X(T \wedge \tau_{2\ell+2}^{\varepsilon}) - X(T \wedge \tau_{2\ell+1}^{\varepsilon}))$ in the resulting expression converges in probability to $\int_0^T \mathfrak{f}(X(t))\,\mathrm{d}S(t)$, just as in the proof of Theorem 4.1. Thus, thanks to (3.3), the second summation converges in probability to $\boldsymbol{\gamma} L^{\|X\|}(T)$, and this implies

$$\sum_{\ell \in \mathbb{N}_0} \big(X\big(T \wedge \tau_{2\ell+1}^{\varepsilon}\big) - X\big(T \wedge \tau_{2\ell}^{\varepsilon}\big)\big) = \sum_{\ell=0}^{N(T,\varepsilon)-1} \varepsilon \mathfrak{f}\big(X\big(\tau_{2\ell+1}^{\varepsilon}\big)\big) + O(\varepsilon) \underset{\varepsilon \downarrow 0}{\longrightarrow} \boldsymbol{\gamma} L^{\|X\|}(T)$$

in probability. By Theorem VI.1.10 in [21] once again, we also have the convergence in probability $\varepsilon N(T, \varepsilon) \to L^{\|X\|}(T)$ as $\varepsilon \downarrow 0$ where $N(T, \varepsilon) := \sharp\{\ell \in \mathbb{N} : \tau_{2\ell}^{\varepsilon} < T\}$. Therefore,

$$\frac{1}{N(T,\varepsilon)} \sum_{\ell=0}^{N(T,\varepsilon)-1} \mathfrak{f}\big(X\big(\tau_{2\ell+1}^{\varepsilon}\big)\big) \underset{\varepsilon \downarrow 0}{\longrightarrow} \boldsymbol{\gamma} \qquad \text{holds in probability, on the event } \big\{L^{\|X\|}(T) > 0\big\}.$$

Now $\|\boldsymbol{\gamma}\| \le 1$ follows from $\|\mathfrak{f}(\cdot)\| \le 1$, since we can select (thanks to (3.5) and (2.9)) a sufficiently large $T \in (0, \infty)$ such that $\mathbb{P}(L^{\|X\|}(T) > 0) > 0$. □

6. Walsh diffusions and the associated martingale problems

We cannot expect pathwise uniqueness, therefore neither can we expect strength, to hold for the equations of (2.8) or (3.3). Indeed, when $U(\cdot)$ is standard Brownian motion, the process $X(\cdot)$ constructed in Theorem 2.1 is the Walsh Brownian motion – a process whose filtration cannot be generated by *any* Brownian motion of *any* dimension; see the celebrated paper by Tsirel'son [26] , as well as Proposition 7.2 below and [18]. In light of these observations, it is natural to ask whether the next best thing, that is, *uniqueness in distribution*, might hold for these equations under appropriate conditions. We try in this section to provide some affirmative answers to this question, when the folded driving semimartingale $S(\cdot)$ is a reflected diffusion; the main results appear in Propositions 6.2 and 6.3.

6.1. *The folded driving semimartingale as a reflected diffusion*

Let us start by considering the canonical space $\Omega_1 := C([0, \infty); [0, \infty))$ of nonnegative, continuous functions on $[0, \infty)$. We endow this space with the usual topology of uniform convergence over compact intervals and with the σ-algebra $\mathcal{F}_1 := \mathcal{B}(\Omega_1)$ of its Borel sets. We consider also the filtration $\mathbb{F}_1 := \{\mathcal{F}_1(t)\}_{0 \le t < \infty}$ generated by its coordinate mapping, i.e., $\mathcal{F}_1(t) = \sigma(\omega_1(s), 0 \le s \le t)$.

Given BOREL-measurable coefficients $\boldsymbol{b} : [0,\infty) \to \mathbb{R}$ and $\boldsymbol{\sigma} : [0,\infty) \to \mathbb{R} \setminus \{0\}$, we define

$$\begin{aligned} K^{\psi}(\cdot;\omega_1) &:= \psi\big(\omega_1(\cdot)\big) - \psi\big(\omega_1(0)\big) - \int_0^{\cdot} \mathcal{G}\psi\big(\omega_1(t)\big) \cdot \mathbf{1}_{\{\omega_1(t)>0\}}\,\mathrm{d}t, \\ \mathcal{G}\psi(r) &:= \boldsymbol{b}(r)\psi'(r) + \frac{1}{2}\boldsymbol{\sigma}^2(r)\psi''(r); \quad r \in [0,\infty),\ \psi \in C_0^2\big([0,\infty);\mathbb{R}\big). \end{aligned} \tag{6.1}$$

6.1.1. *Local submartingale problem for a reflected diffusion*

In the manner of STROOCK & VARADHAN [25], we formulate the *Local Submartingale Problem associated with the pair* $(\boldsymbol{\sigma}, \boldsymbol{b})$ as follows.

Local submartingale problem associated with the pair $(\boldsymbol{\sigma}, \boldsymbol{b})$. For every given $x \in [0,\infty)$, to find a probability measure $\mathbb{Q}^{\bullet}$ on the space $(\Omega_1, \mathcal{F}_1)$, under which:

(i) $\omega_1(0) = x$ and $\int_0^{\infty} \mathbf{1}_{\{\omega_1(t)=0\}}\,\mathrm{d}t = 0$ hold $\mathbb{Q}^{\bullet}$-a.e.; and moreover,
(ii) for every function $\psi \in C^2([0,\infty);\mathbb{R})$ with $\psi'(0+) \geq 0$, the process $K^{\psi}(\cdot)$ in (6.1) is a continuous local submartingale, and a continuous local martingale whenever $\psi'(0+) = 0$, with respect to the filtration $\mathbb{F}_1^{\bullet} = \{\mathcal{F}_1^{\bullet}(t)\}_{0 \le t < \infty}$ with $\mathcal{F}_1^{\bullet}(t) := \mathcal{F}_1^{\circ}(t+)$.

Here we have denoted by $\mathbb{F}_1^{\circ} := \{\mathcal{F}_1^{\circ}(t), 0 \le t < \infty\}$ the augmentation of $\mathbb{F}_1$ under $\mathbb{Q}^{\bullet}$. As usual, we shall say that this problem is *well-posed*, if it admits exactly one solution.

6.2. *A local martingale problem for the planar diffusion*

Consider now the canonical space $\Omega_2 := C([0,\infty);\mathbb{R}^2)$ of $\mathbb{R}^2$-valued continuous functions on $[0,\infty)$, with the σ-algebra $\mathcal{F}_2 := \mathcal{B}(\Omega_2)$ of its BOREL sets. Consider also its coordinate mapping and the natural filtration $\mathbb{F}_2 := \{\mathcal{F}_2(t)\}_{0 \le t < \infty}$ with $\mathcal{F}_2(t) = \sigma(\omega_2(s), 0 \le s \le t)$. We recall the Definitions 4.1, 4.2.

Given a probability measure $\boldsymbol{\mu}$ on $\mathcal{B}(\mathfrak{S})$, and BOREL-measurable functions $\boldsymbol{b} : [0,\infty) \to \mathbb{R}$, $\boldsymbol{\sigma} : [0,\infty) \to \mathbb{R} \setminus \{0\}$ as in Section 6.1, we define for every function $g \in \mathfrak{D}$ the process

$$\begin{aligned} M^{g}(\cdot;\omega_2) &:= g\big(\omega_2(\cdot)\big) - g\big(\omega_2(0)\big) - \int_0^{\cdot} \mathcal{L}g\big(\omega_2(t)\big) \cdot \mathbf{1}_{\{\|\omega_2(t)\|>0\}}\,\mathrm{d}t, \quad \text{where} \\ \mathcal{L}g(x) &:= \boldsymbol{b}\big(\|x\|\big)\partial_r g(x) + \frac{1}{2}\boldsymbol{\sigma}^2\big(\|x\|\big)\partial_{rr}^2 g(x); \quad x \in \mathbb{R}^2. \end{aligned} \tag{6.2}$$

6.2.1. *The local martingale problem*

We formulate now the *Local Martingale Problem associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ as follows.

Local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$. For every fixed x $\in \mathbb{R}^2$, to find a probability measure $\mathbb{Q}$ on the canonical space $(\Omega_2, \mathcal{F}_2)$, such that:

(i) $\omega_2(0) = \mathrm{x}$ holds $\mathbb{Q}$-a.e.;
(ii) the analogue of the "non-stickiness" property (2.13) holds, namely

$$\int_0^{\infty} \mathbf{1}_{\{\omega_2(t)=\mathbf{0}\}}\,\mathrm{d}t = 0, \quad \mathbb{Q}\text{-a.e.}; \tag{6.3}$$

(iii) for every function g in $\mathfrak{D}_+^{\boldsymbol{\mu}}$ (respectively, $\mathfrak{D}^{\boldsymbol{\mu}}$) as in (4.1), the process $M^g(\cdot;\omega_2)$ of (6.2) is a continuous local submartingale (resp., martingale) with respect to the filtration $\mathbb{F}_2^{\bullet} := \{\mathcal{F}_2^{\bullet}(t)\}_{0 \le t < \infty}$.

Here we have set $\mathcal{F}_2^{\bullet}(t) := \mathcal{F}_2^{\circ}(t+)$, and denoted by $\mathbb{F}_2^{\circ} = \{\mathcal{F}_2^{\circ}(t)\}_{0 \le t < \infty}$ the $\mathbb{Q}$-augmentation of the filtration $\mathbb{F}_2$. Again, this problem is called "well-posed" if it admits exactly one solution.

• The theory of the STROOCK & VARADHAN martingale problem is extended in Proposition 6.1 right below, for a continuous planar semimartingale $X(\cdot)$ that satisfies the properties (2.13)-(2.14) and, with coefficients γ_i, $i=1,2$ given through (2.7), the system of stochastic integral equations

$$X_i(\cdot) = X_i(0) + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\big[\boldsymbol{b}\big(\|X(t)\|\big)\,\mathrm{d}t + \boldsymbol{\sigma}\big(\|X(t)\|\big)\,\mathrm{d}W(t)\big] + \gamma_i L^{\|X\|}(\cdot), \quad i=1,2. \tag{6.4}$$

Proposition 6.1 (Stochastic equations for WALSH diffusions).

(a) *For every weak solution* (X,W), $(\Omega,\mathcal{F},\mathbb{P})$, $\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$ *to the system of stochastic equations* (6.4), *we have*

$$\|X(\cdot)\| = \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}}\big(\boldsymbol{b}\big(\|X(t)\|\big)\,\mathrm{d}t + \boldsymbol{\sigma}\big(\|X(t)\|\big)\,\mathrm{d}W(t)\big) + L^{\|X\|}(\cdot); \tag{6.5}$$

and if this weak solution also satisfies the conditions (2.13)–(2.14), *then it induces a solution to the local martingale problem associated with the triple* $(\boldsymbol{\sigma},\boldsymbol{b},\boldsymbol{\mu})$.

(b) *Conversely, every solution to the local martingale problem associated with the triple* $(\boldsymbol{\sigma},\boldsymbol{b},\boldsymbol{\mu})$ *induces a weak solution to the system* (6.4) *which satisfies the conditions* (2.13)–(2.14). *The state process* $X(\cdot)$ *in this weak solution satisfies also the system of stochastic equations* (2.8) *with "folded driver"* $S(\cdot)=\|X(\cdot)\|$.

(c) *Uniqueness holds for the local martingale problem associated with* $(\boldsymbol{\sigma},\boldsymbol{b},\boldsymbol{\mu})$, *if and only if uniqueness in distribution holds for the system of integral equations* (6.4), *subject to the conditions* (2.13), (2.14).

Proof of part (a). We first validate (6.5) for any weak solution to (6.4). From (6.4) we see

$$\int_0^T \big(\big|\mathfrak{f}_i\big(X(t)\big)\boldsymbol{b}\big(\|X(t)\|\big)\big| + \mathfrak{f}_i^2\big(X(t)\big)\boldsymbol{\sigma}^2\big(\|X(t)\|\big)\big)\,\mathrm{d}t < \infty, \quad i=1,2,\ 0\le T<\infty.$$

Since $\mathfrak{f}_1^2(\mathrm{x})+\mathfrak{f}_2^2(\mathrm{x})=1$ and $|\mathfrak{f}_1(\mathrm{x})|+|\mathfrak{f}_2(\mathrm{x})|\ge 1$ hold for any $\mathrm{x}\in\mathbb{R}^2\setminus\{\mathbf{0}\}$, we obtain then

$$\int_0^T \mathbf{1}_{\{\|X(t)\|>0\}}\big(\big|\boldsymbol{b}\big(\|X(t)\|\big)\big| + \boldsymbol{\sigma}^2\big(\|X(t)\|\big)\big)\,\mathrm{d}t < \infty, \quad 0\le T<\infty. \tag{6.6}$$

Let us recall at this point the proof of FREIDLIN–SHEU formula with $g(x)=\|x\|$, $x\in\mathbb{R}^2$. We note that $g_z'(0+)\equiv 1$, and therefore the sum in (5.15) converges to $L^{\|X\|}(T)$ as $\varepsilon\downarrow 0$, without assuming (2.14). We obtain (6.5) for the radial process $\|X(\cdot)\|$. The continuous semimartingale $X(\cdot)$ thus satisfies the system of equations (2.8) with the "folded driver" $S(\cdot)=\|X(\cdot)\|$.

Suppose now that the properties (2.13)–(2.14) are also satisfied by the weak solution we have posited. From Theorem 4.1, for every function $g\in\mathfrak{D}_+^{\boldsymbol{\mu}}$ (resp., $g\in\mathfrak{D}^{\boldsymbol{\mu}}$), the process $M^g(\cdot;X)$ as in (6.2) is then a local submartingale (resp., martingale). The property (6.3) comes from (2.13). Consequently, a solution $\mathbb{Q}$ to the local martingale problem associated with the triple $(\boldsymbol{\sigma},\boldsymbol{b},\boldsymbol{\mu})$ is given by the probability measure $\mathbb{Q}=\mathbb{P}X^{-1}$ induced by the process $X(\cdot)$ on the canonical space $(\Omega_2,\mathcal{F}_2)$. □

Proof of part (b). Conversely, suppose that the local martingale problem of Section 6.2.1 associated with the triple $(\boldsymbol{\sigma},\boldsymbol{b},\boldsymbol{\mu})$ has a solution $\mathbb{Q}$. We recall the notation in (2.7) and define on the canonical space the processes

$$X(\cdot)\equiv\big(X_1(\cdot),X_2(\cdot)\big)' := \big(\|\omega_2(\cdot)\|\mathfrak{f}_1\big(\omega_2(\cdot)\big),\ \|\omega_2(\cdot)\|\mathfrak{f}_2\big(\omega_2(\cdot)\big)\big)'. \tag{6.7}$$

As in Proposition 4.1, we consider the following functions in the family $\mathfrak{D}^{\boldsymbol{\mu}}$ of (4.1):

$$g_i(x) = x_i - \gamma_i\|x\|, \quad x\in\mathbb{R}^2, i=1,2. \tag{6.8}$$

Consider also the functions $g_3\in\mathfrak{D}_+^{\boldsymbol{\mu}}$ and $g_{j,k}\in\mathfrak{D}^{\boldsymbol{\mu}}$ for $1\le j,k\le 3$ defined by

$$g_3(x) := \|x\|, \qquad g_{j,k}(x) := g_j(x)g_k(x); \qquad x=(r,\theta)\in\mathbb{R}^2. \tag{6.9}$$

• We deduce then from (iii) in the definition of the local martingale problem, that the processes $M_i(\cdot) := M^{g_i}(\cdot; X)$ and $M_{j,k}(\cdot) := M^{g_{j,k}}(\cdot; X)$ are continuous local martingales for $1 \le i \le 2$ and $1 \le j, k \le 3$. Following Definition VIII.3.2 in [21], we introduce the operator Γ on $\mathfrak{D}^{\boldsymbol{\mu}}_+ \times \mathfrak{D}^{\boldsymbol{\mu}}_+$ as

$$\Gamma(f,g)(x) := \big(\mathcal{L}(fg) - f\mathcal{L}(g) - g\mathcal{L}(f)\big)(x)\mathbf{1}_{\{\|x\|>0\}} = \boldsymbol{\sigma}^2\big(\|x\|\big)\partial_r g(x)\partial_r f(x)\mathbf{1}_{\{x\neq\mathbf{0}\}} \tag{6.10}$$

for $f, g \in \mathfrak{D}^{\boldsymbol{\mu}}_+$. Here the last equality comes from the expression of the operator $\mathcal{L}$, where $\partial_r f(\cdot)$ and $\partial_r g(\cdot)$ are derivatives of $f(\cdot)$ and $g(\cdot)$, respectively, in the sense of Definition 4.2. In light of Proposition VIII.3.3 in [21], we identify for $1 \le j, k \le 2$ the cross-variation structure

$$\langle M_j, M_k\rangle(\cdot) = \int_0^{\cdot} r_{j,k}(t)\,\mathrm{d}t, \quad r_{j,k}(t) := \boldsymbol{\sigma}^2\big(\|X(t)\|\big)\big(\mathfrak{f}_j\big(X(t)\big) - \gamma_j\big)\big(\mathfrak{f}_k\big(X(t)\big) - \gamma_k\big)\mathbf{1}_{\{\|X(t)\|>0\}}. \tag{6.11}$$

• We also observe that the continuous process

$$N(\cdot) := M^{g_3}(\cdot; X) = \big\|X(\cdot)\big\| - \big\|X(0)\big\| - \int_0^{\cdot} \boldsymbol{b}\big(\|X(t)\|\big)\mathbf{1}_{\{\|X(t)\|>0\}}\,\mathrm{d}t \tag{6.12}$$

is a local submartingale; this way we obtain the semimartingale property of the radial process $\|X(\cdot)\|$. By the DOOB–MEYER decomposition (e.g., [17], Theorem 1.4.10), there exists then an adapted, continuous and increasing process $A(\cdot)$ such that

$$M_3(\cdot) := N(\cdot) - A(\cdot) = \big\|X(\cdot)\big\| - \big\|X(0)\big\| - \int_0^{\cdot} \boldsymbol{b}\big(\|X(t)\|\big)\mathbf{1}_{\{\|X(t)\|>0\}}\,\mathrm{d}t - A(\cdot) \tag{6.13}$$

is a continuous local martingale. *We claim that this increasing process is $A(\cdot) = L^{\|X\|}(\cdot)$, the local time at the origin of the continuous, nonnegative semimartingale $\|X(\cdot)\|$.*

• In order to substantiate this claim, observe first from (6.13) that $L^{\|X\|}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|=0\}}\,\mathrm{d}A(t)$, since $M_3(\cdot)$ is a continuous local martingale. Then it suffices to show

$$A(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|=0\}}\,\mathrm{d}A(t), \quad \text{and} \quad \int_0^{\cdot} \big\|X(t)\big\|\,\mathrm{d}A(t) = 0. \tag{6.14}$$

To do so, let us fix two arbitrary constants c_1, c_2 with $0 < c_1 < c_2$ and define a sequence of stopping times inductively, via $\sigma_0 := \inf\{t \ge 0 : \|X(t)\| = c_2\}$ if $\|X(0)\| < c_2$ and $\sigma_0 := 0$ otherwise; as well as

$$\sigma_{2n+1} := \inf\big\{t \ge \sigma_{2n} : \big\|X(t)\big\| = c_1\big\}, \qquad \sigma_{2n+2} := \inf\big\{t \ge \sigma_{2n+1} : \big\|X(t)\big\| = c_2\big\}; \quad n \in \mathbb{N}_0.$$

We note that $\|X(t)\| \ge c_1$ holds for $t \in (\sigma_{2n}, \sigma_{2n+1})$; and conversely, that $\|X(t)\| > c_2$ implies $t \in (\sigma_{2n}, \sigma_{2n+1})$ for some $n \in \mathbb{N}_0$. Thus, by taking an appropriate smooth function $g_4 \in \mathfrak{D}^{\boldsymbol{\mu}}$ of the form $g_4(r,\theta) = \psi(r)$ where $\psi : [0,\infty) \to [0,\infty)$ is smooth with $\psi(r) = r$ for $r \ge c_1$, one can show that $N(\cdot \wedge \sigma_{2n+1}) - N(\cdot \wedge \sigma_{2n})$ is a continuous local martingale.

Then, since both processes $N(\cdot \wedge \sigma_{2n+1}) - A(\cdot \wedge \sigma_{2n+1})$ and $N(\cdot \wedge \sigma_{2n}) - A(\cdot \wedge \sigma_{2n})$ are continuous local martingales, so is $A(\cdot \wedge \sigma_{2n+1}) - A(\cdot \wedge \sigma_{2n})$. But this last process is of bounded variation, so $A(\cdot \wedge \sigma_{2n+1}) \equiv A(\cdot \wedge \sigma_{2n})$ for every $n \in \mathbb{N}_0$. In other words, the process $A(\cdot)$ is flat on $[\sigma_{2n}, \sigma_{2n+1}]$ for every n. Therefore we have $\int_0^{\infty} \mathbf{1}_{\{\|X(t)\|\in(c_2,\infty)\}}\,\mathrm{d}A(t) \equiv 0$, because $\|X(t)\| \in (c_2,\infty)$ implies $t \in (\sigma_{2n}, \sigma_{2n+1})$ for some $n \in \mathbb{N}_0$. Since $c_2 > 0$ can be chosen arbitrarily small, we obtain (6.14).

• We return to the computation of the cross-variations $\langle M_j, M_3\rangle(\cdot)$ for $1 \le j \le 3$. Recalling (6.9)–(6.10), (6.13) with (6.14), and the proof of Proposition VIII.3.3 in [21], we deduce

$$\langle M_j, M_3\rangle(\cdot) = \int_0^{\cdot} \Gamma(g_j, g_3)\big(X(t)\big)\,\mathrm{d}t - \int_0^{\cdot} (1+\delta_{j3})g_j\big(X(t)\big)\,\mathrm{d}A(t) = \int_0^{\cdot} \Gamma(g_j, g_3)\big(X(t)\big)\,\mathrm{d}t$$

with the KRONECKER delta δ_{j3}. Hence we obtain $\langle M_j, M_3\rangle(\cdot) = \int_0^{\cdot} r_{j,3}(t)\,\mathrm{d}t$ for $1 \le j \le 3$, where

$$r_{j,3}(t) := \boldsymbol{\sigma}^2\big(\|X(t)\|\big)\big(\mathfrak{f}_j\big(X(t)\big) - \gamma_j\big)\mathbf{1}_{\{X(t)\neq\mathbf{0}\}} \quad \text{for } j = 1,2, \quad \text{and} \quad r_{3,3}(t) := \boldsymbol{\sigma}^2\big(\|X(t)\|\big)\mathbf{1}_{\{\|X(t)\|>0\}}.$$

• We have now computed all elements of the (3×3) matrix $(\mathrm{d}\langle M_i, M_k\rangle(t)/\mathrm{d}t)_{1\le i,k\le 3} = (r_{i,k}(t))_{1\le i,k\le 3}$; we observe also that this matrix is of rank 1, on $\{t \ge 0 : X(t) \ne \mathbf{0}\}$. By Theorem 3.4.2 and Proposition 5.4.6 in [17], there exists an extension of the original probability space, and on it

(i) a three-dimensional standard Brownian motion $\widetilde{W}(\cdot) = (\widetilde{W}_1(\cdot), \widetilde{W}_2(\cdot), \widetilde{W}_3(\cdot))'$,
(ii) a one-dimensional standard Brownian motion $W(\cdot)$, and
(iii) measurable, adapted, matrix-valued processes $(\rho_{i,k}(\cdot))_{1\le i,k\le 3}$ with $\int_0^T [\rho_{i,k}(t)]^2\,\mathrm{d}t < \infty$, such that

$$M_i(\cdot) = \sum_{k=1}^{3} \int_0^{\cdot} \rho_{i,k}(t)\,\mathrm{d}\widetilde{W}_k(t) = \int_0^{\cdot} \boldsymbol{\sigma}\big(\|X(t)\|\big)\big(\mathfrak{f}_i\big(X(t)\big) - \gamma_i\big)\mathbf{1}_{\{X(t)\ne\mathbf{0}\}}\,\mathrm{d}W(t), \quad i = 1, 2 \tag{6.15}$$

and $M_3(\cdot) = \int_0^{\cdot} \boldsymbol{\sigma}(\|X(t)\|)\mathbf{1}_{\{X(t)\ne\mathbf{0}\}}\,\mathrm{d}W(t)$. Substituting this into the decomposition $N(\cdot) = M_3(\cdot) + L^{\|X\|}(\cdot)$ and then into (6.12), we obtain the stochastic equation (6.5) for the radial process $\|X(\cdot)\|$. Substituting (6.15), (6.5) into $M_i(\cdot) = M^{g_i}(\cdot)$ expressed as in (6.2) for $i = 1, 2$, we deduce that the process $X(\cdot)$ defined in (6.7) satisfies the system of (6.4). The property (2.13) is exactly (6.3).

• Finally, for every set $A \in \mathcal{B}(\mathfrak{S})$, we consider the functions, similar to what we define in (4.3),

$$g_4(x) := g_5(x) - \|x\|\boldsymbol{\mu}(A), \qquad g_5(x) := \|x\|\mathbf{1}_A\big(\mathfrak{f}(x)\big), \quad x \in \mathbb{R}^2. \tag{6.16}$$

Since $g_4 \in \mathfrak{D}^{\boldsymbol{\mu}}$ and $g_5 \in \mathfrak{D}^{\boldsymbol{\mu}}_+$, we obtain that the continuous process

$$g_4\big(X(\cdot)\big) - g_4\big(X(0)\big) - \int_0^{\cdot} \boldsymbol{b}\big(\|X(t)\|\big)\big(\mathbf{1}_{\{\mathfrak{f}(X(t))\in A\}} - \boldsymbol{\mu}(A)\big)\mathbf{1}_{\{\|X(t)\|>0\}}\,\mathrm{d}t \tag{6.17}$$

is a local martingale; moreover, with $R^A(\cdot) := g_5(X(\cdot))$, we may repeat an argument similar to the one deployed above, and obtain that

$$R^A(\cdot) - R^A(0) - \int_0^{\cdot} \boldsymbol{b}\big(\|X(t)\|\big)\mathbf{1}_{\{\mathfrak{f}(X(t))\in A\}\cap\{\|X(t)\|>0\}}\,\mathrm{d}t - L^{R^A}(\cdot) \tag{6.18}$$

is a continuous local martingale. Furthermore, on account of (6.13), we see that

$$\boldsymbol{\mu}(A)\bigg(\big\|X(\cdot)\big\| - \big\|X(0)\big\| - \int_0^{\cdot} \boldsymbol{b}\big(\|X(t)\|\big)\mathbf{1}_{\{\|X(t)\|>0\}}\,\mathrm{d}t - L^{\|X\|}(\cdot)\bigg) \tag{6.19}$$

is also a continuous local martingale. Subtracting (6.18) from (6.17) and adding (6.19), we deduce that the finite variation process $L^{R^A}(\cdot) - \boldsymbol{\mu}(A)L^{\|X\|}(\cdot)$ is a continuous local martingale, and hence identically zero, i.e., $L^{R^A}(\cdot) \equiv \boldsymbol{\mu}(A)L^{\|X\|}(\cdot)$ as in (2.14).

We conclude from this analysis, that the system of equations (6.4) admits a weak solution with the properties (2.13) and (2.14). This proves part (b); part (c) is now evident. □

Remark 6.1. Looking back at the definition of the above local martingale problem for the planar diffusion, we recall Definition 4.1 and observe that the following statements (i)–(ii) are equivalent:

(i) For every $g \in \mathfrak{D}^{\boldsymbol{\mu}}_+$, the process $M^g(\cdot;\omega_2)$ is a continuous local submartingale;
(ii) For every $g \in \mathfrak{D}^{\boldsymbol{\mu}}$, the process $M^g(\cdot;\omega_2)$ is a continuous local martingale, and the process $M^{g_3}(\cdot;\omega_2)$ is a continuous local submartingale, where $g_3(x) = \|x\| = r$ is defined in (6.9).

Indeed, if (i) holds, $M^{g_3}(\cdot;\omega_2)$ is a continuous local submartingale since $g_3(x) = \|x\|$ is in $\mathfrak{D}^{\mu}_+$. For every $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ we have $g \in \mathfrak{D}^{\boldsymbol{\mu}}_+$ and $-g \in \mathfrak{D}^{\mu}_+$, hence both $M^g(\cdot;\omega_2)$ and $M^{-g}(\cdot;\omega_2) = -M^g(\cdot;\omega_2)$ are continuous local submartingales. Thus $M^g(\cdot;\omega_2)$ is a continuous local martingale, and (ii) follows.

Next, assume that (ii) holds. Every $g \in \mathfrak{D}^{\boldsymbol{\mu}}_+$ can then be decomposed as $g = g_{(1)} + g_{(2)}$, where $g_{(1)}(x) = c\|x\|$ with $c := D_{\boldsymbol{\mu}}g(0) \ge 0$ and $g_{(2)} = g - g_{(1)} \in \mathfrak{D}^{\boldsymbol{\mu}}$. Thus the above condition (ii) implies that $M^{g_{(1)}}(\cdot;\omega_2) = c\|\omega_2(\cdot)\|$ is a local submartingale, and that $M^{g_{(2)}}(\cdot;\omega_2)$ is a local martingale; hence $M^g(\cdot;\omega_2) = M^{g_{(1)}}(\cdot;\omega_2) + M^{g_{(2)}}(\cdot;\omega_2)$ is a local submartingale, and (i) follows.

6.3. *Well-posedness*

We conjecture that, if the local submartingale problem associated with the pair $(\boldsymbol{\sigma}, \boldsymbol{b})$ is well-posed, then the same is true for the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$.

The result that follows settles this conjecture in the affirmative, for the driftless case $\boldsymbol{b} \equiv \mathbf{0}$. Proposition 6.3 then deals with the case of a drift $\boldsymbol{b} = \boldsymbol{\sigma c}$ with $\boldsymbol{c} : \mathbb{R}_+ \to \mathbb{R}$ bounded and measurable.

Proposition 6.2 (Well-posedness for driftless WALSH diffusions). *Suppose that*

(i) *the drift $\boldsymbol{b}$ is identically equal to zero*; *and that*
(ii) *the reciprocal of the dispersion coefficient $\boldsymbol{\sigma} : [0, \infty) \to \mathbb{R} \setminus \{0\}$ is locally square-integrable*; *i.e.*,

$$\int_K \frac{\mathrm{d}y}{\boldsymbol{\sigma}^2(y)} < \infty \quad \textit{holds for every compact set } K \subset [0, \infty). \tag{6.20}$$

Then the local submartingale problem of Section 6.1, associated with the pair $(\boldsymbol{\sigma}, \mathbf{0})$, is well-posed.

Moreover, the local martingale problem of Section 6.2 associated with the triple $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$ is also well-posed; *and uniqueness in distribution holds, subject to the properties in (2.13) and (2.14), for the corresponding system of stochastic integral equations in (6.4) with $\boldsymbol{b} \equiv 0$, namely,*

$$X_i(\cdot) = X_i(0) + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\boldsymbol{\sigma}\big(\big\|X(t)\big\|\big)\,\mathrm{d}W(t) + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2. \tag{6.21}$$

Proof of existence. Let us consider the stochastic integral equation

$$S(\cdot) = r + \int_0^{\cdot} \boldsymbol{\sigma}\big(S(t)\big)\,\mathrm{d}W(t) + L^S(\cdot) \tag{6.22}$$

driven by one-dimensional Brownian motion $W(\cdot)$. It is shown in [24] that, under (6.20), this equation (6.22) has a non-negative, unique-in-distribution weak solution; equivalently, the STROOCK & VARADHAN local submartingale problem [25] associated with $(\boldsymbol{\sigma}, \mathbf{0})$ for $K^{\psi}(\cdot)$ in (6.1) is well-posed.

Let us also verify the property (2.4). From Exercise 3.7.10 in [17] , we get

$$0 = \int_0^{\infty} \mathbf{1}_{\{S(t)=0\}}\,\mathrm{d}\langle S\rangle(t) = \int_0^{\infty} \mathbf{1}_{\{S(t)=0\}}\boldsymbol{\sigma}^2\big(S(t)\big)\,\mathrm{d}t, \quad \text{thus also} \quad \int_0^{\infty} \mathbf{1}_{\{S(t)=0\}}\,\mathrm{d}t = 0$$

because $\boldsymbol{\sigma}(\cdot)$ never vanishes. It follows then from Theorem 2.1 that, on a suitably enlarged probability space, we may construct from this reflected diffusion $S(\cdot)$ a continuous, planar semimartingale $X(\cdot)$ which satisfies $\|X(\cdot)\| = S(\cdot)$, the system of equations (6.21), and the properties (2.10)–(2.14). On the strength of Proposition 6.1(a), the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$ admits a solution. □

Proof of uniqueness. We adopt the idea of proof in Theorem 3.2 of [1]. Suppose there are *two* solutions $\mathbb{Q}_j$, $j = 1, 2$ to this local martingale problem associated with the triple $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$. Let us take an arbitrary set $A \in \mathcal{B}(\mathfrak{S})$) and consider the functions $h_A(\cdot)$ and $g_A(\cdot)$ defined as in (4.3) for the indicator $\phi = \mathbf{1}_A$, namely

$$h_A(x) := \big(\mathbf{1}_{\{\mathfrak{f}(x)\in A\}} - \boldsymbol{\mu}(A)\big) \cdot \mathbf{1}_{\{\|x\|>0\}} = \big(\boldsymbol{\mu}\big(A^c\big)\mathbf{1}_{\{\mathfrak{f}(x)\in A\}} - \boldsymbol{\mu}(A)\mathbf{1}_{\{\mathfrak{f}(x)\in A^c\}}\big) \cdot \mathbf{1}_{\{x\neq\mathbf{0}\}}, \tag{6.23}$$

$$g_A(x) := \|x\| h_A(x), \quad x \in \mathbb{R}^2. \tag{6.24}$$

The above function $g_A(\cdot)$ belongs to the family $\mathfrak{D}^{\boldsymbol{\mu}}$ in (4.1), as does the function $[g_A(\cdot)]^2$. By assumption and Proposition 4.1, the process $M_A(\cdot) := g_A(\omega_2(\cdot))$ is then a $\mathbb{Q}_j$-local martingale, with

$$\langle M_A\rangle(T) = \big\langle g_A\big(\omega_2(\cdot)\big)\big\rangle(T) = \int_0^T \big[h_A\big(\omega_2(t)\big)\big]^2\boldsymbol{\sigma}^2\big(\big\|\omega_2(t)\big\|\big)\,\mathrm{d}t; \quad 0 \le T < \infty, j = 1, 2.$$

Let us also take an arbitrary $C \in \mathcal{B}((0,\infty))$. We shall show that for all $0 \le s < t < \infty$, the conditional probability $C_j := \mathbb{Q}_j(\|\omega_2(t)\| \in C, \mathfrak{f}(\omega_2(t)) \in A | \mathcal{F}_2(s))$ does not depend on $j = 1, 2$. It follows then from standard arguments, that the finite-dimensional distributions of $\omega_2(\cdot)$ are uniquely determined.

• First, let us assume $0 < \boldsymbol{\mu}(A) < 1$. We note that $g_A(x) > 0$, if $\mathfrak{f}(x) \in A$; $g_A(x) < 0$ if $\mathfrak{f}(x) \in A^c$; and $g_A(x) = 0$ if $x = \mathbf{0}$. It is also easy to verify that the process

$$U_A(\cdot) := \int_0^{\cdot} \left(\frac{1}{\boldsymbol{\mu}(A^c)} \cdot \mathbf{1}_{\{g_A(\omega_2(t))>0\}} + \frac{1}{\boldsymbol{\mu}(A)} \cdot \mathbf{1}_{\{g_A(\omega_2(t))\le 0\}} \right) \cdot \frac{\mathrm{d}M_A(t)}{\boldsymbol{\sigma}(\|\omega_2(t)\|)} \tag{6.25}$$

is a continuous $\mathbb{Q}_j$-local martingale with $\langle U_A \rangle(t) = t$ for $t \ge 0$; i.e., a $\mathbb{Q}_j$-Brownian motion for $j = 1, 2$. The probability distribution of the process $M_A(\cdot) = g_A(\omega_2(\cdot))$ is then determined uniquely and *independently of the solution* $\mathbb{Q}_j,\ j = 1, 2$ *to the local martingale problem.*

This is because, under the assumption (6.20) on the dispersion coefficient and thanks to the theory of ENGELBERT & SCHMIDT [5], the stochastic differential equation driven by the Brownian motion $U_A(\cdot)$ and derived from (6.25),

$$\mathrm{d}M_A(t) = \boldsymbol{\varrho}\big(M_A(t)\big)\,\mathrm{d}U_A(t), \quad 0 \le t < \infty \tag{6.26}$$

with $c_0 := \boldsymbol{\mu}(A^c)$, $c_1 := \boldsymbol{\mu}(A)$ and the new dispersion function

$$\boldsymbol{\varrho}(x) := c_0 \cdot \boldsymbol{\sigma}\left(\frac{x}{c_0}\right) \cdot \mathbf{1}_{\{x>0\}} + c_1 \cdot \boldsymbol{\sigma}\left(-\frac{x}{c_1}\right) \cdot \mathbf{1}_{\{x\le 0\}}; \quad x \in \mathbb{R}, \tag{6.27}$$

admits a weak solution, which is unique in the sense of the probability distribution. This follows from Theorem 5.5.7 in [17], and from the fact that the reciprocal of the function $\boldsymbol{\varrho}(\cdot)$ inherits the local square-integrability property (6.20) of the reciprocal of $\boldsymbol{\sigma}(\cdot)$. Moreover, $M_A(\cdot) = g_A(\omega_2(\cdot))$ is strongly Markovian with respect to the filtration $\mathbb{F}_2$ (cf. the proof of Lemma 9.2). Therefore,

$$\begin{aligned} C_j &= \mathbb{Q}_j\big(\big\|\omega_2(t)\big\| \in C, \mathfrak{f}\big(\omega_2(t)\big) \in A | \mathcal{F}_2(s)\big) = \mathbb{Q}_j\big(g_A\big(\omega_2(t)\big) \in \boldsymbol{\mu}\big(A^c\big)C | \mathcal{F}_2(s)\big) \\ &= \mathbb{Q}_j\big(g_A\big(\omega_2(t)\big) \in \boldsymbol{\mu}\big(A^c\big)C | g_A\big(\omega_2(s)\big)\big), \qquad 0 \le s < t < \infty, j = 1, 2. \end{aligned}$$

Since the distribution of the process $g_A(\omega_2(\cdot))$ is uniquely determined, C_j does not depend on $j = 1, 2$.

• Secondly, we consider the case $\boldsymbol{\mu}(A) \in \{0, 1\}$. By Proposition 6.1, $\omega_2(\cdot)$ and $\|\omega_2(\cdot)\|$ solve in the weak sense the equations (6.21) and (6.22), respectively. Hence, $\omega_2(\cdot)$ stays on the same ray on each of its excursions away from the origin. Moreover, the distribution of $\|\omega_2(\cdot)\|$ is uniquely determined, and thus $\|\omega_2(\cdot)\|$ is strongly Markovian with respect to the filtration $\mathbb{F}_2$ (cf. Lemma 9.2 below).

If $\boldsymbol{\mu}(A) = 0$, then $g_A(x) = \|x\| \mathbf{1}_{\{\mathfrak{f}(x)\in A, \|x\|>0\}}$, and the process $M_A(\cdot) = g_A(\omega_2(\cdot))$ is a nonnegative, continuous $\mathbb{Q}_j$-local martingale, thus also a supermartingale – so it stays at the origin $\mathbf{0}$ after hitting it for the first time. It follows that $\mathbb{Q}_j$-a.s., the angular part $\mathfrak{f}(\omega_2(\cdot))$ never again visits the set A, after the radial part $\|\omega_2(\cdot)\|$ first becomes zero. Thus with $\tau_s(\omega_2) := \inf\{u \ge s : \|\omega_2(u)\| = 0\}$, we have

$$C_j = \mathbb{Q}_j\big(\big\|\omega_2(t)\big\| \in C, \mathfrak{f}\big(\omega_2(t)\big) \in A | \mathcal{F}_2(s)\big) = \mathbb{Q}_j\big(\big\|\omega_2(t)\big\| \in C, \tau_s(\omega_2) > t | \big\|\omega_2(s)\big\|\big)\mathbf{1}_A\big(\mathfrak{f}\big(\omega_2(s)\big)\big).$$

The case $\boldsymbol{\mu}(A) = 1$ can be treated similarly. Since the distribution of $\|\omega_2(\cdot)\|$ is uniquely determined and independent of $j = 1, 2$, we conclude that C_j does not depend on $j = 1, 2$, if $\boldsymbol{\mu}(A) = 0$ or 1. □

Proposition 6.3 (Well-posedness for WALSH diffusions with drift). *Under the setting of Proposition 6.2, and in addition to the assumptions imposed there, let us consider another function* $\boldsymbol{c} : \mathbb{R}_+ \to \mathbb{R}$ *which is bounded and measurable. We denote by* $\mathbb{Q}^{(\mathbf{0})}$ *the solution to the local martingale problem of Section 6.2 associated with the triple* $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$.

(i) *For every* $T \in (0,\infty)$, *the local martingale problem associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{\sigma}\boldsymbol{c}, \boldsymbol{\mu})$ *for* $M^g(t), 0 \le t \le T$ *in* (6.2), *is then well posed, and its solution is given by the probability measure* $\mathbb{Q}_T^{(\boldsymbol{c})}$ *with*

$$\left.\frac{\mathrm{d}\mathbb{Q}_T^{(\boldsymbol{c})}}{\mathrm{d}\mathbb{Q}^{(\mathbf{0})}}\right|_{\mathcal{F}_2^{\bullet}(t)} := \exp\left(\int_0^t \boldsymbol{c}\big(\big\|\omega_2(u)\big\|\big)\,\mathrm{d}W(u) - \frac{1}{2}\int_0^t \boldsymbol{c}^2\big(\big\|\omega_2(u)\big\|\big)\,\mathrm{d}u \right); \quad 0 \le t \le T.$$

(ii) *Under the assumptions in* (i), *suppose that* $\mathbb{Q}^{(\boldsymbol{c})}$ *solves the local martingale problem associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{\sigma c}, \boldsymbol{\mu})$. *Then there exists an* $\mathbb{F}_2$-*Brownian motion* $B(\cdot)$, *such that every* $\mathbb{F}_2$-*local martingale* $M(\cdot)$ *with* $M(0) = 0$ *can be represented in the integral form* $M(\cdot) = \int_0^{\cdot} H(t)\,\mathrm{d}B(t)$ *for some* $\mathbb{F}_2$-*progressively measurable and locally square-integrable process* $H(\cdot)$.

Proof. (i) This is a direct consequence of Propositions 6.1, 6.2 and GIRSANOV's change of measure. Indeed, it follows from Proposition 6.1 that, under $\mathbb{Q}^{(\mathbf{0})}$, the coordinate process $\omega_2(\cdot)$ satisfies the system of stochastic integral equations (6.21), subject to (2.13) and (2.14). Because of the boundedness of the function $\boldsymbol{c}(\cdot)$, the measure $\mathbb{Q}_T^{(\boldsymbol{c})}$ just introduced is a probability.

By GIRSANOV's theorem (e.g., [17], Theorem 3.5.1) we see that for every fixed $T \in (0, \infty)$, the process $W^{(\boldsymbol{c})}(u) := W(u) - \int_0^u \boldsymbol{c}(\|\omega_2(t)\|)\,\mathrm{d}t$, $0 \le u \le T$ is standard Brownian motion under this probability measure $\mathbb{Q}_T^{(\boldsymbol{c})}$, and thus the coordinate process $\omega_2(\cdot)$ satisfies on the time-horizon $[0, T]$ the system of stochastic integral equations

$$X_i(\cdot) = \mathrm{x} + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\boldsymbol{\sigma}\big(\|X(t)\|\big)\big[\mathrm{d}W^{(\boldsymbol{c})}(t) + \boldsymbol{c}\big(\|X(t)\|\big)\,\mathrm{d}t\big] + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2.$$

Moreover, since the probability measure $\mathbb{Q}_T^{(\boldsymbol{c})}$ is absolutely continuous with respect to $\mathbb{Q}^{(\mathbf{0})}$, we obtain (2.13) and (2.14) with $X(\cdot)$ replaced by $\omega_2(\cdot)$, a.e. under $\mathbb{Q}_T^{(\boldsymbol{c})}$. Thanks to Proposition 6.1 again, $\mathbb{Q}_T^{(\boldsymbol{c})}$ solves the local martingale problem of Section 6.2 associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{\sigma c}, \boldsymbol{\mu})$.

Conversely, for any solution $\mathbb{Q}_T^{(\boldsymbol{c})}$ to the local martingale problem associated with $(\boldsymbol{\sigma}, \boldsymbol{\sigma c}, \boldsymbol{\mu})$ for $M^g(t)$, $0 \le t \le T$ as in (6.2), the probability measure $\mathbb{Q}^{(\mathbf{0})}$ defined via

$$\frac{\mathrm{d}\mathbb{Q}^{(\mathbf{0})}}{\mathrm{d}\mathbb{Q}_T^{(\boldsymbol{c})}}\bigg|_{\mathcal{F}_2^{\bullet}(t)} := \exp\bigg(-\int_0^t \boldsymbol{c}\big(\|\omega_2(u)\|\big)\,\mathrm{d}W^{(\boldsymbol{c})}(u) - \frac{1}{2}\int_0^t \boldsymbol{c}^2\big(\|\omega_2(u)\|\big)\,\mathrm{d}u\bigg); \quad 0 \le t \le T$$

is seen to solve the local martingale problem of Section 6.2 associated with the triple $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$. Since this problem is well-posed, the same holds for the local martingale problem associated with $(\boldsymbol{\sigma}, \boldsymbol{\sigma c}, \boldsymbol{\mu})$.

(ii) From part (i) we know that $\mathbb{Q}^{(\boldsymbol{c})}|_{\mathcal{F}_2^{\bullet}(T)} = \mathbb{Q}_T^{(\boldsymbol{c})}$, $0 \le T < \infty$, and $B(\cdot) := W^{(\boldsymbol{c})}(\cdot)$ is a standard Brownian motion under $\mathbb{Q}^{(\boldsymbol{c})}$. Since the local martingale problem in part (i) is well-posed, the proof of Theorem 4.1 in [1] can be adapted, to show that the required $H(\cdot)$ exists up to any finite time T (see also [16]), and can thus be defined on all of $[0, \infty)$. □

7. Martingale characterization of the WALSH Brownian motion

We still have to show that, when $U(\cdot) \equiv B(\cdot)$ is standard Brownian motion, the construction of Theorem 2.1 leads to the WALSH Brownian motion as defined, for instance, in [1] or [7]. In the present section we establish this connection; cf. Proposition 7.2.

Following these sources, we may characterize the WALSH Brownian motion $\boldsymbol{W}(\cdot)$ in terms of its FELLER semigroup $\{\mathcal{P}_t, t \ge 0\}$ defined for $f \in C_0(\overline{E})$ via $[\mathcal{P}_t f](0, z) := T_t^+ \overline{f}(0)$ and

$$[\mathcal{P}_t f](r, z) := T_t^+ \overline{f}(r) + \big[T_t^0(f_z - \overline{f})\big](r); \quad r > 0, z \in \mathfrak{S}. \tag{7.1}$$

Here $\{T_t^+, 0 \le t < \infty\}$ is the semigroup of reflected Brownian motion on $[0, \infty)$, and $\{T_t^0, 0 \le t < \infty\}$ the semigroup of Brownian motion on $[0, \infty)$ killed upon reaching the origin. For the sake of simplicity, we use polar coordinates in the punctured plane E of (2.6). Abusing notation slightly, we define also

$$\overline{f}(r) := \int_{\mathfrak{S}} f(r, z)\boldsymbol{\mu}(\mathrm{d}z), \qquad f_z(r) := f(r, z); \quad (r, z) \in [0, \infty) \times \mathfrak{S} = \overline{E}, \tag{7.2}$$

for $f \in C(\overline{E})$. Let us assume that $\boldsymbol{W}(0) = \mathrm{x} \in \mathbb{R}^2$. BARLOW, PITMAN & YOR [1] show that there is a FELLER and strong MARKOV process $\boldsymbol{W}(\cdot)$ with values in $\mathbb{R}^2$, continuous paths, and $\{\mathcal{P}_t, 0 \le t < \infty\}$ as its semigroup. This is the process these authors call "WALSH Brownian motion". They show that the radial part $\|\boldsymbol{W}(\cdot)\|$ is one-dimensional reflecting Brownian motion. For this planar process $\boldsymbol{W}(\cdot)$, Hajri & Touhami [10] derive a version of the Freidlin–Sheu formula, that involves the standard, scalar Brownian motion $\boldsymbol{\beta}^{\boldsymbol{W}}(\cdot) := \|\boldsymbol{W}(\cdot)\| - \|\mathrm{x}\| - L^{\|\boldsymbol{W}\|}(\cdot)$ of the filtration $\mathbb{F}^{\boldsymbol{W}} = \{\mathcal{F}^{\boldsymbol{W}}(t)\}_{0\le t<\infty}$.

Here is an extension of Proposition 3.1 in [1]; it shows that the Walsh Brownian motion with spinning measure $\boldsymbol{\mu}$, defined via the semigroup (7.1), generates a solution to the local martingale problem associated with the triple $(\mathbf{1}, \mathbf{0}, \boldsymbol{\mu})$ (cf. Remark 6.1).

Proposition 7.1 (Properties of Walsh Brownian motion). *Let $\boldsymbol{W}(\cdot)$ be the Walsh Brownian motion defined via the semigroup (7.1) and with spinning measure $\boldsymbol{\mu}$. Then*:

(i) *The process $\|\boldsymbol{W}(\cdot)\|$ is reflecting Brownian motion; and $\boldsymbol{W}(\cdot)$ satisfies the properties in (2.10)–(2.11).*
(ii) *For any $g : \mathbb{R}^2 \to \mathbb{R}$ in the class $\mathfrak{D}^{\boldsymbol{\mu}}$ of (4.1), the continuous process below is a local martingale*:

$$g\big(\boldsymbol{W}(\cdot)\big) - g(\mathrm{x}) - \frac{1}{2}\int_0^{\cdot} \partial^2_{rr} g\big(\boldsymbol{W}(t)\big)\mathbf{1}_{\{\boldsymbol{W}(t)\neq\mathbf{0}\}}\,\mathrm{d}t = \int_0^{\cdot} \partial_r g\big(\boldsymbol{W}(t)\big)\mathbf{1}_{\{\boldsymbol{W}(t)\neq\mathbf{0}\}}\,\mathrm{d}\boldsymbol{\beta}^{\boldsymbol{W}}(t).$$

Proof. The claims of (i) are proved in [1]. Claim (ii) follows by applying the Freidlin–Sheu-type formula of Theorem 1.2 in [10] to the process $g(\boldsymbol{W}(\cdot))$. We also note that, with the notation of (4.3), both processes below are continuous martingales:

$$\mathcal{M}^{\boldsymbol{W}}_{(\phi)}(\cdot) = g_{(\phi)}\big(\boldsymbol{W}(\cdot)\big) - g_{(\phi)}(\mathrm{x}) = \int_0^{\cdot} h_{(\phi)}\big(\boldsymbol{W}(t)\big)\,\mathrm{d}\boldsymbol{\beta}^{\boldsymbol{W}}(t), \qquad \mathcal{N}^{\boldsymbol{W}}_{(\phi)}(\cdot) = \big(\mathcal{M}^{\boldsymbol{W}}_{(\phi)}(\cdot)\big)^2 - \langle\mathcal{M}^{\boldsymbol{W}}_{(\phi)}\rangle(\cdot). \qquad \square$$

Our next result shows that, as we expected all along, Walsh semimartingales driven by Brownian motions $U(\cdot)$ are Walsh Brownian motions defined via the semigroup (7.1).

Proposition 7.2 (Stochastic equations for Walsh Brownian motions). *Let us place ourselves in the context of Theorem 2.1, and suppose that the semimartingale $U(\cdot) \equiv B(\cdot)$ of (2.1) is Brownian motion. Then the planar process $X(\cdot)$ constructed there, has the following properties*:

(i) *It is the unique-in-distribution weak solution, subject to the properties (2.13), (2.14), of the system of stochastic integral equations in (3.3), namely $X_i(\cdot) = \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i(X(t))\,\mathrm{d}B(t) + \gamma_i L^{\|X\|}(\cdot)$, $i = 1, 2$.*
(ii) *It is a Walsh Brownian motion.*
(iii) *Every $\mathbb{F}^X$-local martingale $M(\cdot)$ with $M(0) = 0$ has an integral representation $M(\cdot) = \int_0^{\cdot} H(t)\,\mathrm{d}B(t)$, for some $\mathbb{F}^X$-progressively measurable and locally square-integrable process $H(\cdot)$.*

Proof. The first claim follows from Propositions 6.1, 6.2 with $\boldsymbol{\sigma}(\cdot) \equiv 1$; the second claim, that $X(\cdot)$ is Walsh Brownian motion, is a consequence of Propositions 6.1, 6.2 and 7.1. With $U(\cdot) \equiv B(\cdot)$ a standard Brownian motion, Proposition 4.1 shows that both processes below are continuous local martingales

$$M_{(\phi)}(\cdot) = g_{(\phi)}\big(X(\cdot)\big) - g_{(\phi)}(\mathrm{x}) = \int_0^{\cdot} h_{(\phi)}\big(X(t)\big)\,\mathrm{d}B(t), \qquad N_{(\phi)}(\cdot) = \big[M_{(\phi)}(\cdot)\big]^2 - \int_0^{\cdot} \big[h_{(\phi)}\big(X(t)\big)\big]^2\,\mathrm{d}t$$

(cf. Theorem 3.2 of [1]). The third claim follows from Proposition 6.3. $\square$

8. Angular dependence

Let us admit now bounded, Borel-measurable coefficients $\boldsymbol{b} : \mathbb{R} \times \mathfrak{S} \to \mathbb{R}$ and $\boldsymbol{\sigma} : \mathbb{R} \times \mathfrak{S} \to \mathbb{R} \setminus \{0\}$ which may depend on the angular variable $z = \mathfrak{f}(x) \in \mathfrak{S}$, $x \in \mathbb{R}$ in (6.2). We assume also that $\boldsymbol{\sigma}$ is bounded away from zero, and consider the local martingale problem of Section 6.2 but now with the infinitesimal generator re-defined as

$$\mathcal{L}^* g(x) := \boldsymbol{b}\big(\|x\|, \mathfrak{f}(x)\big)G'(x) + \frac{1}{2}\boldsymbol{\sigma}^2\big(\|x\|, \mathfrak{f}(x)\big)G''(x); \quad x \in \mathbb{R}^2, g \in \mathfrak{D}. \tag{8.1}$$

For every given, fixed $z \in \mathfrak{S}$, we set $\boldsymbol{\sigma}_z(r) := \boldsymbol{\sigma}(r, z)$ and define the scale function $\boldsymbol{p}_\theta(\cdot)$ by

$$\boldsymbol{p}_\theta(r) = \boldsymbol{p}(r, z) := \int_0^r \exp\biggl(-2\int_0^\xi \frac{\boldsymbol{b}(\zeta, z)}{\boldsymbol{\sigma}^2(\zeta, z)}\,\mathrm{d}\zeta\biggr)\,\mathrm{d}\xi, \quad r \in [0, \infty),$$

as well as its inverse $\boldsymbol{q}_z(r) = \boldsymbol{q}(r, z)$ in the radial component with $\boldsymbol{q}_z(\boldsymbol{p}_z(r)) = r$. We note that these functions satisfy $\boldsymbol{p}_z(0) = 0 = \boldsymbol{q}_z(0)$ and $\boldsymbol{p}_z'(0+) = 1 = \boldsymbol{q}_z'(0+)$; that $\boldsymbol{p}_z(\cdot)$ has an absolutely continuous, strictly positive derivative; that the second derivative $\boldsymbol{p}_z''(\cdot)$ exists almost everywhere; and that both of these derivatives are bounded. Therefore, by the generalized Itô rule, we see that Theorem 4.1 holds also for the function $\boldsymbol{p}_z(\cdot)$, which may not be in the class $\mathfrak{D}$; the same is true for the function $\boldsymbol{q}_z(\cdot)$.

We consider an auxiliary diffusion coefficient

$$\widetilde{\boldsymbol{\sigma}}_z(r) \equiv \widetilde{\boldsymbol{\sigma}}(r, z) := \boldsymbol{p}_\theta'\bigl(\boldsymbol{q}_z(r)\bigr)\boldsymbol{\sigma}_z\bigl(\boldsymbol{q}_z(r)\bigr), \quad 0 < r < \infty \tag{8.2}$$

and $z \in \mathfrak{S}$, and write $\widetilde{\boldsymbol{\sigma}}(y) \equiv \widetilde{\boldsymbol{\sigma}}(r, z)$ for $y = (r, z) \in [0, \infty) \times \mathfrak{S}$. We introduce also the stochastic clock

$$\mathcal{Q}(\cdot) := \int_0^\cdot \frac{\mathrm{d}u}{[\widetilde{\boldsymbol{\sigma}}(\|X(u)\|, Z(u))]^2} \quad \text{and its inverse} \quad \mathcal{T}(t) := \inf\bigl\{v \geq 0 : \mathcal{Q}(v) > t\bigr\}; \quad 0 \leq t < \infty.$$

Here $X(\cdot) = Z(\cdot)S(\cdot)$ is a Walsh semimartingale as constructed as in (5.2), starting from a one-dimensional Brownian motion $U(\cdot) = B(\cdot)$ in Proposition 7.2. In particular, $X(\cdot)$ is Walsh Brownian motion; whereas $Z(\cdot) = \mathfrak{f}(X(\cdot))$. We consider now the time-changed, rescaled version $Y(\cdot) = (Y_1(\cdot), Y_2(\cdot))' := \boldsymbol{q}(X(\mathcal{T}(\cdot)))$ of this Walsh Brownian motion $X(\cdot)$, expressed in polar coordinates via

$$\bigl\|Y(\cdot)\bigr\| = \boldsymbol{q}\bigl(\bigl\|X\bigl(\mathcal{T}(\cdot)\bigr)\bigr\|, \mathfrak{f}\bigl(X\bigl(\mathcal{T}(\cdot)\bigr)\bigr)\bigr), \qquad \mathfrak{f}\bigl(Y(\cdot)\bigr) = \mathfrak{f}\bigl(X\bigl(\mathcal{T}(\cdot)\bigr)\bigr) = Z\bigl(\mathcal{T}(\cdot)\bigr). \tag{8.3}$$

In terms of this rescaling, we have the representation

$$\mathcal{T}(\cdot) = \int_0^\cdot \bigl(\boldsymbol{p}_z'(r)\boldsymbol{\sigma}_z(r)\bigr)^2\Big|_{z=\mathfrak{f}(Y(t)), r=\|Y(t)\|}\,\mathrm{d}t \tag{8.4}$$

for the inverse clock. The resulting process $Y(\cdot)$ turns out to be a Walsh semimartingale with angular dependence in its local characteristics $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, as follows.

Proposition 8.1. *The process $Y(\cdot)$ defined via (8.3) satisfies the integral equations*

$$\begin{aligned} Y(\cdot) &= Y(0) + \int_0^\cdot \mathfrak{f}\bigl(Y(t)\bigr)\bigl[\boldsymbol{b}\bigl(\bigl\|Y(t)\bigr\|, \mathfrak{f}\bigl(Y(t)\bigr)\bigr)\,\mathrm{d}t + \boldsymbol{\sigma}\bigl(\bigl\|Y(t)\bigr\|, \mathfrak{f}\bigl(Y(t)\bigr)\bigr)\,\mathrm{d}W(t)\bigr] + \boldsymbol{\gamma}L^{\|Y\|}(\cdot), \\ \bigl\|Y(\cdot)\bigr\| &= \bigl\|Y(0)\bigr\| + \int_0^\cdot \mathbf{1}_{\{\|Y(t)\|>0\}}\bigl(\boldsymbol{b}\bigl(\bigl\|Y(t)\bigr\|, \mathfrak{f}\bigl(Y(t)\bigr)\bigr)\,\mathrm{d}t + \boldsymbol{\sigma}\bigl(\bigl\|Y(t)\bigr\|, \mathfrak{f}\bigl(Y(t)\bigr)\bigr)\,\mathrm{d}W(t)\bigr) + L^{\|Y\|}(\cdot) \end{aligned} \tag{8.5}$$

as well as the properties $\int_0^\cdot \mathbf{1}_{\{Y(t)=0\}}\,\mathrm{d}t \equiv 0$ and $L^{R_*^A}(\cdot) \equiv \boldsymbol{\mu}(A)L^{\|Y\|}(\cdot), \forall A \in \mathcal{B}(\mathfrak{S})$. Furthermore, it induces a solution to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ and $\mathcal{L}^*$ in (8.1).

In the above expressions $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)'$ is defined in (2.6), $W(\cdot)$ is one-dimensional Brownian motion, and the “thinned” process $R_*^A(\cdot) := \|Y(\cdot)\| \cdot \mathbf{1}_A(\mathfrak{f}(Y(\cdot))$ is defined for $A \in \mathcal{B}(\mathfrak{S})$.

Proof. Applying the Freidlin–Sheu formula in Theorem 4.1 to $\boldsymbol{q}(X(\cdot))$, we obtain

$$\begin{aligned} \bigl\|Y(\cdot)\bigr\| = \boldsymbol{q}(\mathrm{x}) &+ \int_0^{\mathcal{T}(\cdot)} \partial_r\boldsymbol{q}\bigl(X(u)\bigr)\mathbf{1}_{\{X(u)\neq\mathbf{0}\}}\,\mathrm{d}B(u) + D_{\boldsymbol{\mu}}\boldsymbol{q}(0) \cdot L^{\|X\|}\bigl(\mathcal{T}(\cdot)\bigr) \\ &+ \int_0^{\mathcal{T}(\cdot)} \frac{\partial_{rr}^2\boldsymbol{q}(X(u))}{2}\mathbf{1}_{\{X(u)\neq 0\}}\,\mathrm{d}u. \end{aligned} \tag{8.6}$$

Here by direct calculation

$$\partial_r \boldsymbol{q}(x) := \boldsymbol{q}_z'(r) = \frac{1}{\boldsymbol{p}_z'(\boldsymbol{q}_z(r))}, \qquad \partial_{rr}^2 \boldsymbol{q}(x) := \boldsymbol{q}_z''(r) = \frac{2\boldsymbol{b}(\boldsymbol{q}_z(r), z)}{\boldsymbol{\sigma}^2(\boldsymbol{q}_z(r), \theta) \cdot (\boldsymbol{p}_z'(\boldsymbol{q}_z(r)))^2} \tag{8.7}$$

hold for every $x = (r, z) \in [0, \infty) \times \mathfrak{S}$, where r is not in a set of Lebesgue measure zero that depends on $z \in \mathfrak{S}$. Thanks to the P. LÉVY Theorem, the continuous local martingale

$$W(\cdot) := \int_0^{\mathcal{T}(\cdot)} \frac{\mathrm{d}B(u)}{\widetilde{\boldsymbol{\sigma}}(\|X(u)\|, \mathfrak{f}(X(u)))}$$

is one-dimensional standard Brownian motion. Since $\mathrm{Leb}(\{t : X(t) = \mathbf{0}\}) = \mathrm{Leb}(\{t : S(t) = 0\}) = 0$ a.s. and $\boldsymbol{q}(\mathbf{0}) = 0$ from the construction, we obtain

$$\mathrm{Leb}\big(\{t : \|Y(t)\| = \boldsymbol{q}\big(X\big(\mathcal{T}(t)\big)\big) = \mathbf{0}\}\big) = \mathrm{Leb}\big(\mathcal{T}^{-1}\{t : X(t) = \mathbf{0}\}\big) = 0 \quad \text{a.s.} \tag{8.8}$$

In conjunction with the definitions (8.2) and (8.3), we obtain now the representations

$$\int_0^{\mathcal{T}(\cdot)} \partial_r \boldsymbol{q}\big(X(u)\big)\mathbf{1}_{\{X(u)\neq\mathbf{0}\}}\,\mathrm{d}B(u) = \int_0^{\cdot} \mathbf{1}_{\{\|Y(u)\|>0\}}\boldsymbol{\sigma}\left(\big\|Y(u)\big\|, \mathfrak{f}\big(Y(u)\big)\right)\mathrm{d}W(u) \tag{8.9}$$

(on the strength of Proposition 3.4.8 in [17]), as well as

$$\begin{aligned}\int_0^{\mathcal{T}(\cdot)} \partial_{rr}^2 \boldsymbol{q}\big(X(u)\big)\mathbf{1}_{\{X(u)\neq0\}}\,\mathrm{d}u &= \int_0^{\cdot} \partial_{rr}^2 \boldsymbol{q}\big(X\big(\mathcal{T}(u)\big)\big)\frac{\mathrm{d}\mathcal{T}(u)}{\mathrm{d}u}\mathbf{1}_{\{X(\mathcal{T}(u))\neq\mathbf{0}\}}\,\mathrm{d}u \\ &= 2\int_0^{\cdot} \mathbf{1}_{\{\|Y(u)\|>0\}}\boldsymbol{b}\big(\big\|Y(u)\big\|, \mathfrak{f}\big(Y(u)\big)\,\mathrm{d}u\end{aligned} \tag{8.10}$$

(by time-change). From these considerations and (8.8) we also obtain the identification of local time

$$L^{\|Y\|}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{Y(u)=0\}}\,\mathrm{d}\|Y\|(u) = D_{\boldsymbol{\mu}}\boldsymbol{q}(0) \cdot L^{\|X\|}\big(\mathcal{T}(\cdot)\big), \tag{8.11}$$

thus also the dynamics for the radial part of the process $Y(\cdot)$, namely

$$\big\|Y(\cdot)\big\| = \big\|Y(0)\big\| + \int_0^{\cdot} \mathbf{1}_{\{\|Y(t)\|>0\}}\big(\boldsymbol{b}\big(\big\|Y(t)\big\|, \mathfrak{f}\big(Y(t)\big)\big)\,\mathrm{d}t + \boldsymbol{\sigma}\left(\big\|Y(t)\big\|, \mathfrak{f}\big(Y(t)\big)\right)\mathrm{d}W(t)\big) + L^{\|Y\|}(\cdot).$$

• Recalling (8.3), and applying the FREIDLIN–SHEU formula in Theorem 4.1 to the process $Y_i(\cdot) = \boldsymbol{q}(X(\mathcal{T}(\cdot)))\mathfrak{f}_i(X(\mathcal{T}(\cdot)))$, we obtain

$$\begin{aligned}Y_i(\cdot) = \mathrm{y}_i &+ \int_0^{\mathcal{T}(\cdot)} \partial_r \boldsymbol{q}\big(X(u)\big)\mathfrak{f}_i\big(X(u)\big)\mathbf{1}_{\{X(u)\neq\mathbf{0}\}}\,\mathrm{d}B(u) + \frac{1}{2}\int_0^{\mathcal{T}(\cdot)} \partial_{rr}^2 \boldsymbol{q}\big(X(u)\big)\mathfrak{f}_i\big(X(u)\big)\mathbf{1}_{\{X(u)\neq\mathbf{0}\}}\,\mathrm{d}u \\ &+ D_{\boldsymbol{\mu}}[\boldsymbol{q}\mathfrak{f}_i](0) \cdot L^{\|X\|}\big(\mathcal{T}(\cdot)\big); \quad i = 1, 2.\end{aligned}$$

Hence, combining with (8.9)–(8.11) and $\mathfrak{f}(\mathbf{0}) = 0$ and $\boldsymbol{q}_z'(0+) = 1$, we obtain the dynamics (8.5).

• Furthermore, for every $g \in \mathfrak{D}$ by another application of the FREIDLIN–SHEU formula in Theorem 4.1 to $g(Y(\cdot)) = g(\boldsymbol{q}(r, z), z)|_{r=\|X(\mathcal{T}(\cdot))\|, z=\mathfrak{f}(X(\mathcal{T}(\cdot)))}$ with $\boldsymbol{q}_z(0+) = 0$, we derive

$$\begin{aligned}g\big(Y(T)\big) = g(\mathrm{y}) &+ \int_0^T \mathbf{1}_{\{Y(t)\neq\mathbf{0}\}}\left(\boldsymbol{b}\big(\big\|Y(t)\big\|, \mathfrak{f}\big(Y(t)\big)\big)\partial_r g\big(Y(t)\big) + \frac{1}{2}\boldsymbol{\sigma}^2\big(\big\|Y(t)\big\|, \mathfrak{f}\big(Y(t)\big)\big)\partial_{rr}^2 g\big(Y(t)\big)\right)\mathrm{d}t \\ &+ \int_0^T \mathbf{1}_{\{Y(t)\neq\mathbf{0}\}}\partial_r g\big(Y(t)\big)\boldsymbol{\sigma}\left(\big\|Y(t)\big\|, \mathfrak{f}\big(Y(t)\big)\right)\mathrm{d}W(t) + D_{\boldsymbol{\mu}} g(0) \cdot L^{\|Y\|}(T), \quad 0 \le T < \infty,\end{aligned} \tag{8.12}$$

in conjunction with (8.7)–(8.11) and $\boldsymbol{q}_z'(0+) = 1$. When $g \in \mathfrak{D}^{\boldsymbol{\mu}}$, we can apply this to $M^g(\cdot; Y)$ in (6.2) – now redefined with the operator $\mathcal{L}^*$ of (8.1) – to conclude that $M^g(\cdot; Y)$ is equal to the local martingale

$$g\big(Y(\cdot)\big) - g(\mathrm{y}) - \int_0^{\cdot} \mathcal{L}^* g\big(Y(t)\big)\mathbf{1}_{\{Y(t)\neq\mathbf{0}\}}\,\mathrm{d}t = \int_0^{\cdot} \partial_r g\big(Y(t)\big)\mathbf{1}_{\{Y(t)\neq\mathbf{0}\}}\boldsymbol{\sigma}\big(\|Y(t)\|, \mathfrak{f}\big(Y(t)\big)\big)\,\mathrm{d}W(t).$$

Therefore, $Y(\cdot)$ induces a solution to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ and the second-order differential operator $\mathcal{L}^*$ in (8.1). Other properties of $Y(\cdot)$ are now verified readily. □

Proposition 8.2. *Under the assumptions and with the notation of this section, the local martingale problem of Section* 6.2, *associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ *and the operator* $\mathcal{L}^*$ *in* (8.1), *is well-posed.*

Proof. Existence of a solution to this local martingale problem is established by Proposition 8.1. To prove uniqueness, we can reverse the steps of the construction in Proposition 8.1, as follows. Consider *any* solution of the local martingale problem of Section 6.2, associated with the triple and $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ and the operator $\mathcal{L}^*$, and the coordinate process $Y(\cdot) := \omega_2(\cdot)$ on the canonical space for that problem. We introduce the time change $\mathcal{T}(\cdot)$ as in (8.4), along with its inverse $\mathcal{Q}(\cdot)$; as well as the time-changed, rescaled version $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ of the process $Y(\cdot)$, defined in polar coordinates via

$$\|X(\cdot)\| := \boldsymbol{p}\big(\big\|Y\big(\mathcal{Q}(\cdot)\big)\big\|, \mathfrak{f}\big(Y\big(\mathcal{Q}(\cdot)\big)\big)\big), \qquad \mathfrak{f}\big(X(\cdot)\big) := \mathfrak{f}\big(Y\big(\mathcal{Q}(\cdot)\big)\big). \tag{8.13}$$

Using Proposition 6.1 (rather, its obvious generalization to coefficients with angular dependence) and Theorem 4.1, we have for the planar process $Y(\cdot)$ the appropriate FREIDLIN–SHEU-formula. With this at hand, the planar process $X(\cdot)$ is seen to be a WALSH Brownian motion with spinning measure $\boldsymbol{\mu}$, in a manner similar to that in the proof of Proposition 8.1. The path $t \mapsto X(t)$ is, with probability one, continuous in the topology induced by the tree metric (3.2), and hence so is the path $t \mapsto Y(t)$. In terms of this WALSH Brownian motion, we can express the time change $\mathcal{Q}(\cdot)$ as $\mathcal{Q}(\cdot) = \int_0^{\cdot} [\widetilde{\boldsymbol{\sigma}}(\|X(u)\|, \mathfrak{f}(X(u))]^{-2}\,\mathrm{d}u$.

The crucial step now, is to note that the process $Y(\cdot)$ can be written as $Y(t) = \Psi_t(X(\cdot))$. Here $\Psi_{\cdot}$ is a measurable mapping defined by $\Psi_t(\omega_2) = \boldsymbol{q}(\Pi_{\mathcal{T}(t;\omega_2)}(\omega_2))$, in terms of the measurable projection mapping $\Pi_t(\omega_2) := \omega_2(t)$ and the continuous time change

$$\mathcal{T}(t;\omega_2) := \inf\left\{v \ge 0 : \int_0^v \frac{\mathrm{d}u}{[\widetilde{\boldsymbol{\sigma}}(\|\omega_2(u)\|, \mathfrak{f}(\omega_2(u)))]^2} > t\right\}, \quad 0 \le t < \infty.$$

Since the distribution of the WALSH Brownian motion $X(\cdot)$ is uniquely determined (see Section 7), the distribution of $Y(\cdot)$ is also determined uniquely from these considerations.

We conclude that the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ is well-posed. □

9. The time-homogeneous strong MARKOV property

From Section 7, we know that the unique solution to the well-posed local martingale problem associated with the triple $(\mathbf{1}, \mathbf{0}, \boldsymbol{\mu})$ induces a WALSH Brownian motion, which is a time-homogeneous strong MARKOV process as shown in [1]. We generalize this result in Section 9.1, by showing that every solution to a well-posed local martingale problem as in Section 6.2, associated with a triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, induces a time-homogeneous strong MARKOV process.

Next, we pick up the thread of Proposition 6.1(a), and try to see what we can say about solutions to *the system of stochastic equations* (6.4) *for given* $(\gamma_1, \gamma_2) \in \mathbb{R}^2$, *subject only to the non-stickiness condition* (2.13). We find that for some such solutions there is no spinning measure $\boldsymbol{\mu}$ such that the "thinning condition" (2.14) is satisfied. We show that the time-homogeneous strong MARKOV property can be used to rule out these solutions. Then for every solution with an appropriate version of this property, we prove the existence of a spinning measure $\boldsymbol{\mu}$ for which the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ is solved by the distribution of the state process $X(\cdot)$ in the solution.

In this spirit we obtain in Section 9.2 a similar conclusion as in Part (a) of Proposition 6.1, but with the notable difference that here $\boldsymbol{\mu}$ is not given in advance; its existence is established in the proof of Theorem 9.1, the next major result of this work. As a corollary of this result, we show in Section 9.3 that with $\boldsymbol{b} = \mathbf{0}, \boldsymbol{\sigma} = \mathbf{1}$ the equations (6.4), subject to (2.13) and to the time-homogeneous strong MARKOV property, characterize WALSH Brownian motions.

Throughout this section, we shall always refer to Section 6.1 for local submartingale problems associated with pairs $(\boldsymbol{\sigma}, \boldsymbol{b})$ (corresponding to one-dimensional reflected diffusions), and to Section 6.2 for local martingale problems associated with triples $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ (corresponding to planar diffusions).

9.1. *On well-posed local martingale problems*

Definition 9.1. Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$, we say that an adapted and continuous process $X(\cdot)$ with values in some Euclidean space $\mathbb{R}^d$ is *time-homogeneous strongly Markovian* with respect to it if, for every stopping time T of $\mathbb{F}$, real number $t \ge 0$, and set $\Gamma \in \mathcal{B}(C[0,\infty)^d)$, the identity

$$\mathbb{P}\big(X(T+\cdot) \in \Gamma \,|\, \mathcal{F}(T)\big) = \mathbb{P}\big(X(T+\cdot) \in \Gamma \,|\, X(T)\big) = \mathfrak{g}\big(X(T)\big) \quad \text{holds } \mathbb{P}\text{-a.e. on } \{T < \infty\}.$$

Here $\mathfrak{g} : \mathbb{R}^d \to \mathbb{R}$ is some bounded, measurable function that may depend on Γ, but not on T.

Here we consider only continuous processes, and work on the space $C[0,\infty)^d$ of $\mathbb{R}^d$-valued continuous functions. Clearly, every strong MARKOV process with a one-parameter transition semigroup is time-homogeneous strongly Markovian. Also, a diffusion is time-homogeneous strongly Markovian under every probability measure in the system (Definition 5.1, Chapter IV of [14]). We shall show here that every solution to a well-posed local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ induces a time-homogeneous strongly Markovian process. This is an extension of Theorem 5.4.20 in [17] in the context of Section 6.2. Its proof given here is in the same context.

Proposition 9.1. *Suppose that the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ is well-posed, and let $\mathbb{Q}^{\mathrm{x}}$ be its solution with $\omega_2(0) = \mathrm{x}$, $\mathbb{Q}^{\mathrm{x}}$-a.e. Then for every stopping time T of $\mathbb{F}_2$, $C \in \mathcal{F}_2$, and $\mathrm{x} \in \mathbb{R}^2$, the process $\omega_2(\cdot)$ satisfies the property*

$$\mathbb{Q}^{\mathrm{x}}\big(\theta_T^{-1} C \,|\, \mathcal{F}_2(T)\big)(\omega_2) = \mathbb{Q}^{\omega_2(T)}(C), \quad \mathbb{Q}^{\mathrm{x}}\text{-a.e. on } \{T < \infty\},$$

where θ_T is the shift operator $(\theta_T(\omega_2))(\cdot) := \omega_2(T(\omega_2) + \cdot)$. In particular, $\omega_2(\cdot)$ is time-homogeneous strongly Markovian with respect to $(\Omega_2, \mathcal{F}_2, \mathbb{Q}^{\mathrm{x}})$ and the filtration $\mathbb{F}_2$, for every $\mathrm{x} \in \mathbb{R}^2$.

We shall need a countable determining class for our local martingale problem, so we introduce it next. A crucial result in this regard, Lemma 9.1 below, is proved in an Appendix, Section A.

Definition 9.2. We shall denote by $\mathfrak{E} \subseteq \mathfrak{D}_+^{\boldsymbol{\mu}}$ the collection that consists of

(i) the functions $g_A(x) := \|x\|(\mathbf{1}_A(\mathfrak{f}(x)) - \boldsymbol{\mu}(A))$, $A \subset \mathfrak{S}$ as in (6.16), where $\arg(A)$ is of the form $[a, b)$ and a, b are rational numbers; and of
(ii) the following functions in $\mathfrak{D}_+^{\boldsymbol{\mu}}$ used in the proof of Part (b) of Proposition 6.1: namely, $g_1, g_2, g_{i,k}$, $1 \le i, k \le 2$ in (6.8); $g_{1,1}^{\circ}, g_{2,2}^{\circ}, g_3$ in (6.9); as well as, for every rational $c_1 > 0$, a function $g_4 \in \mathfrak{D}^{\boldsymbol{\mu}}$ of the form $g_4(r, \theta) = \psi(r)$ where $\psi : [0,\infty) \to [0,\infty)$ is smooth with $\psi(r) = r$ for $r \ge c_1$.

In particular, $\mathfrak{E}$ is a *countable collection*.

Lemma 9.1. *Suppose $\mathbb{Q}$ is a probability measure on $(\Omega_2, \mathcal{F}_2)$ with $\omega_2(0) = \mathrm{x}$, $\mathbb{Q}$-a.e., under which $M^g(\cdot; \omega_2)$ is a continuous local martingale (resp., submartingale) of the filtration $\mathbb{F}_2$ for every function $g \in \mathfrak{D}^{\boldsymbol{\mu}} \cap \mathfrak{E}$ (resp., $\mathfrak{E}$). Then this is also true for every function $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ (resp., $\mathfrak{D}_+^{\boldsymbol{\mu}}$).*

Proof of Proposition 9.1. We proceed as in [17], proof of Theorem 5.4.20, including Lemma 5.4.18 and Lemma 5.4.19. It is easy to check that all the arguments there apply to our context (with some standard localization and application of optional sampling to submartingales), except for the final step of the proof of Lemma 5.4.19. To get through it, we only need to find a countable collection $\mathfrak{E} \subset \mathfrak{D}_+^{\boldsymbol{\mu}}$ with the property that, in order to show that $M^g(\cdot; \omega_2)$ is a continuous local martingale (resp., submartingale) for every function $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ (resp., $\mathfrak{D}_+^{\boldsymbol{\mu}}$), it suffices to have these properties for all functions in $\mathfrak{E}$. We appeal now to Lemma 9.1, and the proof of Proposition 9.1 follows. □

9.2. *Time-homogeneous strongly Markovian solutions to* (6.4), *under only* (2.13)

Let us recall Part (a) of Proposition 6.1. Suppose that we do not specify a measure $\boldsymbol{\mu}$ in advance, and that the "thinning" condition (2.14) is not imposed. In particular, *with given* BOREL*-measurable functions* $\boldsymbol{b} : [0, \infty) \to \mathbb{R}$, $\boldsymbol{\sigma} : [0, \infty) \to \mathbb{R} \setminus \{0\}$ *and real numbers* $\gamma_i, i = 1, 2$, *we consider the system of stochastic equations* (6.4) *subject only to the non-stickiness condition* (2.13).

From Part (b) of Proposition 6.1 we know that, for a probability measure $\boldsymbol{\mu}$ on $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ with

$$\gamma_i = \int_{\mathfrak{S}} z_i \, \boldsymbol{\mu}(\mathrm{d}z), \quad i = 1, 2, \tag{9.1}$$

every solution to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ induces a solution to the system (6.4), subject to (2.13). *But can we obtain all the solutions of* (6.4), (2.13) *in this way*?

The answer is negative: There are usually several probability measures $\boldsymbol{\mu}$ satisfying (9.1), so we can construct a solution to (6.4) that satisfies (2.13) and features two different "spinning measures", both satisfying (9.1). Then this solution is not related to that of a local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, for *any* $\boldsymbol{\mu}$. The construction will be given in detail at the end of this subsection (Remark 9.4).

Interestingly, if we restrict our scope to solutions with some appropriate time-homogeneous strong MARKOV properties, then each solution to (6.4) subject to the non-stickiness condition (2.13) *is* related to that of a local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, for some $\boldsymbol{\mu}$ that depends on this solution. This is the following main result of this subsection; its proof is given in an Appendix, Section B.

Theorem 9.1 (Strongly Markovian solutions of (6.4), (2.13)). *Let us consider a weak solution* $(X(\cdot), W(\cdot))$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ *of the system* (6.4) *for some given constants* γ_1, γ_2, *namely*

$$X_i(\cdot) = X_i(0) + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\big[\boldsymbol{b}\big(\|X(t)\|\big)\,\mathrm{d}t + \boldsymbol{\sigma}\big(\|X(t)\|\big)\,\mathrm{d}W(t)\big] + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2.$$

(i) *If both* $X(\cdot)$ *and* $\|X(\cdot)\|$ *are time-homogeneous, strongly Markovian processes with respect to* $\mathbb{F}^X = \{\mathcal{F}^X(t)\}_{0 \le t < \infty}$, *and if the condition* (2.13) *holds, then there exists a probability measure* $\boldsymbol{\mu}$ *on* $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ *such that* $X(\cdot)$ *induces a solution to the local martingale problem associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$.

(ii) *If, in addition, the state process of this weak solution satisfies the analogue*

$$\mathbb{P}\big(L^{\|X\|}(\infty) > 0\big) > 0 \tag{9.2}$$

of the condition (3.5), *then the measure* $\boldsymbol{\mu}$ *in* (i) *is uniquely determined by* $X(\cdot)$ *and must satisfy* (9.1).

Remark 9.1. The existence and determination of $\boldsymbol{\mu}$ in Theorem 9.1 are reminiscent of what happens for the skew Brownian Motion, where one can "read off" from the HARRISON & SHEPP equation [11] what the skewness parameter is. The measure $\boldsymbol{\mu}$ here, however, cannot be determined only from the equation (6.4), as one can usually find many $\boldsymbol{\mu}$'s satisfying (9.1), given γ_1, γ_2. Rather, $\boldsymbol{\mu}$ can be gleaned by observing the paths of a given solution, as shown in (B.1) and Proposition B.1 in the proof of Theorem 9.1.

Under appropriate conditions on $(\boldsymbol{\sigma}, \boldsymbol{b})$ in (6.4), we will only need $X(\cdot)$ itself to be time-homogeneous and strongly Markovian with respect to $\mathbb{F}^X$ in Theorem 9.1(i). The following lemma guarantees this.

Lemma 9.2. *With the setting and assumptions of Theorem* 9.1, *suppose that the local submartingale problem of Section* 6.1 *associated with the pair* $(\boldsymbol{\sigma}, \boldsymbol{b})$ *is well-posed.*

Then $\|X(\cdot)\|$ *is time-homogeneous strongly Markovian with respect to* $\mathbb{F}^X$.

Proof. We obtain the equation (6.5) for the radial part $\|X(\cdot)\|$ from Proposition 6.1(i). Applying ITÔ's formula to it in the context of Section 6.1, we see that for every function $\psi \in C^2([0, \infty); \mathbb{R})$ with $\psi'(0+) \ge 0$ (resp. $\psi'(0+) = 0$) , the process $K^{\psi}(\cdot; \|X(\cdot)\|)$ is a continuous local submartingale (resp. martingale) with respect to the filtration $\mathbb{F}$. This process is also adapted to $\mathbb{F}^X$ and $\mathcal{F}^X(t) \subseteq \mathcal{F}(t)$ holds for all $t \geq 0$, so the statement in the last sentence still holds with $\mathbb{F}$ replaced by $\mathbb{F}^X$.

Following the idea of Lemma 5.4.18 and Lemma 5.4.19 in [17], we denoteby $\mathbb{Q}_\omega(A) = \mathbb{Q}(\omega; A) : \Omega \times \mathcal{F} \mapsto [0, 1]$ the regular conditional probability for $\mathcal{F}$ given $\mathcal{F}^X(T)$, where T is a bounded stopping time of $\mathbb{F}^X$. For every $\omega \in \Omega$, we define the probability measure $\mathbb{P}_\omega$ on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ via $\mathbb{P}_\omega(F) := \mathbb{Q}_\omega(\|X(T + \cdot)\| \in F), \forall F \in \mathcal{B}(C[0,\infty))$.

With this notation and the conclusion in the first paragraph of this proof, we can follow the arguments in the aforementioned two lemmas to show that for a.e. $\omega \in \Omega$, the probability measure $\mathbb{P}_\omega$ solves the local submartingale problem associated with the pair $(\boldsymbol{\sigma}, \boldsymbol{b})$, starting at $\|X(T,\omega)\|$. Combining this with the well-posedness of the local submartingale problem, we prove Lemma 9.2 by applying the proof of Theorem 5.4.20 in [17]. □

Remark 9.2. Just as in the proof of Proposition 9.1, the above argument needs a "countable representatives" result like Lemma 9.1. Here it suffices to take functions of the form $f(x) = x$, $g(x) = x^2$, and for every $n \in \mathbb{N}$ a function $f_n(\cdot)$ such that $f_n'(0+) = 0$ and $f_n(x) = x$ for $x \geq (1/n)$.

In conjunction with Lemma 9.2, Theorem 9.1 has the following corollary.

Corollary 9.1. *Suppose that the conditions* (2.13), (9.2) *are satisfied by a weak solution* $(X(\cdot), W(\cdot))$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ *of the system of equations* (6.4), *for some given real numbers* γ_1, γ_2. *Suppose also that the local submartingale problem associated with the pair* $(\boldsymbol{\sigma}, \boldsymbol{b})$ *is well-posed.*

If the state process $X(\cdot)$ *of this weak solution is time-homogeneous and strongly Markovian with respect to* $\mathbb{F}^X$, *then it determines a probability measure* $\boldsymbol{\mu}$ *on* $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ *which satisfies* (9.1), *and is such that* $X(\cdot)$ *induces a solution to the local martingale problem associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$.

9.3. *The case of* WALSH *Brownian motion*

Let us specialize now the system of equations (6.4) to the case $\boldsymbol{b} = \mathbf{0}$, $\boldsymbol{\sigma} = \mathbf{1}$ as in Proposition 7.2, namely

$$X_i(\cdot) = \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\, \mathrm{d}W(t) + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2. \tag{9.3}$$

We shall show that, when $\gamma_1^2 + \gamma_2^2 \leq 1$, this system, coupled with the time-homogeneous strong MARKOV property, characterizes WALSH Brownian motions under the non-stickiness condition (2.13). We note that in the statement and proof of the next proposition, neither (γ_1, γ_2) nor $\boldsymbol{\mu}$ are specified in advance; rather, we view them as related via (9.1).

Proposition 9.2 (A new characterization of WALSH Brownian motions). *Assume that* $Z(\cdot)$ *is a continuous, adapted planar process on some filtered probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}(t)\}_{0 \leq t < \infty}$. *Then the following two assertions are equivalent*:

(i) $Z(\cdot)$ *is a* WALSH *Brownian motion, defined via the semigroup* (7.1), *for some spinning measure* $\boldsymbol{\mu}$.
(ii) *For some pair of real numbers* (γ_1, γ_2) *with* $\gamma_1^2 + \gamma_2^2 \leq 1$, *there exists a weak solution* $(X(\cdot), W(\cdot))$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ *to the system of equations* (9.3), *such that* $X(\cdot)$: *is time-homogeneous strongly Markovian with respect to* $\mathbb{F}^X$; *satisfies the condition* (2.13); *and has the same distribution as* $Z(\cdot)$.

When these assertions hold, the measure $\boldsymbol{\mu}$ *of the statement* (i), *and the coefficients* γ_1, γ_2 *of the statement* (ii), *satisfy the relationship* (9.1).

Proof. (i) ⇒ (ii). On the strength of Propositions 7.1 and 6.1, the process $Z(\cdot)$ induces a weak solution of (9.3) subject to (2.13), where γ_1, γ_2 are given by (9.1) and therefore satisfy $\gamma_1^2 + \gamma_2^2 \leq 1$. Since $Z(\cdot)$ is time-homogeneous strongly Markovian with respect its own filtration, so is this solution.

(ii) $\Rightarrow$ (i). Appealing to Proposition 6.2, we see that the local submartingale problem associated with the pair $(\mathbf{1}, \mathbf{0})$ is well-posed. By Proposition 6.1(i) we obtain that the radial part of $X(\cdot)$ satisfies

$$\|X(\cdot)\| = \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}} \, \mathrm{d}W(t) + L^{\|X\|}(t) = \|X(0)\| + W(\cdot) + L^{\|X\|}(\cdot), \tag{9.4}$$

with the help of the non-stickiness condition (2.13). Therefore, $\|X(\cdot)\|$ is the SKOROKHOD reflection of the Brownian motion $\|X(0)\| + W(\cdot)$, and satisfies $\mathbb{P}(L^{\|X\|}(\infty) = \infty) = 1$, so the condition (9.2) follows.

Now from Corollary 9.1, the weak solution posited in (ii) induces a solution to the local martingale problem associated with the triple $(\mathbf{1}, \mathbf{0}, \boldsymbol{\mu})$, for some probability measure $\boldsymbol{\mu}$ that satisfies (9.1). Propositions 7.1 and 6.2 show that $X(\cdot)$ is a WALSH Brownian motion with spinning measure $\boldsymbol{\mu}$, and so is $Z(\cdot)$. □

Remark 9.3 (Similarities and differences). Proposition 9.2 and Theorem 3.1 show that the system of equations (9.3), with the condition $\gamma_1^2 + \gamma_2^2 \le 1$ on the coefficients, is a two-dimensional analogue of the HARRISON & SHEPP equation [11] for the skew Brownian motion. But with the following caveat:

The equations (9.3), (2.13) characterize WALSH Brownian motions only when we restrict attention to time-homogeneous strongly Markovian processes. *If this restriction is not imposed, there will be solutions to the system* (9.3) *that are not* WALSH *Brownian motions.* Such solutions are discussed in the next remark.

Furthermore, the equation (9.3) does not describe a unique WALSH Brownian motion, but may be satisfied by many such motions with different spinning measures (cf. Remark 9.1). By contrast, we can read off the flipping probability from the coefficient in the equation for the one-dimensional skew Brownian motion. The construction in Remark 9.4 right below is actually based on this observation.

Remark 9.4 (A solution to the system of equations (9.3) that features two different spinning measures). Consider the system of equations (9.3) with $\gamma_1 = \gamma_2 = 0$ and $\mathrm{x} = (0, 0)$, and note that both measures

$$\boldsymbol{\mu}_1 = \frac{1}{2}\delta_{(1,0)} + \frac{1}{2}\delta_{(-1,0)} \quad \text{and} \quad \boldsymbol{\mu}_2 = \frac{1}{2}\delta_{(0,1)} + \frac{1}{2}\delta_{(0,-1)}$$

satisfy (9.1). Let $X(\cdot)$ be a WALSH Brownian motion that solves the system (9.3) with $X(0) = (0, 0)$, $\gamma_1 = \gamma_2 = 0$, spinning measure $\boldsymbol{\mu}_1$ and driving Brownian motion $B(\cdot)$. Let $Y(\cdot)$ be another WALSH Brownian motion that solves (9.3) with $Y(0) = (1, 0)$, $\gamma_1 = \gamma_2 = 0$, spinning measure $\boldsymbol{\mu}_2$ and driver $\widetilde{B}(\cdot) := B(\tau_{(1,0)} + \cdot)$, another Brownian motion. Now define $\tau_{(1,0)} := \inf\{t \ge 0 : X(t) = (1, 0)\}$ and

$$Z(t) := X(t), 0 \le t < \tau_{(1,0)}, \quad \text{and} \quad Z(\tau_{(1,0)} + t) := Y(t), \quad \forall t \ge 0.$$

The so-defined process $Z(\cdot)$ solves (9.3) with $Z(0) = (0, 0)$, $\gamma_1 = \gamma_2 = 0$ and driving Brownian motion $B(\cdot)$, but is *not* a WALSH Brownian motion: it switches from $\boldsymbol{\mu}_1$ to $\boldsymbol{\mu}_2$ after time $\tau_{(1,0)}$. It is also not time-homogeneous strongly Markovian, by virtue of either Proposition 9.2 or elementary observations.

10. Examples

Example 10.1 (TSIREL'SON's triple point). When $\gamma_i = 0$, $i = 1, 2$, the equations (2.8) and (3.3) for $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ become, respectively,

$$X_i(T) = x_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\, \mathrm{d}S(t) \quad \text{and} \quad X_i(T) = x_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\, \mathrm{d}U(t); \quad i = 1, 2.$$

This is the case when the common probability distribution $\boldsymbol{\mu}$ of the I.I.D. random variables $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots\}$ in (5.1) has zero expectation, namely $\mathbb{E}[\boldsymbol{\xi}_1] = \mathbf{0}$. For instance, when $\boldsymbol{\mu}$ assigns equal weights of $1/3$ to three points at angles $2\pi\ell/3$, $\ell = 0, 1, 2$ on the unit circumference $\mathfrak{S}$ that trisect it.

If, in addition, $U(\cdot) = W(\cdot)$ is Brownian motion, and thus the SKOROKHOD reflection $S(\cdot) = W(\cdot) + \max_{0 \le s \le \cdot}(-W(s))^+$ in (2.2) is a reflecting Brownian motion, we deduce from Section 3.2 that the corresponding planar process $X(\cdot)$ is a martingale, to wit $X_i(T) = \mathrm{x}_i + \int_0^T \mathfrak{f}_i(X(t))\, \mathrm{d}W(t)$, $i = 1, 2$.

It was conjectured by BARLOW, PITMAN & YOR [1], and shown in the landmark paper by TSIREL'SON [26] (cf. [29], [18]), that the natural filtration of this martingale $X(\cdot)$ is not generated by *any* Brownian motion of *any* dimension.

Example 10.2 (WALSH's Brownian motion with polar drifts). Let us look at the case $\boldsymbol{\sigma}(\cdot) \equiv 1$ and $\boldsymbol{c}(\cdot) \equiv -\lambda$ for some $\lambda > 0$ in Proposition 6.3. The driving one-dimensional semimartingale $U(\cdot)$ for $X(\cdot)$ is Brownian motion with negative drift $-\lambda$ and with instantaneous reflection at the origin. From Proposition 6.1, the process $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ satisfies

$$X_i(T) = \mathrm{x}_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\big(-\lambda\, \mathrm{d}t + \mathrm{d}W(t)\big) + \gamma_i L^{\|X\|}(T), \quad 0 \le T < \infty$$

for $i = 1, 2$, where $W(\cdot)$ is one-dimensional standard Brownian motion.

Moreover, following Proposition 8.1, we may replace the constant drifts by *drifts exhibiting angular dependence*. Suppose that $\boldsymbol{\sigma}^2(r, z) = 1$ and $\boldsymbol{b}(r, z) = \boldsymbol{\lambda}(z)$ for some measurable function $\boldsymbol{\lambda} : \mathfrak{S} \to (0, \infty)$. The resulting process $Y(\cdot)$ in Proposition 8.1 has the dynamics

$$Y(T) = \mathrm{y} + \int_0^T \mathfrak{f}\big(Y(t)\big)\big(-\boldsymbol{\lambda}\big(\mathfrak{f}\big(Y(t)\big)\big)\, \mathrm{d}t + \mathrm{d}W(t)\big) + \boldsymbol{\gamma} L^{\|Y\|}(T), \quad 0 \le T < \infty.$$

Since the driving semimartingale is positive recurrent in $\mathbb{R}_+$, if $\int_{\mathfrak{S}} (1/\boldsymbol{\lambda}(z))\boldsymbol{\mu}(\mathrm{d}z) < +\infty$, the degenerate planar process $X(\cdot)$ is positive recurrent. Its stationary distribution is expressed in polar coordinates as

$$\left(\int_{\mathfrak{S}} \frac{\boldsymbol{\mu}(\mathrm{d}z)}{2\boldsymbol{\lambda}(z)}\right)^{-1} e^{-2\boldsymbol{\lambda}(z)r}\, \mathrm{d}r\, \boldsymbol{\mu}(\mathrm{d}z); \quad r > 0, z \in \mathfrak{S}$$

by the distribution of occupation times and the excursion theory of SALMINEN, VALLOIS & YOR [23]. If $\boldsymbol{\lambda}(\cdot) \equiv \lambda$(constant), then the stationary distribution reduces to $(2\lambda e^{-2\lambda r}\, \mathrm{d}r)\boldsymbol{\mu}(\mathrm{d}z)$, $r > 0$, $z \in \mathfrak{S}$.

Example 10.3 (WALSH semimartingale driven by BESSEL processes). Suppose that $R^2(\cdot)$ is a squared BESSEL process with dynamics $\mathrm{d}R^2(t) = \delta\, \mathrm{d}t + 2\sqrt{R^2(t)}\, \mathrm{d}W(t)$, where $\delta \in (1, 2)$ and $W(\cdot)$ is one-dimensional standard Brownian.

We take the square root $|R(\cdot)|$ of this process as the driving semimartingale: $U(\cdot) = |R(\cdot)| = S(\cdot)$ in Theorem 2.1. This process $S(\cdot)$ does not accumulate local time at the origin, i.e., $L^S(\cdot) \equiv 0$ holds for $\delta \in (1, 2)$, hence the resulting planar process $X(\cdot)$ of Theorem 2.1 has the dynamics

$$X_i(T) = \mathrm{x}_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\left(\frac{\delta - 1}{2\|X(t)\|} \cdot \mathbf{1}_{\{\|X(t)\neq 0\|\}}\, \mathrm{d}t + \mathrm{d}W(t)\right), \quad 0 \le T < \infty$$

for $i = 1, 2$. (Note that when $\delta \ge 2$, the process $R(\cdot)$ never reaches the origin; when $\delta = 1$, the process $X(\cdot)$ becomes WALSH Brownian motion; when $\delta \in (0, 1)$, the semimartingale property is violated.)

Furthermore, and by analogy with Example 10.2, given a measurable function $\boldsymbol{\delta} : \mathfrak{S} \to (1, 2)$ we may use the time-change technique with the dispersion $\boldsymbol{\sigma}^2(r, z) = 4r$ and the drift $\boldsymbol{b}(r, z) = \boldsymbol{\delta}(z)$ and consider the WALSH semimartingale $Y(\cdot)$ driven by angular dependent, squared-BESSEL process

$$Y(T) = \mathrm{y} + \int_0^T \mathfrak{f}\big(Y(t)\big)(\boldsymbol{\delta}\big(\mathfrak{f}\big(Y(t)\big)\, \mathrm{d}t + 2\sqrt{\|Y(t)\|}\, \mathrm{d}W(t)\big), \quad 0 \le T < \infty.$$

Here, the process $\|Y(\cdot)\|$ does not accumulate local time at the origin. The corresponding scale function, inverse function and stochastic clock are given by $\boldsymbol{p}_z(r) = r^{(2-\boldsymbol{\delta}(z))/2}$, $\boldsymbol{q}_z(r) = r^{2/(2-\boldsymbol{\delta}(z))}$, and

$$\mathcal{T}(\cdot) = \int_0^{\cdot} \big((2 - \boldsymbol{\delta}(z))^2 r^{-(\boldsymbol{\delta}(z)-1)}\big)\Big|_{r=\|Y(t)\|, z=\mathfrak{f}(Y(t))} \mathrm{d}t,$$

respectively. It can be shown that the stochastic clock does not explode (cf. Lemma 3.1 of [3], Proposition XI.1.11 of [21], pages 285–289 of [22] and Appendix A.1 of [13]).

From this process $Y(\cdot)$ we may define now the WALSH semimartingale $\Xi(\cdot) = (\Xi_1(\cdot), \Xi_2(\cdot))'$ with $\Xi_i(\cdot) := \mathfrak{f}_i(Y(\cdot))\|Y(\cdot)\|^{1/2}$, $i = 1, 2$ driven by a BESSEL process with angular dependence, which satisfies the vector integral equation derived from (8.12), namely,

$$\Xi(T) = \Xi(0) + \int_0^T \mathfrak{f}\big(\Xi(t)\big)\bigg(\frac{\delta(\mathfrak{f}(\Xi(t))) - 1}{2\|\Xi(t)\|}\mathbf{1}_{\{\|\Xi(t)\neq 0\|\}}\,\mathrm{d}t + \mathrm{d}W(t)\bigg), \quad 0 \le T < \infty.$$

Appendix A: The proof of Lemma 9.1

We denote by $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ (resp. $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+}$) the collection of functions g in $\mathfrak{D}^{\boldsymbol{\mu}}$ (resp. $\mathfrak{D}^{\boldsymbol{\mu}}_+$) such that $M^g(\cdot) = M^g(\cdot;\omega_2)$ is a continuous local martingale (resp. submartingale) of the filtration $\mathbb{F}_2$, under $\mathbb{Q}$. Then we have $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}} \supseteq \mathfrak{D}^{\boldsymbol{\mu}} \cap \mathfrak{E}$ and $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+} \supseteq \mathfrak{E}$ by assumption. The goal here is to show $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}} = \mathfrak{D}^{\boldsymbol{\mu}}$ and $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+} = \mathfrak{D}^{\boldsymbol{\mu}}_+$.

Recalling that $\mathfrak{E}$ contains the functions in Definition 9.2(ii), we can follow the proof of Part (b) of Proposition 6.1 and show that there exists a one-dimensional standard Brownian motion $W(\cdot)$ on an extension of the filtered probability space $(\Omega_2, \mathcal{F}_2, \mathbb{Q})$, $\mathbb{F}_2$ such that (6.4), (6.5) hold with $X(\cdot)$ given by (6.7), or simply $X(\cdot) := \omega_2(\cdot)$. It is clear, therefore, that $\int_0^t \mathbf{1}_{\{\|\omega_2(u)\|>0\}}(|\boldsymbol{b}(\|\omega_2(u)\|)| + \boldsymbol{\sigma}^2(\|\omega_2(u)\|))\,\mathrm{d}u < \infty$ holds for all $0 \le t < \infty$, $\mathbb{Q}$-a.s. Hence, it is not hard to validate the following two observations.

First observation. $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ is a linear space.

Second observation. Suppose $\{g_n\}_{n\in\mathbb{N}} \subseteq \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ and $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ satisfy that as $n \uparrow \infty$, $g_n(x) \to g(x)$, $\forall x \in \mathbb{R}^2$ and $\partial_r g_n(x) \to \partial_r g(x)$, $\partial^2_{rr} g_n(x) \to \partial^2_{rr} g(x)$, $\forall x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and $(g_n, g, \partial_r g_n, \partial_r g, \partial^2_{rr} g_n, \partial^2_{rr} g)$ are all uniformly bounded on every compact subset of $\mathbb{R}^2$. Then we have $g \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$.

• Returning to our argument, we know that for the functions of Definition 9.2, the process $M^{g_A}(\cdot;\omega_2)$ is a local martingale for any interval $\arg(A)$ ($A \subseteq \mathfrak{S}$) of the form $[a, b)$, where a, b are rationals. Thus the same is true when A is the disjoint union of such intervals, by linearity. These sets form an algebra. By the second observation and monotone class arguments, the same is also true for every BOREL subset A of $\mathfrak{S}$. Now for any two disjoint BOREL subsets A, B of $\mathfrak{S}$ we define

$$g_{A,B}(x) := \|x\|\big(\boldsymbol{\mu}(A)\mathbf{1}_{\{\mathfrak{f}(x)\in B\}} - \boldsymbol{\mu}(B)\mathbf{1}_{\{\mathfrak{f}(x)\in A\}}\big) \quad \text{and note} \quad g_{A,B}(x) = \boldsymbol{\mu}(A)g_B(x) - \boldsymbol{\mu}(B)g_A(x),$$

thus $g_{A,B} \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ by linearity. Starting from this and using linearity and induction, we show that if $h : \mathfrak{S} \to \mathbb{R}$ is simple and satisfies $D_{\boldsymbol{\mu}}h(0) = 0$, then the mapping $x \mapsto \|x\| \cdot h(\mathfrak{f}(x))$ is in $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$. Using the second observation, we see that this statement is still true when "simple" is replaced by "bounded and measurable".

Let us recall now that, we have obtained the existence of a one-dimensional Brownian motion $W(\cdot)$ on an extension of the filtered probability space $(\Omega_2, \mathcal{F}_2, \mathbb{Q})$, $\mathbb{F}_2$, along with (6.4) and (6.5), where $X(\cdot) := \omega_2(\cdot)$. By defining $S(\cdot) := \|X(\cdot)\|$, we can follow the proof of Theorem 4.1 to establish for any given function $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ with $g'_z(0+) \equiv 0$ the following FREIDLIN–SHEU-type semimartingale decomposition:

$$\begin{aligned} g\big(\omega_2(\cdot)\big) = g(\mathrm{x}) &+ \int_0^\cdot \mathbf{1}_{\{\|\omega_2(t)\|>0\}}\Big(\boldsymbol{b}\big(\|\omega_2(t)\|\big)\partial_r g\big(\omega_2(t)\big) + \frac{1}{2}\boldsymbol{\sigma}^2\big(\|\omega_2(t)\|\big)\partial^2_{rr} g\big(\omega_2(t)\big)\Big)\,\mathrm{d}t \\ &+ \int_0^\cdot \mathbf{1}_{\{\|\omega_2(t)\|>0\}}\boldsymbol{\sigma}\big(\|\omega_2(t)\|\big)\partial_r g\big(\omega_2(t)\big)\,\mathrm{d}W(t). \end{aligned}$$

The condition (2.14) is *not* needed here; and neither are terms involving local time.

This is because the use of (2.14) in the proof of Theorem 4.1 comes only when proving the convergence to local time as in (5.15). But this property holds here trivially, courtesy of $g'_z(0+) \equiv 0$. It follows from the above decomposition of FREIDLIN–SHEU-type that, if $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ satisfies $g'_z(0+) \equiv 0$, then $g \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$.

Finally, we observe that every $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ can be decomposed as $g = g^{(1)} + g^{(2)}$, where the function $x \mapsto g^{(1)}(x) := \|x\| \cdot g_z'(0+)$ is in $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ by the first paragraph of this bullet, and $g^{(2)} := g - g^{(1)} \in \mathfrak{D}^{\boldsymbol{\mu}}$ satisfies $(g_z^{(2)})'(0+) \equiv 0$. With the considerations above, we see $g \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$, thus $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}} = \mathfrak{D}^{\boldsymbol{\mu}}$.

We decompose then every function $g \in \mathfrak{D}_+^{\boldsymbol{\mu}}$ as $g = g_{(1)} + g_{(2)}$, where $g_{(1)}(x) := c\|x\|$ with a constant $c := D_{\boldsymbol{\mu}} g(0) \geq 0$ and $g_{(2)} := g - g_{(1)} \in \mathfrak{D}^{\boldsymbol{\mu}}$ (cf. Remark 6.1). Here $M^{g_{(2)}}(\cdot)$ is a local martingale, and $M^{g_{(1)}}(\cdot) = cM^{g_3}(\cdot)$ is a local submartingale (cf. Definition 9.2(ii)). Thus $M^g(\cdot)$ is also a local submartingale and $g \in \widetilde{\mathfrak{D}_+^{\boldsymbol{\mu}}}$. We conclude then $\widetilde{\mathfrak{D}_+^{\boldsymbol{\mu}}} = \mathfrak{D}_+^{\boldsymbol{\mu}}$, and the proof of Lemma 9.1 is complete.

Appendix B: The proof of Theorem 9.1

We shall first identify the measure $\boldsymbol{\mu}$ from $X(\cdot)$, using the time-homogeneous strong MARKOV property of this process. Then we establish a FREIDLIN–SHEU-type formula for $X(\cdot)$, so as to relate this process to a solution of the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$. Recall the definition in (5.4).

Proof of Theorem 9.1(i), Part A. Let us start by assuming that, with probability one, all these stopping times $\{\tau_m^\varepsilon, m \in \mathbb{N}_0\}$ in (5.4) are finite. Then for every $\varepsilon > 0$, $\ell \in \mathbb{N}_0$, we define

$$\boldsymbol{\mu}_\ell^\varepsilon(B) := \mathbb{P}\big(\mathfrak{f}\big(X\big(\tau_{2\ell+1}^\varepsilon\big)\big) \in B\big), \quad \forall B \in \mathcal{B}(\mathfrak{S}). \tag{B.1}$$

Proposition B.1. *The measure $\boldsymbol{\mu}_\ell^\varepsilon$ just introduced does not depend on either ε or ℓ, so we can define $\boldsymbol{\mu} := \boldsymbol{\mu}_\ell^\varepsilon, \forall \varepsilon > 0, \ell \in \mathbb{N}_0$. Furthermore, $\{\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))\}_{\ell \in \mathbb{N}_0}$ is a sequence of independent random variables with common distribution $\boldsymbol{\mu}$, for every fixed $\varepsilon > 0$.*

Proof. *Step 1.* We shall show in this step that $\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))$ is independent of $\mathcal{F}^X(\tau_{2\ell}^\varepsilon)$ for any $\varepsilon > 0$, $\ell \in \mathbb{N}_0$, and that the random variables $\{\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))\}_{\ell \in \mathbb{N}_0}$ are I.I.D. for any fixed $\varepsilon > 0$. By assumption and Definition 9.1, we have for every $\varepsilon > 0$, $\ell \in \mathbb{N}_0$, $B \in \mathcal{B}(\mathfrak{S})$, the identity

$$\mathbb{P}\big(\mathfrak{f}\big(X\big(\tau_{2\ell+1}^\varepsilon\big)\big) \in B | \mathcal{F}^X\big(\tau_{2\ell}^\varepsilon\big)\big) = \mathbb{P}\big(X\big(\tau_{2\ell}^\varepsilon + \cdot\big) \in A_1 | \mathcal{F}^X\big(\tau_{2\ell}^\varepsilon\big)\big) = \mathbb{P}\big(X\big(\tau_{2\ell}^\varepsilon + \cdot\big) \in A_1 | X\big(\tau_{2\ell}^\varepsilon\big)\big).$$

Here

$$A_1 := \big\{\omega \in C[0, \infty)^2 : \mathfrak{f}\big(\omega\big(\tau_1^\varepsilon(\omega)\big)\big) \in B, \omega(0) = 0\big\} \in \mathcal{B}\big(C[0, \infty)^2\big),$$

and the above conditional probability also equals $\mathfrak{h}_1(X(\tau_{2\ell}^\varepsilon))$, for some bounded measurable function $\mathfrak{h}_1 : \mathbb{R}^2 \to \mathbb{R}$ that depends only on A_1. Now because $X(\tau_{2\ell}^\varepsilon) \equiv 0$, this conditional probability is a constant that is irrelevant to $\tau_{2\ell}^\varepsilon$, in particular, to ℓ. We deduce that $\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))$ is independent of $\mathcal{F}^X(\tau_{2\ell}^\varepsilon)$, and its distribution does not depend on ℓ. Therefore, the random variables in $\{\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))\}_{\ell \in \mathbb{N}_0}$ are I.I.D.

Step 2. On the strength of Step 1, we can define $\boldsymbol{\mu}^\varepsilon := \boldsymbol{\mu}_\ell^\varepsilon, \forall \ell \in \mathbb{N}_0$. We shall show in this step that

$$\boldsymbol{\mu}^{\varepsilon_1}(B) = \boldsymbol{\mu}^{\varepsilon_2}(B), \quad \forall B \in \mathcal{B}(\mathfrak{S}), \varepsilon_1 > \varepsilon_2 > 0.$$

Since $\|X(\tau_1^{\varepsilon_1})\| = \varepsilon_1 > \varepsilon_2$, and $\|X(\cdot)\| \leq \varepsilon_2$ on every $[\tau_{2\ell}^{\varepsilon_2}, \tau_{2\ell+1}^{\varepsilon_2}]$, we see that for a.e. $\omega \in \Omega$ there exists a unique $\ell_2 \in \mathbb{N}_0$ (depending on ω), such that $\tau_{2\ell_2+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell_2+2}^{\varepsilon_2}$. Then we can partition $\Omega = \bigcup_{\ell \in \mathbb{N}_0} F_\ell$, $F_\ell := \{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}$, where the right-hand side is a disjoint union. On the event F_ℓ, $\tau_1^{\varepsilon_1}$ and $\tau_{2\ell+1}^{\varepsilon_2}$ are on the same excursion interval of $\|X(\cdot)\|$ and hence we have $\mathfrak{f}(X(\tau_1^{\varepsilon_1})) = \mathfrak{f}(X(\tau_{2\ell+1}^{\varepsilon_2}))$.

With the considerations above and on the strength of Lemma B.1 below, we can write

$$\boldsymbol{\mu}^{\varepsilon_1}(B) = \sum_{\ell \in \mathbb{N}_0} \mathbb{P}\Big(\big\{\mathfrak{f}\big(X\big(\tau_1^{\varepsilon_1}\big)\big) \in B\big\} \cap F_\ell\Big) = \sum_{\ell \in \mathbb{N}_0} \mathbb{P}(E_\ell \cap \widetilde{F}_\ell \cap G_\ell) = \boldsymbol{\mu}^{\varepsilon_2}(B) \sum_{\ell \in \mathbb{N}_0} \mathbb{P}(F_\ell) = \boldsymbol{\mu}^{\varepsilon_2}(B),$$

where $E_\ell := \{\mathfrak{f}(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\}$, $\widetilde{F}_\ell := \{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\}$, $G_\ell := \{\max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \varepsilon_1\}$, $\ell \in \mathbb{N}_0$. This way we complete Step 2, and Proposition B.1 is proved. $\square$

Lemma B.1. *For every $\varepsilon_1 > \varepsilon_2 > 0$, $\ell \in \mathbb{N}_0$, $B \in \mathcal{B}(\mathfrak{S})$, the three events E_ℓ, $\widetilde{F}_\ell$ and G_ℓ just defined in the proof of Proposition B.1 are independent and satisfy $\widetilde{F}_\ell \cap G_\ell = F_\ell$.*

Proof. Fix $\varepsilon_1 > \varepsilon_2 > 0$, $\ell \in \mathbb{N}_0$, $B \in (\mathfrak{S})$. It is fairly clear that $F_\ell \subseteq \{\tau_{2\ell}^{\varepsilon_2} < \tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\} \cap G_\ell \subseteq \widetilde{F}_\ell \cap G_\ell$. Conversely, if $\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$, then since $\|X(t)\| \le \varepsilon_2$ for $t \in [\tau_{2\ell}^{\varepsilon_2}, \tau_{2\ell+1}^{\varepsilon_2}]$, we have $\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$. On the event $\widetilde{F}_\ell \cap G_\ell$, there exists $t \in (\tau_{2\ell+1}^{\varepsilon_2}, \tau_{2\ell+2}^{\varepsilon_2}) \subset (\tau_0^{\varepsilon_1}, \tau_{2\ell+2}^{\varepsilon_2})$ such that $\|X(t)\| \ge \varepsilon_1$, and hence $\tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}$. This implies $G_\ell \cap \widetilde{F}_\ell \subseteq F_\ell$. Thus $F_\ell = \widetilde{F}_\ell \cap G_\ell$.

By Step 1, proof of Proposition B.1, E_ℓ is independent of $\mathcal{F}^X(\tau_{2\ell}^{\varepsilon_2})$. Since $\widetilde{F}_\ell \in \mathcal{F}^X(\tau_{2\ell}^{\varepsilon_2})$, E_ℓ and $\widetilde{F}_\ell$ are independent, and both belong to $\mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})$. Since $\|X(\cdot)\|$ is time-homogeneous strongly Markovian,

$$\mathbb{P}\big(G_\ell | \mathcal{F}^X\big(\tau_{2\ell+1}^{\varepsilon_2}\big)\big) = \mathbb{P}\big(\big\|X\big(\tau_{2\ell+1}^{\varepsilon_2} + \cdot\big)\big\| \in A_2 | \mathcal{F}^X\big(\tau_{2\ell+1}^{\varepsilon_2}\big)\big) = \mathbb{P}\big(\big\|X\big(\tau_{2\ell+1}^{\varepsilon_2} + \cdot\big)\big\| \in A_2 \big| \big\|X\big(\tau_{2\ell+1}^{\varepsilon_2}\big)\big\|\big),$$

where $A_2 := \{\omega \in C[0,\infty) : \omega(\cdot) \text{hits} \varepsilon_1 \text{before hitting} 0 \text{ with } \omega(0) = \varepsilon_2\} \in \mathcal{B}(C[0,\infty))$. This conditional probability is a measurable function of $\|X(\tau_{2\ell+1}^{\varepsilon_2})\| \equiv \varepsilon_2$, and therefore G_ℓ is independent of $\mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})$. Combining this observation with the last paragraph, we complete the proof of Lemma B.1. □

- We establish now the FREIDLIN–SHEU formula for $X(\cdot)$ in this setting: *For every $g \in \mathfrak{D}$, we have*

$$g\big(X(\cdot)\big) = g(\mathrm{x}) + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}} \Big(\boldsymbol{b}\big(\big\|X(t)\big\|\big)\partial_r g\big(X(t)\big) + \frac{1}{2}\boldsymbol{\sigma}^2\big(\big\|X(t)\big\|\big)\partial_{rr}^2 g\big(X(t)\big)\Big)\,\mathrm{d}t$$
$$+ \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}} \boldsymbol{\sigma}\big(\big\|X(t)\big\|\big)\partial_r g\big(X(t)\big)\,\mathrm{d}W(t) + D_{\boldsymbol{\mu}} g(0) \cdot L^{\|X\|}(T). \tag{B.2}$$

With the considerations at the start of this section and $S(\cdot) := \|X(\cdot)\|$, we can proceed exactly as the proof of Theorem 4.1, *except for the step of proving* (5.15)*, because now we cannot rely on* (2.14). But this convergence still holds: setting $N(T,\varepsilon) := \sharp\{\ell \in \mathbb{N} : \tau_{2\ell}^{\varepsilon} < T\}$ and $h(z) := g_z'(0+)$, we have

$$\sum_{\{\ell : \tau_{2\ell+1}^{\varepsilon} < T\}} \varepsilon g'_{Z(\tau_{2\ell+1}^{\varepsilon})}(0+) = \varepsilon N(T,\varepsilon) \cdot \frac{1}{N(T,\varepsilon)} \sum_{\ell=0}^{N(T,\varepsilon)-1} h\big(Z\big(\tau_{2\ell+1}^{\varepsilon}\big)\big) + O(\varepsilon),$$

where $\{Z(\tau_{2\ell+1}^{\varepsilon})\}_{\ell \in \mathbb{N}_0}$ are I.I.D. with common distribution $\boldsymbol{\mu}$ by Proposition B.1. By the strong law of large numbers, $\lim_{N\to\infty} \sup_{n \ge N} |(\frac{1}{n}\sum_{\ell=0}^{n-1} h(Z(\tau_{2\ell+1}^{\varepsilon}))) - \int_{\mathfrak{S}} h(z)\boldsymbol{\mu}(\mathrm{d}z)| = 0$ holds a.e., thus also in probability. Moreover, this convergence in probability is uniform in ε, because the distribution of $Z(\tau_{2\ell+1}^{\varepsilon})$ does not depend on ε. Now, it is not hard to see that we have the convergence in probability

$$\frac{1}{N(T,\varepsilon)} \sum_{\ell=0}^{N(T,\varepsilon)-1} h\big(Z\big(\tau_{2\ell+1}^{\varepsilon}\big)\big) \underset{\varepsilon\downarrow 0}{\longrightarrow} \int_{\mathfrak{S}} h(z)\boldsymbol{\mu}(\mathrm{d}z) = D_{\boldsymbol{\mu}} g(0), \quad \text{on the event } \Big\{\lim_{\varepsilon\downarrow 0} N(T,\varepsilon) = \infty\Big\}.$$

On the complement $\{\lim_{\varepsilon\downarrow 0} N(T,\varepsilon) < \infty\}$ of this event, we have $\varepsilon N(T,\varepsilon) \underset{\varepsilon\downarrow 0}{\overset{L}{\longrightarrow}} {}^{\|X\|}(T) = 0$, and thus

$$\sum_{\{\ell : \tau_{2\ell+1}^{\varepsilon} < T\}} \varepsilon g'_{Z(\tau_{2\ell+1}^{\varepsilon})}(0+) \underset{\varepsilon\downarrow 0}{\longrightarrow} D_{\boldsymbol{\mu}} g(0) L^{\|X\|}(T) = 0, \quad \text{in probability.}$$

This establishes our claim, and derives the FREIDLIN–SHEU Formula (B.2) for the state process $X(\cdot)$ of the posited weak solution. With (B.2) just established, and (2.13) valid by assumption, we see that $X(\cdot)$ generates a probability measure on $(C[0,\infty)^2, \mathcal{B}(C[0,\infty)^2))$ which solves the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, where $\boldsymbol{\mu}$ is defined as in Proposition B.1. This proves Part (i) of Theorem 9.1, assuming that the stopping times $\{\tau_m^{\varepsilon}\}_{m \in \mathbb{N}_0, \varepsilon > 0}$ are all finite with probability one. □

Proof of Theorem 9.1(i), Part B. When $\{\tau_m^\varepsilon\}_{m\in\mathbb{N}_0,\varepsilon>0}$ can be infinite, we proceed as follows.

Step 1: If $\mathbb{P}(\tau_0^\varepsilon<\infty)=0$, then $L^{\|X\|}(\cdot)\equiv 0$ and (B.2) holds for any $\boldsymbol{\nu}$. Thus the conclusion of Part (ii) of Theorem 9.1 is true for any probability measure $\boldsymbol{\mu}$ on $(\mathfrak{S},\mathcal{B}(\mathfrak{S}))$. If $\mathbb{P}(\tau_0^\varepsilon<\infty)>0$, we know from (2.13) that $X(\cdot)$ can reach the origin and leave it with positive probability, so we can pick up a ε_0 such that $\mathbb{P}(\tau_1^{\varepsilon_0}<\infty)>0$. Then for every $\varepsilon\in(0,\varepsilon_0]$, $\ell\in\mathbb{N}_0$, we define the probability measure $\boldsymbol{\mu}_\ell^\varepsilon$ by

$$\boldsymbol{\mu}_\ell^\varepsilon(B):=\mathbb{P}\big(\mathfrak{f}\big(X\big(\tau_{2\ell+1}^\varepsilon\big)\big)\in B\,|\,\tau_{2\ell+1}^\varepsilon<\infty\big),\quad \forall B\in\mathcal{B}(\mathfrak{S}).$$

This is well-defined for $\ell\in\mathbb{N}_0$, by our choice of ε_0 and the strong MARKOV property of $X(\cdot)$.

Step 2: It is straightforward but heavier in notation, to follow the steps of Proposition B.1 and Lemma B.1 and check that $\boldsymbol{\mu}_\ell^\varepsilon$ does not depend on either ε or ℓ; so we can define $\boldsymbol{\mu}:=\boldsymbol{\mu}_\ell^\varepsilon, \forall\varepsilon>0,\ell\in\mathbb{N}_0$. Now we enlarge the original probability space by means of a countable collection of $\mathfrak{S}$-valued I.I.D. random variables $\{\boldsymbol{\xi}_\ell^\varepsilon\}_{\varepsilon\in\mathbb{Q}^+,\ell\in\mathbb{N}_0}$ with common distribution $\boldsymbol{\mu}$, and independent of the σ-algebra $\mathcal{F}$. For every $\varepsilon\in\mathbb{Q}^+$, $\ell\in\mathbb{N}_0$, we define the $\mathfrak{S}$-valued $\widetilde{\mathfrak{f}}(X(\tau_{2\ell+1}^\varepsilon)):=\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))\mathbf{1}_{\{\tau_{2\ell+1}^\varepsilon<\infty\}}+\boldsymbol{\xi}_\ell^\varepsilon\mathbf{1}_{\{\tau_{2\ell+1}^\varepsilon=\infty\}}$.

It is again straightforward but tedious, to check that the random variables $\{\widetilde{\mathfrak{f}}(X(\tau_{2\ell+1}^\varepsilon))\}_{\ell\in\mathbb{N}_0}$ are independent with common distribution $\boldsymbol{\mu}$, for any $\varepsilon\in\mathbb{Q}^+$. Then in the same way as in Part A, we can argue the convergence in (5.15) along rationals. The proof of Part (i) of Theorem 9.1 is now complete. □

Proof of Theorem 9.1(ii). Under the assumptions for Parts (i) and (ii), let $\boldsymbol{\mu}$ be some probability measure for which the conclusion (i) holds. Then by Proposition 6.1(b), we know that $X(\cdot)$ also solves (6.4) with γ_i replaced by $\int_{\mathfrak{S}} z_i\,\boldsymbol{\mu}(\mathrm{d}z)$. Thus we must have $\gamma_i=\int_{\mathfrak{S}} z_i\,\boldsymbol{\mu}(\mathrm{d}z)$, which is (9.1), on the strength of $\mathbb{P}(L^{\|X\|}(\infty)>0)>0$. Moreover (2.14) also holds, namely $L^{R^A}(\cdot)\equiv\boldsymbol{\mu}(A)L^{\|X\|}(\cdot),\quad \forall A\in\mathcal{B}([0,2\pi))$, with $R^A(\cdot)=\|X(\cdot)\|\cdot\mathbf{1}_A(\mathfrak{f}(X(\cdot)))$. Thanks to $\mathbb{P}(L^{\|X\|}(\infty)>0)>0$ again, we see from the above relationship that $X(\cdot)$ uniquely determines $\boldsymbol{\mu}$. The proof of Theorem 9.1 is now complete. □

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