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Pathwise stochastic calculus with local times[1]

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Abstract. We study a notion of local time for a continuous path, defined as a limit of suitable discrete quantities along a general sequence of partitions of the time interval. Our approach subsumes other existing definitions and agrees with the usual (stochastic) local times a.s. for paths of a continuous semimartingale. We establish pathwise version of the Tanaka–Meyer, change of variables and change of time formulae. We provide equivalent conditions for existence of pathwise local time. Finally, we study in detail how the limiting objects, the quadratic variation and the local time, depend on the choice of partitions. In particular, we show that an arbitrary given non-decreasing process can be achieved a.s. by the pathwise quadratic variation of a standard Brownian motion for a suitable sequence of (random) partitions; however, such degenerate behaviour is excluded when the partitions are constructed from stopping times.

Résumé. Nous étudions la notion de temps local pour un processus continu, défini comme la limite de fonctions discrètes le long d'une suite de partitions de l'intervalle de temps. Notre approche englobe les définitions déjà existantes et coincide p.s. avec la définition (stochastique) usuelle des temps locaux pour les trajectoires de semimartingales continues. Nous établissons une version trajectorielle des formules de changement de variables, de temps et de Tanaka–Meyer ansi que plusieures conditions équivalentes pour l'existence du temps local trajectoire par trajectoire. Finalement, nous proposons une étude détaillée de la façon dont le processus limite, son temps local et sa variation quadratique, dépendent du choix de la suite de partitions. Nous montrons en particulier qu'un processus non-décroissant donné peut toujours être obtenu p.s. comme la variation quadratique d'un mouvement Brownien standard le long d'une suite appropriée de partitions (aléatoires). De tels comportements pathologiques sont cependant exclus quand les partitions sont construites à l'aide de temps d'arrêt.

MSC: 60G17; 60H05

Keywords: Pathwise local-time; Itô–Tanaka formula; Random partitions; Brownian variation

1. Introduction

In a seminal paper, Föllmer [9] pioneered a non-probabilistic approach to stochastic calculus. For a function x of real variable, he introduced a notion of quadratic variation $\langle x\rangle_t$ along a sequence of partitions $(\pi_n)_n$ and proved the associated Itô's formula for $f \in C^2$:

$$f(x_t) - f(x_0) = \int_0^t f'(x_s)\,dx_s + \frac{1}{2}\int_0^t f''(x_s)\,d\langle x\rangle_s, \tag{1.1}$$

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where the integral $\int_0^t f'(x_s)\,dx_s$ is defined as the limits of non-anticipative Riemann sums, shown to exist whenever $\langle x\rangle_t$ exists. Föllmer also observed that a path of a semimartingale a.s. admits quadratic variation in the pathwise sense and the usual stochastic integral agrees with his pathwise integral a.s.

The underlying motivation behind our current study was to extend the pathwise stochastic integral and its Itô's formula to functions f which are not in C^2. This question arose from applications in mathematical finance (see Davis et al. [6]) but, we believe, is worth pursuing for its own sake. It led us to develop pathwise stochastic calculus featuring local times, which is the first main contribution of our work. We define a notion of local time $L_t^x(u)$ for a continuous function x, prove the associated Tanaka–Meyer formula and show that a path of a continuous semimartingale X a.s. admits pathwise local time $L_t^{X(\omega)}(u)$ which then agrees with the usual (stochastic) local time. Our contribution should be seen in the context of three previous connected works. First, our results are related to Bertoin [1], who showed similar results for a large class of Dirichlet processes; see also Coutin, Nualart and Tudor [5] (who consider fractional Brownian motion with Hurst index $H > 1/3$) and Sottinen and Viitasaari [22] (who consider a class of Gaussian processes). Second, related results appeared in the unpublished diploma thesis of Wuermli [26]. Our approach is similar, however the proof in [26] was complicated and applied only to square integrable martingales. We also have a slightly different definition of local time which includes continuity in time and importantly we consider convergence in L^p for $p \in [1,\infty)$ instead of just $p = 2$. This allows us to capture the tradeoff between the generality of paths considered and the scope of applicability of the Tanaka–Meyer formula. Indeed, as the term $\int_{\mathbb{R}} L_t^\pi(u) f''(du)$ suggests, there is a natural duality between L_t^π and f'', so the smaller the space to which L_t^π belongs, the more general f'' one can take.

The latter idea was already exploited in our third main reference, the recent paper by Perkowski and Prömel [17] (and, to a much lesser extent, by Feng and Zhao [8]), in which L_t^π belongs to the space of continuous functions, and thus f'' can be a general measure (i.e. f' has bounded variation and f is the difference of two arbitrary convex functions), recovering the Tanaka–Meyer formula in full generality (the authors also consider the case where L_t^π is continuous and has bounded p-variation, so that f' can be any function with bounded q-variation). In particular, the conclusion of their main theorem (Theorem 3.5 in [17]) on the existence of $L_t^{X(\omega)}(u)$ for a.e. ω is stronger than ours; however, since their local time has to be continuous, extra conditions are required (see Section 6 below), whereas our Theorem 6.1 applies to a general semimartingale X.

Further, we investigate several questions not considered in [1,26] and [17], as we explain now. The main advantage of our definition, as compared with these previous works, is that we are able to characterise the existence of pathwise local time with a number of equivalent conditions (see Theorem 3.1). This feature seems to be entirely new. It allowed us in particular to build an explicit example of a path which admits a quadratic variation but no local time. Also, while [6] and [26] (following [9]) consider partitions π_n whose mesh is going to zero (which are well suited for changing variables), the results [1, Theorem 3.1] and [17, Theorem 3.5] of existence of pathwise local time consider Lebesgue[2] partitions. Since neither type of partition is a special case of the other, this makes the results in these papers hard to compare. We solve this conundrum by proving our existence result (Theorem 6.1) for a general type of partition, which subsumes both types considered above. Finally, we prove that the existence of $L_t(u)$ is preserved by a C^1 change of variables (improving on [6, Proposition B.6]) and by time changes, and that $g \to \int_0^t g(x_s)\,dx_s$ is continuous (similarly[3] to [17]).

Finally, we investigate how the limiting objects, quadratic variation and local time, depend on the choice of partitions. We show that for a path which oscillates enough, with a suitable choice of partitions, its quadratic variation can attain essentially any given non-decreasing function. From this, taking care of null sets and measurability issues, one can deduce that for a Brownian motion W and a given increasing $[0,\infty]$-valued measurable process A with $A_0 = 0$ there exist refining partitions $(\pi_n)_n$ made of *random* times such that

$$\langle W\rangle_t^{\pi_n} := \sum_{t_j\in\pi_n} (W_{t_{j+1}\wedge t} - W_{t_j\wedge t})^2 \to A_t \quad \text{a.s. for all } t \geq 0.$$

This result illustrates, in the most stark way possible, the dependence of the pathwise quadratic variation (and thus of the pathwise local time) on the partitions $(\pi_n)_n$. This may push the reader to dismiss the notion of pathwise

[2]Which we define in (2.2).

[3]Note that due to difference in definitions of local time, we use different topologies.

quadratic variation (and local time). However, it worth recalling that if we restrict ourselves to partitions constructed from *stopping* times, the limit of $\langle W\rangle_t^{\pi_n}$ exists and is independent of the choice of partitions, and it always equals t. Analogously, our Theorem 6.1 states that, if we only consider partitions constructed from stopping times, the pathwise local time $L_t^{X(\omega)}(u)$ of a semimartingale X exists and is independent of the choice of partitions, and it coincides with the classical local time.

As already known by Lévy, $\inf_\pi \langle x\rangle_1^\pi = 0$ for every continuous function x and $\sup_\pi \langle W(\omega)\rangle_1^\pi = \infty$ for a.e. ω. Our analysis builds on these facts and answers in particular two questions: whether for the general path $x = W(\omega)$ one can make $\langle x\rangle_1^{\pi_n}$ converge to any chosen $C = C(\omega) \in \mathbb{R}$, and what dependence in t we can expect for $\lim_n \langle x\rangle_t^{\pi_n}$. Specifically it is clear that it must be an increasing function, and we wondered whether it is automatically continuous; indeed, while we followed Föllmer [9], who carefully *required* that $\lim_n \langle x\rangle_t^{\pi_n}$ be continuous,[4] several authors who cite [9] do not (see for example [1,6,21]) and our results show that this is a significant omission.

The plan for rest of the paper is as follows. In Section 2 we introduce most of the notations and definitions, and recall parts of [9]. In Section 3 we identify several conditions equivalent to the existence of pathwise local time, prove the Tanaka–Meyer formula and the continuity of $g \to \int_0^t g(x_s)\,dx_s$. In Section 4 we consider change of variable and time, and in Section 5 we extend Tanaka–Meyer formula from the case of a Sobolev function f to the case where f is a difference of convex functions. In Section 6 we prove that a path of a semimartingale a.s. admits the pathwise local time, and relate this to the downcrossing representation of semimartingale local time proved by Lévy. Finally in Section 7 we state the results about dependence of quadratic variation on the sequence of partitions, including the convergence $\langle W\rangle_t^{\pi_n} \to A_t$ mentioned above. We only give the proof for one path avoiding the (non-trivial) technicalities related to measurability and null sets.

2. Pathwise stochastic calculus

In this section we introduce most notations and definitions used throughout the article, and we revisit the part of [9] which deals with continuous functions, slightly refining its results to include uniformity in t and more general partitions.

By measure we mean sigma-additive positive measure; a Radon measure will be the difference of two measures which are finite on compact sets. With $|\mu|$ we will denote the total-variation measure relative to a 'real measure' μ (i.e. μ is the difference of two measures), and with $\max(\mu, 0)$ the measure $(\mu + |\mu|)/2$ (i.e. the positive part in the Hahn–Jordan decomposition of μ). We will say that $g_n \to g$ *fast in* $L^p(\mu)$ if $\sum_{n\in\mathbb{N}} \|g_n - g\|_{L^p(\mu)}^p < \infty$; this trivially implies that $g_n \to g$ a.s. and in $L^p(\mu)$. We will denote by $\mathcal{B}$ (resp. $\mathcal{B}_T$) the Borel sets of $[0,\infty)$ (resp. $[0,T]$). For a continuous function $x = (x_s)_{s\geq 0}$, $\underline{x}_t$ and $\overline{x}_t$ are respectively the minimum and maximum of x_s over $s \in [0,t]$. We set $x_\infty = 0$, denote by δ_t the Dirac measure at t, and by π a partition of $[0,\infty)$, i.e. $\pi = (t_k)_{k\in\mathbb{N}}$ where $t_k \in [0,\infty]$, $t_0 = 0$, $t_k < t_{k+1}$ if $t_{k+1} < \infty$, and $\lim_{k\to\infty} t_k = \infty$. For such x and π, we set

$$O_t(x,\pi) := \max\Big\{\big|x(b) - x(a)\big| : a, b \in [t_k, t_{k+1}) \cap [0,t] \text{ for some } k \in \mathbb{N}\Big\}. \tag{2.1}$$

Föllmer works with a sequence of finite partitions $(\pi_n)_n$ whose step on compacts converges to zero. This excludes very commonly used partitions: the *Lebesgue partitions*, i.e., those of the form $\pi_P = \pi_P(x) = (t_k)_{k\in\mathbb{N}}$,

$$\text{where} \quad t_0 := 0,\, t_{k+1} := \inf\{t > t_k : x_t \in P, x_t \neq x_{t_k}\} \tag{2.2}$$

for some P partition of $\mathbb{R}$, i.e., $P = (p_k)_{k\in\mathbb{Z}}$ with

$$p_k \in [-\infty,\infty], \qquad \lim_{k\to\pm\infty} p_k = \pm\infty, \qquad p_0 = 0 \quad \text{and} \quad p_k < p_{k+1} \quad \text{if } p_k, p_{k+1} \in \mathbb{R}.$$

We will work instead with partitions π_n such that $O_T(x,\pi_n) \to 0$ for all $T < \infty$; these are very flexible, as they subsume both Lebesgue partitions and the ones used by Föllmer. Moreover they allow us to obtain time-change

[4] More precisely [9] deals with càdlàg x and requires that μ_n (defined later in (2.3)) converge weakly to a measure μ which assigns mass $(\Delta x_t)^2$ to the singleton $\{t\}$; if x is continuous this implies continuity of $\langle x\rangle_\cdot^\Pi$.

results, and have the additional advantage that one can always pass to refinements (since if $\pi \subseteq \pi'$ then $O_T(x,\pi) \geq O_T(x,\pi')$).

While our aim in this paper is to develop a pathwise, non-probabilistic, theory, it is often the case that we want to consider paths that arise as sample functions of some stochastic process. Such processes are assumed to be defined on some underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ satisfying the 'usual conditions'. We denote by $\int_0^t H_s\, dX_s$ the stochastic integral of a predictable and locally-bounded integrand H with respect to a continuous semimartingale $X = (X_t)_{t\geq 0}$. Inequalities between random variables are tacitly supposed to hold for $\mathbb{P}$-almost every ω. A sequence of partitions of $[0,\infty)$ made of random (resp. stopping) times will be called a *random* (resp. *optional*) *partition* of $[0,\infty)$; more precisely if $\pi = (\tau_k)_{k\in\mathbb{N}}$, where τ_k are $[0,\infty]$-valued random variables such that $\tau_0 = 0$, $\tau_k \leq \tau_{k+1}$ with $\tau_k < \tau_{k+1}$ on $\{\tau_{k+1} < \infty\}$, and $\lim_{k\to\infty} \tau_k = \infty$, then π is called a random partition, and if moreover $\{\tau_k \leq t\} \in \mathcal{F}_t$ for all k, t then π is an optional[5] partition.

Definition 2.1. Given a continuous function $x : [0,\infty) \to \mathbb{R}$ and a sequence of partitions $\Pi = (\pi_n)_n$ such that $O_T(x,\pi_n)$ converges to zero as $n\to\infty$ for all $T<\infty$, we will say that x *has quadratic variation* (sampled along Π) if the measures

$$\mu_n := \sum_{t_j\in\pi_n} (x_{t_{j+1}} - x_{t_j})^2 \delta_{t_j} \tag{2.3}$$

converge vaguely[6] to a measure *without atoms* μ as $n\to\infty$. We will write $x\in\mathcal{Q}$ if $O_T(x,\pi_n)\to 0$ for all $T<\infty$ and x has quadratic variation.

Recall that μ_n converges weakly to a non-atomic measure μ iff its cumulative distribution function converges pointwise to a continuous function, and thus iff

$$\langle x\rangle_t^{\pi_n} := \sum_{t_j\in\pi_n} (x_{t_{j+1}\wedge t} - x_{t_j\wedge t})^2$$

converges pointwise to a *continuous* (increasing) function $\langle x\rangle_t$; the cumulative distribution function of μ is then $\langle x\rangle$, and is called the quadratic variation of x. Such convergence is then always uniform in t, and more generally for every $T>0$ and continuous function $f : [\underline{x}_T, \overline{x}_T] \to \mathbb{R}$

$$\sum_{t_j\in\pi_n} f(x_{t_i})(x_{t_{j+1}\wedge t} - x_{t_j\wedge t})^2 \longrightarrow \int_0^t f(x_s)\, d\langle x\rangle_s \quad \text{uniformly in } t\in[0,T]: \tag{2.4}$$

indeed if $t_j \leq t < t_{j+1}$ the sum on the left of (2.4) differs from $\int_0^t f\, d\mu_n$ by at most

$$\left| f(x_{t_j})\big((x_{t_{j+1}} - x_{t_j})^2 - (x_t - x_{t_j})^2\big)\right| \leq 2\|f\|_\infty O_T(x,\pi_n)^2,$$

and $\int_0^t f\, d\mu_n$ converges to $\int_0^t f\, d\mu$ uniformly in $t\leq T$ as the following simple observation applied to the positive and negative parts of f shows.

Scholium 2.2 (Polya). Let $F, F_n : [0,T]\to\mathbb{R}$ be càdlàg increasing. If F is continuous and $F_n\to F$ pointwise then the convergence is uniform in $t\in[0,T]$.

Note that a priori μ, $\langle x\rangle$ and $\mathcal{Q}$ depend on $\Pi = (\pi_n)_n$; when we want to stress this dependence, we will write μ^Π, $\langle x\rangle^\Pi$, $\mathcal{Q}^\Pi$. Note also that the series in (2.4) is in fact a finite sum, since every partition is finite on compacts.

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[5] The terminology is justified by the fact that τ is a stopping time iff $\mathbf{1}_{\{\tau\leq\cdot\}}$ is an optional process.

[6] Meaning that $\int f\, d\mu_n \to \int f\, d\mu$ for every continuous function f with compact support.

We introduce now some more notation which will be used throughout and in particular in Section 7 and its proofs. Given numbers $a \le s \le t \le b$ and a finite partition π of $[a, b]$ (meaning $\pi = (t_i)_{i=0}^k$ with $t_0 = a$, $t_i < t_{i+1}$ for all i, and $t_k = b$) we set

$$\langle x \rangle^{\pi}_{(s,t]} := \sum_i (x_{(t_{k+1} \wedge t) \vee s} - x_{(t_k \wedge t) \vee s})^2; \tag{2.5}$$

if $s = a$ and $t = b$ the latter expression simplifies and we denote it with

$$\langle x \rangle_{\pi} := \langle x \rangle^{\pi}_{(a,b]} = \sum_i (x_{t_{k+1}} - x_{t_k})^2.$$

Notice that if π is a partition of $[a, b] \ni s, t$ then

$$\langle x \rangle^{\pi}_{(a,b]} = \langle x \rangle^{\pi}_{(a,s]} + \langle x \rangle^{\pi}_{(s,b]} \quad \text{if } s \in \pi; \tag{2.6}$$

in particular

$$\langle x \rangle^{\pi}_{(s,t]} \le \langle x \rangle^{\pi}_{(a,b]} \quad \text{if } s, t \in \pi, \tag{2.7}$$

and if $\tilde{\pi}$ is a partition of $[b, c]$ then

$$\langle x \rangle^{\pi \cup \tilde{\pi}}_{(a,c]} = \langle x \rangle^{\pi}_{(a,b]} + \langle x \rangle^{\tilde{\pi}}_{(b,c]}. \tag{2.8}$$

We shall now see that the quadratic variation sampled along optional partitions $(\pi_n)_n$ exists on a.e. path of a semimartingale and that a.e. it does not depend on $(\pi_n)_n$. This is essentially the usual result on the existence of the quadratic variation for a semimartingale.

Proposition 2.3. *Let X be a continuous semimartingale and $[X]_t := X_t^2 - 2\int_0^t X_s \, dX_s$. If $\Pi = (\pi_n)_n$ are optional partitions such that $O_T(X, \pi_n) \to 0$ a.s. for all $T < \infty$ then there exists some subsequence $(n_k)_k$ such that, for each ω outside a $\mathbb{P}$-null set and setting $\Pi' := (\pi_{n_k})_k$, we have $X(\omega) \in \mathcal{Q}^{\Pi'}$ and $\langle X(\omega) \rangle^{\Pi'} = [X](\omega)$.*

Proof. Write $\pi_n = (\tau_j^n)_j$, take $H^n := \sum_j X_{\tau_j^n} \mathbf{1}_{(\tau_j^n, \tau_{j+1}^n]}$ and notice that

$$2\int_0^t X \, dX + [X]_t = X_t^2 = 2\int_0^t H^n \, dX_t + \langle X \rangle_t^{\pi_n}. \tag{2.9}$$

Since H^n converges pointwise to X and is bounded by a locally bounded predictable process,[7] the stochastic dominated convergence theorem gives that $\int_0^\cdot H^n \, dX$ converges to $\int_0^\cdot X \, dX$ uniformly on compacts in probability, which implies the thesis. □

We now show that one can identify some of the subsubsequences along which the previous statement holds; in particular this holds when π_n is the Lebesgue partition π_{D_n} corresponding to $D_n := 2^{-n}\mathbb{N}$ (the dyadics of order n). Given $p \in [1, \infty)$ we denote by $\mathcal{S}^p$ the set of continuous semimartingales $X = (X_t)_{t \in [0,T]}$ which satisfy

$$\|X\|_{\mathcal{S}^p} := \left\| [M]_T^{1/2} \right\|_{L^p} + \left\| \int_0^T d|V|_t \right\|_{L^p} < \infty,$$

where $X = M + V$ is the canonical semimartingale decomposition of X, $[M]_t := M_t^2 - 2\int_0^t M_s \, dM_s$ is the quadratic variation of M and $|V|_t$ is the variation of $(V_s)_{s \in [0,t]}$. We recall the inequality

$$\left\| \sup_{t \le T} |X_t| \right\|_{L^p(\mathbb{P})} \le C_p \|X\|_{\mathcal{S}^p}, \tag{2.10}$$

[7] For example $|H_t^n| \le X_t^*$ with $X_t^* := \sup_{s \le t} |X_s|$.

which holds for local martingales (this being one side of the celebrated Burkholder–Davis–Gundy inequalities) and thus trivially extends to $X \in \mathcal{S}^p$. We will also use without further mention that if H is locally-bounded and predictable then the canonical decomposition of $\int_0^\cdot H\,dX$ is $\int_0^\cdot H\,dM + \int_0^\cdot H\,dV$ and so

$$\left\|\int_0^\cdot H\,dX\right\|_{\mathcal{S}^p} = \left\|\left(\int_0^T H_t^2\,d[M]_t\right)^{1/2}\right\|_{L^p} + \left\|\int_0^T |H_t|\,d|V|_t\right\|_{L^p}.$$

Proposition 2.4. *If in Proposition 2.3 we make the stronger assumption that $\sum_n O_T(X,\pi_n) < \infty$ a.s. for all $T < \infty$ then $X(\omega) \in \mathcal{Q}^\Pi$ and $\langle X(\omega)\rangle^\Pi = [X](\omega)$ for a.e. ω.*

Proof. Fix a compact time interval $[0,T]$ on which we will work. By prelocalizing we can assume that $X \in \mathcal{S}^4$ (see[8] Émery [7, Théorème 2]) and passing to an equivalent probability we can moreover[9] assume that $K := \sum_n O_T(X,\pi_n) \in L^4$. Take H^n as in Proposition 2.3, $K^n := H^n - X$ and notice that $\sup_t |K_t^n| \le O_T(X,\pi_n)$, so $\sum_n \sup_t |K_t^n| \le K$ and in particular $\sum_n \sup_t |K_t^n|^2 \le K^2 \in L^2$. Using (2.10) and $(a+b)^2 \le 2(a^2+b^2)$ gives that $\sup_t |\int_0^t K^n\,dX|$ converges to zero fast in L^2 if

$$\sum_n \mathbb{E}\left(\int (K^n)^2\,d[M] + \left(\int K^n\,d|V|\right)^2\right)$$

is finite, which is true since it is bounded above by $\mathbb{E}(K^2([M]_T + |V|_T^2))$, which is finite by Hölder inequality since $K \in L^4$, $X \in \mathcal{S}^4$. Since $\sup_{t\le T} |\int_0^t K^n\,dX| \to 0$ fast in L^2 and thus a.s., (2.9) yields $\sup_{t\le T} |\langle X\rangle_t^{\pi_n} - [X]_t| \to 0$ a.s. □

Theorem 2.5 (Föllmer [9]). *If $x \in \mathcal{Q}$, $g \in C^1$ and $t \in [0,\infty)$ the limit*

$$\lim_n \sum_{t_j \in \pi_n} g(x_{t_j})(x_{t_{j+1}\wedge t} - x_{t_j \wedge t}) \tag{2.11}$$

exists uniformly on compacts and defines a continuous function of t denoted $\int_0^t g(x_s)\,dx_s$. This integral satisfies Itô's formula: for $f \in C^2(\mathbb{R})$,

$$f(x_t) - f(x_0) = \int_0^t f'(x_s)\,dx_s + \frac{1}{2}\int_0^t f''(x_s)\,d\langle x\rangle_s. \tag{2.12}$$

Notice that the series defining the Föllmer integral in (2.11) and later in this paper are in fact finite sums, since every partition is finite on compacts.

Proof of Theorem 2.5. By using the second order Taylor's expansion write

$$\sum_{t_j \in \pi_n} f(x_{t_{j+1}\wedge t}) - f(x_{t_j\wedge t}) \tag{2.13}$$

as

$$\sum_{t_j \in \pi_n} f'(x_{t_j})(x_{t_{j+1}\wedge t} - x_{t_j\wedge t}) + \frac{1}{2}\sum_{t_j \in \pi_n} f''(x_{t_j})(x_{t_{j+1}\wedge t} - x_{t_j\wedge t})^2 + C_n(t), \tag{2.14}$$

where the correction term $C_n(t)$ is bounded by

$$\sum_{t_j \in \pi_n} \phi\big(|x_{t_{j+1}\wedge t} - x_{t_j\wedge t}|\big)(x_{t_{j+1}\wedge t} - x_{t_j\wedge t})^2 \tag{2.15}$$

[8] This statement also appears in [18, Chapter 5, Theorem 14], without proof.
[9] If $X \in \mathcal{S}^4(\mathbb{P})$ and $(d\mathbb{Q}/d\mathbb{P})(\omega) := C\exp(-K(\omega))$ then $K \in L^4(\mathbb{Q})$, and $X \in \mathcal{S}^4(\mathbb{Q})$ since $d\mathbb{Q}/d\mathbb{P} \in L^\infty(\mathbb{P})$.

for some increasing function ϕ which is continuous at 0 and such that $\phi(0)=0$. Since $x \in \mathcal{Q}^{(\pi_n)_n}$, the term (2.15) converges to 0 (for $t=T$, and thus also uniformly in $t \le T$). Since (2.4) states that the second term of (2.14) converges to the last term of (2.12) uniformly on compacts, by difference the first term of (2.14) also converges, uniformly on compacts; moreover (2.12) holds since the telescopic sum (2.13) equals $f(x_t)-f(x_0)$. □

Remark that Föllmer [9] considers sums of the form

$$\sum_{\pi_n \ni t_j \le t} g(x_{t_j})(x_{t_{j+1}}-x_{t_j}) \quad \text{and} \quad \sum_{\pi_n \ni t_j \le t} g(x_{t_j})(x_{t_{j+1}}-x_{t_j})^2,$$

whereas we consider

$$\sum_{t_j \in \pi_n} g(x_{t_j})(x_{t_{j+1}\wedge t}-x_{t_j\wedge t}) \quad \text{and} \quad \sum_{t_j \in \pi_n} g(x_{t_j})(x_{t_{j+1}\wedge t}-x_{t_j\wedge t})^2. \tag{2.16}$$

Since the difference between these two sums is

$$g(x_{t_i})(x_{t_{i+1}}-x_t) \quad \text{and} \quad g(x_{t_i})\big((x_{t_{i+1}}-x_{t_i})^2-(x_{t_{i+1}}-x_t)^2\big)$$

(where $i := \max\{j : \pi_n \ni t_j \le t\}$), which goes to zero as $O_T(x,\pi_n) \to 0$, these expressions are equal in the limit. The reason we prefer to use (2.16) is that it involves only non-anticipative quantities (i.e. their value of time t does not depend on the value of x at later times), which better fits with the theory of stochastic integration and thus allows us to obtain formulae like (2.9) and (6.5).

3. Pathwise local time

As already suggested in [9], there should be an extension of 'Itô formula' valid also when f'' is not a continuous functions, as it is in (2.12). In the theory of continuous semimartingales, such an extension proceeds via local times and the Tanaka–Meyer formula; what follows is a pathwise version.

If f'_- is the left-derivative of a convex function f, and f'' is the second derivative of f in the sense of distributions (i.e. the unique positive Radon measure which satisfies $f''([a,b)) = f'_-(b) - f'_-(a)$) we obtain for $a \le b$

$$\begin{aligned} f(b)-f(a) &= \int_a^b f'_-(y)\,dy = \int_a^b \left(f'_-(a) + \int_{[a,y)} f''(du) \right) dy \\ &= f'_-(a)(b-a) + \int_{[a,b)} (b-u) f''(du), \quad \text{so that} \end{aligned}$$

$$f(b)-f(a) = f'_-(a)(b-a) + \int_{-\infty}^{\infty} \mathbf{1}_{[a\wedge b, a\vee b)}(u)|b-u| f''(du) \quad \forall a,b \in \mathbb{R}.$$

So if given a function x and a partition $\pi = (t_j)_j$, we set for $u,v \in \mathbb{R}$

$$[\![u,v]\!) := \begin{cases} [u,v), & \text{if } u \le v, \\ [v,u), & \text{if } u > v, \end{cases}$$

and define the discrete local time (along π) as

$$L_t^{\pi}(u) := 2 \sum_{t_j \in \pi} \mathbf{1}_{[\![x_{t_j\wedge t}, x_{t_{j+1}\wedge t}]\!)}(u) |x_{t_{j+1}\wedge t} - u|, \tag{3.1}$$

then, if f equals the *difference of* two convex functions, we have the following discrete Tanaka–Meyer formula

$$f(x_t) - f(x_0) = \sum_{t_j \in \pi} f'_-(x_{t_j})(x_{t_{j+1}\wedge t} - x_{t_j \wedge t}) + \frac{1}{2} \int_{\mathbb{R}} L_t^{\pi}(u) f''(du). \tag{3.2}$$

A simple but important remark is that only the values of f in the compact interval $[\underline{x}_t, \overline{x}_t]$ are relevant. Note that $L^\pi_t(u) = 0$ for $u \notin [\underline{x}_t, \overline{x}_t]$ and $L^\pi_t(\cdot)$ is càdlàg, thus it is bounded; in particular $L^\pi_t(\cdot)$ is f''-integrable.

In the remainder of this section we will restrict our attention to those functions whose second derivative is not a general Radon measure but instead one which admits a density with respect to the Lebesgue measure. Thus, the underlying measure space will be $\mathbb{R}$ with its Borel sets, endowed with the Lebesgue measure $\mathcal{L}^1(du)$ (sometimes denoted simply by du). We will consider $L^{\pi_n}_t(\cdot)$ as a function in L^p, and denote by $W^{k,p}$ the (Sobolev) space of functions whose kth derivative in the sense of distributions is in L^p, so that $W^{1,p}$ is the set of absolutely continuous functions whose classical derivative (which exists a.e.) belongs to $W^{0,p} = L^p$, and $W^{2,p}$ is the set of C^1 functions whose classical derivative belongs to $W^{1,p}$. The following is our main theorem in this section.

Theorem 3.1. *Let x be continuous function and fix a sequence of partitions $\Pi = (\pi_n)_n$ such that $O_t(x, \pi_n) \to 0$ as $n \to \infty$ for all $t \in [0, \infty)$. Then, for $1 < p < \infty$, $q = p/(p-1)$, the following are equivalent*:

(1) $\sum_{t_j \in \pi_n} g(x_{t_j})(x_{t_{j+1}\wedge t} - x_{t_j \wedge t})$ *converges for every $g \in W^{1,q}$ and $t \in [0, \infty)$ to a continuous function of t, which we denote by $\int_0^t g(x_s)\, dx_s$.*

(2) $\sum_{t_j \in \pi_n} g(x_{t_j})(x_{t_{j+1}\wedge \cdot} - x_{t_j \wedge \cdot})$ *converges uniformly on compacts for every $g \in W^{1,q}$.*

(3) $(L^{\pi_n}_t)_n$ *converges weakly in L^p to some L_t for all $t \in [0, \infty)$, and $[0, \infty) \ni t \mapsto L_t \in L^p$ is continuous if L^p is endowed with the weak topology.*

(4) *For all $t \in [0, \infty)$ there exists $L_t \in L^p$ s.t. $\int_\mathbb{R} L^{\pi_n}_\cdot(u)h(u)\, du \to \int_\mathbb{R} L_\cdot(u)h(u)\, du$ uniformly on compacts for every $h \in L^q$.*

(5) *$x \in \mathcal{Q}^\Pi$ and for all $M \in [0, \infty)$ there exists $t \geq M$ such that $(L^{\pi_n}_t)_n$ is bounded in L^p.*

If the above conditions are satisfied then $(L^{\pi_n}_t)_n$ is bounded in L^p for all $t \in [0, \infty)$, and for all $f \in W^{2,q}$ and $t \in [0, \infty)$

$$f(x_t) - f(x_0) = \int_0^t f'(x_s)\, dx_s + \frac{1}{2}\int_\mathbb{R} L_t(u) f''(u)\, du, \tag{3.3}$$

and for all Borel bounded h

$$\int_0^t h(x_s)\, d\langle x\rangle_s = \int_\mathbb{R} L_t(u)h(u)\, du. \tag{3.4}$$

The statements above hold for $p = \infty$, $q = 1$ if the weak topology on L^p is replaced by the weak topology on L^∞. Moreover, they also hold for $p = 1$, $q = \infty$ if in item (5) boundedness in L^p is replaced by equintegrability.*

In Theorem 3.1 we slightly modify[10] the setting of [26] in order to obtain a stronger theorem with equivalent conditions; the main novelty is that item (5) implies the others. In particular we can exactly describe the difference between functions that only admit (pathwise) quadratic variation and the ones that also have local time. In Example 3.6 below we show that the two notions are strictly different and give an explicit construction of a path which admits quadratic variation but not a pathwise local time.

We will henceforth denote by $\mathcal{L}_p$ the space of continuous functions x for which the equivalent conditions of Theorem 3.1 hold. We will call $L_t(u)$ the *pathwise local time of x* at time t at level u. Observe that $\mathcal{L}_p$ and $L_t(u)$ a priori depend on $\Pi = (\pi_n)_n$, and that $L_t(u)$ depends on x. We will write $\mathcal{L}^\Pi_p$, $L^\Pi_t(u)$, $L^x_t(u)$ or $L^{x,\Pi}_t(u)$ only when we want to highlight these dependencies; as in the remainder of this section $\Pi = (\pi_n)_n$ will be fixed, we will never do that, and we will simply write L^n_t for $L^{\pi_n}_t$.

Notice that since $L^n_t(u) = 0$ for $u \notin [\underline{x}_t, \overline{x}_t]$, we can consider $L^n_t(u)$ as an element of $L^p(\mu)$ with μ being the restriction of the Lebesgue measure to $[\underline{x}_t, \overline{x}_t]$. In particular, Theorem 3.1 holds if $W^{k,q}$ is replaced with $W^{k,q}_{\mathrm{loc}}$. Moreover, if $\hat{p} \leq p$, since μ is finite, $L^p(\mu)$ embeds continuously in $L^{\hat{p}}(\mu)$, and so $\mathcal{L}_p \subseteq \mathcal{L}_{\hat{p}}$ and the limits of $(L^{\pi_n}_t)_n$ in the weak L^p and $L^{\hat{p}}$ topology coincide, so L_t does not really depend on p.

[10] Indeed [26] does not require $t \mapsto L_t$ to be continuous, and considers strong convergence in L^2 instead of weak convergence in L^p.

Note also that for $x \in \mathcal{L}_1$, using standard regularisation techniques, we can define a modification $(l_t)_t$ of the pathwise local time $(L_t)_t$ which is càdlàg and increasing in t for a.e. u. The occupation time formula then extends to all Borel bounded h

$$\int_0^T h(t, x_t)\, d\langle x\rangle_t = \int_{\mathbb{R}} \int_0^T h(t, u)\, dl_t(u)\, du. \tag{3.5}$$

Finally, we show that if $x \in \mathcal{L}_p$ the Föllmer integral is a continuous linear functional on $W^{1,q}$. This fact could have been used to define Föllmer's integral for $g \in W^{1,q}$ as the continuous extension of the Föllmer's integral for $g \in C^1$ defined in Theorem 2.5, as done in [1]. Note that the following result would not hold if we only assumed uniform convergence on compacts of g_n to g.

Proposition 3.2. *Let $p \in (1, \infty]$ (resp. $p = 1$) with conjugate exponent q. If $x \in \mathcal{L}_p$, $g_n, g \in W^{1,q}$, $g_n(x_0) \to g(x_0)$ and $g_n' \to g'$ in the weak (resp. weak*) topology of L^q, then $\int_0^t g_n(x_s)\, dx_s \to \int_0^t g(x_s)\, dx_s$ for all $t \in [0, \infty)$, and the convergence is uniform on compacts if moreover $|g_n'| \to |g'|$ weakly (resp. weakly*) in L^q.*

Proof. Define $f(u) := \int_{x_0}^u g(y)\, dy$ and analogously f_n from g_n, and notice that $f_n(u) \to f(u)$ for all $u \in \mathbb{R}$, so Tanaka–Meyer formula (3.3) gives the thesis. If moreover $|g_n'| \to |g'|$ weakly in L^q then since the positive part $\max(h, 0)$ of h equals $(h + |h|)/2$, Polya's Scholium 2.2 shows local uniformity of the convergence

$$\int_{\mathbb{R}} L_t(u) \max\big(g_n'(u), 0\big)\, du \to \int_{\mathbb{R}} L_t(u) \max\big(g'(u), 0\big)\, du;$$

working analogously with the negative parts we get the thesis. □

In the rest of this section we establish Theorem 3.1 via a series of lemmas; if not explicitly stated otherwise, p is assumed to be in $(1, \infty)$.

Lemma 3.3. *$x \in \mathcal{Q}$ iff $\int_{\mathbb{R}} L_t^n(u)\, du$ converges to a continuous function ψ_t of $t \in [0, \infty)$. In this case the convergence is uniform on compacts and $\langle x\rangle = \psi$.*

Proof. Applying formula (3.2) with $f(x) = x^2 \in L^{\mathbf{1}}([\underline{x}_t, \overline{x}_t])$ we obtain

$$\sum_{t_j \in \pi_n} x_{t_{j+1}\wedge t}^2 - x_{t_j\wedge t}^2 - 2x_{t_j}(x_{t_{j+1}\wedge t} - x_{t_j\wedge t}) = \int_{\mathbb{R}} L_t^n(u)\, du. \tag{3.6}$$

The statement follows rewriting the left side of (3.6) as $\sum_{t_j \in \pi_n} (x_{t_{j+1}\wedge t} - x_{t_j\wedge t})^2$. □

Given $x \in \mathcal{Q}$, ν_t will denote the *occupation measure of* $(x_s)_{s\le t}$ (along Π), defined on the Borel sets of $[0, t]$ by $\nu_t(A) := \int_0^t \mathbf{1}_A(x_s)\, d\langle x\rangle_s$.

Lemma 3.4. *If $x \in \mathcal{Q}$ and $t \in [0, \infty)$ the following are equivalent.*

(1) *For every $g \in W^{1,q}$ the following sequence converges*

$$\sum_{t_j \in \pi_n} g(x_{t_j})(x_{t_{j+1}\wedge t} - x_{t_j\wedge t}). \tag{3.7}$$

(2) *The sequence $(L_t^n(\cdot))_n$ converges in the weak topology of L^p (to a quantity which we denote by $L_t(\cdot)$).*

The above conditions imply that $(L_t^n(\cdot))_n$ is bounded in L^p and (3.3) holds. Conversely, if $(L_t^n(\cdot))_n$ is bounded in L^p and $x \in \mathcal{Q}$ then items (1) and (2) hold, and ν_t has a density L_t with respect to $\mathcal{L}^1$.

Proof. The equivalence between items (1) and (2), and the fact that these imply (3.3), follows immediately applying (3.2) with $f(u) := \int_{x_*}^u g(y)\,dy$. That item (2) implies the boundedness of $(L_t^n(\cdot))_n$ follows from Banach–Steinhaus theorem. For the opposite implication notice that since $x \in \mathcal{Q}$ we can use Theorem 2.5, which together with (3.2) shows that

$$\exists \lim_n \int_{\mathbb{R}} L_t^n(u)h(u)\,du = \int_0^t h(x_s)\,d\langle x\rangle_s = \int h\,d\nu_t \quad \text{for all } h \in C^0. \tag{3.8}$$

Since L^p is reflexive (see [3, Theorem 4.10]), its unit ball is sequentially compact in the weak topology [3, Theorem 3.18], so we can get convergence of L_t^n along some subsequence (of any subsequence) to some L_t and all we have to show is that the limit does not depend on the subsequence. Considering $(L_t^n)_n$ as elements of the measure space $([\underline{x}_t, \overline{x}_t], \mathcal{L}^1)$ we have that $C^0 \subseteq L^p$, so $\int g(u)\nu_t(du) = \int g(u)L_t(u)\,du$ for all continuous g. Thus $L_t(u)\,du = \nu_t(du)$; in particular the limit L_t does not depend on the subsequence, proving item (2). □

Lemma 3.5. *If the equivalent conditions* (1) *and* (2) *of Lemma* 3.4 *are satisfied for all* $t \in [0,\infty)$, *the following conditions are equivalent.*

(1) *For every* $g \in W^{1,q}$ *the function* $\int_0^t g(x_s)\,dx_s$ *is continuous in* $t \in [0,\infty)$.
(2) *For every* $g \in W^{1,q}$ *the convergence in* (3.7) *is uniform on compacts.*
(3) *The map* $[0,\infty) \ni t \mapsto L_t(\cdot) \in L^p$ *is continuous in the weak topology of* L^p.
(4) *For every* $h \in L^q$ *the convergence* $\int_{\mathbb{R}} L_t^n(u)h(u)\,du \to \int_{\mathbb{R}} L_t(u)h(u)\,du$ *is uniform on compacts.*

Proof. The identity (3.2) shows that items (2) and (4) are equivalent. The identity (3.3) shows that (1) and (3) are equivalent. Trivially item (2) implies item (1). Finally Scholium 2.2 shows that item (3) implies item (4). □

Proof of Theorem 3.1. If item (5) holds, since the last term in the decomposition (4.3) is bounded by $O_t(x,\pi)$ and the two sums are increasing in t, $(L_t^n)_n$ is bounded in L^p for all $t \in [0,\infty)$; moreover Lemma 3.4 shows that for all $h \in L^q$

$$\exists \lim_n \int_{\mathbb{R}} L_t^n(u)h(u)\,du = \int L_t(u)h(u)\,du = \int h\,d\nu_t = \int_0^t h(x_s)\,d\langle x\rangle_s. \tag{3.9}$$

Since (3.9) shows that $\int L_t(u)h(u)\,du$ is a continuous function of t, Lemma 3.5 implies that item (3) holds. That item (3) implies $x \in \mathcal{Q}$ follows applying Lemma 3.3 since $\mathbf{1}_{[\underline{x}_t,\overline{x}_t]} \in L^p$ and $L_t^n = 0$ outside $[\underline{x}_t, \overline{x}_t]$. Lemma 3.4 states that ν_t has a density L_t; thus, formula (3.4) holds. All other assertions follow directly from Lemmas 3.4 and 3.5.

If $p = 1$ or $p = \infty$ the proofs hold with the following minor modification in the part of the proof of Lemma 3.4 which deals with the sequential compactness of $(L_t^n)_n$. If $p = \infty$, the unit ball of L^∞ is sequentially compact since it is compact (and metrizable) in the weak* topology because of Banach–Alaoglu theorem (and since L^1 is separable), see [3, Theorem 3.16] (and see [3, Theorems 3.28 and 4.13]). If $p = 1$, since $L_t^n \geq 0$, Lemma 3.3 implies that $(L_t^n)_n$ is bounded in L^1, so if $(L_t^n)_n$ is equintegrable then it is weakly sequentially compact (by the Dunford–Pettis theorem, see [3, Theorem 4.30]). □

We end this section with the following

Example 3.6. There exists a function that admits pathwise quadratic variation but no pathwise local time.

Put differently, we show that the inclusion $\mathcal{L}_1^\Pi \subset \mathcal{Q}^\Pi$ can be strict. This proves that the additional requirement in (5) in Theorem 3.1 is not automatic and is indeed needed. More precisely, we now construct a continuous function $x : [0,1] \to \mathbb{R}$ and a sequence of refining partitions $\Pi = (\pi_n)_n$ of $[0,1]$ whose mesh is going to zero and such that $\langle x\rangle_t^{\pi_n}$ converges for all t to the (continuous!) Cantor function $c(t)$ (a.k.a. the Devil's staircase) but $(L_1^{\pi_n})_n$ does not converge weakly in $L^1(du)$; in particular x has no pathwise local time along $(\pi_n)_n$, no matter which definition we use.[11] Our construction was inspired by a remark by Bertoin on page 194 of [1].

[11] Meaning that if one replaced the weak topology of L^1 with any stronger topology (e.g. the weak/strong topology of L^p, or the uniform topology as done in [17, Definition 2.5]) one would still not obtain convergence of $L_t^{\pi_n}$.

Divide $[0,1]$ into three equal subintervals and remove the middle one $I_1^1 := (\frac{1}{3}, \frac{2}{3})$. Divide each of the two remaining closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ into three equal subintervals and remove the middle ones $I_1^2 := (\frac{1}{3^2}, \frac{2}{3^2})$ and $I_2^2 := (\frac{7}{3^2}, \frac{8}{3^2})$. Continuing in this fashion, at each step i we remove the middle intervals $I_1^i, \dots, I_{2^{i-1}}^i$, each of length $1/3^i$. The Cantor set is defined as

$$C := [0,1] \Big\backslash \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} I_j^i,$$

and the function which we will consider is $x(t) := \sqrt{2 \min_{s \in C} |s-t|}$. To construct our partitions π_n of $[0,1]$ we define first a refining sequence $(\pi_{n,j}^i)_n$ of Lebesgue partitions of I_j^i setting $\pi_{n,j}^i = (t_{n,j}^{k,i})_{k=0}^{2^{ni+1}}$ with $t_{n,j}^{0,i} = \inf I_j^i$ (so that $x(t_{n,j}^{0,i}) = 0$), and

$$t_{n,j}^{k+1,i} := \inf\Big\{ t > t_{n,j}^{k,i} : x_t \in \Big(2^{-ni} \sup_{t \in I_j^i} x(t)\Big)\mathbb{Z}, x_t \neq x_{t_{n,j}^{k,i}} \Big\}, \tag{3.10}$$

so that $t_{n,j}^{2^{ni+1},i} = \sup I_j^i$ and $|x_{t_{n,j}^{k+1,i}} - x_{t_{n,j}^{k,i}}| = 2^{-ni} \sup_{t \in I_j^i} x(t) = 1/(\sqrt{3} 2^n)^i$ and so

$$\langle x \rangle_{I_j^i}^{\pi_{n,j}^i} := \sum_{k=0}^{2^{ni+1}-1} (x_{t_{n,j}^{k+1,i}} - x_{t_{n,j}^{k,i}})^2 = 2^{ni+1} 3^{-i} 2^{-2ni} = 3^{-i} 2^{1-ni}. \tag{3.11}$$

Then define our refining sequence $(\pi_n)_n$ of partitions of $[0,1]$ whose mesh is going to zero setting $\pi_n := \{0,1\} \cup \bigcup_{i=1}^n \bigcup_{j=1}^{2^{i-1}} \pi_{n,j}^i$ and we set $\varepsilon_n := \frac{2}{2^n 3}$ so that as $n \to \infty$

$$\langle x \rangle_1^{\pi_n} = \sum_{i=1}^{n} \sum_{j=1}^{2^{i-1}} \langle x \rangle_{I_j^i}^{\pi_{n,j}^i} = \sum_{i=1}^{n} (\varepsilon_n)^i = \frac{1-(\varepsilon_n)^{n+1}}{1-\varepsilon_n} \to 1. \tag{3.12}$$

Now, the Cantor function c is defined on $[0,1]$ to be the only continuous extension of the function f which is defined on the set $D := \{0,1\} \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} \bar{I}_j^i$ in this way:[12] $f(0)=0$, $f(1)=1$, and each time we remove the middle third I_j^i from a parent interval J_j^i, f is defined on the closure $\bar{I}_j^i$ of I_j^i to be the average of its values at the extremes of J_j^i (so $f = 1/2$ on $\bar{I}_1^1$, $f = 1/4$ on $\bar{I}_1^2$ and $f = 3/4$ on $\bar{I}_2^2$ etc.).

Since the difference between $\langle x \rangle_t^{\pi_n}$ and the increasing function $\sum_{\pi_n \ni t_j \le t} (x_{t_{j+1}} - x_{t_j})^2$ is going to zero for all t as $n \to \infty$, and since c is continuous and increasing, to conclude that $\langle x \rangle_t^{\pi_n} \to c(t)$ for all t it is enough to show it for all t in the dense set D, see also Lemma 7.2 below. We already know this for $t=1$ and (trivially) for $t=0$. Since $\langle x \rangle_{(0,\frac{1}{3}]}^{\pi_n} = \langle x \rangle_{(\frac{2}{3},1]}^{\pi_n}$, and (3.11) shows that $\langle x \rangle_{(\frac{1}{3},\frac{2}{3}]}^{\pi_n} \to 0$, (3.12) and (2.6) give that $\langle x \rangle_t^{\pi_n} \to 1/2 = c(t)$ for all $t \in [\frac{1}{3}, \frac{2}{3}] = \bar{I}_1^1$. Analogously $\langle x \rangle_{(0,\frac{1}{9}]}^{\pi_n} = \langle x \rangle_{(\frac{2}{9},\frac{1}{3}]}^{\pi_n}$, and (3.11) shows that $\langle x \rangle_{(\frac{1}{9},\frac{2}{9}]}^{\pi_n} \to 0$, so $\langle x \rangle_{\frac{1}{3}}^{\pi_n} \to 1/2$ and (2.6) give that $\langle x \rangle_t^{\pi_n} \to 1/2^2 = c(t)$ for all $t \in [\frac{1}{9}, \frac{2}{9}] = \bar{I}_1^2$. In this way we see that $\langle x \rangle_t^{\pi_n} \to c(t)$ for all $t \in D$ and thus for all $t \in [0,1]$.

To conclude, let us prove that the pathwise local time $L_1^{\pi_n}(u)$ converges to 0 for all $u \neq 0$, so that $(L_1^{\pi_n})_n$ does not converge weakly in $L^1(du)$ (because otherwise, by the Dunford–Pettis theorem [3, Theorem 4.30], it would be uniformly integrable and would thus converge *to zero* strongly in $L^1(du)$, whereas we know that $\int_0^1 L_1^{\pi_n}(u)\,du = \langle x \rangle_1^{\pi_n} \to c(1) = 1$). Since $x(t) \ge 0$ for all t, $L_1^{\pi_n}(u) = 0$ for all n if $u < 0$. For each i, j the function $(x(t))_{t \in \pi_n \cap I_j^i}$ crosses[13] each level $u > 0$ at most twice, and since $\sup\{x(t) : t \notin \bigcup_{i=1}^k \bigcup_{j=1}^{2^{i-1}} I_j^i\} = 1/\sqrt{3^{k+1}}$ is strictly smaller than

[12] Such extension exists and is unique since f is continuous and D is dense in $[0,1]$.

[13] Meaning that either $x_{t_k} \le u < x_{t_{k+1}}$ or $x_{t_{k+1}} \le u < x_{t_k}$ where $(t_k)_k := \pi_n \cap I_j^i$.

any $u > 0$ for big enough $i = i(u)$, the number of times $(x(t))_{t\in\pi_n}$ crosses level $u > 0$ is bounded above independently of n; since $O_1(x, \pi_n) = 1/(\sqrt{3}n) \to 0$ as $n \to \infty$, this implies that $L_1^{\pi_n}(u) \to 0$.

4. Change of variables and time-change

In applications to the study of variance derivatives, for example [6], one starts with a continuous positive price function S, and the 'variance' is defined as the quadratic variation of the *log price* $x = \log S$. In this connection it is useful to be able to change variables, and to relate for example the local time of $\log(x)$ with the one of x. We recall that, although being a semimartingale is preserved only by C^2 transformations, possessing a quadratic variation (in the sense of Definition 2.1) is more generally invariant under C^1 transformations; indeed $f \in C^1$ and $x \in \mathcal{Q}^\Pi$ imply $f(x) \in \mathcal{Q}^\Pi$ and $\langle f(x)\rangle_t^\Pi = \int_0^t f'(x_s)^2 \langle x\rangle_s^\Pi$ (see [21, Proposition 2.2.10]). We prove below a similar result for the pathwise local time (if f is monotone), extending the C^2 case treated in [6]; then we show that time-change preserves the pathwise local time.

For Propositions 4.1 and 4.2 we consider a fixed sequence of partition $(\pi_n)_n$ such that $O_t(x, \pi_n) \to 0$ as $n \to \infty$ for all $t \in [0, \infty)$.

Proposition 4.1. *Let $x \in \mathcal{L}_p$ and let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and strictly monotone. Then $f(x) \in \mathcal{L}_p$ and the pathwise local times of x and $f(x)$ are related by*

$$L_t^{f(x)}\big(f(u)\big) = \big|f'(u)\big| L_t^x(u). \tag{4.1}$$

In Proposition 4.1 one considers the same sequence of partitions $(\pi_n)_n$ for x and for $f(x)$. This seems to be problematic, since ideally we would like Proposition 4.1 to hold also for Lebesgue partitions, and clearly if P is a partition of $\mathbb{R}$ then $\pi_P(f(x))$ differs from $\pi_P(x)$. However Proposition 4.1 does apply to suitably chosen Lebesgue partitions since $\pi_{f(P)}(f(x)) = \pi_P(x)$ if f is strictly increasing.

To prove Proposition 4.1 and better understand the behavior of L^π, let $t_J := \max\{t_j \in \pi : t_j \le t\}$ and

$$\begin{aligned} \pi_t^U(u) &:= \{t_j \in \pi : x_{t_j} \le u < x_{t_{j+1}}, t_{j+1} \le t\}, \\ \pi_t^D(u) &:= \{t_j \in \pi : x_{t_{j+1}} \le u < x_{t_j}, t_{j+1} \le t\}, \end{aligned} \tag{4.2}$$

and notice that, since all the terms in (3.1) with $t < t_j$ are equal to zero,

$$L_t^\pi(u)/2 = \sum_{t_j\in\pi_t^U(u)} (x_{t_{j+1}} - u) + \sum_{t_j\in\pi_t^D(u)} (u - x_{t_{j+1}}) + \mathbf{1}_{[x_{t_J}, x_t)}(u)|x_t - u|. \tag{4.3}$$

Proof of Proposition 4.1. Since including t in any partition π does not change the value of $L_t^\pi(u)$ and insures that the last term in (4.3) is zero, we assume without loss of generality that our partitions contain t. If f is strictly increasing and $a \le b$ then

$$x_a \le u < x_b \quad \text{iff } f(x_a) \le f(u) < f(x_b),$$

and thus (4.3) implies that $L_t^{f(x),\pi}(f(u))/2$ equals

$$\sum_{t_j\in\pi_t^U(u)} \big(f(x_{t_{j+1}}) - f(u)\big) + \sum_{t_j\in\pi_t^D(u)} \big(f(u) - f(x_{t_{j+1}})\big). \tag{4.4}$$

If $t_j \in \pi_t^U(u)$, since $f \in C^1$ there exists $z_j(u) \in (u, x_{t_{j+1}})$ such that

$$f(x_{t_{j+1}}) - f(u) = f'\big(z_j(u)\big)(x_{t_{j+1}} - u),$$

so we can write the first sum in (4.4) as

$$\sum_{t_j\in\pi_t^U(u)} \big(f'\big(z_j(u)\big)-f'(u)\big)(x_{t_{j+1}}-u)+f'(u)\sum_{t_j\in\pi_t^U(u)}(x_{t_{j+1}}-u). \tag{4.5}$$

Treating analogously the second sum in (4.4) we get that

$$L_t^{f(x),\pi}\big(f(u)\big)-f'(u)L_t^{x,\pi}(u) \tag{4.6}$$

is bounded by

$$2\sum_{t_j\in\pi_t^U(u)\cup\pi_t^D(u)} \big|f'\big(z_j(u)\big)-f'(u)\big||x_{t_{j+1}}-u|. \tag{4.7}$$

Now define

$$R_t(g,\pi):=\max\big\{\big|g(c)-g(d)\big|:c,d\in[\underline{x}_t,\overline{x}_t],|c-d|\le O_t(x,\pi)\big\}. \tag{4.8}$$

Clearly (4.7) with $\pi=\pi_n$ is bounded by $R_t(f',\pi_n)L_t^{x,\pi_n}(u)$, so since L_t^{x,π_n} converges to L_t^x and $R_t(f',\pi_n)\to 0$ we get that (4.6) with $\pi=\pi_n$ converges to 0, proving the thesis.

If f is strictly decreasing then the argument is the same save for the sign change, which comes from the fact that upcrossings are now transformed in downcrossings and conversely, so $x_{t_{j+1}}-u$ needs to be replaced by $u-x_{t_{j+1}}$. □

Proposition 4.2. *Let $\tau:[0,\infty)\to[0,\infty)$ be an increasing càdlàg function such that x_τ is continuous and $\tau(0)=0$. Given $\Pi=(\pi_n)_n$, let $\tau(\Pi):=(\tau_{\pi_n})_n$ where, given $\pi=(t_j)_j$, τ_π denotes the partition $(\tau_{t_j})_j$. If $O_{\tau_t}(x,\tau_{\pi_n})\to 0$ for all $t\in[0,\infty)$ and $x\in\mathcal{L}_p^{\tau(\Pi)}$ then $O_t(x\circ\tau,\pi_n)\to 0$ for all $t\in[0,\infty)$, $x_\tau\in\mathcal{L}_p^{\Pi}$ and the pathwise local times are related by*

$$L_t^{x\circ\tau,\Pi}(u)=L_{\tau_t}^{x,\tau(\Pi)}(u).$$

Moreover if τ is bijective[14] *then $x\circ\tau$ is continuous, $O_{\tau_t^{-1}}(x\circ\tau,\tau_\pi^{-1})=O_t(x,\pi)$ for any partition π, and if P is a partition of $\mathbb{R}$ then the Lebesgue partitions of $x\circ\tau$ and x satisfy $\pi_P(x\circ\tau)=\tau^{-1}_{\pi_P(x)}$.*

Proof. Even if τ is not strictly increasing, the identity

$$\big\{\tau_s:s\in[t_i,t_{i+1})\cap[0,t]\big\}=[\tau_{t_i},\tau_{t_{i+1}})\cap[0,\tau_t]\cap\tau\big([0,\infty)\big),$$

holds, and it trivially implies that $O_t(x\circ\tau,\pi_n)\le O_{\tau_t}(x,\tau_{\pi_n})$, with equality if τ is a bijection. Trivially $L_t^{x\circ\tau,\pi}(u)=L_{\tau_t}^{x,\tau_\pi}(u)$ holds for every partition π, and everything else follows easily. □

Note that Propositions 4.1 and 4.2 hold (with the same proof) with other definitions of existence of the pathwise local time; for example if one replaced the weak topology of L^p for $p\in[1,\infty)$ (resp. the weak* topology on L^∞) with the strong one in item (3) of Theorem 3.1, or if one considered Definition 2.5 in [17].

5. Extension to convex functions

The choice of how to define the existence of the pathwise local time is intrinsically linked to the class of functions for which one is able to establish the pathwise Tanaka–Meyer formula (3.3). To establish it for all convex functions one needs to restrict significantly the set of paths for which the local time exists; nonetheless, in [17] it is shown that this approach works for general enough paths (namely, for the 'typical path' in the sense of Vovk [25]).

It is natural to ask if the above can be extended even further, to all continuous functions. As already remarked in [9], the next proposition shows that, if one wants to consider a generic path of a local martingale, the answer is no –

[14] I.e. if τ is strictly increasing, continuous and such that $\tau(0)=0$, $\lim_{t\to\infty}\tau(t)=\infty$.

to define stochastic integrals in a pathwise manner for more general integrands one has to consider partitions which depend both on the integrator and the integrand as in [2, Theorem 7.14] and [12].

Proposition 5.1 (Stricker [23]). *Let $x \in C[0,T]$. If for every sequence of partitions $(\pi_n)_n$ with $O_T(x,\pi_n) \to 0$ and every bounded continuous function f on $\mathbb{R}$ the Riemann sums $\sum_{t_i \in \pi_n} f(x_{t_i})(x_{t_i+1} - x_{t_i})$ converge, then x has finite variation.*

In what follows we take a different route from [17] to further extend Föllmer's integral and Tanaka–Meyer formula beyond $f \in C^2$. We consider f which is a difference of two convex functions and write f'_- for its left-continuous derivative and f'' for the second distributional derivative. In a way somewhat reminiscent of [1, Proposition 1.2], we define $\int_0^t f'_-(x_s)\,dx_s$ as the limit of $\int_0^t f'_n(x_s)\,dx_s$, where f_n are some special C^2 functions converging to f and $\int_0^t f'_n(x_s)\,dx_s$ is defined in Theorem 2.5 as a limit of Riemann sums.

We now fix $\Pi = (\pi_n)_n$ such that $O_t(x,\pi_n) \to 0$ as $n \to \infty$ for all $t \in [0,\infty)$, and we consider a function g which is C^2, positive and with compact support in $[0,\infty)$, and such that $\int_{\mathbb{R}} g(x)\,dx = 1$. We will then approximate the target fuction f with $f_n := g_n * f$, where $*$ denotes the convolution between a function and a measure (or a function), g_n is the mollifier $g_n(u) := ng(nu)$. Recall that, if $x \in \mathcal{L}_1$, $L_t(\cdot)$ is seen an element of $L^1(du)$; the following theorem assumes that there exists a modification of L_t which is càdlàg in u, i.e., a function $\tilde{L}_t(u)$ càdlàg in u and such that, for each t, the set $\{u : \tilde{L}_t(u) \neq L_t(u)\}$ has zero Lebesgue measure; this is not an unreasonable assumption, as it is satisfied by a.e. path of a semimartingale (indeed the local time of a continuous semimartingale has a modification which is jointly càdlàg in u and continuous in t).

Theorem 5.2. *Assume that $x \in \mathcal{L}_1$ and there exists a modification $L_t(u)$ of the pathwise local time which is càdlàg in u for all t. If f is convex then f_n is C^2 and for all $t \in [0,\infty)$ the Föllmer integral $\int_0^t f'_n(x_s)\,dx_s$ converges to a finite limit, denoted by $\int_0^t f'_-(x_s)\,dx_s$, which is independent of the choice of g and satisfies*

$$f(x_t) - f(x_0) = \int_0^t f'_-(x_s)\,dx_s + \frac{1}{2}\int_{\mathbb{R}} L_t(u) f''(du). \tag{5.1}$$

Moreover if $L_t(u)$ is jointly càdlàg in u and continuous in t then the convergence is uniform on compacts and $t \mapsto \int_0^t f'_-(x_s)\,dx_s$ is continuous.

Theorem 5.2 allows us to define[15] $\int_0^t g_-(x_s)\,dx_s$ for any function g of finite variation on compacts, since then $f(u) := \int_{x_0}^u g(y)\,dy$ is the difference of convex functions. We now study the continuity properties of $g \mapsto \int_0^\cdot g_-(x_s)\,dx_s$.

Proposition 5.3. *Let $x \in \mathcal{L}_1$ and assume that there exists a modification $L_t(u)$ of the pathwise local time which is continuous in u. If g_n, and g are functions of finite variation on compacts, $g_n(x_0) \to g(x_0)$ and $g'_n \to g'$ weakly (seen as measures), then $\int_0^t g_n(x_s)\,dx_s \to \int_0^t g(x_s)\,dx_s$ for all $t \in [0,\infty)$. Moreover, if $|g'_n| \to |g'|$ weakly then the convergence is uniform on compacts.*

Proof. Define $f(u) := \int_{x_0}^u g(y)\,dy$ and analogously f_n from g_n, and notice that $f_n(u) \to f(u)$ for all $u \in \mathbb{R}$, so Tanaka–Meyer formula (5.1) gives the thesis. If moreover $|g'_n| \to |g'|$ weakly then, since the positive part $\max(h,0)$ of h equals $(h + |h|)/2$, Polya's Scholium 2.2 shows local uniformity of the convergence

$$\int_{\mathbb{R}} L_t(u) \max\bigl(g'_n, 0\bigr)(du) \to \int_{\mathbb{R}} L_t(u) \max\bigl(g', 0\bigr)(du);$$

working analogously with the negative parts we get the thesis. □

[15] As pointed out to us by Föllmer [10] another possible definition of $\int_0^t f'_-(x_s)\,dx_s$ for non-smooth convex f is as the limit of $\int_0^t f'_k(x_s)\,dx_s$ for *any* $(f_k)_k \subseteq C^2$ such that $f''_k(x)\,dx$ (considered as a measure) converges weakly to $f''(x)\,dx$. It follows from (3.3) that this definition makes sense (i.e. the limit exists and is independent of the approximating sequence $(f_k)_k$), and agrees with ours, if $L_t(u)$ has a modification which is continuous in u.

It is then natural to ask for which paths the above given definition of $\int_0^t f'_-(x_s)\,dx_s$ coincides with the one used in Theorem 3.1 for $f \in W^{2,q}$. The answer is that the limit of the Riemann sums $\sum_{t_j\in\pi_n} f'_-(x_{t_j})(x_{t_{j+1}\wedge t} - x_{t_j\wedge t})$ exists and equals $\int_0^t f'_-(x_s)\,dx_s$ iff $\int L_t^{x,\pi_n}(u) f''(du)$ converges to $\int L_t^x(u) f''(du)$, as it follows from (3.2) and (5.1). In particular this holds if $x \in \mathcal{L}_p \subseteq \mathcal{L}_1$, so the definition of the Föllmer's integral given in Theorem 5.2 is indeed an extension of the one given in Theorem 3.1.

Proof of Theorem 5.2. Since f is uniformly continuous on compacts, $f_n \to f$ pointwise. Thus, if we can prove that $\int_{\mathbb{R}} L_t\,df_n'' \to \int_{\mathbb{R}} L_t\,df''$, the thesis follows applying (3.3) to f_n and taking limits; indeed (5.1) shows that $\int_0^t f'_-(x_s)\,dx_s$ does not depend on g. Define $\hat{g}_n(u) := g_n(-u)$ and apply Fubini's theorem and the identity $f_n'' = g_n * f''$ to get that

$$\int_{\mathbb{R}} L_t\,df_n'' = \int_{\mathbb{R}} (\hat{g}_n * L_t)\,df''. \tag{5.2}$$

Since L_t is zero outside $[\underline{x}_t, \overline{x}_t]$ and g has compact support, L_t, g and $\hat{g}_n * L_t$ are all 0 outside a common compact interval $[-A, A]$. In particular since $L_t(\cdot)$ is càdlàg it is bounded; since $\sup_u |\hat{g}_n * L_t(u)| \le \sup_u |L_t(u)|$, the thesis follows from the dominated convergence theorem and (5.2) if we prove that $\hat{g}_n * L_t(u) \to L_t(u)$ for all u. Notice that

$$(\hat{g}_n * L_t - L_t)(u) = \int_{\mathbb{R}} \hat{g}_n(y)\big(L_t(u-y) - L_t(u)\big)\,dy. \tag{5.3}$$

Since $L_t(\cdot)$ is right continuous, for every $\varepsilon > 0$ and u there is an n such that

$$\big|L_t(u-y) - L_t(u)\big| < \varepsilon \quad \text{if } y \in [-A/n, 0]; \tag{5.4}$$

since $g = 0$ outside $[0, A]$, the integral on the right side of (5.3) is actually over $[-A/n, 0]$, so $|\hat{g}_n * L_t(u) - L_t(u)| < \varepsilon$.

Finally if $L_t(u)$ is jointly càdlàg in u and continuous in t then n such that (5.4) holds can be chosen as to hold simultaneously for all t in any given compact set. This implies that the convergence is uniform on compacts, and so $\int_0^{\cdot} f'_-(x_s)\,dx_s$ is continuous. □

6. Upcrossing representations of local time

In this section we will consider a continuous semimartingale $X = (X_t)_t$ (with $t \in [0,\infty)$ or $t \in [0,T]$) with canonical semimartingale decomposition $X = M + V$ and with (classical[16]) local time $\ell_t(u)$ which is (jointly) continuous in t and càdlàg in u (such a version exists, see [13, Chapter 3, Theorem 7.1]). Some of our results specialise to the case where ℓ is jointly continuous in t and u, which happens for example when dV is absolutely continuous with respect to $d[M]$: as explained[17] in [27, Example 2.2.3], since (6.4) below gives that $Y := \int_0^{\cdot} \mathbf{1}_{\{X_s=u\}}\,dM_s = 0$, so that if $dV = \sigma\,d[M]$ then using (6.4) again we get that $\ell_\cdot(u) - \ell_\cdot(u-) = 2\int_0^{\cdot} \mathbf{1}_{\{X_s=u\}}\sigma_s\,d[M]_s = 2\int_0^{\cdot} \sigma_s\,d[Y]_s = 0$. In particular $dV = \sigma\,d[M]$ holds if $V = 0$, i.e. if X is a local martingale (under the original probability $\mathbb{P}$ or a $\mathbb{Q}$ such that $\mathbb{P} \ll \mathbb{Q}$).

The following is the main theorem of this section. It essentially says that the pathwise local time sampled along optional partitions $(\pi_n)_n$ exists on a.e. path of a semimartingale, and that a.e. it equals the (classical) local time (in particular, it does not depend on $(\pi_n)_n$).

Theorem 6.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is the difference of two convex functions, that π_n are optional partitions such that $O_T(X, \pi_n) \to 0$ a.s. and that $X = (X_t)_{t\in[0,\infty)}$ is a continuous semimartingale. If X has a jointly continuous local*

[16] We refer to the semimartingale local time, i.e. the one for which the Tanaka–Meyer formula holds; this is in general different from the parallel notion of local time for Markov processes.

[17] We thank an anonymous referee for this observation and reference.

time ℓ, or if f is C^1, then there exists a subsequence $(n_k)_k$ such that, for ω outside a $\mathbb{P}$-null set (which may depend on f''),

$$\sup_{t\le T}\bigl|L_t^{X(\omega),\pi_{n_k}(\omega)}(u)-\ell_t(\omega,u)\bigr|\to 0 \quad \text{in } L^p\bigl(\bigl|f''\bigr|(du)\bigr) \text{ as } k\to\infty \tag{6.1}$$

simultaneously for all $p\in[1,\infty)$, $T<\infty$.

Note that applying Theorem 6.1 with $f(x)=x^2/2\in C^1$ gives in particular that a.e. path of a continuous semimartingale is in $\mathcal{L}_p$ for all $p<\infty$; indeed, $L_t^{X,\pi_{n_k}}(u)\to\ell_t(u)$ strongly (and thus weakly) in $L^p(du)$ a.s., locally uniformly in t.

The previous theorem follows from the following technical statement.

Theorem 6.2. *Let π_n be optional partitions such that $O_T(X,\pi_n)\to 0$ a.s., $p\in[1,\infty)$, $T<\infty$, $X\in\mathcal{S}^p$, μ be a sigma-finite positive Borel measure on $\mathbb{R}$, and define*

$$h^{\pi_n}(u):=\Bigl\|\sup_{t\le T}\bigl|L_t^{X(\omega),\pi_n(\omega)}(u)-\ell_t(\omega,u)\bigr|\Bigr\|_{L^p(\mathbb{P}(d\omega))},\quad u\in\mathbb{R}. \tag{6.2}$$

Then $h^{\pi_n}(\cdot)$ is bounded and $(h^{\pi_n}(\cdot))_n$ converges pointwise (resp. μ a.e.) to 0 if ℓ is jointly continuous (resp. if μ is a measure with no atoms).

The fact that $(h^{\pi_n}(\cdot))_n$ converges pointwise to zero was given an involved proof[18] in [26] in the case where X is a continuous martingale bounded in L^2, $p=2$ and π_n are deterministic partitions such that $\|\pi_n\|\to 0$. In the case where X is in a class of continuous Dirichlet processes which includes $\mathcal{S}^2$ semimartingales and the partitions are of Lebesgue-type, it is shown in [1, Theorem 2.5 and Proposition 2.7] that $L_t^{X(\omega),\pi_n(\omega)}(u)\to\ell_t(\omega,u)$ weakly in $L^1(d\mathbb{P}\times du)$ for each t.

Moreover, Lemieux [14, Theorem 2.4] has derived a version of Theorem 6.1 where the $L^p(|f''|(du))$ convergence is replaced by the uniform convergence, in the special case where the partitions are of Lebesgue-type. For the case of continuous local martingales one can also consult Perkins [16] or Chacon et al. [4, Theorem 2 and Remark 2], or Perkowski and Prömel [17, Theorem 3.5 and Remark 3.6], who actually prove convergence not only for $\mathbb{P}$ a.e. ω but even quasi surely with respect to the set of all local martingale measures. Although our approach yields a weaker type of convergence, it has a simple proof and it works for continuous semimartingales and general optional partitions such that $O_T(X,\pi_n)\to 0$ a.s.

In the special case of Lebesgue partitions $\pi=\pi_{\varepsilon\mathbb{Z}}$, Theorem 6.2 closely relates to the downcrossing representation of local time conjectured by Lévy (proved by Itô and McKean for Brownian motion, extended by El Karoui to semimartingales, and found in [19, Theorem VI.1.10]), which states that, for an $X\in\mathcal{S}^p$,

$$\lim_{\varepsilon\to 0}\Bigl\|\sup_{t\le T}\bigl|\varepsilon D_t^\varepsilon(\omega,0)-\ell_t(\omega,0)\bigr|\Bigr\|_{L^p(\mathbb{P}(d\omega))}=0,$$

where $D_t^\varepsilon(\omega,0)$ (defined in (6.3) below) is the number of downcrossings at level 0. Indeed, as we now explain, Lévy's representation above is equivalent to the fact that $h^{\pi_{\varepsilon_n\mathbb{Z}}}(0)\to 0$ whenever $0<\varepsilon_n\to 0$.

Given a continuous path $x=(x_s)_{s\le t}$ and $a<b$, we set $\sigma_0^{a,b}:=0$, $\tau_0^{a,b}=\inf\{t:x_t=b\}$ and, for $k\ge 1$, we define

$$\begin{aligned}&\sigma_k^{a,b}:=\inf\bigl\{t>\tau_{k-1}^{a,b}:x_t=a\bigr\},\qquad \tau_k^{a,b}:=\inf\bigl\{t>\sigma_k^{a,b}:x_t=b\bigr\},\\&D_t^\varepsilon(u):=\max\bigl\{k:\sigma_k^{u,u+\varepsilon}\le t\bigr\}.\end{aligned} \tag{6.3}$$

It turns out that the downcrossings $D_t^\varepsilon(u)$ of $(X_s)_{s\le t}$ from $u+\varepsilon$ to u are closely related to the local time along $\pi_{\varepsilon\mathbb{Z}}$. Indeed, the upcrossings $U_t^\varepsilon(u):=\max\{k:\tau_k^{u+\varepsilon,u}\le t\}$ of $(X_s)_{s\le t}$ from u to $u+\varepsilon$ differ from $D_t^\varepsilon(u)$ by at most 1, so

[18] The uniformity in t, not stated in [26], follows easily by Doob's L^2-inequality since (6.5) shows that $(L_t^{\pi_n,X}(u)-\ell_t(u))_t$ is a L^2 bounded martingale for each u,n.

using (4.3) we get that

$$L_t^{\pi_{\varepsilon\mathbb{Z}}}(u)/2 = U_t^{\pi_{\varepsilon\mathbb{Z}}}(u)(\varepsilon - u) + D_t^{\pi_{\varepsilon\mathbb{Z}}}(u)u + \mathbf{1}_{[x_{t_J},x_t)}(u)|x_t - u|.$$

The last term is bounded by $O_t(x,\pi_{\varepsilon\mathbb{Z}}) \le \varepsilon$ and, considering $u=0$, we get that

$$\left|L_t^{\pi_{\varepsilon\mathbb{Z}}}(0)/2 - \varepsilon D_t^{\varepsilon}(0)\right| \le 2\varepsilon,$$

which concludes the proof of equivalence.

We recall the following fact, for which we refer to [19, Chapter 6, Theorem 1.7]:

$$2\int_0^{\cdot} \mathbf{1}_{\{X_s=u\}}\,dX_s = 2\int_0^{\cdot} \mathbf{1}_{\{X_s=u\}}\,dV_s = \ell_\cdot(u) - \ell_\cdot(u-) \quad \text{a.s.}, \forall u \in \mathbb{R}. \tag{6.4}$$

Proof of Theorem 6.2. Consider the convex function $f(x) := |x-u|$ and let $\operatorname{sign}(x-u)$ be its left-derivative and $2\delta_u$ its second (distributional) derivative. Subtracting from the discrete-time Tanaka–Meyer formula (3.2) its continuous-time stochastic counterpart we get that

$$0 = \int_0^t \big(H_s^{\pi_n}(u) - H_s(u)\big)\,dX_s + \big(L_t^{\pi_n,X}(u) - \ell_t(u)\big)/2, \tag{6.5}$$

where using $\pi_n = (\tau_i^n)_i$ we define the predictable processes

$$H_s^{\pi_n}(u) := \sum_i \operatorname{sign}(X_{\tau_i^n} - u)\mathbf{1}_{(\tau_i^n,\tau_{i+1}^n]}(s) \quad \text{and} \quad H_s(u) := \operatorname{sign}(X_s - u).$$

Now $h^{\pi_n}(u) \to 0$ follows from (2.10) and (6.5) if we show that $\int_0^{\cdot} H_s^{\pi_n}(u)\,dX_s \to \int_0^{\cdot} H_s(u)\,dX_s$ in $\mathcal{S}^p$. To this end notice that

$$\left|H_s^{\pi_n}(u) - H_s(u)\right| \le K_s^{\pi_n}(u) := 2 \times \mathbf{1}_{\{O_s(X,\pi_n) \ge |X_s - u|\}}, \tag{6.6}$$

and that since $X_\cdot$ and $O_\cdot(X,\pi_n)$ are continuous adapted processes, $K_\cdot^{\pi_n}(u)$ is predictable, so it is enough to prove that $\int_0^{\cdot} K_s^{\pi_n}(u)\,dX_s \to 0$ in $\mathcal{S}^p$. Since $O_T^{\pi_n} \to 0$ a.s. implies that $K_t^{\pi_n}(u) \to 0$ a.s. on $\{X_t \neq u\}$ for all $t \le T$, and since $K^{\pi_n} \le 2$, the thesis follows from the (deterministic) dominated convergence theorem if $\| \int_0^{\cdot} \mathbf{1}_{\{X_s=u\}}\,dX_s\|_{\mathcal{S}^p} = 0$, which by (6.4) holds for all u if ℓ is continuous. Since Minkowski inequality for integrals says that

$$\left\| \int_0^T \mathbf{1}_{\{X_s=u\}}\,d|V|_s \right\|_{L^p(\mu)} \le \int_0^T \|\mathbf{1}_{\{X_s=u\}}\|_{L^p(\mu)}\,d|V|_s = \int_0^T \big(\mu(\{X_s\})\big)^{1/p}\,d|V|_s,$$

which is zero for μ which has no atoms. Using (6.4), considering $L^p(\mu\otimes\mathbb{P})$ norm and using Fubini, we conclude that $\| \int_0^{\cdot} \mathbf{1}_{\{X_s=u\}}\,dX_s\|_{\mathcal{S}^p} = 0$ for μ a.e. u, and so $h^{\pi_n} \to 0$ μ a.e. Finally (2.10), (6.5) and (6.6) imply that

$$h^{\pi_n}(u) \le C_p \left\| \int_0^{\cdot} 2\big(H_s^{\pi}(u) - H_s(u)\big)\,dX_s \right\|_{\mathcal{S}^p} \le 4C_p\|X\|_{\mathcal{S}^p} \quad \text{for all } u \in \mathbb{R},$$

concluding the proof. □

Proof of Theorem 6.1. Let $(\tau_m)_m$ a sequence of stopping times which prelocalizes X to $\mathcal{S}^p$ (see Emery [7, Theoreme 2]), i.e. $\tau_m \uparrow \infty$ a.s. and $X^{\tau_m-} \in \mathcal{S}^p$ for all m. Let $\mu_i(A) := |f''|(A\cap[-i,i])$ and set

$$G_n(\omega,T,u) := \sup_{t\le T}\left|L_t^{X(\omega),\pi_n(\omega)}(u) - \ell_t(u,\omega)\right|$$

and $G_n^m := \mathbf{1}_{\{T<\tau_m\}}G_n$. Since μ_i is a finite measure, Theorem 6.2 implies that, as $n\to\infty$, G_n^m converges to 0 in $L^p(\mathbb{P}\times\mu_i)$, for all $m,i\in\mathbb{N}$ and $T\ge 0$. Passing to a subsequence (without relabelling) we can get convergence fast in $L^p(\mathbb{P} \times \mu_i)$ and so, for ω outside a $\mathbb{P}$-null set $N^{p,T}_{i,m}$, $G^m_n(\omega, T, \cdot)$ converges to 0 in $L^p(\mu_i)$. Then along a diagonal subsequence we obtain that $G^m_n(\omega, T, \cdot)$ converges to 0 in $L^p(\mu_i)$ for all $i, m, p, T \in \mathbb{N} \setminus \{0\}$ for every ω outside the null set $N_{f''} := \bigcup_{i,m,T,p\in\mathbb{N}\setminus\{0\}} N^{p,T}_{i,m}$. Since $G_n = G^m_n$ on $\{T < \tau_m\}$, $G_n \to 0$ in $L^p(\mu_i)$ for all $i, p, T \in \mathbb{N} \setminus \{0\}$ for every ω outside $N_{f''}$. Since outside a compact set $G_n(\omega, T, \cdot) = 0$ for all n, convergence in $L^p(\mu_i)$ for arbitrarily big i, p implies convergence in $L^p(|f''|)$ for all $p \in [1, \infty)$. Since $G_n(\omega, \cdot, u) = 0$ is increasing, convergence for arbitrarily big T implies convergence for all $T \in [0, \infty)$. □

7. Dependence on the partitions

In this section we investigate the extent to which the pathwise quadratic variation $\langle x\rangle^\Pi := \lim_n \langle x\rangle^{\pi_n}$ depends on the sequence of partitions $\Pi := (\pi_n)_n$. Instead of constructing explicit examples we show that, for functions with a highly oscillatory behaviour, the pathwise quadratic variation depends in the most extreme way possible on $(\pi_n)_n$. We then build on this fact and extend the result to a stochastic setting considering paths of a Brownian motion. We only give a detailed proof of the main statement for a fixed path. Extending this to an a.s. statement in a Brownian setting involves thorny technicalities which arise from the dependence in ω (i.e. tracking the null sets and ensuring measurability) and we only briefly outline the necessary steps.[19]

Our work builds on two facts already mentioned (without proof) by Lévy in [15, p. 190]: that $\inf_\pi \langle x\rangle^\pi_1 = 0$ for *every* continuous function x and that for a.e. path $B(\omega)$ of a Brownian motion $\sup_\pi \langle B(\omega)\rangle^\pi_1 = \infty$. Proofs can be found in Freedman [11, p. 47–48]; the second fact can be found in a strengthened form and with an alternative proof in Taylor [24, Corollary in Section 4]. Our first result combines and generalises the above: we show that, with a suitable choice of $(\pi_n)_n$, the pathwise quadratic variation may be equal to an arbitrary given increasing process a. Notice that we do not even assume a is right-continuous. We will denote with D_n the dyadics of order n in $[0, 1]$, i.e $D_n := [0, 1] \cap \mathbb{N}2^{-n}$.

Theorem 7.1. *Let $x : [0, 1] \to \mathbb{R}$ be a continuous function such that for every $0 \le c < d \le 1$ there exist partitions $(\hat{\pi}_n)_n$ of $[c, d]$ such that $\lim_n \langle x\rangle^{\hat{\pi}_n}_{(c,d]} = \infty$. Then, if $a : [0, 1] \to [0, \infty)$ is an increasing function such that $a_0 = 0$, there exist refining partitions $(\pi_n)_n$ of $[0, 1]$ such that $D_n \subseteq \pi_n$ for all n and*

$$\langle x\rangle^{\pi_n}_t \to a_t \quad \text{for all } t \in [0, 1] \text{ as } n \to \infty, \tag{7.1}$$

and the convergence is uniform if $(a_t)_t$ is continuous. Moreover, given arbitrary partitions $(\bar{\pi}_n)_n$, one can choose the $(\pi_n)_n$ such that $\bar{\pi}_n \subseteq \pi_n$ for all n.

To prove that convergence occurs at all times simultaneously, we will need the following simple lemma.

Lemma 7.2. *Let $a : [0, 1] \to [0, \infty)$ be increasing, $x : [0, 1] \to \mathbb{R}$ be continuous and $(\pi_n)_n$ be partitions of $[0, 1]$ such that $O(x, \pi_n) \to 0$, and assume that*

$$\langle x\rangle^{\pi_n}_t \to a(t) \quad \text{for all } t \in F \subseteq [0, 1] \text{ as } n \to \infty.$$

If F is dense in $[0, 1]$ and contains the times of jump of a then $\langle x\rangle^{\pi_n}_\cdot \to a$. pointwise on $[0, 1]$, and if a is continuous the convergence is uniform.

Proof. Although $\langle x\rangle^{\pi_n}$ is not necessarily an increasing function, it differs from the increasing function $a^n(t) := \mu_n([0, t])$ (where μ_n is as in (2.3)) by at most $O(x, \pi_n)$, and so it is enough to prove the statement with a^n replacing $\langle x\rangle^{\pi_n}$. By hypothesis $\langle x\rangle^{\pi_n}_t \to a(t)$ for all t at which a is not continuous. If a is continuous at t then for each $\varepsilon > 0$ there exist $s_1, s_2 \in F$ s.t. $s_1 < t < s_2$ and $a(s_2) - a(s_1) < \varepsilon$, and so $a(t) - \varepsilon \le a(s_1) = \lim_n \langle x\rangle^{\pi_n}_{s_1} \le \liminf_n \langle x\rangle^{\pi_n}_t$ and analogously $\limsup_n \langle x\rangle^{\pi_n}_t \le a(t) + \varepsilon$. Letting $\varepsilon \downarrow 0$ we see that $\lim_n \langle x\rangle^{\pi_n}_t$ exists and equals $a(t)$. Scholium 2.2 concludes the proof. □

[19] An interested reader can find the full detailed proof in an earlier version of our work at arXiv:1508.05984v2.

Proof of Theorem 7.1. Note that, as observed already by Freedman [11], given $k \in \mathbb{N} \setminus \{0\}$ and π we can build a partition $\pi' \supseteq \pi$ such that $\langle x\rangle_t^{\pi'} = \langle x\rangle_t^{\pi}/k$; indeed it is enough to do so on each subinterval of π, so we can assume that $\pi = \{c, d\}$. If $x(c) = x(d)$ take $\pi' := \{c, d\}$; if $x(c) \neq x(d)$ we define $\pi' = (t_i)_{i=0}^k$ setting $t_0 := c$ and

$$t_i := \min\big\{t \in [c, d] : x(t) = x(c) + \big(x(d) - x(c)\big)i/k\big\} \quad \text{for } i = 1, \dots, k;$$

indeed $\langle x\rangle_t^{\pi'} = \sum_{i=0}^{k-1}((x(d) - x(c))/k)^2 = \langle x\rangle_t^{\pi}/k$ holds. We will denote by $F(\pi, k)$ the partition π' built with the above construction starting from π and k.

We now fix a $t \in [0, 1]$ and prove the existence of some π' such that $|\langle x\rangle_t^{\pi'} - a(t)| \leq 1/2^n$. To do so we take $i \in \mathbb{N}$ such that $a(t) \in [i/2^n, (i+1)/2^n)$ and show that there exists π such that $\langle x\rangle_t^{\pi} \in [i, i+1)$ and then take $\pi' = F(\pi, 2^n)$. Note that we automatically know such π exists when $i = 0$ (by taking $\pi = F(\tilde\pi, k)$ where $\tilde\pi$ is an arbitrary partition and k a big enough integer), so let us now consider the case $i \geq 1$. By assumption, there exist a partition $\hat\pi$ such that $\langle x\rangle_t^{\hat\pi} \geq i^2$. Now using Freedman's construction with k equal to be the integer part of $\langle x\rangle_t^{\hat\pi}/i$ we obtain $\pi := F(\hat\pi, k)$ such that $\langle x\rangle_t^{\pi} = \langle x\rangle_t^{\hat\pi}/k \in [i, i + i/k) \subseteq [i, i + 1)$, concluding our proof of the existence of π and thus of some π' such that $|\langle x\rangle_t^{\pi'} - a(t)| \leq 1/2^n$.

Now, given any $\tilde\pi_n$, by applying the above reasoning to the increments of a on each subinterval of $\tilde\pi_n$, we can find a $\pi_n \supseteq \tilde\pi_n$ such that $|\langle x\rangle_t^{\pi_n} - a(t)| \leq 1/2^n$ simultaneously for all $t \in \tilde\pi_n$. If we define $(\tilde\pi_n, \pi_n)$ by induction setting $\tilde\pi_0 := \{0, 1\} =: \pi_0$ and taking $\tilde\pi_n$ to be the union of $D_n \cup \bar\pi_n$ with the times when a has a jump of size bigger than $1/n$ and with $\bigcup_{k<n} \pi_k$, and then building π_n from $\tilde\pi_n$ as explained at the beginning of this paragraph, we obtain a refining π_n which contains $D_n \cup \bar\pi_n$ and such that $\langle x\rangle_t^{\pi_n} \to a(t)$ holds for any dyadic time and any time of jump of a, and thus holds simultaneously for all t by Lemma 7.2. □

One can apply Theorem 7.1 to the paths of Brownian motion; indeed, as Lévy first remarked, for a.e. ω there exist partitions $\pi_n = \pi_n(\omega)$ s.t. $\lim_n \langle B(\omega)\rangle_{(0,1]}^{\pi_n} = \infty$. However, to obtain an interesting result one needs to show that the partitions can be chosen *in a measurable way*. This requires first to correspondingly strengthen Levy's result in the following way.

Lemma 7.3. *If $0 \leq c < d$ there exist* random *partitions π^n of $[c, d]$ such that*

$$\langle B\rangle_{(c,d]}^{\pi^n} \to \infty \quad a.s. \text{ as } n \to \infty.$$

To prove Lemma 7.3, one needs to revisit the proof of [24, Theorem 1 and its Corollary in Section 4] and delve into the proof of the existence of a Vitali subcover to show how one can choose a measurable one (on a set of probability arbitrarily close to 1); although this essentially follows from an application of the section theorem, the proof is not simple and we do not include it here.

Having established Lemma 7.3, one can follow the logic of the proof of Theorem 7.1 and with laborious but entirely elementary proofs[20] one can check measurability to obtain a similar result for the paths of Brownian motion, which we state below. To slightly generalize Theorem 7.1 to include the case of a positive but potentially non-finite process A, we identify $[0, 1]$ (with the Euclidean topology) with $[0, \infty]$ using the bijection $1 - \exp(-x)$ (where $e^{-\infty} := 0$), and thus we endow $[0, \infty]$ with the distance $d(a, b) := |e^{-a} - e^{-b}|$, which makes it homeomorphic to $[0, 1]$ and for all $M < \infty$ satisfies $d(a, b) \leq |a - b| \leq C\, d(a, b)$ for all $a, b \in [0, M]$ and some $C = C(M)$. Of course, if $A_\cdot(\omega)$ is finite valued the convergence under the Euclidean distance $|a - b|$ is equivalent to the convergence under the distance $d(a, b)$. In all that follows, if $\pi = (\tau_n)_n$ is a random partition we denote by $\pi(\omega)$ the sequence $(\tau_n(\omega))_{n\in\mathbb{N}}$. Given sets C, D which depend on ω, we write that $C \subseteq D$ if $C(\omega) \subseteq D(\omega)$ for a.e. ω, and in particular we say that a sequence of random partition $(\pi_n)_{n\in\mathbb{N}}$ is *refining* if $\pi_n \subseteq \pi_{n+1}$ for all n.

Theorem 7.4. *Let B be a Brownian motion and A a jointly-measurable*[21] *increasing process with values in* $[0, \infty]$ *and such that $A_0 = 0$. Then, there exist refining random partitions π_n of* $[0, \infty)$ *such that $\mathbb{N}2^{-n} \subseteq \pi_n$ and for all ω*

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[20] The proofs rest entirely on Borel Cantelli's lemma and on the fact that, given a càdlàg adapted process D, its jumps of size bigger than a given constant are stopping times, see [20, Theorem 3.1].

[21] Of course any *càdlàg* increasing process A is jointly-measurable. However this is not true for general increasing processes: for example if $A := Y\mathbf{1}_{\{\tau\}} + \mathbf{1}_{(\tau,\infty)}$ where τ is an exponentially distributed random time and Y is a non-measurable function with values in $(0, 1)$ then A is a process with respect to the completed sigma-algebra (A_t is measurable since $A_t = 0$ a.e.), yet A is clearly not jointly-measurable.

outside a null set

$$\left\langle B(\omega)\right\rangle_t^{\pi_n(\omega)} \to A_t(\omega) \quad \text{for all } t \in [0,\infty) \text{ as } n \to \infty; \tag{7.2}$$

if $0 \le c < d < \infty$ *the convergence in* (7.2) *is uniform on* $t \in [c,d]$ *for the* ω*'s at which* $(A_t(\omega))_{t\in[c,d]}$ *is continuous. Moreover, given arbitrary random partitions* $(\bar{\pi}_n)_n$ *one can choose* $\pi_n = (\tau_n^i)_i$ *such that* $\bar{\pi}_n \subseteq \pi_n$ *for all* n *and, if* A *is adapted,* $\tau_n^i + 2^{-n}$ *is a stopping time for each* i, n.

It is insightful to contrast this result with the well known fact that, if $(\pi_n)_n$ is a sequence of *optional* partitions such that $O_T(B,\pi_n) \to 0$ a.e., $\langle B(\omega)\rangle_t^{\pi_n(\omega)}$ converges to t uniformly on compacts in probability (see the proof of Proposition 2.3). The random times $(\tau_n^i)_i$ making up π_n do not need to look far into the future to break the convergence of $\langle B(\omega)\rangle_t^{\pi_n(\omega)}$ to t: as the theorem states, one could take the $(\tau_n^i)_i$ to look only an arbitrarily small amount of time into the future. Notice that the random times making up π_n are bounded (since $(\tau_n^i)_i = \pi_n \supseteq \mathbb{N}2^{-n}$ implies $\tau_n^i \le i/2^n$).

Although we stated Theorem 7.4 only for Brownian motion, it holds for any continuous stochastic process with an oscillatory behavior wild enough to have infinite 2-variation on any interval, in the sense that for every $0 \le c < d < \infty$ there exist random partitions π_n of $[c,d]$ along which the quadratic variation of B converges a.s. to infinity. In particular our proof of Lemma 7.3 shows that Theorem 7.4 applies whenever B is a continuous adapted[22] process for which there exists some continuous strictly increasing function ψ such that $\psi(h)/h^2 \to 0$ as $h \downarrow 0$ and

$$\text{for every } t \ge 0 \quad \limsup_{h\downarrow 0} \frac{\psi(|B_{t+h} - B_t|)}{h} \ge 1 \quad \text{a.s.} \tag{7.3}$$

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