ANNALES
DE L'INSTITUT
HENRI
POINCARÉ

PROBABILITÉS
ET STATISTIQUES

www.imstat.org/aihp

A new computation of the critical point for the planar random-cluster model with $q \geq 1$

Hugo Duminil-Copin, Aran Raoufi and Vincent Tassion

Département de Mathématiques, Université de Genève, Genève, Switzerland.
E-mail: hugo.duminil@unige.ch; aran.raoufi@unige.ch; vincent.tassion@unige.ch

Abstract. We present a new computation of the critical value of the random-cluster model with cluster weight $q \geq 1$ on $\mathbb{Z}^2$. This provides an alternative approach to the result in (*Probab. Theory Related Fields* **153** (2012) 511–542). We believe that this approach has several advantages. First, most of the proof can easily be extended to other planar graphs with sufficient symmetries. Furthermore, it invokes RSW-type arguments which are not based on self-duality. And finally, it contains a new way of applying sharp threshold results which avoid the use of symmetric events and periodic boundary conditions. Some of the new methods presented in this paper have a larger scope than the planar random-cluster model, and may be useful to investigate sharp threshold phenomena for more general dependent percolation processes in arbitrary dimensions.

Résumé. Nous proposons une nouvelle preuve du fait que le point critique du modèle de percolation de Fortuin–Kasteleyn sur le réseau carré vaut $\sqrt{q}/(1 + \sqrt{q})$ lorsque $q \geq 1$. Cette preuve est une alternative à la stratégie implémentée dans (*Probab. Theory Related Fields* **153** (2012) 511–542). Cette approche a plusieurs avantages. Tout d'abord, la grande majorité des arguments peuvent être généralisés aux autres graphes planaires (ayant suffisamment de symétries). De plus, elle n'invoque pas d'argument de type RSW basés sur l'auto-dualité du modèle. Enfin, elle repose sur une nouvelle façon d'appliquer les théorèmes de seuil qui n'utilise pas la symmétrie des évènements de croisement et les conditions de bord périodiques. Certaines de ces nouvelles méthodes ont un champs d'application qui dépasse largement le cas du modèle de percolation de Fortuin–Kasteleyn et pourrait être utile pour prouver la décroissance exponentielles d'autres modèles de percolation dépendante.

1. Introduction

The random-cluster model is one of the most classical generalization of Bernoulli percolation and electrical networks. This model was introduced by Fortuin and Kasteleyn in [8] and has since then been the object of intense study, both physically and mathematically.

A *percolation configuration* on a graph $G = (V_G, E_G)$ (V_G is the vertex set and E_G the edge set) is an element $\omega = (\omega_e : e \in E_G)$ in $\{0, 1\}^{E_G}$. An edge e is said to be *open* (in ω) if $\omega_e = 1$, otherwise it is *closed*. A configuration ω can be seen as a subgraph of G with vertex set $V_\omega := V_G$ and edge set $E_\omega := \{e \in E_G : \omega_e = 1\}$.

Let $p \in [0, 1]$ and $q \geq 1$. We will work with the *random-cluster measure* $\phi^1_{p,q}$ *on the square lattice with weights* (p, q) *and wired boundary conditions*. Let us briefly recall its definition here.

The *square lattice* $\mathbb{Z}^2$ is defined to be the graph with

$$V_{\mathbb{Z}^2} := \big\{(x, y) : x, y \in \mathbb{Z}\big\},$$

$$E_{\mathbb{Z}^2} := \big\{\big\{(x, y), (x', y')\big\} \subset \mathbb{Z}^2 \text{ such that } |x - x'| + |y - y'| = 1\big\}.$$

Let G be a finite subgraph of $\mathbb{Z}^2$ and let $\phi^1_{G,p,q}$ be the measure on percolation configurations ω on G defined by

$$\phi^1_{G,p,q}(\omega) = \frac{p^{|E_\omega|}(1 - p)^{|E_G \setminus E_\omega|} q^{k_1(\omega)}}{Z^1(G, p, q)},$$

where $k_1(\omega)$ is the number of connected components of the percolation configuration $\overline{\omega}$ on $\mathbb{Z}^2$ defined by $\overline{\omega}_e = \omega_e$ if $e \in E_G$, and 1 otherwise, and $Z^1(G, p, q)$ is a normalizing constant to make the total mass of the measure equal to 1. Then, $\phi^1_{p,q}$ is the probability measure on percolation configurations on $\mathbb{Z}^2$ defined as the weak limit of the $\phi^1_{G,p,q}$ as G exhausts $\mathbb{Z}^2$. We refer to [12] for a justification that this limit exists.

The random-cluster model on $\mathbb{Z}^2$ is known to undergo a phase transition for $q \geq 1$. Let $\{0 \leftrightarrow \infty\}$ be the event that 0 is in an infinite connected component of ω. There exists $p_c = p_c(q)$ such that $\phi^1_{p,q}[0 \leftrightarrow \infty]$ is equal to 0 for $p < p_c$, and is strictly positive if $p > p_c$. We give a new proof of the following result, which was originally obtained in [1].

Theorem 1. *Let $q \geq 1$, the critical value $p_c = p_c(q)$ is equal to $\sqrt{q}/(1 + \sqrt{q})$. Furthermore, for $p < p_c$, there exists $c = c(p, q) > 0$ such that for any $x \in \mathbb{Z}^2$,*

$$\phi^1_{p,q}[0 \leftrightarrow x] \leq \exp\big[-c\|x\|\big], \tag{1.1}$$

where $\|(a, b)\| = \max\{|a|, |b|\}$, and $\{0 \leftrightarrow x\}$ denotes the event that there exists a path from 0 to x in ω.

As in [1], our strategy is based on the study of the crossing probabilities, but we use more generic arguments. We will first prove Theorem 2 below, then we will deduce Theorem 1 by using some self-duality properties specific to the random-cluster model on the square lattice.

Let $\mathcal{C}_h(a, b)$ be the event that there exits a sequence of vertices $v_0, \ldots, v_k$ in $[\![-a, a]\!] \times [\![-b, b]\!]$ such that $v_0 \in \{-a\} \times [\![-b, b]\!]$, $v_k \in \{a\} \times [\![-b, b]\!]$, and for any $0 \leq i < k$ the vertex v_i is a neighbor of the vertex v_{i+1} and $\omega_{v_i, v_{i+1}} = 1$. This event corresponds to the existence of a "crossing from left to right" in the box $[\![-a, a]\!] \times [\![-b, b]\!]$.

Theorem 2. *Let $q \geq 1$ and $p \in [0, 1]$. If*

$$\inf_{n \geq 1} \phi^1_{p,q}\big[\mathcal{C}_h(n, n)\big] > 0, \tag{$\mathcal{A}$}$$

then for any $\delta \in (0, 1 - p]$, there exists $c > 0$ such that for every $n \geq 0$,

$$\phi^1_{p+\delta,q}\big[\mathcal{C}_h(2n, n)\big] \geq 1 - e^{-cn}.$$

Remark 3. *If $\phi^1_{p,q}[0 \leftrightarrow \infty] > 0$, then ($\mathcal{A}$) holds: because of the symmetry of the lattice*

$$\phi^1_{p,q}\big[0 \leftrightarrow \{-n\} \times [\![-n, n]\!]\big] > \phi^1_{p,q}[0 \leftrightarrow \infty]/4,$$

and the same holds for $\phi^1_{p,q}[0 \leftrightarrow \{n\} \times [\![-n, n]\!]]$. FKG inequality (see the next section) then implies ($\mathcal{A}$).

We isolated Theorem 2 because its proof does not involve duality arguments. Although we present the proof for $\mathbb{Z}^2$, the argument is valid for any quasi-periodic planar lattice with sufficient symmetries. (The required symmetry is invariance under $\pi/2$ rotation and reflections with respect to horizontal and vertical axes.) Theorem 2 is combined with a duality argument presented in Section 5 to compute the critical value and prove that the phase transition is sharp on the square lattice. This duality argument is not valid if we only assume the symmetries mentioned above, and we refer to [6] for a proof of sharpness of the phase transition for models with such symmetries.

The proof of Theorem 2 is divided into three steps, each one corresponding to a proposition below.

Proposition 1 (RSW-type result). *Let $q \geq 1$. If ($\mathcal{A}$) holds for $\phi^1_{p,q}$, then*

$$\limsup_{n \to \infty} \phi^1_{p,q}\big[\mathcal{C}_h(3n, n)\big] > 0. \tag{$\mathcal{B}$}$$

The proof of this proposition is based on a Russo–Seymour–Welsh (RSW) type argument used in [17] in the context Voronoi percolation. Interestingly, this part of the argument uses the FKG inequality only. The cost is that we obtain that the limsup only is strictly positive, instead of the infimum. Nevertheless, as we will see this will be sufficient for our purpose.

Proposition 2 (Sharp threshold for crossing probabilities). *Let $q \geq 1$ and $p \in [0, 1]$. There exists $c = c(p, q) > 0$ such that for any $\delta \in (0, 1 - p]$, and for any $n \geq 1$,*

$$\phi^1_{p+\delta,q}\big[\mathcal{C}_h(2n, n)\big] \geq 1 - \frac{1}{\phi^1_{p,q}[\mathcal{C}_h(3n, n)]} n^{-c\delta}. \tag{1.2}$$

This type of statement has been widely used in the recent studies of phase transition. It is based on sharp threshold results going back to [4,14] (we provide more details before the proof). The novelty of the proof of the proposition above lies in the fact that we do not need to symmetrize the event to which we wish to apply the sharp threshold. More precisely, in [1], a similar sharp threshold result is obtained by first proving the result on the torus, and then bootstrapping it to the plane. Here we present a new method based on the sharp-threshold theorem of [10] that allows us to circumvent this difficulty.

Proposition 3 (Bootstraping to exponential decay). *Let $q \geq 1$ and $p \in [0, 1]$. If*

$$\limsup_{n \to \infty} \phi^1_{p,q}\big[\mathcal{C}_h(2n, n)\big] = 1, \tag{$\mathcal{C}$}$$

then for any $\delta \in (0, 1 - p]$, there exists $c = c(\delta, p, q) > 0$ such that for any $n \geq 1$,

$$\phi^1_{p+\delta,q}\big[\mathcal{C}_h(2n, n)\big] \geq 1 - e^{-cn}.$$

These three propositions imply Theorem 2 readily. Indeed, ($\mathcal{A}$) at p and Proposition 1 applied at p imply ($\mathcal{B}$) at p. Proposition 2 applied to p and $\delta/2$ implies ($\mathcal{C}$) at $p + \delta/2$. Proposition 3 applied to $p + \delta/2$ and $\delta/2$ concludes the proof.

Notation

From now on, we fix $q \geq 1$ and write ϕ_p instead of $\phi^1_{p,q}$, and $\phi_{G,p}$ instead of $\phi^1_{G,p,q}$. Furthermore, for an automorphism T of the square lattice and an event A, we define the *image* of A by T as the set

$$B := \big\{\omega : T^{-1}(\omega) \in A\big\},$$

where $T^{-1}(\omega)_e := \omega_{T^{-1}(e)}$. Note that ϕ_p is symmetric under any automorphism T of the lattice, and therefore A and the image of A by T have the same probability. *We will often refer to this fact without mentioning it.* Since we will use it extensively, we also introduce $\tau_x A$ to be the image of A by the translation τ_x of vector x.

Further motivations

Our first motivation for introducing the new methods in this article is to provide a short and self-contained proof of Theorem 2. Our writing choices are in this perspective and we did not seek for generality. Nevertheless, we would like to mention that the methods used to prove Propositions 2 and 3 are not restricted to the planar random-cluster model. They can be be applied in higher dimension to investigate sharp threshold phenomena of crossing events. For example, our methods could be used to prove the following result, which is analogous to Proposition 2 and valid in

any dimension $d \geq 2$. Consider the wired random cluster $\phi^1_{\mathbb{Z}^d, p, q}$ in dimension $d \geq 2$ (it can defined in the same way as in dimension 2). For $1 \leq m \leq n$, consider the event $\mathcal{A}(m, n)$ that there exists an open path from the box $[-m, m]^d$ to the boundary of $[-n, n]^d$. For every $p \in (0, 1)$, there exists a constant $c > 0$ such that for any $\delta > 0$ and $2 \leq 2m \leq n$,

$$\phi^1_{\mathbb{Z}^d, p+\delta, q}[\mathcal{A}(2m, n)] \geq 1 - \frac{1}{\phi^1_{\mathbb{Z}^d, p, q}[\mathcal{A}(m, n)]} m^{-c\delta}. \tag{1.3}$$

Our methods can also find applications for other dependent percolation measures (for example the class of monotonic measures introduced in [11]).

Organization

The paper is organized as follows. The next three sections correspond respectively to the proofs of the last three propositions. The last section is devoted to the proof of Theorem 1. We included bibliographical comments and discussions on the scope of the proofs and the comparison with existing arguments at the end of each section.

2. Proof of Proposition 1

Below, we will make extensive use of the FKG inequality (see [12, Theorem 3.8]) which states that

$$\phi_p[A \cap B] \geq \phi_p[A]\phi_p[B], \tag{FKG}$$

for any two increasing events A and B. We recall that an event A is increasing if for every $\omega \in A$ and $\omega' \geq \omega$ (for the product ordering on $\{0, 1\}^{E_{\mathbb{Z}^2}}$), we also have $\omega' \in A$.

Since we will use it repeatedly, let us recall a classical fact. Let $k, \ell > 0$, and $n, m \geq 1$. If $m \geq n/k$, then

$$\phi_p[\mathcal{C}_h(\ell n, n)] \geq \phi_p[\mathcal{C}_h(n + m, n)]^{2k\ell}. \tag{2.1}$$

In order to obtain this inequality, apply the FKG inequality to the events $\tau_{(2mj,0)}\mathcal{C}_h(n + m, n)$ for $2j \in [\![-k\ell, k\ell]\!]$ and $\tau_{(2mj,0)}\widetilde{\mathcal{C}}_h(n, n)$ with $2j \in [\![-k\ell + 1, k\ell - 1]\!]$, where $\widetilde{\mathcal{C}}_h(n, n)$ is the image of $\mathcal{C}_h(n, n)$ by the rotation of angle $\pi/2$ around the origin.

For $n \geq 1$ and $-n \leq \alpha \leq \beta \leq n$, define $\mathcal{H}_n(\alpha, \beta)$ to be the event (illustrated on Figure 1) that there exists an open path in $[\![-n, n]\!]^2$ from $\{-n\} \times [\![-n, n]\!]$ to $\{n\} \times [\![\alpha, \beta]\!]$.

The symmetry with respect to the x-axis implies that

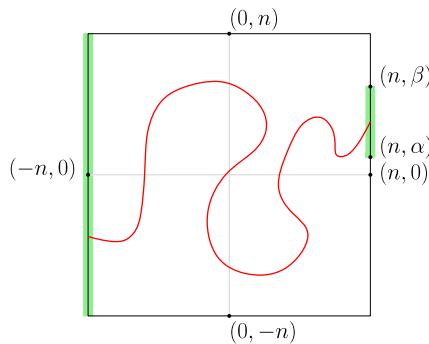$$\phi_p[\mathcal{H}_n(0, n)] \geq \frac{1}{2}\phi_p[\mathcal{C}_h(n, n)].$$

Fig. 1. A diagrammatic representation of the event $\mathcal{H}_n(\alpha, \beta)$.
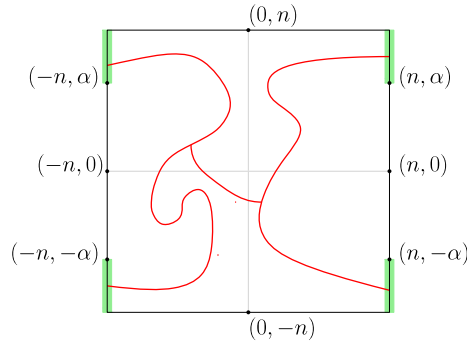
Fig. 2. A diagrammatic representation of the event $\mathcal{X}_n(\alpha)$.

Therefore, $(\mathcal{A})$ implies that

$$c_0 = \inf_{n \in \mathbb{N}} \phi_p\big[\mathcal{H}_n(0, n)\big] > 0. \tag{$\mathcal{A}'$}$$

Fix p such that $(\mathcal{A})$ is satisfied. Let $h_n(\alpha) := \phi_p[\mathcal{H}_n(\alpha, n)]$. Note that h_n is strictly decreasing and $h_n(0) \geq c_0$. Define $\alpha_n := \max\{\alpha \leq n : h_n(\alpha) \geq c_0/2\}$ so that

$$\phi_p\big[\mathcal{H}_n(\alpha, n)\big] \geq c_0/2 \quad \text{for every } 0 \leq \alpha \leq \alpha_n \quad \text{and} \tag{2.2}$$

$$\phi_p\big[\mathcal{H}_n(0, \alpha)\big] \geq c_0/2 \quad \text{for every } \alpha_n \leq \alpha \leq n. \tag{2.3}$$

Equation (2.2) follows directly from the definition of α. To show Equation (2.3), first notice that one can assume $\alpha_n \leq \alpha < n$, and then observe that in this case

$$\phi_p\big[\mathcal{H}_n(0, \alpha)\big] \geq \underbrace{\phi_p\big[\mathcal{H}_n(0, n)\big]}_{\geq c_0} - \underbrace{\phi_p\big[\mathcal{H}_n(\alpha + 1, n)\big]}_{= h_n(\alpha+1) \leq c_0/2} \geq c_0/2. \tag{2.4}$$

For $0 \leq \alpha \leq n$, let $\mathcal{X}_n(\alpha)$ be the event (illustrated on Figure 2) that there exists a connected component of ω in $[\![-n, n]\!]^2$ that intersects the line segments $\{-n\} \times [\![-n, -\alpha]\!]$, $\{-n\} \times [\![\alpha, n]\!]$, $\{n\} \times [\![-n, -\alpha]\!]$, and $\{n\} \times [\![\alpha, n]\!]$.

Let $\mathcal{H}^1$, $\mathcal{H}^2$, $\mathcal{H}^3$ and $\mathcal{H}^4$ be the events obtained from $\mathcal{H}_n(\alpha, n)$ by taking the successive images by the orthogonal symmetries with respect to the x and y-axis. Let $\mathcal{F}$ be the image of $\mathcal{C}_h(n, n)$ by the rotation of angle $\pi/2$ around the origin. Then for $\alpha \leq \alpha_n$, we find

$$\begin{aligned}
\phi_p\big[\mathcal{X}_n(\alpha)\big] &\geq \phi_p\big[\mathcal{H}^1 \cap \mathcal{H}^2 \cap \mathcal{H}^3 \cap \mathcal{H}^4 \cap \mathcal{F}\big] \\
&\overset{\text{(FKG)}}{\geq} \phi_p\big[\mathcal{H}_n(\alpha, n)\big]^4 \cdot \phi_p\big[\mathcal{C}_h(n, n)\big] \\
&\overset{(2.2),(\mathcal{A})}{\geq} \left(\frac{c_0}{4}\right)^4 \cdot c_0.
\end{aligned}$$

As a consequence,

$$c_1 := \inf_{n \in \mathbb{N}} \phi_p\big[\mathcal{X}_n(\alpha_n)\big] > 0. \tag{$\mathcal{A}''$}$$

We now divide the proof in three cases:

Case 1: $\alpha_{n+\lfloor n/2 \rfloor} \geq 2\alpha_n$ for n large enough. In such case, α_n would not be bounded by n for every n, which is absurd. Thus, this case does not occur.

Case 2: $\alpha_n \geq n/2$ for infinitely many n. In such case, pick n so that $\alpha_n \geq n/2$. Set $\mathcal{X}$ to be the image of $\mathcal{X}_n(\alpha_n)$ by rotation of angle $\pi/2$ around the origin. Thus

$$\phi_p\big[\mathcal{C}_h(3n, n)\big] \;\geq\; \phi_p\left[\bigcap_{i=-3}^{3} \tau_{(in,0)}\mathcal{X}\right]$$

$$\overset{\text{(FKG)}}{\geq} \phi_p\big[\mathcal{X}_n(\alpha_n)\big]^7 \overset{(\mathcal{A}'')}{\geq} c_1^7.$$

As a consequence, the existence of infinitely such n implies $(\mathcal{B})$.

Case 3: $\alpha_n < \lfloor n/2 \rfloor$ and $\alpha_{n+\lfloor n/2 \rfloor} < 2\alpha_n$ for infinitely many n. Fix n satisfying the two previous inequalities. To lighten the notation, set $m := \lfloor n/2 \rfloor$ and $N := n + m$. Consider the two square boxes

$$R := (-m, -\alpha_n) + [\![-N, N]\!]^2,$$

$$R' := (m, -\alpha_n) + [\![-N, N]\!]^2.$$

Let $\mathcal{E} = \tau_{(-m,-\alpha_n)}\mathcal{H}_N(0, 2\alpha_n)$ and $\mathcal{E}'$ be the image of $\mathcal{E}$ by the reflection with respect to the y-axis. Then, $[\![-n, n]\!]^2$ is included in $R \cap R'$ since $\alpha_n \leq m$ and it follows that $\mathcal{X}_n(\alpha_n) \cap \mathcal{E} \cap \mathcal{E}' \subset \mathcal{C}_h(N + 2m, N)$ (see Figure 3). This, together with $\alpha_N < 2\alpha_n$, implies

$$\phi_p\big[\mathcal{C}_h(N + 2m, N)\big] \;\geq\; \phi_p\big[\mathcal{X}_n(\alpha_n) \cap \mathcal{E} \cap \mathcal{E}'\big]$$

$$\overset{\text{(FKG)}}{\geq} \phi_p\big[\mathcal{X}_n(\alpha_n)\big] \cdot \phi_p\big[\mathcal{H}_N(0, 2\alpha_n)\big]^2$$

$$\overset{(\mathcal{A}''),(2.3)}{\geq} c_1 \cdot \left(\frac{1}{2}c_0\right)^2. \tag{2.5}$$

We can apply (2.1) to N and $2m$ to deduce that $\phi_p[\mathcal{C}_h(3N, N)] \geq c_2$ for some $c_2 > 0$ which depends on c_1 and c_0 only. Therefore, the existence of infinity many such N implies $(\mathcal{B})$.
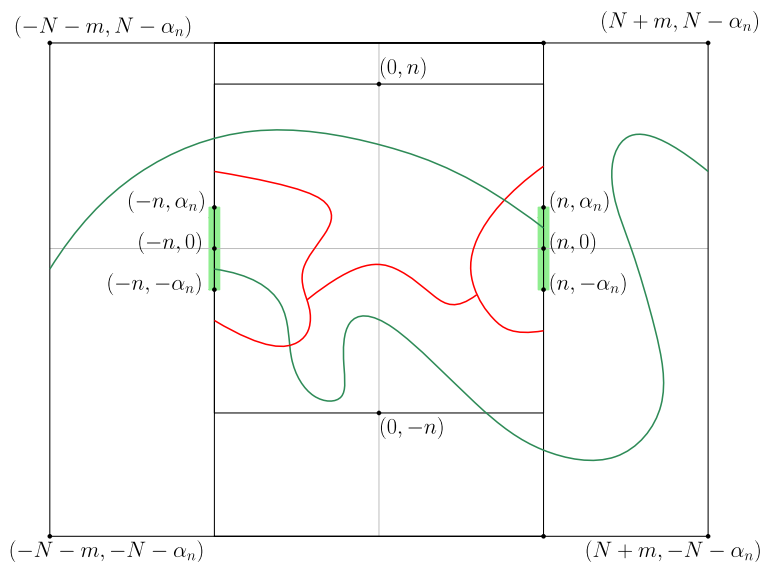

Fig. 3. The simultaneous occurrence of $\mathcal{X}_n(\alpha_n)$, $\mathcal{E}$ and $\mathcal{E}'$ implies the existence of a horizontal crossing in $[\![-(N + 2m), (N + 2m)]\!] \times [\![-N, N]\!]$.

Remarks and comments.

1. In terms of the percolation model, as mentioned in [17], the proof only requires the FKG inequality. In particular, it does not involve independence, duality, or a conditioning with respect to the highest crossing.
2. In terms of the graph, planarity is clearly essential. The proof also requires the graph to have several symmetries: namely the axial symmetry with respect to the axis, the invariance under translation and the symmetry under the rotation of angle $\pi/2$ around the origin.
3. The fact that we prove a result for the limsup only is a draw back, but this will not be relevant for the rest of the proof.
4. One may prove stronger bounds on crossing probabilities, see e.g. [5,7].

3. Proof of Proposition 2

Our goal is to prove a sharp threshold for the probability of an open path from left to right. The starting point of the proof of such a statement is usually the following simple differential equality [12, Theorem 3.12]. Let G be a finite subgraph of $\mathbb{Z}^2$ and A an increasing event depending on the states of edges in G only. We have for every $p \in (0, 1)$,

$$\frac{d}{dp}\phi_{G,p}[A] = \frac{1}{p(1-p)}\sum_{e \in E_G} J_{A,G,p}(e),$$ (3.1)

where $J_{A,G,p}(e) := \phi_{G,p}[\mathbf{1}_A \omega_e] - \phi_{G,p}[A]\phi_{G,p}[\omega_e]$.

In order to prove a sharp threshold result, we will use the following result, which is a straightforward consequence of [10, Theorem 5.1] (the original result concerns a more general class of measures than that of the random-cluster model). There exists a constant $c = c(p, q) > 0$ such that uniformly in G and A,

$$\sum_{e \in E_G} J_{A,G,p}(e) \geq c\phi_{G,p}[A]\big(1 - \phi_{G,p}[A]\big) \log\left(\frac{c}{\max\{J_{A,G,p}(e) : e \in E_G\}}\right).$$ (3.2)

In order to avoid confusion, let us mention that (3.2) is usually stated in terms of the notion of influence of an edge e which is, up to constant, related to $J_{A,p}(e)$. At this point, (3.1) together with (3.2) imply

$$\frac{d}{dp}\phi_{G,p}[A] \geq c \cdot \phi_{G,p}[A]\big(1 - \phi_{G,p}[A]\big) \cdot f_{G,p}(A),$$ (3.3)

where

$$f_{G,p}(A) := \max\left\{\log\left(\frac{c}{\max\{J_{A,G,p}(e) : e \in E_G\}}\right), \sum_{e \in E_G} J_{A,G,p}(e)\right\}.$$

We plan to apply this inequality to our context by proving that for our event, the influence of any edge e is small. We face a tiny technical difficulty: we are working directly in infinite volume with ϕ_p. For this reason, we introduce the following lemma which states an integrate infinite-volume version of (3.3) (its proof can be skipped in a first reading). Let $J_{A,p}(e) := \phi_p[\mathbf{1}_A \omega_e] - \phi_p[A]\phi_p[\omega_e]$ and

$$f_p(A) := \max\left\{\log\left(\frac{c}{\sup\{J_{A,p}(e) : e \in E_{\mathbb{Z}^2}\}}\right), \sum_{e \in E_{\mathbb{Z}^2}} J_{A,p}(e)\right\} \in [0, \infty].$$

Lemma 4. *For every $p \in (0, 1)$, $\delta \in [0, 1 - p)$, and every increasing event A depending on finitely many edges,*

$$\log\left(\frac{\phi_{p+\delta}[A]}{1 - \phi_{p+\delta}[A]} \cdot \frac{1 - \phi_p[A]}{\phi_p[A]}\right) \geq c \int_p^{p+\delta} f_s(A)\, ds.$$ (3.4)

Proof. For a positive integer n, let $\Lambda_n := [\![-n, n]\!]^2$. Choosing $G = \Lambda_n$ and integrating (3.3) between p and $p + \delta$ gives

$$\log\left(\frac{\phi_{\Lambda_n, p+\delta}[A]}{1 - \phi_{\Lambda_n, p+\delta}[A]} \cdot \frac{1 - \phi_{\Lambda_n, p}[A]}{\phi_{\Lambda_n, p}[A]}\right) \geq c \int_p^{p+\delta} f_{\Lambda_n, s}(A)\, ds.$$

The definition of the infinite-volume measure implies that the left-hand side converges to the left-hand side of (3.4). Thus, Fatou's lemma implies

$$\log\left(\frac{\phi_{p+\delta}[A]}{1 - \phi_{p+\delta}[A]} \cdot \frac{1 - \phi_p[A]}{\phi_p[A]}\right) \geq c \int_p^{p+\delta} \liminf_{n \to \infty} f_{\Lambda_n, s}(A)\, ds,$$

so that it suffices to show that for any increasing event A depending on finitely many edges,

$$\liminf_{n \to \infty} f_{\Lambda_n, s}(A) \geq f_s(A). \tag{3.5}$$

First, fix $k \leq n$ and observe that

$$f_{\Lambda_n, s}(A) \geq \sum_{e \in E_{\Lambda_n}} J_{A, \Lambda_n, s}(e) \geq \sum_{e \in E_{\Lambda_k}} J_{A, \Lambda_n, s}(e)$$

since $J_{A, \Lambda_n, s}(e) \geq 0$ by the FKG inequality. The definition of the infinite-volume measure implies that $J_{A, \Lambda_n, s}(e)$ tends to $J_{A, s}(e)$ as n tends to infinity. Letting n and then k tend to infinity implies

$$\liminf_{n \to \infty} f_{\Lambda_n, s}(A) \geq \sup_{k \geq 1} \sum_{e \in E_{\Lambda_k}} J_{A, s}(e) = \sum_{e \in E_{\mathbb{Z}^2}} J_{A, s}(e). \tag{3.6}$$

At this point, (3.5) (and therefore the claim) would follow from

$$\liminf_{n \to \infty} f_{\Lambda_n, s}(A) \geq \log\left(\frac{c}{\sup\{J_{A, s}(e) : e \in E_{\mathbb{Z}^2}\}}\right), \tag{3.7}$$

which we prove now. Let $k \leq n$ and $e \notin E_{\Lambda_k}$, the domain Markov property and the comparison between boundary conditions (see [12]) imply that

$$J_{A, \Lambda_n, s}(e) = \left(\phi_{\Lambda_n, s}[\mathbf{1}_A | \omega_e = 1] - \phi_{\Lambda_n, s}[A]\right)\phi_{\Lambda_n, s}[\omega_e]$$
$$\leq \phi_{\Lambda_k, s}[A] - \phi_s[A].$$

Now, since $\sup\{J_{A, s}(e) : e \in E_{\mathbb{Z}^2}\} > 0$, we can use the definition of the infinite-volume measure to choose k large enough so that

$$\phi_{\Lambda_k, s}[A] - \phi_s[A] \leq \sup\{J_{A, s}(e) : e \in E_{\mathbb{Z}^2}\}.$$

We deduce that

$$\sup_{n \geq k} \sup\{J_{A, \Lambda_n, p}(e) : e \in E_{\Lambda_n} \setminus E_{\Lambda_k}\} \leq \sup\{J_{A, s}(e) : e \in E_{\mathbb{Z}^2}\}.$$

But the definition of the infinite-volume measure also implies that

$$\limsup_{n \to \infty} \sup\{J_{A, \Lambda_n, p}(e) : e \in E_{\Lambda_k}\} \leq \sup\{J_{A, s}(e) : e \in E_{\mathbb{Z}^2}\}.$$

Combining the last two displayed inequalities gives

$$\liminf_{n\to\infty} f_{\Lambda_n,s}(A) \geq \log\left(\frac{c}{\limsup_{n\to\infty}\sup\{J_{A,\Lambda_n,p}(e) : e \in E_{\Lambda_n}\}}\right)$$

$$\geq \log\left(\frac{c}{\sup\{J_{A,s}(e) : e \in E_{\mathbb{Z}^2}\}}\right)$$

which is (3.7). □

The previous lemma will be combined with the following lemma. Let $A_k = A_k(n) := \mathcal{C}_h(k, n)$.

Lemma 5. *For any n large enough,*

$$\sum_{k=2n}^{3n} f_p(A_k) \geq \frac{1}{6} n \log n.$$

Before proving this lemma, let us show how it can be used to conclude the proof. Applying the above inequality to the right-hand side of (3.4), we obtain

$$\sum_{k=2n}^{3n} \log\left(\frac{\phi_{p+\delta}[A_k]}{1-\phi_{p+\delta}[A_k]} \cdot \frac{1-\phi_p[A_k]}{\phi_p^1[A_k]}\right) \geq \frac{c\delta}{6} n \log n.$$

As a consequence, there exists $k \in [\![2n, 3n]\!]$ such that

$$\log\left(\frac{\phi_{p+\delta}[A_k]}{1-\phi_{p+\delta}[A_k]} \cdot \frac{1-\phi_p[A_k]}{\phi_p[A_k]}\right) \geq \frac{c\delta}{6} \log n,$$

or after applying the exponential,

$$\frac{1}{1-\phi_{p+\delta}[A_k]\phi_p[A_k]} \geq \frac{\phi_{p+\delta}[A_k]}{1-\phi_{p+\delta}[A_k]} \cdot \frac{1-\phi_p[A_k]}{\phi_p[A_k]} \geq n^{c\delta/6}. \tag{3.8}$$

The fact that $A_{3n} \subset A_k \subset A_{2n}$ implies that

$$\phi_{p+\delta}[A_{2n}] \geq \phi_{p+\delta}[A_k] \overset{(3.8)}{\geq} 1 - \frac{1}{\phi_p[A_k]} n^{-c\delta/6} \geq 1 - \frac{1}{\phi_p[A_{3n}]} n^{-c\delta/6},$$

which is the claim of Proposition 2. We therefore only need to prove Lemma 5.

Proof of Lemma 5. Let $e \in E_{\mathbb{Z}^2}$. We have

$$J_{A_k,p}(e) = \phi_p[\mathbf{1}_{A_k}\omega_e] - \phi_p[A_k]\phi_p[\omega_e]$$

$$= \phi_p[\mathbf{1}_{\tau_{(1,0)}A_k}\omega_{\tau_{(1,0)}e}] - \phi_p[A_k]\phi_p[\omega_{\tau_{(1,0)}e}]$$

$$= J_{A_k,p}(\tau_{(1,0)}e) + \phi_p\big[(\mathbf{1}_{\tau_{(1,0)}A_k} - \mathbf{1}_{A_k})\omega_{\tau_{(1,0)}e}\big].$$

Since both $\mathbf{1}_{\tau_{(1,0)}A_k}$ and $\mathbf{1}_{A_k}$ are equal to 1 (respectively 0) on A_{k+1} (respectively the complement of $\tau_{(1,0)}A_{k-1}$), we deduce that

$$\big|J_{A_k,p}(e) - J_{A_k,p}(\tau_{(1,0)}e)\big| \leq \phi_p[A_{k-1}] - \phi_p[A_{k+1}]. \tag{3.9}$$

Since the sequence of events (A_k) is decreasing, we deduce that

$$\sum_{k=2n}^{3n} \phi_p[A_{k-1}] - \phi_p[A_{k+1}] = \phi_p[A_{2n-1}] + \phi_p[A_{2n}] - \phi_p[A_{3n}] - \phi_p[A_{3n+1}] \leq 2,$$

and therefore there exists a set $\mathcal{K} \subset [\![2n, 3n]\!]$ of cardinality at least $n/2$ such that for any $k \in \mathcal{K}$ and any $e \in E_{\mathbb{Z}^2}$,

$$\left| J_{A_k, p}(e) - J_{A_k, p}(\tau_{(1,0)}e) \right| \overset{(3.9)}{\leq} \phi_p[A_{k-1}] - \phi_p[A_{k+1}] \leq \frac{4}{n}.$$

Now, for $k \in \mathcal{K}$, one has that

$$f_p(A_k) \geq \frac{1}{3} \log n \tag{3.10}$$

as can be seen by dividing into the two following cases:

Case 1: For every $e \in E_{\mathbb{Z}^2}$, $J_{A_k, p}(e) \leq \frac{c}{n^{1/3}}$. Then the definition of f gives

$$f_p(A_k) \geq \frac{1}{3} \log n.$$

Case 2: There exists $e \in E_{\mathbb{Z}^2}$ such that $J_{A_k, p}(e) \geq \frac{c}{n^{1/3}}$. In such case, $J_{A_k, p}(\tau_{(s,0)}e) \geq \frac{c}{2n^{1/3}}$ for any edge $\tau_{(s,0)}e$ with $s \leq \frac{c}{8}n^{2/3}$. Therefore,

$$f_p(A_k) \geq \sum_{s=0}^{\lfloor \frac{c}{8}n^{2/3} \rfloor} J_{A_k, p}(\tau_{(s,0)}e) \geq \sum_{s=0}^{\lfloor \frac{c}{8}n^{2/3} \rfloor} \frac{c}{2} n^{-1/3} \geq \frac{c^2}{16} n^{1/3} \geq \frac{1}{3} \log n.$$

provided n is large enough.

Summing over every $k \in [\![2n, 3n]\!]$ gives

$$\sum_{k=2n}^{3n} f_p(A_k) \geq \sum_{k \in \mathcal{K}} f_p(A_k) \overset{(3.10)}{\geq} \frac{1}{6} n \log n. \qquad \square$$

Remarks and comments.

1. Sharp threshold theorems first emerged in the context of Boolean functions (see e.g. [4] and references therein). A weak version of the method of sharp threshold appears in the work of Russo and it was used it in the context of percolation [15,16]. Inequality (3.2) was first used for percolation by Bollobàs and Riordan [2,3]. It has since then found many other applications, thanks in particular to the generalization [11] to the non-Bernoulli case. It was in particular instrumental in [1,6].
2. Lemma 5 should hold without averaging, hence showing that $\mathcal{C}_h(k, n)$ always satisfies a sharp threshold. Unfortunately, in order to prove such a result, one should prove that $\phi_p[\mathcal{C}_h(k, n)]$ and $\phi_p[\mathcal{C}_h(k + 1, n)]$ are always polynomially close to each others. Note that if such a result would be true, we could apply it to $k = 3n$ and obtain Proposition 2 with $3n$ instead of $2n$ on the left-hand side.
3. Historically, sharp threshold theorems were often proved by using events which are invariant under translations. Such strategies required to work on a torus. In the case of models with dependencies, translating results obtained with periodic boundary conditions to results in the plane were often very difficult. Lemma 5 enables to avoid this difficulty.
4. We expect Lemma 5 to have further applications to the theory of sharp thresholds. Indeed, we assumed very little about the events A_n, namely that they were forming a decreasing sequence of events and that $A_n \Delta \tau_{(1,0)}(A_n) \subset A_{n-1} \setminus A_{n+1}$. Other sequences of events satisfy similar properties (for instance $B_n = \{[\![-n, n]\!]^2 \leftrightarrow \infty\}$ or even $C_n = \{E_n \leftrightarrow F\}$ where E_n is an increasing sequence of sets with $F \cap E_n = \varnothing$).

5. The idea of proving that all $J_{A,p}(e)$ are small was already present in [11]. Following their strategy, $J_{A_n,p}(e)$ would be bounded by the probability of having an open path going to distance $2n$. This bound could be used to prove that $p_c \leq \sqrt{q}/(1 + \sqrt{q})$, but would be insufficient to prove Theorem 1. Furthermore, the proof of Lemma 5 is sufficiently elementary to represent a good alternative to the strategy proposed in [11].

4. Proof of Proposition 3

In this section, A^c denotes the complement of the event A. For an increasing event A, let $H_{A^c}(\omega)$ be the Hamming distance from ω to A^c in $\{0, 1\}^{E_G}$, defined as the minimal number of edges that need to be turned to closed in order to go from ω to a configuration in A^c. We use the following inequality, stated as a lemma.

Lemma 6. *Let* $p \in [0, 1]$, $\delta \in [0, 1 - p]$, *and* A *an increasing event depending on finitely many edges. Then*

$$\phi_{p+\delta}[A] \geq 1 - \exp\left(-4\delta\phi_p[H_{A^c}]\right). \tag{4.1}$$

Proof. Let G be a finite subgraph of $\mathbb{Z}^2$ such that A is measurable with respect to the state of the edges of G. Let $|\omega| = \sum_{e \in E_G} \omega_e$. The facts that $\mathbf{1}_{A^c} H_{A^c} = 0$ and $|\omega| - H_{A^c}$ is increasing imply that

$$
\begin{aligned}
\frac{d}{dp} \log\left(1 - \phi_{G,p}[A]\right) &\overset{(3.1)}{=} -\frac{\phi_{G,p}[\mathbf{1}_A|\omega|] - \phi_{G,p}[A]\phi_{G,p}[|\omega|]}{p(1-p)(1 - \phi_{G,p}[A])} \\
&= \frac{\phi_{G,p}[\mathbf{1}_{A^c}|\omega|] - \phi_{G,p}[A^c]\phi_{G,p}[|\omega|]}{p(1-p)\phi_{G,p}[A^c]} \\
&= \frac{\phi_{G,p}[\mathbf{1}_{A^c}(|\omega| - H_{A^c})] - \phi_{G,p}[A^c]\phi_{G,p}[|\omega|]}{p(1-p)\phi_{G,p}[A^c]} \\
&\overset{(FKG)}{\leq} \frac{\phi_{G,p}[A^c]\phi_{G,p}[|\omega| - H_{A^c}] - \phi_{G,p}[A^c]\phi_{G,p}[|\omega|]}{p(1-p)\phi_{G,p}[A^c]} \\
&\leq -4\phi_{G,p}[H_{A^c}]. \tag{4.2}
\end{aligned}
$$

Integrating (4.2) between p and $p + \delta$ and then taking the exponential, one obtain

$$\frac{1 - \phi_{G,p+\delta}[A]}{1 - \phi_{G,p}[A]} \leq \exp\left(-4\int_p^{p+\delta} \phi_{G,s,q}^1[H_{A^c}]\,ds\right) \leq \exp\left(-4\delta\phi_{G,p}[H_{A^c}]\right).$$

In the second inequality we used that $\phi_{G,s}[H_{A^c}]$ is increasing in s. We conclude the proof by using $\phi_{G,p}[A] \geq 0$ and by letting G tend to infinity. $\qquad\square$

In the of Proposition 3, a key ingredient will be the following lemma, which can be seen as a generalization of (2.1). For a rectangle $R = [\![-a, a]\!] \times [\![-b, b]\!]$, let

$$N(a, b) := \max\left\{n \geq 1 : \exists n \text{ disjoint paths in } R \text{ from } \{-a\} \times [\![-b, b]\!] \text{ to } \{a\} \times [\![-b, b]\!]\right\}.$$

Note that $N(a, b)$ is equal to H_{A^c} where A is the event that there exists a path from left to right in R (this follows from Menger's mincut-maxflow theorem).

As in (2.1), we have

$$\phi_p\left[N(\ell n, n) \geq 1\right] \geq \phi_p\left[N(2n, n) \geq 1\right]^{2(\ell-2)+1}$$

for every $p \in [0, 1]$, $n \geq 1$, $\ell \geq 2$. We prove below that this argument also works for several disjoint paths, and we can replace the "1"s in the two events estimated above by an arbitrary number.

Lemma 7. *Let $0 \leq p \leq 1$, $n, u \geq 1$, $\ell \geq 2$. We have*

$$\phi_p\big[\mathsf{N}(\ell n, n) \geq u\big] \geq \phi_p\big[\mathsf{N}(2n, n) \geq u\big]^{2(\ell-2)+1}.$$

Proof. We only prove it for $\ell = 3$. The more general statement above follows by induction. Consider the rectangles $R_j = [\![(-3 + j)n, (1 + j)n]\!] \times [\![-n, n]\!]$, $j = 0, 2$, and $R_1 = [\![-n, n]\!] \times [\![-n, 3n]\!]$. Let E be the event that both R_0 and R_2 are crossed horizontally by u disjoint paths, and R_1 is crossed vertically by u disjoint paths. By the FKG inequality, we have

$$\phi_p[E] \geq \phi_p\big[\mathsf{N}(2n, n) \geq u\big]^3.$$

Now, observe that on the event E, there must exist at least u disjoint paths crossing horizontally the rectangle $R = [\![-3n, 3n]\!] \times [\![-n, n]\!]$. Indeed, if we close less than u edges in R then R_0 and R_2 remains crossed horizontally and R_1 remains crossed vertically, and therefore R is also crossed horizontally. Therefore, by Menger's Mincut-Maxflow theorem, there must exist at least u disjoint paths crossing horizontally the rectangle R. Thus we obtain

$$\phi_p\big[\mathsf{N}(3n, n) \geq u\big] \geq \phi_p[E] \geq \phi_p[p\big[\mathsf{N}(2n, n) \geq u\big]^3. \qquad \square$$

We are ready to proceed with the proof of Proposition 3. We assume that

$$\limsup_{n \to \infty} \phi_p\big[\mathcal{C}_h(2n, n)\big] = 1, \tag{4.3}$$

holds, and fix $\delta > 0$.

We choose $i_0 \geq 1$ large enough such that for every $i \geq i_0$,

$$1 - \exp\left(-\frac{\delta 2^{i-9}}{(i+1)^4}\right) \geq \frac{1}{2}. \tag{4.4}$$

Then, by (4.3), we can pick $n_0 \geq 1$ such that

$$\phi_p\big[\mathcal{C}_h(2n_0, n_0)\big]^{4 \cdot 2^{2i_0}} \geq \frac{1}{2}. \tag{4.5}$$

The proof of Proposition 3 is based on the following lemma.

Lemma 8. *For $i \geq 0$ set $p_i = p + \delta \sum_{1 \leq j \leq i} \frac{1}{i^2}$, $c_i = 1 - \frac{1}{4} \sum_{1 \leq j \leq i} \frac{1}{i^2}$ and $K_i = 2^i n_0$. For every $i \geq i_0$, we have*

$$\phi_{p_i}\big[\mathsf{N}(2K_i, K_i) \geq c_i 2^i\big] \geq 1/2. \tag{4.6}$$

Before proving this lemma, let us explain how we finish the proof. First, observe that $p_i \leq p + 2\delta$ and $c_i \geq 1/2$ for any $i \geq 0$. Therefore, by monotonicity, (4.6) implies for every $i \geq i_0$,

$$\phi_{p+2\delta}\big[\mathsf{N}(2K_i, K_i)\big] \geq 2^{i-2}.$$

Since $\mathsf{N}(2K_i, K_i)(\omega)$ is equal to $H_{\mathcal{C}_h^c(2K_i, K_i)}$, Lemma 6 implies that for $i \geq i_0$,

$$\phi_{p+3\delta}\big[\mathcal{C}_h(2K_i, K_i)\big] \geq 1 - e^{-\delta 2^i}.$$

Let $n \geq 2^{i_0} n_0$ and choose i such that $2^i n_0 \leq n < 2^{i+1} n_0$. We have

$$\phi_{p+3\delta}\big[\mathcal{C}_h(2n, n)\big] \geq \phi_{p+3\delta}\big[\mathcal{C}_h(4K_i, K_i)\big] \overset{(2.1)}{\geq} \big(1 - e^{-\delta 2^i}\big)^5 \geq 1 - e^{-c'n}$$

for some constant $c' > 0$ small enough. This implies for some constant $c > 0$, for every $n \geq 1$

$$\phi_{p+3\delta}\big[\mathcal{C}_h(2n, n)\big] \geq 1 - e^{-cn}.$$

Since δ is arbitrary, Proposition 3 follows readily. We now can concentrate on the proof of Lemma 8.

Proof of Lemma 8. We prove the result by induction on $i \geq i_0$. For $i = i_0$, (2.1) implies that

$$\phi_p\big[\mathcal{C}_h(2K_{i_0}, n_0)\big] \geq \phi_p\big[\mathcal{C}_h(2n_0, n_0)\big]^{4 \cdot 2^{i_0}}.$$

Then, by considering the translates of $\mathcal{C}_h(2K_{i_0}, n_0)$ by the vector $(0, (2j-1)n_0)$ with $-2^{i_0-1} < j \leq 2^{i_0-1}$, we deduce that

$$\phi_p\big[\mathsf{N}(2K_{i_0}, Ki_0) \geq 2^{i_0}\big] \overset{(\text{FKG})}{\geq} \phi_p\big[\mathcal{C}_h(2n_0, n_0)\big]^{4 \cdot 2^{2i_0}} \overset{(4.5)}{\geq} \frac{1}{2}.$$

Let us move to the induction step. Let $i \geq i_0$ such that (4.6) holds. First, by Lemma 7, we have

$$\phi_{p_i}\big[\mathsf{N}(2K_{i+1}, K_{i+1}) \geq c_i 2^{i+1}\big] \geq \frac{1}{2^{10}}.$$

Note that $(\mathsf{N}(2K_{i+1}, K_{i+1}) - c_{i+1}2^{i+1}) \vee 0$ is exactly equal to $H_{\mathsf{N}(2K_{i+1}, K_{i+1}) < c_{i+1}2^i}$. Hence, the above equation implies

$$\phi_{p_i}[H_{\mathsf{N}(2K_{i+1}, K_{i+1}) < c_{i+1}2^{i+1}}] \geq \frac{1}{2^{10}}(c_i - c_{i+1})2^{i+1} = \frac{1}{2^{11}(i+1)^2}2^i.$$

Lemma 6 applied to $A = \{\mathsf{N}(2K_{i+1}, K_{i+1}) \geq c_{i+1}2^{i+1}\}$ implies that

$$\phi_{p_{i+1}}\big[\mathsf{N}(2K_{i+1}, K_{i+1}) \geq c_{i+1}2^{i+1}\big] \geq 1 - \exp\left(-\frac{(p_{i+1} - p_i)2^{i-9}}{(i+1)^2}\right) = 1 - \exp\left(-\frac{\delta 2^{i-9}}{(i+1)^4}\right) \overset{(4.4)}{\geq} \frac{1}{2}.$$

which concludes the proof. $\qquad\square$

Remarks and comments.

1. The argument is inspired by a similar yet less powerful argument introduced in [6].
2. In [1], exponential decay was proved in two steps. First, the cluster of the origin was proved to have finite moments of any order. Then Theorem 5.60 of [12] implied the proof. Note that Theorem 5.60 uses the Domain Markov property and is based on a non-trivial theorem of Kesten [9]. The argument presented here avoids the use of the domain Markov property and is self-contained.
3. Equation (4.2) is usually stated (see [12, Theorem 2.56] or [13]) in terms of the number H_A of edges that must be switched to open in ω to be in A (this is the Hamming distance to A in $\{0, 1\}^{E_G}$), and reads

$$\frac{d}{dp}\log\big(\phi_{G,p}[A]\big) \geq \frac{\phi_{G,p}[H_A]}{p(1-p)}, \tag{4.7}$$

which, when integrated between $p - \delta$ and p, gives

$$\phi_{G,p-\delta}[A] \leq \exp\big(-4\delta\phi_{G,p}[H_A]\big).$$

This inequality is useful to prove that a probability is close to 0, while (4.1) is useful to prove that the probability is close to 1.

5. Proof of Theorem 1

Consider the dual lattice $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$. Every edge e of $\mathbb{Z}^2$ crosses exactly one edge of $(\mathbb{Z}^2)^*$ in its middle. We denote this edge by e^*. Now, let ω^* be the configuration on $(\mathbb{Z}^2)^*$ defined by $\omega^*_{e^*} = 1 - \omega_e$.

Consider the measure ϕ_p^0 to be the random-cluster measure with edge-weights p and q, and free boundary conditions on $\mathbb{Z}^2$. The only properties of ϕ_p^0 that we will use are the following (we refer to [12] for a definition of the free boundary conditions and the two following properties):

Stochastic domination. For any increasing event A, $\phi_p^0[A] \leq \phi_p^1[A]$.

Duality. If ω is sampled according to ϕ_p^1, then ω^* is sampled according to the measure $\phi_{p^*}^0$ translated by the vector $(\frac{1}{2}, \frac{1}{2})$, where

$$\frac{pp^*}{(1-p)(1-p^*)} = q.$$

Let $\mathcal{E}$ be the event that there is an open path in ω from left to right in $[\![0, 2n+1]\!] \times [\![0, 2n]\!]$. Also introduce $\mathcal{E}^*$ be the event that there is an open path in ω^* from top to bottom in $[\![\frac{1}{2}, 2n + \frac{1}{2}]\!] \times [\![-\frac{1}{2}, 2n + \frac{1}{2}]\!]$. Since either $\mathcal{E}$ or $\mathcal{E}^*$ occur, we deduce that

$$1 = \phi_p^1[\mathcal{E}] + \phi_p^1[\mathcal{E}^*] \overset{\text{Duality}}{=} \phi_p^1[\mathcal{E}] + \phi_{p^*}^0[\mathcal{E}].$$

For $p = \sqrt{q}/(1 + \sqrt{q})$, $p^* = p$ and therefore the stochastic domination implies that

$$\phi_{\sqrt{q}/(1+\sqrt{q})}^1 [\mathcal{C}_h(n, n)] \geq \phi_{\sqrt{q}/(1+\sqrt{q})}^1[\mathcal{E}] \geq \frac{1}{2}.$$

Theorem 2 implies that for any $p > \sqrt{q}/(1 + \sqrt{q})$, there exists $c > 0$ such that for every $n \geq 1$,

$$\phi_p^1[\mathcal{C}_h(2n, n)] \geq 1 - e^{-cn}. \tag{5.1}$$

For every k, define $\mathcal{A}_k$ to be $\mathcal{C}_h(2^{k+1}, 2^k)$ if k is even, and its image by the rotation of angle $\pi/2$ around the origin if k is odd. If all the $\mathcal{A}_\ell$ occur simultaneously for every ℓ such that $2^\ell \geq n/2$, then $[\![-n, n]\!]^2$ is connected to infinity. Thus, there exists $c' > 0$ such that for every $n \geq 1$,

$$\phi_p^1\big[[\![-n, n]\!]^2 \longleftrightarrow \infty\big] \overset{\text{(FKG)}}{\geq} \prod_{\ell \in \mathbb{N}: 2^\ell \geq n/2} \phi_p^1[\mathcal{A}_\ell] \overset{(5.1)}{\geq} \prod_{\ell \in \mathbb{N}: 2^\ell \geq n/2} \left(1 - e^{-c2^\ell}\right) \geq 1 - e^{-c'n}.$$

This implies that $p_c \leq \sqrt{q}/(1 + \sqrt{q})$.

Now, fix $p < \sqrt{q}/(1 + \sqrt{q})$. Let x with $\|x\| = n$ and let A_x be the event that x is connected to 0 in $[\![-n, n]\!]^2$. We wish to bound the probability of A_x. Without loss of generality, we assume that the first coordinate of x equals n. Then, define the event $B = \tau_{(-n, 0)} A_x \cap A_x$ and $\tilde{B}$ its symmetric with respect to the y axis. We have that

$$\phi_p^0[B \cap \tilde{B}] \overset{\text{(FKG)}}{\geq} \phi_p^0[B]^2 \overset{\text{(FKG)}}{\geq} \phi_p^0[A_x]^4.$$

Now, let $C_1 = \tau_{(0, 2n)} B \cap \tilde{B}$ and C_2, C_3 and C_4 the images by the rotations of angles $\pi/2$, π and $3\pi/2$ around the origin. We deduce that

$$\phi_p^0[C_1 \cap C_2 \cap C_3 \cap C_4] \overset{\text{(FKG)}}{\geq} \phi_p^0[B \cap \tilde{B}]^4 \geq \phi_p^0[A_x]^{16}.$$

Yet, on $C_1 \cap C_2 \cap C_3 \cap C_4$, we have that $[\![-n, n]\!]^2$ is not connected to infinity in ω^*. Using duality (note that $p^* \geq \sqrt{q}/(1 + \sqrt{q})$), the previous inequality implies that

$$\phi_p^0[A_x]^{16} \leq 1 - \phi_{p^*}^1\big[[\![-n, n]\!]^2 \leftrightarrow \infty\big] \overset{(5.1)}{\leq} e^{-cn}, \tag{5.2}$$

for a constant $c > 0$ depending on p and q only.

Now, if 0 is connected to x, then 0 must be connected to a vertex on the boundary of the box of size $\|x\|$ *inside* the box itself. Hence, (5.2) implies that for any x,

$$\phi_p^0[0 \leftrightarrow x] \le 8\|x\|e^{-c\|x\|/16}.$$

The previous inequality implies (1.1) for every $p < \sqrt{q}/(1+\sqrt{q})$ satisfying $\phi_p^1 = \phi_p^0$. This can be extended to every $p < \sqrt{q}/(1+\sqrt{q})$ using the fact that the set of values of p for which $\phi_p^1 \ne \phi_p^0$ is at most countable (see [12, Theorem 4.58]). Thus (1.1) holds for every $p < \sqrt{q}/(1+\sqrt{q})$. In particular it shows that $p_c \ge \sqrt{q}/(1+\sqrt{q})$.

Remarks and comments. The inequality $p_c(q) \ge \sqrt{q}/(1+\sqrt{q})$ was already proved using an argument due to Zhang in [12]. However, we did not use this inequality in the present argument.

Acknowledgements

References

[1] V. Beffara and H. Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for $q \ge 1$. *Probab. Theory Related Fields* **153** (3–4) (2012) 511–542. MR2948685

[2] B. Bollobás and O. Riordan. The critical probability for random Voronoi percolation in the plane is $1/2$. *Probab. Theory Related Fields* **136** (3) (2006) 417–468. MR2257131

[3] B. Bollobás and O. Riordan. A short proof of the Harris–Kesten theorem. *Bull. Lond. Math. Soc.* **38** (3) (2006) 470–484. MR2239042

[4] J. Bourgain, J. Kahn, G. Kalai, Y. Katznelson and N. Linial. The influence of variables in product spaces. *Israel J. Math.* **77** (1–2) (1992) 55–64. MR1194785

[5] H. Duminil-Copin. *Parafermionic Observables and Their Applications to Planar Statistical Physics Models. Ensaios Matematicos* **25**. Brazilian Mathematical Society, 2013. MR3184487

[6] H. Duminil-Copin and I. Manolescu. The phase transitions of the planar random-cluster and Potts models with q larger than 1 are sharp. *Probab. Theory Related Fields* **164** (3-4) (2016) 865–892.

[7] H. Duminil-Copin, V. Sidoravicius and V. Tassion. Absence of infinite cluster for critical Bernoulli percolation on slabs. *Comm. Pure Appl. Math.* **69** (7) (2016) 1397–1411.

[8] C. M. Fortuin and P. W. Kasteleyn. On the random-cluster model. I. Introduction and relation to other models. *Physica* **57** (1972) 536–564. MR0359655

[9] A. Gandolfi and H. Kesten. Greedy lattice animals. II. Linear growth. *Ann. Appl. Probab.* **4** (1) (1994) 76–107. MR1258174

[10] B. Graham and G. Grimmett. Sharp thresholds for the random-cluster and Ising models. *Ann. Appl. Probab.* **21** (1) (2011) 240–265. MR2759201

[11] B. T. Graham and G. R. Grimmett. Influence and sharp-threshold theorems for monotonic measures. *Ann. Probab.* **34** (5) (2006) 1726–1745. MR2271479

[12] G. Grimmett. *The Random-Cluster Model. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **333**. Springer-Verlag, Berlin, 2006. MR2243761

[13] G. R. Grimmett and M. S. T. Piza. Decay of correlations in random-cluster models. *Comm. Math. Phys.* **189** (2) (1997) 465–480. MR1480028

[14] J. Kahn, G. Kalai and N. Linial. The influence of variables on Boolean functions. In *Foundations of Computer Science, 1988, 29th Annual Symposium on* 68–80. IEEE, 1988.

[15] L. Russo. An approximate zero–one law. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **61** (1) (1982) 129–139. MR0671248

[16] M. Talagrand. On Russo's approximate zero–one law. *Ann. Probab.* **22** (1994) 1576–1587.

[17] V. Tassion. Crossing probabilities for Voronoi percolation. *Ann. Probab.* **44** (5) (2016) 3385–3398.