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Long time dynamics and disorder-induced traveling waves in the stochastic Kuramoto model

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Abstract. The aim of the paper is to address the long time behavior of the Kuramoto model of mean-field coupled phase rotators, subject to white noise and quenched frequencies. We analyse the influence of the fluctuations of both thermal noise and frequencies (seen as a disorder) on a large but finite population of N rotators, in the case where the law of the disorder is symmetric. On a finite time scale $[0, T]$, the system is known to be self-averaging: the empirical measure of the system converges as $N \to \infty$ to the deterministic solution of a nonlinear Fokker–Planck equation which exhibits a stable manifold of synchronized stationary profiles for large interaction. On longer time scales, competition between the finite-size effects of the noise and disorder makes the system deviate from this mean-field behavior. In the main result of the paper we show that on a time scale of order $\sqrt{N}$ the fluctuations of the disorder prevail over the fluctuations of the noise: we establish the existence of disorder-induced traveling waves for the empirical measure along the stationary manifold. This result is proved for fixed realizations of the disorder and emphasis is put on the influence of the asymmetry of these quenched frequencies on the direction and speed of rotation of the system. Asymptotics on the drift are provided in the limit of small disorder.

Résumé. Le but de ce travail est d'étudier le comportement en temps long du modèle de Kuramoto, défini par un système de rotateurs en interaction de type champ-moyen, perturbé par un bruit blanc et possédant des fréquences aléatoires gelées. Nous analysons l'influence des fluctuations induites par le bruit et les fréquences (vues comme un désordre pour le modèle) sur une population de N rotateurs (N grand mais fini), dans le cas où la loi du désordre est symétrique. Sur un intervalle de temps borné $[0, T]$, le système est auto-moyennant: la mesure empirique du système converge pour $N \to \infty$ vers la solution déterministe d'une équation de Fokker–Planck non linéaire possédant une variété stable de solutions stationnaires synchronisées pour une interaction suffisamment grande. Sur une échelle de temps plus grande, les effets de taille finie dûs à la présence du bruit et du désordre induisent une déviation macroscopique du système par rapport à ce comportement de champ-moyen. Le résultat principal de cet article montre que, sur une échelle de temps d'ordre $\sqrt{N}$, les fluctuations induites par le désordre l'emportent sur celles données par le bruit: nous montrons que le désordre induit l'existence de fronts pour la dynamique de la mesure empirique se propageant le long de la variété stationnaire. Ce résultat est valide pour une réalisation gelée du désordre. L'accent est mis sur l'influence de l'asymétrie des fréquences sur la direction et la vitesse de propagation du front et nous donnons une asymptotique de cette vitesse dans la limite de faible désordre.

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1. Introduction

1.1. *Long time dynamics of mean-field interacting particle systems*

The macroscopic behavior of numerous stochastic interacting particle systems appearing in physics or biology is usually described by nonlinear partial differential equations. In this context, systems of diffusions in all-to-all interactions, that is *mean-field particle systems* [32,33], have attracted much attention in the past years, since they are relevant in many situations from statistical physics (synchronization of oscillators [1,27,41]) to biology (emergence of synchrony in neural networks [3,9]) and have provided particle approximations for various PDEs (see [10,31] and references therein). From a statistical physics point of view, a natural extension of these models concerns similar particle systems in a random environment, that is when the particles obey to the influence of an additional randomness, or *disorder*, representing inhomogeneous behaviors between particles. Such a modeling is particularly relevant in a biological context, where each particle/diffusion captures the state of one single individual (activity of a neuron, phase in a circadian rythm) and the disorder models intrinsic dynamical behavior for each individual (e.g. inhibition or excitation in populations of heterogeneous neurons [3,9]).

The aim of the paper is to address the influence of the disorder on the long time dynamics of a large but finite population of mean-field interacting diffusions with noise. A crucial aspect in this perspective is the notion of *self-averaging*: in the limit of a large number of individuals and/or on a long time scale (in a way that needs to be made precise), is the macroscopic behavior of the system the same for every typical realization of the disorder? If not, is it possible to quantify the influence of the fluctuations of the random environment on the behavior of the system?

It appears that the analysis of such mean-field systems differs significantly depending on the time scale one considers. On a time scale of order 1 (w.r.t. the size of the population), it is now well-known that the macroscopic behavior of mean-field particle systems are well described by nonlinear PDEs of McKean–Vlasov type [20,33]. A vast literature exists on the links between the microscopic system and its mean-field limit (fluctuations, large deviations and finite time dynamics) mostly in the non-disordered case (see e.g. [18,34,43] and references therein) but also for disordered systems [16,28].

When one considers longer time scales (w.r.t. the size of the population) and for a large but finite number of particles, some randomness remains in the system so that Brownian fluctuations generally induce microscopic dynamics that may differ significantly from the dynamics of the mean-field equation. For mean-field systems without disorder, a vast literature exists concerning fluctuations induced by thermal noise. In this respect, the notion of *uniform propagation of chaos* has been addressed for several mean-field models by many authors (see e.g. [8,31] for the granular media equation or [25,39] for ranked-based models). In case the mean-field PDE admits an isolated stable fixed point, due to large deviation phenomena, the finite-size system exits from any neighborhood of the fixed point at exponential times in N (N being the size of the population) [17,35], whereas in case of an unstable fixed point, the system escapes at a time scale of order $\log N$ [38]. Fewer results exist in the case where the mean-field PDE admits a whole stable curve of stationary solutions. In [7,15], the effect of thermal noise is considered for the mean-field plane rotators model [6] which is known to admit in the limit as $N \to \infty$ a stable circle of stationary solutions. In this case, the finite size particle system has Brownian fluctuations on time scales of order N.

In the case of disordered systems, we are not aware of any similar analysis on long time dynamics of mean-field interacting particles. The present work could be seen as a first result in this direction. In particular, we provide in Theorem 2.4 a rigorous and quantitative justification to a phenomenon already observed by Balmforth and Sassi [4] on the basis of numerical simulations.

1.2. *The stochastic Kuramoto model with disorder*

We address in this paper the long time behavior of the Kuramoto model with noise and disorder, which describes the evolution of a population of rotators (the jth rotator being defined by its phase $\varphi_j^\omega(t) \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$), given by the system of $N \geq 1$ stochastic differential equations of mean-field type

$$\mathrm{d}\varphi_j^\omega(t) = \delta\omega_j \,\mathrm{d}t - \frac{K}{N}\sum_{l=1}^{N} \sin\bigl(\varphi_j^\omega(t) - \varphi_l^\omega(t)\bigr)\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t), \quad j = 1, \dots, N, t \geq 0, \tag{1.1}$$

where $(B_j)_{j=1,\dots,N}$ is a family of standard independent Brownian motions, K, σ and δ are positive parameters. In particular, $\delta > 0$ is a scaling parameter. The main result will be stated for small $\delta > 0$, as it relies on perturbation results of the case where $\delta = 0$.

The Kuramoto model [1,27,41] is the main prototype for synchronization phenomena and, due to its mathematical tractability, has been studied in details in the past years [6,14,21,22].

Remark 1.1. *Note that* (1.1) *is invariant by rotation*: *if* $(\varphi_j^\omega(t))_{j=1,\dots,N}$ *solves* (1.1), *then so does* $(\varphi_j^\omega(t)+\alpha)_{j=1,\dots,N}$ *for all* $\alpha \in \mathbb{R}$. *Moreover, by the change of variables* $t \to t/\sigma^2$, *one can get rid of the coefficient* σ *in front of the Brownian motions* (*up to the obvious modifications* $\delta \to \delta/\sigma^2$ *and* $K \to K/\sigma^2$). *Hence, with no loss of generality, we suppose* $\sigma = 1$ *in the following*.

Following the point of view adopted at the beginning of this introduction, the system (1.1) presents two types of noise: in addition to the thermal noise (B_j), the *disorder* in (1.1) is given by a sequence $(\omega_j)_{j=1,\dots,N}$ of i.i.d random variables with distribution λ, independent from the Brownian motions. Each ω_j represents an intrinsic inhomogeneous frequency for the rotator φ_j^ω. The index ω in the notation φ_j^ω is used to emphasize the dependency of the system in the disorder.

A crucial aspect in the understanding of the dynamics of (1.1) concerns the (possible lack of) symmetry of the sequence $(\omega_j)_{j\geq 1}$. First note that, by the obvious change of variables $\varphi_j^\omega(t) \mapsto \varphi_t^\omega(t) - \mathbb{E}(\omega)t$ in (1.1), it is always possible to assume that the expectation of the disorder $\mathbb{E}(\omega) = \int_{\mathbb{R}} \omega\lambda(d\omega)$ is zero (otherwise, we observe macroscopic traveling waves with speed $\mathbb{E}(\omega)$). The asymmetry of the disorder can be given at different scales. The most simple situation corresponds to a *macroscopic asymmetry*, that is when the law λ itself is asymmetric. With no loss of generality, we can for example assume that, on a macroscopic level, a majority of rotators will be associated to a positive frequency whereas a minority will have negative frequencies. In the limit of an infinite population, this asymmetry makes the whole system rotate at a constant speed that depends only on the law λ and this rotation is noticeable at the scale of the nonlinear Fokker–Planck equation (1.3) associated to (1.1). This case has been the object of a previous paper (see [21], Theorem 2.2 and Section 2.2 below).

The present paper is concerned with the situation where the law of the disorder is symmetric. Here, the previous argument cannot be applied since in the limit as $N \to \infty$, the population is equally balanced between positive and negative frequencies: the macroscopic speed of rotation found in [21], Theorem 2.2 vanishes. Hence, the analysis of long time dynamics of (1.1) requires a deeper understanding of the *microscopic asymmetry* of the disorder, that is the finite-size fluctuations of the disorder w.r.t. the thermal noise. An informal description of the dynamics of (1.1) is the following (see Figure 2 below): if the constant K is sufficiently large, the mean-field coupling term leads to synchronization of the whole system along a nontrivial density. Even if λ is symmetric, finite-size fluctuations of the sample $(\omega_j)_{j=1,\dots,N}$ make it *not* symmetric so that the fluctuations of the disorder compete with the fluctuations of the Brownian motions $(B_j)_{j=1,\dots,N}$ and make the whole system rotate with speed and direction depending on the fixed realization of the disorder (ω_j) (and not only on the law λ itself). The main point of the paper is to give a rigorous meaning to this phenomenon, noticed numerically in [4]: we will show that at times of order $\sqrt{N}$, the dynamics of (1.1) deviates from its mean-field limit, with the appearance of synchronized traveling waves induced by the finite-size fluctuations of the disorder. We refer to Section 1.6 below for a precise description of this phenomenon.

We present in the following subsections some well-known properties of (1.1) which are needed to state our result. We describe in particular its infinite population limit on bounded time intervals and the existence of stationary measures for the limit system in case of symmetric disorder.

1.3. *Mean-field limit on bounded time intervals*

All the statistical information of (1.1) is contained in the empirical measure $(\mu_{N,t}^\omega)_{t\geq 0} \in C([0,\infty), \mathcal{M}_1(\mathbb{T}\times\mathbb{R}))$ ($\mathcal{M}_1$ being the set of probability measures endowed with its weak topology) defined as

$$\mu_{N,t}^\omega := \frac{1}{N}\sum_{j=1}^N \delta_{(\varphi_j^\omega(t),\omega_j)}, \quad t \geq 0. \tag{1.2}$$

When the distribution λ of the disorder satisfies $\int |\omega|\lambda(\mathrm{d}\omega) < \infty$ and the initial condition $\mu^{\omega}_{N,0}$ converges weakly to some p_0 when $N \to \infty$, it is easy to see ([16,28]) that the empirical measure (1.2) converges weakly on bounded time intervals (that is in $C([0,T], \mathcal{M}_1(\mathbb{T}\times\mathbb{R}))$ for all $T \geq 0$) to a deterministic limit measure whose density p_t with respect $\ell \otimes \lambda$ (where ℓ denotes the Lebesgue measure on $\mathbb{T}$) satisfies the following system of nonlinear Fokker–Planck PDEs:

$$\partial_t p_t(\theta,\omega) = \frac{1}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta\big(p_t(\theta,\omega)\big(\langle J * p_t\rangle_\lambda(\theta) + \delta\omega\big)\big), \quad \omega \in \mathrm{Supp}(\lambda), \theta \in \mathbb{T}, t \geq 0, \tag{1.3}$$

where

$$J(\theta) := -K\sin(\theta), \tag{1.4}$$

and $\langle\cdot\rangle_\lambda$ represents the integration with respect to λ: $\langle J * u\rangle_\lambda(\theta) = \int_{\mathbb{R}}\int_{\mathbb{T}} J(\psi)u(\theta-\psi,\omega)\,\mathrm{d}\psi\,\lambda(\mathrm{d}\omega)$. We insist on the fact that in (1.3), ω is a real number in the support of λ, while in (1.1) and (1.2), it is an index emphasizing the dependency in the disorder of the system.

Some properties of system (1.3) are detailed in [21]. In particular, if λ-almost surely, $p_0(\cdot,\omega)$ is a probability measure then (1.3) admits a unique solution p_t for all $t > 0$ such that λ-almost surely, $p_t(\cdot,\omega)$ is also a probability measure, with positive density with respect to the Lebesgue measure and is an element of $C^\infty((0,\infty)\times\mathbb{T},\mathbb{R})$.

1.4. *Symmetric disorder*

As already mentioned, we consider the case where the law λ of the disorder is *symmetric*. We restrict our analysis to finite disorder: fix $d \geq 1$ and suppose that the frequencies $(\omega_j)_{j\geq 1}$ take their values in $\{\omega^{-d}, \omega^{-(d-1)}, \dots, \omega^{d-1}, \omega^d\}$, where $\omega^i = -\omega^{-i}$ for all $i = 0,\dots,d$. We denote as $(\lambda^i \in [0,1], i = -d,\dots,d)$ the probability of drawing each ω^i and assume that $\lambda^i = \lambda^{-i}$ for all $i = 1,\dots,d$. From now on, the law of the disorder λ is identified with $(\lambda^{-d},\dots,\lambda^d)$. Note that we may suppose in the following that $\omega_0 = 0 \notin \mathrm{Supp}(\lambda)$. The result still holds with obvious changes in notations.

Under this hypothesis, almost surely, for sufficiently large N, each possible value ω^i of the disorder appears at least once and we can rewrite (1.1) by regrouping the rotators into $(2d+1)$ sub-populations: for all $i = -d,\dots,d$, denote as N^i the number of rotators $(\varphi^i_j(t))_{j=1,\dots,N^i}$ with frequency ω^i. Obviously, $N = \sum_{i=-d}^d N^i$ and the system (1.1) becomes

$$\mathrm{d}\varphi^i_j(t) = \delta\omega^i\,\mathrm{d}t - \frac{K}{N}\sum_{k=-d}^{d}\sum_{l=1}^{N^k}\sin\big(\varphi^i_j(t) - \varphi^k_l(t)\big)\,\mathrm{d}t + \mathrm{d}B^i_j(t), \quad j = 1,\dots,N^i, i = -d,\dots,d. \tag{1.5}$$

In this framework, the empirical measure $\mu^{\omega}_{N,t}$ in (1.2) can be identified with $\mu_{N,t} := (\mu^{-d}_{N,t},\dots,\mu^{d}_{N,t})$, where μ^i_N is the empirical measure of the rotators with frequency ω^i:

$$\mu^i_{N,t} = \frac{1}{N^i}\sum_{j=1}^{N^i}\delta_{\varphi^i_j(t)}, \quad t \geq 0, i = -d,\dots,d, \tag{1.6}$$

and its mean-field limit (1.3) can be identified with $p_t = (p_t^{-d},\dots,p_t^d)$, solution to

$$\partial_t p^i_t(\theta) = \frac{1}{2}\partial^2_\theta p^i_t(\theta) - \partial_\theta\left(p^i_t(\theta)\left(\sum_{k=-d}^{d}\lambda^k J * p^k_t(\theta) + \delta\omega^i\right)\right), \quad t \geq 0, i = -d,\dots,d. \tag{1.7}$$

1.5. *Stationary solutions and phase transition*

A remarkable aspect of the Kuramoto model is that one can compute semi-explicitly the stationary solutions of (1.7), when λ is symmetric (see e.g. [40]): each stationary solution to (1.7) is the rotation of a profile $q = (q^{-d}, \dots, q^d)$ (i.e. given by $q(\cdot + \alpha)$ for some $\alpha \in \mathbb{T}$) of the form

$$q^i(\theta) = \frac{S_\delta^i(\theta, 2Kr)}{Z_\delta^i(2Kr)}, \tag{1.8}$$

where for each $i = -d, \dots, d$, $q^i(\cdot)$ is a probability density on $\mathbb{T}$, $S_\delta^i(\theta, 2Kr)$ is given by

$$\begin{aligned} S_\delta^i(\theta, x) = e^{x\cos\theta + 2\delta\omega^i\theta}\bigg[&\big(1 - e^{4\pi\delta\omega^i}\big)\int_0^\theta e^{-x\cos u - 2\delta\omega^i u}\,\mathrm{d}u \\ &+ e^{4\pi\delta\omega^i}\int_0^{2\pi} e^{-x\cos u - 2\delta\omega^i u}\,\mathrm{d}u\bigg], \end{aligned} \tag{1.9}$$

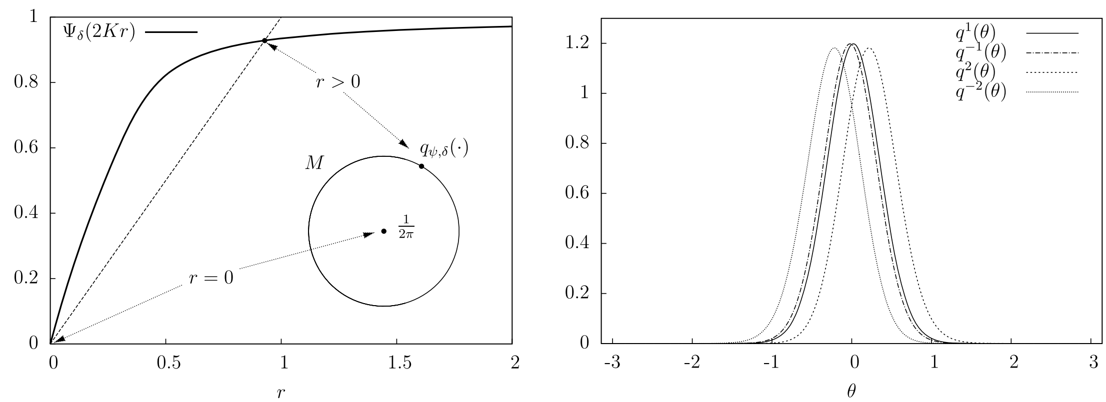
$Z_\delta^i(2Kr)$ is a normalization constant and r is a solution of the fixed-point problem

$$r = \Psi_\delta(2Kr), \tag{1.10}$$

with

$$\Psi_\delta(x) = \sum_{k=-d}^{d} \lambda^k \frac{\int_0^{2\pi}\cos(\theta) S_\delta^k(\theta, x)\,\mathrm{d}\theta}{Z_\delta^k(x)}. \tag{1.11}$$

We refer to [40] or [29], p. 75 for more details on this calculation. Computing the solution to the fixed-point relation (1.10) enables to exhibit a phase transition for (1.7): the value $r = 0$ always solves (1.10) and corresponds to the uniform stationary solution $q \equiv (1/2\pi, \dots, 1/2\pi)$. It is the only stationary solution to (1.7) as long as $K \le K_c$, for a certain critical parameter $K_c = K_c(\delta, (\omega^i)_i, (\lambda^i)_i) > 1$. This characterizes the absence of synchrony in case of small interaction. When $K > K_c$, this flat profile coexists with circles of synchronized solutions corresponding to positive fixed-points in (1.10): each solution $r > 0$ to (1.10) gives rise to a nontrivial stationary profile q given by (1.8) and to the circle of all its translation $q(\cdot + \alpha)$, by invariance by rotation of the system (see Figure 1).



(A) Correspondance between fixed-points of $\Psi_\delta(\cdot)$ and stationary solutions to (1.7).

(B) A synchronized profile with $d = 2$, $q = (q^{-2}, q^{-1}, q^1, q^2)$.

Fig. 1. Fixed-point function $\Psi_\delta(\cdot)$ and stationary profiles when $K = 5$, $d = 2$, $\omega_1 = 1$, $\omega_2 = 10$ and $\delta = 0.1$.

However, several circles may coexist when $K > K_c$ and these circles may not be locally stable (even the characterization of these circles in full generality is unclear, see e.g. [29], §2.2.2). To ensure uniqueness and stability of a circle of non-trivial profiles, fix $K > 1$ and restrict to small values of δ: it is indeed proved in [21], Lemma 2.3 that there exists $\delta_1 = \delta_1(K) > 0$ such that for all $\delta \le \delta_1$, the fixed-point problem (1.10) admits a unique positive solution r_δ. We denote by $q_{0,\delta}$ the corresponding profile given by (1.8) with $r = r_\delta$, by $q_{\psi,\delta}$ its rotation of angle $\psi \in \mathbb{T}$ (i.e. $q_{\psi,\delta}(\cdot) := q_{0,\delta}(\cdot - \psi)$) and by M the corresponding circle of stationary profiles (see Figure 1):

$$M := \{q_{\psi,\delta} : \psi \in \mathbb{T}\}. \tag{1.12}$$

It is proved in [21], Theorems 2.2 and 2.5 that the circle M is stable under the evolution (1.7): the solution of (1.7) starting from an initial condition sufficiently close to M converges to a element $q_{\psi,\delta}$ of M as $t \to \infty$. More details about this stability are given in Section 2.3. Whenever it is clear from the context, we will use the notations q_δ or q_ψ instead of $q_{\psi,\delta}$, depending on the parameter we want to emphasize.

1.6. *Long time behavior*

Simulations of (1.5) (Figure 2) suggest an initial transition of the system from an incoherent state to a synchronized one, during which the empirical measures of the rotators approaches the circle M of synchronized stationary profiles. Secondly, the empirical measure remains close to M and travels at first order at constant speed (which is random, depending on the realization of the disorder, see Figure 3) along M on the time scale $N^{1/2}t$. Let us give some intuition of this phenomenon: to fix ideas, consider the case where $d = 1$, $\omega^1 = -\omega^{-1} = 1$ and $\lambda^{-1} = \lambda^1 = \frac{1}{2}$. This corresponds to the simplest decomposition in (1.5) between two subpopulations, one naturally rotating clockwise ($\omega_i = +1$) and the second rotating anti-clockwise ($\omega_i = -1$). One can imagine that fluctuations in the finite sample $(\omega_1, \dots, \omega_N) \in \{\pm 1\}^N$ may lead, for example, to a majority of $+1$ with respect to -1, so that the rotators with positive frequency induce a global rotation of the whole system in the direction of the majority. When N is large, this asymmetry is small, typically of order $N^{-1/2}$ and is not sufficient to make the empirical measure drift away from the attracting manifold M, but induces a small drift that becomes macroscopic at times of order $N^{1/2}$.

The purpose of the paper is precisely to prove the existence of this random traveling wave and show that it is indeed an effect of the fluctuations of the disorder. Our approach consists in a precise analysis of the dynamics of the empirical measure (1.6), which involves both disorder and thermal noise. One of the main difficulties is to control the thermal noise term and prove that it does not play any role at first order on the $N^{1/2}$-time scale.

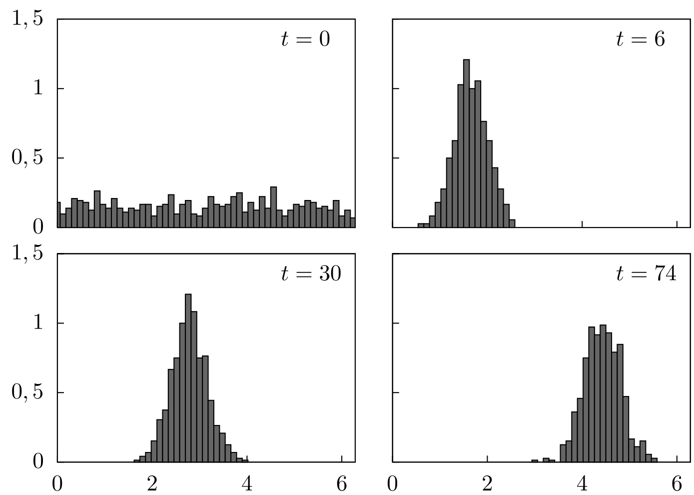


Fig. 2. Evolution of the marginal of the empirical measure (1.2) on $\mathbb{T}$ for a fixed choice of the disorder ($N = 600$, $\lambda = \frac{1}{2}(\delta_{-1} + \delta_1)$, $K = 6$). Starting from uniformly distributed rotators on $\mathbb{T}$ ($t = 0$), the empirical measure converges to a synchronized profile on the manifold M ($t = 6$) and then moves (here to the right) at a constant speed, on a time scale compatible with $N^{1/2}$.

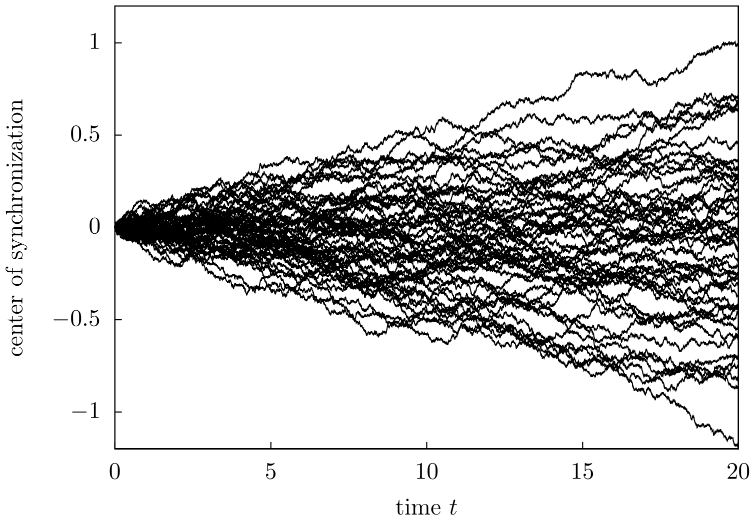


Fig. 3. Trajectories of the center of synchronization for different realizations of the disorder ($\lambda = \frac{1}{2}(\delta_{-0.5} + \delta_{0.5})$, $K = 4$, $N = 400$).

2. Main results and strategy of proof

2.1. *The result*

Admissible sequence of disorder

We stress the fact that the random traveling waves described above is essentially a *quenched* phenomenon, that is, true for a fixed realization of the disorder $(\omega_i)_{i\geq 1}$. In particular, the result does not really depend on the underlying mechanism that produced the sequence $(\omega_i)_{i\geq 1}$, it only depends on the asymmetry of this sequence. We prove our result for any *admissible* sequence of disorder $(\omega_i)_{i\geq 1}$, defined as follows.

Definition 2.1. *Fix a sequence $(\omega_i)_{i\geq 1}$ taking values in $\{\omega^{-d}, \omega^{-(d-1)}, \dots, \omega^{d-1}, \omega^d\}$ and for all $N \geq 1$, define the empirical proportions of frequencies in the N-sample $(\omega_1, \dots, \omega_N)$*

$$\lambda_N^k := \frac{N^k}{N}, \quad k = -d, \dots, d, \tag{2.1}$$

where N^k is the number of rotators with frequencies equal to ω^k (recall Section 1.4). Define also the fluctuation process associated to $(\omega_i)_{i\geq 1}$ by $\xi_N := (\xi_N^{-d}, \dots, \xi_N^d)$, where

$$\xi_N^k := N^{1/2}\big(\lambda_N^k - \lambda^k\big), \quad k = -d, \dots, d, N \geq 1, \tag{2.2}$$

where $(\lambda^{-d}, \dots, \lambda^d)$ is given in Section 1.4. Note that $\sum_{k=-d}^d \xi_N^k = 0$ for all $N \geq 1$. We say that the sequence $(\omega_i)_{i\geq 1}$ is admissible *if the following holds*

(1) Law of large numbers: *for all $k = -d, \dots, d$, λ_N^k converges to λ^k, as $N \to \infty$.*
(2) Central limit behavior: *for all $\zeta > 0$, there exists N_0 (possibly depending on the sequence $(\omega_i)_{i\geq 1}$) such that for all $N \geq N_0$,*

$$\max_{k=-d,\dots,d} \big|\xi_N^k\big| \leq N^{\zeta}.$$

Remark 2.2 (Admissibility for i.i.d. variables). *An easy application of the Borel–Cantelli Lemma shows that any independent and identically distributed sequence of disorder $(\omega_i)_{i\geq 1}$ with law λ is almost surely admissible, in the sense of Definition 2.1.*

Main result
From now on, we fix once and for all an admissible sequence $(\omega_i)_{i\geq 1}$ in the sense of Definition 2.1. A convenient framework for the analysis of the dynamics of (1.6) and (1.7) corresponds to the space H_d^{-1}, dual of the space H_d^1, which is the closure of the set of vectors $(u^{-d},\dots,u^d)$ of regular functions u^k with zero mean value on $\mathbb{T}$ under the norm

$$\|u\|_{1,d} := \left(\sum_{k=-d}^{d} \lambda^k \int_{\mathbb{T}} \big(\partial_\theta u^k(\theta)\big)^2 \,\mathrm{d}\theta\right)^{1/2}. \tag{2.3}$$

Remark 2.3. *If u is a vector of probability measures on $\mathbb{T}$, then u naturally belongs to H_d^{-1}, since the family of vectors given by $a_{n,k}(\theta)=(0,\dots,0,\frac{\sqrt{2}\cos(n\theta)}{n\sqrt{\lambda^k}},0,\dots,0)$ and $b_{n,k}(\theta)=(0,\dots,0,\frac{\sqrt{2}\sin(n\theta)}{n\sqrt{\lambda^k}},0,\dots,0)$ form an orthonormal basis of H_d^1 and for each such vector u*

$$\|u\|_{-1,d} = \sqrt{\sum_{k=-d}^{d}\sum_{n=1}^{\infty}\big(\langle u,a_{n,k}\rangle^2+\langle u,b_{n,k}\rangle^2\big)} \leq \pi\sqrt{\frac{2}{3}\sum_{k=-d}^{d}\big(\lambda^k\big)^{-1}}. \tag{2.4}$$

A similar argument also shows that, for any such vector of probability u and for any bounded function w, then $w\cdot u\in H_d^{-1}$. More details on the construction of H_d^{-1} are given in Appendix A.

The main result of the paper is the following.

Theorem 2.4. *For all $K>1$, there exists $\delta(K)$ such that, for all $\delta\leq\delta(K)$, there exists a linear form $b:\mathbb{R}^{2d+1}\to\mathbb{R}$ (depending on K, δ, the probability distribution λ and the possible values of the disorder ω^i) and a real number $\varepsilon_0>0$ such that the following holds: for any admissible sequence $(\omega_i)_{i\geq 1}$, any vector of probability measures p_0 satisfying $\mathrm{dist}_{H_d^{-1}}(p_0,M)\leq\varepsilon_0$ such that for all $\varepsilon>0$,*

$$\mathbf{P}\big(\|\mu_{N,0}-p_0\|_{-1,d}\geq\varepsilon\big)\to 0, \quad \textit{as } N\to\infty, \tag{2.5}$$

there exists $\theta_0\in\mathbb{T}$ (depending on p_0) and a constant c such that for each finite time $t_f>0$ and all $\varepsilon>0$, denoting $t_0^N=cN^{-1/2}\log N$, we have

$$\mathbf{P}\Big(\sup_{t\in[t_0^N,t_f]}\|\mu_{N,N^{1/2}t}-q_{\theta_0+b(\xi_N)t}\|_{-1,d}\geq\varepsilon\Big)\to 0, \quad \textit{as } N\to\infty. \tag{2.6}$$

Moreover, $\xi\mapsto b(\xi)$ has the following expansion in δ: for all ξ such that $\sum_{k=-d}^{d}\xi^k=0$, we have

$$b(\xi)=\delta\sum_{k=-d}^{d}\xi^k\omega^k+O\big(\delta^2\big). \tag{2.7}$$

Theorem 2.4 is simply saying that, on a time scale of order $N^{1/2}$, the empirical measure (1.6) is asymptotically close to a synchronized profile $q\in M$, traveling at speed $b(\xi_N)$ along M. This drift depends on the asymmetry ξ_N of the quenched disorder $(\omega_i)_{i\geq 1}$. In (2.6), t_0^N represents the time necessary for the system to get sufficiently close to the manifold M.

Some particular cases and extensions
First remark that the situation where the sample of the disorder $(\omega_i)_{i=1,\dots,N}$ is perfectly symmetric corresponds to $\xi_N^{-i}=\xi_N^i$ for all $i=1,\dots,d$. In this case, the drift in (2.6) vanishes:

Proposition 2.5. *If for all $i=1,\dots,d$, $\xi^{-i}=\xi^i$, then $b(\xi)=0$.*

In particular, if one chooses the disorder in such a way that $(\omega_i)_{i=1,\dots,N}$ is always symmetric (e.g. choose an even number of particles N and define each ω_i to be alternatively ± 1), the drift is always zero. We believe in this case that one would need to look at larger time scales of order N to see the first order of the expansion of the empirical measure μ_N. Proof of Proposition 2.5 is given in Section 7.1.

In case the sequence $(\omega_i)_{i\geq 1}$ is i.i.d. with law λ, a standard Central Limit Theorem shows that the drift $b(\xi_N)$ converges in law to a Gaussian distribution $\mathcal{N}(0, v^2)$, where v^2 depends on K, δ, the probability distribution λ and the possible values of the disorder ω^i.

Proposition 2.6. *The following asymptotic of* v^2 *holds when* $\delta \to 0$:

$$v^2 = \delta \sum_{k=-d}^{d} \lambda^k \left(\omega^k\right)^2 + O\left(\delta^2\right). \tag{2.8}$$

Proof of Proposition 2.6 is given in Section 7.2.

Remark 2.7. *Without much modification in the proof, the result can be easily extended to sequences* $(\omega_i)_{i\geq 1}$ *with fluctuations of order different from* $\sqrt{N}$, *that is when for some* $a \in (0, 1)$,

$$\xi_N^a \to \xi^a, \quad \text{as } N \to \infty, \tag{2.9}$$

for some vector ξ^a *where* $\xi_N^a := N^a(\lambda_N - \lambda)$. *In this case, the correct time renormalization is* N^a *and we obtain a result of the type*

$$\mathbf{P}\Big(\sup_{t\in[t_0^N, t_f]} \|\mu_{N,N^a t} - q_{\theta_0 + b(\xi^a)t}\|_{-1,d} \geq \varepsilon\Big) \to 0, \quad \text{as } N \to \infty, \tag{2.10}$$

for a time t_0^N *of order* $N^{-a}\log(N)$. *Here, we only treat the case* $a = 1/2$ *for simplicity. For smaller fluctuations of size* N^{-a} *with* $a \geq 1$, *the time renormalization should be of order* N. *Since at this scale the effects of the thermal noise appear, the limit phase dynamics should be of diffusive type and a precise analysis of the different terms and symmetries that occur would be necessary to get the proper drift in this case.*

2.2. *Links with existing models*

Symmetric versus non-symmetric disorder

This work is the natural continuation of [21], Theorems 2.2 and 2.5 in the case of a symmetric disorder. The purpose of [21] was to analyze the dynamics of the nonlinear Fokker–Planck equation (1.3) for both symmetric and asymmetric law of the disorder. The main point is that understanding (1.3) is not sufficient in itself for the analysis of the finite size system (1.1) in the symmetric case, since it does not account for the finite-size effects of the disorder that are crucial here.

As already mentioned, in the case where λ is asymmetric, one observes macroscopic travelling waves with deterministic drift at the scale of the nonlinear Fokker–Planck equation (1.3). It is reasonable to think that an analysis similar to what has been done in this paper would also show the existence of a finite order correction to this deterministic drift for a large but finite system with quenched disorder.

Some previous results already suggested the possibility of these disorder-induced traveling waves in the Kuramoto model. Namely, the purpose of previous work [28] was to prove a quenched central limit theorem for the empirical measure (1.2) around its mean-field limit (1.3) on a finite time horizon $[0, T]$. The main result of [28] is the quenched convergence as $N \to \infty$ of the fluctuation process $\eta_{N,t}^{\omega} := \sqrt{N}(\mu_{N,t}^{\omega} - p_t)$ to η_t^{ω} solution to a linear SPDE, which is still disorder dependent. In [30], it is shown that the limiting fluctuation process has a linear behavior for large times: $\frac{\eta_t^{\omega}}{t} \to v(\omega)$, as $t \to \infty$, for some disorder dependent distribution $v(\omega)$. Putting these two results together, one obtains informally that, as $N \to \infty$ and $t \to \infty$,

$$\mu_{N,t}^{\omega} \approx q_t + \frac{\eta_t^{\omega}}{\sqrt{N}} \approx q_t + \frac{t v(\omega)}{\sqrt{N}}. \tag{2.11}$$

Of course, the previous expansion of $\mu^{\omega}_{N,t}$ around its mean-field limit p_t is only formal since the convergence as $N \to \infty$ of $\eta^{\omega}_{N,t}$ to η^{ω} on $[0, T]$ is not uniform in T. But (2.11) suggests that the correct time scale in order to capture the influence of the disorder is precisely $\sqrt{N}$, that is the time scale we consider in this paper.

The case $\delta = 0$

This paper uses techniques previously developed in [7] in the context of the stochastic Kuramoto model without disorder, that is when one takes $\delta = 0$ in (1.1):

$$\mathrm{d}\varphi_j(t) = -\frac{K}{N}\sum_{l=1}^{N}\sin\bigl(\varphi_j(t) - \varphi_l(t)\bigr)\,\mathrm{d}t + \mathrm{d}B_j(t), \qquad j = 1, \dots, N, \tag{2.12}$$

associated in the limit $N \to \infty$ to the mean-field PDE

$$\partial_t p_t(\theta) = \frac{1}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta\bigl(p_t(\theta) J * p_t(\theta)\bigr). \tag{2.13}$$

Similarly to (1.7) in Section 1.5, evolution (2.13) generates a stable circle M_0 of stationary synchronized profiles when $K > K_c(0) = 1$ (see Section B.1 for further details). The model (2.12)–(2.13) has been the subject of a series of recent papers [6,7,22,23], addressing the linear and nonlinear stability of the circle of synchronized profiles M_0 as well as the long time dynamics of the microscopic system (2.12). The analysis of (2.13) strongly relies on the reversibility of (2.12) (with the existence of a proper Lyapunov functional, see [6] for more details), whereas reversibility is lost when $\delta > 0$.

Concerning the long time behavior of (2.12), it is shown in [7] that under very general hypotheses on the initial condition, the empirical measure of (2.12) first approaches the circle M_0 exponentially fast (that corresponds to the synchronization of the system (2.12) along a stationary profile solving (2.13)) and then stays close to M_0 for a long time with high probability, while the phase of its projection on M_0 performs a Brownian motion as $N \to \infty$ which corresponds to a macroscopic effect of the thermal noise. The persistence of proximity of the empirical measure to M for long times and the convergence of this phase to a Brownian motion were in fact already established in the unpublished Ph.D. thesis [15] the authors of [7] were not aware of, using in particular moderate deviations estimates of the mean field process. Note that the techniques of [15] do not apply here, since a similar analysis would involve moderate (or large) deviations in a quenched set-up, result that, to the best of our knowledge, has not been proven so far (for *averaged* large deviations, see [16]).

A significant difference between [7,15] and the present analysis is that the Brownian excursions in [7,15] occur on a time scale of order N whereas it is sufficient to look at times of order $N^{1/2}$ to see the traveling waves in the disordered case. This will entail significant simplifications in the analysis of (1.5), since the detailed analysis on the thermal noise performed in [7,15] will not be required here.

Note also that, contrary to [7], we do not prove the first step of the phenomenon described in Figure 2, that is the initial approach of the system to a neighborhood of the manifold M in an exponentially short time, regardless of the initial condition. This result would require a global stability result for the system of PDEs (1.7) which has not been proved for the moment, due to the absence of any Lyapunov functional for (1.7) when $\delta > 0$. We prove our result for initial conditions belonging to some macroscopic neighborhood of M (see Section 6 for more details).

SPDE models with vanishing noise

This paper is related to previous works in the context of SPDE models for phase separation. In [12,19], the authors studied the Allen–Cahn model with symmetric bistable potential and vanishing noise. They showed that for an initial data close a profile connecting the two phase, the interface performs a Brownian motion. Some techniques initially introduced in these works, as the discretization of the dynamics in an iterative scheme, were developed in [7] in the context of the Kuramoto model without disorder (making use of Sobolev spaces with negative exponents to deal with empirical measures) and will have a central role in our analysis (see Section 2.6). The results of [12,19] have been extended in [11] by considering small asymmetries in the potential which induce a drift in the interface dynamics and by considering macroscopically finite volumes [5], with effect a repulsion at the boundary for the phase. Stochastic interface motions have also been recently studied in the context of the Cahn–Hilliard model with vanishing colored noise [2]. In this model, the limit behavior of the interface is given by a SDE (or system of SDE's in the case of several interfaces) with drift and diffusion coefficients depending on coloration of the noise and on the length of the interface.

2.3. *Linear stability of stationary solutions*

In the whole paper, we suppose that $K > 1$ and that $\delta > 0$ is smaller than some $\delta(K) > 0$. This critical value $\delta(K)$ is determined by $\delta(K) = \min(\delta_1(K), \delta_2(K))$, where $\delta_1(K)$ ensures the existence of a unique circle M of stationary solutions (recall Section 1.5) and where $\delta_2(K)$ comes from the stability analysis of this circle (see Appendix B for more details).

More precisely, our result relies deeply on the linear stability of the dynamical system induced by the limit system of PDEs (1.7) in the neighborhood of the circle of stationary profiles M. For $\psi \in \mathbb{T}$, $\delta > 0$, consider the operator $L_{\psi,\delta}$ of the linearized evolution around $q_{\psi,\delta} \in M$ given by

$$(L_{\psi,\delta}u)^i = \frac{1}{2}\partial_\theta^2 u^i - \delta\omega^i \partial_\theta u^i - \partial_\theta\left(u^i \sum_{k=-d}^{d} \lambda^k\big(J * q_{\psi,\delta}^k\big) + q_{\psi,\delta}^i \sum_{k=-d}^{d} \lambda^k\big(J * u^k\big)\right), \tag{2.14}$$

for all $i = -d, \dots, d$ with domain

$$\left\{u = \big(u^{-d}, \dots, u^d\big) : u^i \in C^2(\mathbb{T}) \text{ and } \int_{\mathbb{T}} u^i(\theta)\,\mathrm{d}\theta = 0, \forall i = -d, \dots, d\right\}. \tag{2.15}$$

Due to the invariance by rotation of the model (1.7), $L_{\psi,\delta}$ is linked to $L_{0,\delta}$ in an obvious way: $L_{\psi,\delta}u_\psi(\cdot) = L_{0,\delta}u(\cdot)$, where $u_\psi(\cdot) = u(\cdot - \psi)$, so that the operators $(L_{\psi,\delta})_{\psi\in\mathbb{T}}$ obviously share the same spectral properties. For any operator L, the usual notations $\sigma(L)$ (resp. $\rho(L)$ and $R(z, L)$) will be used for the spectrum of L (resp. its resolvent set and its resolvent operator for $z \in \rho(L)$).

One can prove (see [21], Theorem 2.5 and Appendix B below) that for all $0 \le \delta \le \delta_2(K)$, the following holds: $L_{\psi,\delta}$ is closable in H_d^{-1}, sectorial, that is there exists $\varphi \in (\frac{\pi}{2}, \pi)$ such that

$$\rho(L_{\psi,\delta}) \supset \big\{z \in \mathbb{C}, \big|\arg(z)\big| < \varphi\big\},$$

$$\big\|R(z, L_{\psi,\delta})\big\|_{H_d^{-1}} \le \frac{M}{|z|}, \quad \text{for every } z \in \mathbb{C}, \big|\arg(z)\big| < \varphi,$$

$L_{\psi,\delta}$ has 0 for eigenvalue, associated to the eigenvector $\partial_\theta q_{\psi,\delta}$, which belongs to the tangent space of M in $q_{\psi,\delta}$ (this reflects the fact that the dynamics induced by (1.7) on M is trivial) and that the rest of the spectrum is negative, separated from the eigenvalue 0 by a spectral gap $\gamma_L > 0$. More details about these questions are given in Appendix B.

The fact that the eigenvalue 0 is isolated from the rest of the spectrum $\sigma(L_{\psi,\delta}) \setminus \{0\}$ implies that H_d^{-1} can be decomposed into a direct sum $T_{\psi,\delta} \oplus N_{\psi,\delta}$, where $T_{\psi,\delta} = \mathrm{Span}(\partial_\theta q_{\psi,\delta})$ such that the spectrum of the restriction of $L_{\psi,\delta}$ to $N_{\psi,\delta}$ (resp. $T_{\psi,\delta}$) is $\sigma(L_{\psi,\delta}) \setminus \{0\}$ (resp. $\{0\}$). We denote by $P_{\psi,\delta}^0$ the projection on $T_{\psi,\delta}$ along $N_{\psi,\delta}$ and $P_{\psi,\delta}^s = 1 - P_{\psi,\delta}^0$. Both $P_{\psi,\delta}^0$ and $P_{\psi,\delta}^s$ commute with $L_{\psi,\delta}$. In particular, for all $\psi \in \mathbb{T}$, $\delta > 0$, there exists a linear form $\mathrm{p}_{\psi,\delta}$ satisfying, for all $u \in H_d^{-1}$

$$P_{\psi,\delta}^0 u = \mathrm{p}_{\psi,\delta}(u)\partial_\theta q_{\psi,\delta}. \tag{2.16}$$

We also denote by C_P and C_L positive constants such that for all $u \in H_d^{-1}$, $t > 0$:

$$\big\|P_{\psi,\delta}^0 u\big\|_{-1,d} \le C_P\|u\|_{-1,d}, \tag{2.17}$$

$$\big\|P_{\psi,\delta}^s u\big\|_{-1,d} \le C_P\|u\|_{-1,d}, \tag{2.18}$$

$$\big\|e^{tL_{\psi,\delta}} P_{\psi,\delta}^s u\big\|_{-1,d} \le C_L e^{-\gamma_L t}\big\|P_{\psi,\delta}^s u\big\|_{-1,d}, \tag{2.19}$$

$$\big\|e^{tL_{\psi,\delta}} u\big\|_{-1,d} \le C_L\left(1 + \frac{1}{\sqrt{t}}\right)\|u\|_{-2,d}. \tag{2.20}$$

Inequality (2.19) is a consequence of [24], Theorem 1.5.3, p. 30 and (2.20) is proved in Proposition B.7 in Appendix B. Once again, we will often drop the dependency in the parameters ψ or δ in $P_{\psi,\delta}^0$ and $P_{\psi,\delta}^s$ for simplicity of notations.

A consequence of the contraction (2.19) along the space $N_{\psi,\delta}$ is that M is locally stable with respect to the evolution given by (1.7) (see for example exercise 6* of the Chapter 6 of [24], or Theorem 2.2 of [21] for our particular model): for any p_0 in a neighborhood of M, there exists $\psi \in \mathbb{T}$ such that the solution of (1.7) converges to $q_{\psi,\delta}$ exponentially fast (with rate given by γ_L).

2.4. *Dynamics of the empirical measure*

The starting point of the proof of Theorem 2.4 is to write the semi-martingale decomposition (see Proposition 3.1) of the difference between the empirical measure $\mu_{N,t}$ defined in (1.6) and any element of $q_{\psi,\delta} \in M$. Namely, define the process $t \mapsto \nu_{N,t}$, $t \geq 0$ by

$$\nu_{N,t}^i := \mu_{N,t}^i - q_{\psi,\delta}^i, \quad i = -d, \dots, d. \tag{2.21}$$

The point is to write a mild formulation of this semi-martingale decomposition that makes sense in the space H_d^{-1} (recall that $\mu_{N,t}$ and $\nu_{N,t}$ belong to H_d^{-1} due to (2.4)). This mild formulation involves in particular the semi-group $e^{tL_{\psi,\delta}}$ of the operator $L_{\psi,\delta}$ (2.14) so that one can take advantage of the contraction properties of this semi-group in the neighborhood of the manifold M.

Proposition 2.8. *For all $K > 1$, for all $0 \leq \delta \leq \delta(K)$, the process $(\nu_{N,t})_{t\geq 0}$ defined by (2.21) satisfies the following stochastic partial differential equation in $\mathcal{C}([0,+\infty), H_d^{-1})$, written in a mild form*:

$$\nu_{N,t} = e^{tL_{\psi,\delta}}\nu_{N,0} + \int_0^t e^{(t-s)L_{\psi,\delta}}\big(D_N - \partial_\theta R_N(\nu_{N,s})\big)\,\mathrm{d}s + Z_{N,t}, \quad N \geq 1, t \geq 0, \tag{2.22}$$

where

$$D_N = D_{N,\psi,\delta} := -\partial_\theta\left(q_{\psi,\delta}\sum_{k=-d}^{d}\big(\lambda_N^k - \lambda^k\big)\big(J * q_{\psi,\delta}^k\big)\right), \tag{2.23}$$

$$\begin{aligned} R_N(\nu_{N,s}) &= R_{N,\psi,\delta}(\nu_{N,s}) \\ &:= \left(\sum_{k=-d}^{d}\big(\lambda_N^k - \lambda^k\big) J * q_{\psi,\delta}^k\right)\nu_{N,s} + q_{\psi,\delta}\sum_{k=-d}^{d}\big(\lambda_N^k - \lambda^k\big)\big(J * \nu_{N,s}^k\big) \\ &\quad + \left(\sum_{k=-d}^{d}\lambda_N^k J * \nu_{N,s}^k\right)\nu_{N,s}, \end{aligned} \tag{2.24}$$

and $Z_{N,t} = Z_{N,t,\psi}$ is the limit in H_d^{-1} as $t' \nearrow t$ of $Z_{N,t,t'}$ defined by

$$Z_{N,t,t'}(h) = \sum_{i=-d}^{d}\frac{\lambda^i}{N^i}\sum_{j=1}^{N^i}\int_0^{t'}\partial_\theta\big[\big(e^{(t-s)L_\psi^*}h\big)^i\big]\big(\varphi_j^i(s)\big)\,\mathrm{d}B_j^i(s), \tag{2.25}$$

that we denote

$$Z_{N,t}(h) = \sum_{i=-d}^{d}\frac{\lambda^i}{N^i}\sum_{j=1}^{N^i}\int_0^{t}\partial_\theta\big[\big(e^{(t-s)L_\psi^*}h\big)^i\big]\big(\varphi_j^i(s)\big)\,\mathrm{d}B_j^i(s), \tag{2.26}$$

and where all the terms in (2.22) make sense as elements of $\mathcal{C}([0,\infty), H_d^{-1})$.

The proof of Proposition 2.8 may be found in Section 3. The term $Z_{N,t}$ in (2.22) represents the effect of the thermal noise on the system. The term involving D_N is the one that produces the drift we are after on the time scale

$N^{1/2}t$, when the empirical measure $\mu_{N,t}$ is close to the manifold M. To make this drift appear, we rely on an iterative procedure, as explained in Section 2.6.

2.5. *Moving closer to the manifold M*

We place ourselves in the framework of Theorem 2.4: we fix $\varepsilon_0 > 0$ and suppose the existence of a probability measure $p_0 \in H_d^{-1}$ such that $\mathrm{dist}_{H_d^{-1}}(p_0, M) \leq \varepsilon_0$ with $\mathbf{P}(\|\mu_{N,0} - p_0\|_{-1,d} \geq \varepsilon) \to 0$ as $N \to \infty$, for all $\varepsilon > 0$. The constant ε_0 will be chosen small enough in Section 6.

The first step in proving our result is to show that the empirical measure $\mu_{N,t}$ reaches a neighborhood of size $N^{-1/2}$ in a time of order $\log N$. We use the projection defined in the following lemma, whose proof can be found in Appendix C, along with several regularity results.

Lemma 2.9. *There exists $\sigma > 0$ such that for all h such that $\mathrm{dist}_{H_d^{-1}}(h, M) \leq \sigma$, there exists a unique phase $\psi =: \mathrm{proj}_M(h) \in \mathbb{T}$ such that $P_\psi^0(h - q_\psi) = 0$ and the mapping $h \mapsto \mathrm{proj}_M(h)$ is $\mathcal{C}^\infty$.*

From now on, we fix a sufficiently small constant ζ, more precisely satisfying

$$\zeta < \frac{1}{8}. \tag{2.27}$$

We prove the following result:

Proposition 2.10. *Under the above hypotheses, there exists a phase $\theta_0 \in \mathbb{T}$, an event B^N such that $\mathbf{P}(B^N) \to 1$ and a constant $c > 0$ such that for all $\varepsilon > 0$, for N sufficient large, on the event B^N, the projection $\psi_0 = \psi_0^N = \mathrm{proj}_M(\mu_{N,N^{1/2}t_0^N})$ is well-defined and*

$$\|\mu_{N,N^{1/2}t_0^N} - q_{\psi_0}\|_{-1,d} \leq N^{-1/2+2\zeta}, \tag{2.28}$$

and

$$|\psi_0 - \theta_0| \leq \varepsilon, \tag{2.29}$$

where $t_0^N = cN^{-1/2}\log N$.

We refer to Section 6 for a proof of this result. Since it relies on a discretization scheme similar to the one we introduce in the next paragraph, we leave the details to Section 6.

2.6. *Dynamics on the manifold M*

We now place ourselves on the event B^N (see Proposition 2.10), so that on the time $N^{1/2}t_0^N$ we have $\|\mu_{N,N^{1/2}t_0^N} - q_{\psi_0}\|_{-1,d} \leq N^{-1/2+2\zeta}$ where $\psi_0 = \mathrm{proj}_M(\mu_{N,N^{1/2}t_0^N})$. The point is to analyse the dynamics of (2.22) on a time scale of order $N^{1/2}$, using the knowledge we have on stability of the manifold M (recall (1.12)). The following iterative scheme we introduce is similar to ones used in [7,12].

The iterative scheme

We divide the evolution of the dynamics (2.22) in time intervals $[T_n, T_{n+1}]$ with $T_n = N^{1/2}t_0^N + nT$ where T is a constant independent of N, satisfying $T \geq 1$ and

$$e^{-\gamma_L T} \leq \frac{1}{4C_L C_P}, \tag{2.30}$$

where the constants C_L and C_P where introduced in Section 2.3. This hypothesis means in some sense that the size of the times intervals T is chosen large enough so that, starting at time T_n, the contracting properties of the limit dynamics given by (1.7) around M can be observed at time T_{n+1}. This will be useful in Section 5.

We define these time intervals until we reach the final time $N^{1/2}t_f$, and the number of steps n_f is thus given by

$$n_f := \left\lfloor \frac{N^{1/2}}{T}\big(t_f - t_0^N\big)\right\rfloor + 1. \tag{2.31}$$

The intuition of this discretization is the following: if for a certain $n = 0, 1, \dots, n_f$, the process $\mu_{T_n} = \mu_{N,T_n}$ is close enough to the manifold M, we can define the phase α_n of its projection on M by:

$$\alpha_n := \mathrm{proj}_M(\mu_{N,T_n}). \tag{2.32}$$

This projection is in particular well defined when $\mathrm{dist}_{H_d^{-1}}(\mu_{N,T_n}, M) \le \sigma$, where the constant $\sigma > 0$ is given by Lemma 2.9. To be allowed to define these projections for all $n = 0, 1, \dots, n_f$, we stop the process before it escapes too far from M if required. To do so, we introduce the stopping time

$$\iota := \inf\big\{u \in \big[t_0^N N^{1/2}, t_f N^{1/2}\big], \mathrm{dist}_{H_d^{-1}}(\mu_{N,u}, M) \ge \sigma\big\}, \tag{2.33}$$

and then consider the random phases ψ_{n-1} defined as

$$\psi_n := \mathrm{proj}_M(\mu_{N,T_n\wedge\iota}), \quad n = 0, \dots, n_f. \tag{2.34}$$

We will not work directly with $\mu_{N,u\wedge\iota}$, but rather study the behaviour of the difference $\mu_{N,u\wedge\iota} - q_{\psi_{n-1}}$ on each interval $[T_{n-1}, T_n]$. We consider thus the family of processes $\nu_{N,n,t}$ defined for $n = 1, \dots, n_f$ and $t \in [0, T]$ by

$$\nu_{N,n,t} := \mu_{N,(T_{n-1}+t)\wedge\iota} - q_{\psi_{n-1}}. \tag{2.35}$$

To have stopping times better adapted to the iteratively defined processes $\nu_{N,n,t}$ (with value in $[0, T]$ instead of $[t_0^N N^{1/2}, t_f N^{1/2}]$), we will also use, rather than the stopping time ι, the following stopping couple (where the infimum corresponds to the lexicographic order):

$$(n_\tau, \tau) = \inf\big\{(n, t) \in \{1, \dots, n_f\} \times [0, T] : \mathrm{dist}_{H_d^{-1}}(\mu_{N,T_{n-1}+t}, M) \ge \sigma\big\}, \tag{2.36}$$

and the sequence of stopping times $(\tau^n, n = 1 \dots n_f)$:

$$\tau^n := \begin{cases} T & \text{if } n < n_\tau, \\ \tau & \text{if } n = n_\tau, \\ 0 & \text{if } n > n_\tau. \end{cases} \tag{2.37}$$

It is clear that with these new notations $\iota = T_{n^\tau - 1} + \tau$. Moreover, using (2.22), (2.23) and (2.24), we see that for $n = 1, \dots, n_f$ the process $\nu_{N,n,t}$ satisfies the mild equation

$$\begin{aligned}\nu_{N,n,t} = e^{(t\wedge\tau^n)L_{\psi_{n-1}}}\nu_{N,n,0} - \int_0^{t\wedge\tau^n} e^{(t\wedge\tau^n - s)L_{\psi_{n-1}}}\big(D_{N,\psi_{n-1}} + R_{N,\psi_{n-1},\delta}(\nu_{N,n})\big)\,\mathrm{d}s \\ + Z_{N,n,t\wedge\tau^n},\end{aligned} \tag{2.38}$$

where $Z_{N,n,t}$ is defined as

$$Z_{N,n,t}(h) := \sum_{i=-d}^{d} \frac{\lambda^i}{N^i} \sum_{j=1}^{N^i} \int_0^t \partial_\theta\big[\big(e^{(t-s)L^*_{\psi_{n-1}}}h\big)^i\big]\big(\varphi_j^i(T_{n-1}+s)\big)\,\mathrm{d}B_j^i(T_{n-1}+s). \tag{2.39}$$

Remark in particular that when $n > n_\tau$ (so when the process has been stopped), (2.38) simply becomes $\nu_{N,n,t} = \nu_{N,n,0}$ since $\tau^n = 0$.

Remark 2.11. *Whenever there is no confusion, we will drop for simplicity the dependence in N and δ of the discretized processes defined above. Hence, we will write $\nu_{n,t}$ in place of $\nu_{N,n,t}$, $D_{\psi_{n-1}}$ in place of $D_{N,\psi_{n-1},\delta}$ (recall (2.23)), $R_{n,\psi_{n-1}}$ in place of $R_{N,\psi_{n-1},\delta}(\nu_{N,n})$ (recall (2.24)) and $Z_{n,t}$ in place of $Z_{N,n,t}$.*

Controlling the noise and a priori bound on the fluctuation process
A key point in the analysis of (2.38) is to show that one can control the behavior of the noise part $Z_{n,t}$ in (2.38) along the discretization introduced in the last paragraph. More precisely, for ζ chosen according to (2.27) and some positive constant C_Z and defining the event

$$A^N = A^N(C_Z) := \left\{ \sup_{1\le n\le n_f} \sup_{0\le t\le T} \|Z_{n,t}\|_{-1,d} \le C_Z \sqrt{\frac{T}{N}} N^{\zeta} \right\}, \tag{2.40}$$

the purpose of Section 4 is precisely to prove that $\mathbf{P}(A^N)$ tends to 1 as $N \to \infty$. With the knowledge of (2.40), one can prove that the process ν_n remains a priori bounded: using that the sequence of the disorder $(\omega_i)_{i\ge 1}$ is admissible (recall Definition 2.1), we prove in Proposition 5.1, Section 5, that on the event $A^N \cap B^N$,

$$\sup_{1\le n\le n_f} \sup_{t\in[0,T]} \|\nu_{n,t}\|_{-1,d} = O\big(N^{-1/2+2\zeta}\big), \tag{2.41}$$

as $N \to \infty$.

Expansion of the dynamics on the manifold M
The last step of the proof consists in looking at the rescaled dynamics of the phase of the projection of the empirical measure on M, that is the process

$$\Psi_t^N := \psi_{n_t}, \tag{2.42}$$

where $(\psi_n)_{0\le n\le n_f}$ is given by (2.34) and

$$n_t := \left\lfloor \frac{N^{1/2}}{T}\big(t - t_0^N\big) \right\rfloor. \tag{2.43}$$

Namely, we prove in Propositions 5.2 and 5.3 that, with high probability as $N \to \infty$, the following expansion holds:

$$\Psi_t^N = \psi_0 + b(\xi_N)t + O\big(N^{-1/4+2\zeta}\big), \tag{2.44}$$

where b is the linear form of Theorem 2.4 and that $\mu_{N,N^{1/2}t}$ is close to $q_{\Psi_t^N}$ with high probability.

2.7. *Organization of the rest of the paper*

Section 3 is devoted to prove the mild formulation described in Paragraph 2.4. The control of the discretized noise term in (2.39) is addressed in Section 4. The dynamics on the manifold M and the approach to the manifold are studied in Sections 5 and 6 respectively. The asymptotics of the drift as $\delta \to 0$ is studied in Section 7. We compile in Appendices A–D several spectral estimates and expansions in small δ used throughout the paper.

3. Proof of the mild formulation

Define $\mathbf{L}^2_{0,d}$ as the closure of the space of regular test functions $(u^{-d}, \dots, u^d)$ such that $\int_{\mathbb{T}} u^k = 0$ for all $k = -d, \dots, d$ under the norm

$$\|u\|_{0,d} := \left(\sum_{k=-d}^{d} \lambda^k \int_{\mathbb{T}} u^k(\theta)^2 \,\mathrm{d}\theta \right)^{1/2}, \tag{3.1}$$

and the space H_d^α ($\alpha \geq 0$) closure of the same set of test functions under the norm (denoting $\|\cdot\|_0$ the L^2-norm on $\mathbb{T}$)

$$\|u\|_{\alpha,d} := \left\|(-\Delta_d)^{\alpha/2} u\right\|_{0,d} = \left(\sum_{k=-d}^{d} \lambda^k \left\|(-\Delta)^{\alpha/2} u^k\right\|_0^2\right)^{1/2}, \tag{3.2}$$

where Δ_d denotes the Laplacian on $\mathbb{T}^{2d+1}$. We denote by $H_d^{-\alpha}$ the dual space of H_d^α. We also write, for any bounded signed measure m on $\mathbb{T}$, the usual distribution bracket as

$$\langle m, f\rangle := \int_{\mathbb{T}} f(\theta) m(\mathrm{d}\theta),$$

and for any vector $(m^1, \dots, m^d)$ of such measures

$$\langle m, F\rangle_d := \sum_{i=-d}^{d} \lambda^i \langle m^i, F^i\rangle = \sum_{i=-d}^{d} \lambda^i \int_{\mathbb{T}} F^i(\theta) m^i(\mathrm{d}\theta),$$

the corresponding bracket weighted w.r.t. the disorder. Obviously, when the above measure coincide with an L^2 function, this expression coincides with the L^2 scalar product $\langle\cdot,\cdot\rangle_{2,d}$ associated to (3.1).

This section is devoted to prove Proposition 2.8. We begin first with a weak formulation of the SPDE (2.22).

Proposition 3.1. *For all $K > 1$, for all $0 \leq \delta \leq \delta(K)$, for any $(t,\theta) \mapsto F_t(\theta) = (F_t^{-d}(\theta), \dots, F_t^d(\theta)) \in \mathcal{C}^{1,2}([0,+\infty) \times \mathbb{T}, \mathbb{R})$ such that $\int_{\mathbb{T}} F_t(\theta)\,\mathrm{d}\theta = 0$,*

$$\begin{aligned}\langle \nu_{N,t}, F_t\rangle_d = \langle \nu_{N,0}, F_0\rangle_d &+ \int_0^t \left\langle \nu_{N,s}, \partial_s F_s + L_{\psi,\delta}^* F_s\right\rangle_d \mathrm{d}s + \int_0^t \langle D_N, F_s\rangle_d \,\mathrm{d}s \\ &+ \int_0^t \left\langle R_N(\nu_{N,s}), \partial_\theta F_s\right\rangle_d \mathrm{d}s + M_{N,t}^F, \quad N \geq 1, t \geq 0,\end{aligned} \tag{3.3}$$

where D_N, $R_N(\nu_N)$ are respectively defined in (2.23) and (2.24) and

$$M_{N,t}^F := \sum_{i=-d}^{d} \frac{\lambda^i}{N^i} \sum_{j=1}^{N^i} \int_0^t \partial_\theta F_s^i\big(\varphi_j^i(s)\big)\,\mathrm{d}B_j^i(s). \tag{3.4}$$

In (3.3), the operator $L_{\psi,\delta}^*$ is the dual in $\mathbf{L}_{0,d}^2$ of the operator $L_{\psi,\delta}$ in (2.14):

$$\begin{aligned}\big(L_{\psi,\delta}^* v\big)^i = \frac{1}{2}\partial_\theta^2 v^i + \delta\omega^i \partial_\theta v^i + \big(\partial_\theta v^i\big) \sum_{k=-d}^{d} \lambda^k J * q_{\psi,\delta}^k - \int_{\mathbb{T}} \left(\big(\partial_\theta v^i\big) \sum_{k=-d}^{d} \lambda^k J * q_{\psi,\delta}^k\right) \mathrm{d}\theta \\ - \sum_{k=-d}^{d} \lambda^k J * \big(q_{\psi,\delta}^k \partial_\theta v^k\big).\end{aligned} \tag{3.5}$$

We refer to Appendix B (see in particular Propositions B.3 and B.4) for a detailed analysis of the spectral properties of the operator $L_{\psi,\delta}$ and its dual $L_{\psi,\delta}^*$. All we need to retain here is that when δ is small, the operator $L_{\psi,\delta}$ is sectorial in H_d^{-1} and generates a $\mathcal{C}_0$-semi-group $t \mapsto e^{tL_{\psi,\delta}}$ in this space. Moreover, on the space $\mathbf{L}_{0,d}^2$, one has that $(e^{tL_{\psi,\delta}})^* = e^{tL_{\psi,\delta}^*}$. Since the phase ψ is not relevant in this paragraph, we write for simplicity q_δ, L_δ instead of $q_{\psi,\delta}$ and $L_{\psi,\delta}$.

Proof of Proposition 3.1. Note that, using the definition of $J(\cdot)$ in (1.4) and of the empirical measure $\mu_{N,t}$ in (1.6), the system (1.5) may be rewritten as

$$\mathrm{d}\varphi_j^i(t) = \delta\omega^i\,\mathrm{d}t + \sum_{k=-d}^{d} \lambda_N^k J * \mu_t^k\big(\varphi_j^i(t)\big)\,\mathrm{d}t + \mathrm{d}B_j^i(t), \quad i = -d, \dots, d. \tag{3.6}$$

Consider $(t,\theta) \mapsto F_t(\theta) = (F_t^i(\theta))_{i=-d,\dots,d} \in C^{1,2}([0,+\infty) \times \mathbb{T}, \mathbb{R})^{2d+1}$ such that for all $t \geq 0$, $\int_{\mathbb{T}} F_t(\theta)\,\mathrm{d}\theta = 0$. An application of Itô Formula to (1.5) gives, for $i = -d, \dots, d$, $j = 1, \dots, N^i$, $t \geq 0$,

$$\begin{aligned} F_t^i\big(\varphi_j^i(t)\big) &= F_0^i\big(\varphi_j^i(0)\big) + \int_0^t \partial_s F_s^i\big(\varphi_j^i(s)\big)\,\mathrm{d}s + \frac{1}{2}\int_0^t \partial_\theta^2 F_s^i\big(\varphi_j^i(s)\big)\,\mathrm{d}s \\ &\quad + \int_0^t \partial_\theta F_s^i\big(\varphi_j^i(s)\big)\left(\delta\omega^i + \sum_{k=-d}^{d} \lambda_N^k J * \mu_{N,s}^k\big(\varphi_j^i(s)\big)\right)\mathrm{d}s + \int_0^t \partial_\theta F_s^i\big(\varphi_j^i(s)\big)\,\mathrm{d}B_j^i(s). \end{aligned}$$

After summation over $j = 1, \dots, N^i$, we obtain, for $i = -d, \dots, d$,

$$\begin{aligned} \big\langle \mu_{N,t}^i, F_t^i \big\rangle &= \big\langle \mu_{N,0}^i, F_0^i \big\rangle + \int_0^t \left\langle \mu_{N,s}^i, \partial_s F_s^i + \frac{1}{2}\partial_\theta^2 F_s^i + \partial_\theta F_s^i \left(\delta\omega^i + \sum_{k=-d}^{d} \lambda_N^k \big(J * \mu_{N,s}^k\big)\right)\right\rangle \mathrm{d}s \\ &\quad + \frac{1}{N^i}\sum_{j=1}^{N^i} \int_0^t \partial_\theta F_s^i\big(\varphi_j^i(s)\big)\,\mathrm{d}B_j^i(s). \end{aligned} \tag{3.7}$$

Replacing $\mu_{N,t}^i$ by $\nu_{N,t}^i + q_\delta^i$ in (3.7) (recall (2.21)), we obtain

$$\begin{aligned} \big\langle \nu_{N,t}^i, F_t^i \big\rangle + \big\langle q_\delta^i, F_t^i \big\rangle &= \big\langle \nu_{N,0}^i, F_0^i \big\rangle + \big\langle q_\delta^i, F_0^i \big\rangle \\ &\quad + \int_0^t \left\langle \nu_{N,s}^i + q_\delta^i, \partial_s F_s^i + \frac{1}{2}\partial_\theta^2 F_s^i + \partial_\theta F_s^i \left(\delta\omega^i + \sum_{k=-d}^{d} \lambda_N^k J * \big(\nu_{N,s}^k + q_\psi^k\big)\right)\right\rangle \mathrm{d}s \\ &\quad + \frac{1}{N^i}\sum_{j=1}^{N^i} \int_0^t \partial_\theta F_s^i\big(\varphi_j^i(s)\big)\,\mathrm{d}B_j^i(s) \\ &= \big\langle \nu_{N,0}^i, F_0^i \big\rangle + \int_0^t \left\langle \nu_{N,s}^i, \partial_s F_s^i + \frac{1}{2}\partial_\theta^2 F_s^i + \partial_\theta F_s^i \left(\delta\omega^i + \sum_{k=-d}^{d} \lambda_N^k J * q_\delta^k\right)\right\rangle \mathrm{d}s \\ &\quad + \int_0^t \left\langle \nu_{N,s}^i, \partial_\theta F_s^i \sum_{k=-d}^{d} \lambda_N^k \big(J * \nu_{N,s}^k\big)\right\rangle \mathrm{d}s + \int_0^t \left\langle q_\delta^i, \partial_\theta F_s^i \sum_{k=-d}^{d} \lambda_N^k \big(J * \nu_{N,s}^k\big)\right\rangle \mathrm{d}s \\ &\quad + \big\langle q_\delta^i, F_0^i \big\rangle + \int_0^t \left\langle q_\delta^i, \partial_s F_s^i + \frac{1}{2}\partial_\theta^2 F_s^i + \partial_\theta F_s^i \left(\delta\omega^i + \sum_{k=-d}^{d} \lambda_N^k J * q_\delta^k\right)\right\rangle \mathrm{d}s \\ &\quad + \frac{1}{N^i}\sum_{j=1}^{N^i} \int_0^t \partial_\theta F_s^i\big(\varphi_j^i(s)\big)\,\mathrm{d}B_j^i(s). \end{aligned} \tag{3.8}$$

Since by definition q_δ is a stationary solution to (1.7), one easily sees that

$$\big\langle q_\delta^i, F_t^i \big\rangle = \big\langle q_\delta^i, F_0^i \big\rangle + \int_0^t \big\langle q_\delta^i, \partial_s F_s^i \big\rangle \mathrm{d}s, \quad i = -d, \dots, d, t \geq 0, \tag{3.9}$$

and

$$0 = \left\langle \frac{1}{2}\partial_\theta^2 q_\delta^i - \delta\omega^i \partial_\theta q_\delta^i - \partial_\theta\left(q_\delta^i \sum_{k=-d}^{d} \lambda^k J * q_\delta^k\right), F_s^i \right\rangle$$
$$= \left\langle q_\delta^i, \frac{1}{2}\partial_\theta^2 F_s^i + \delta\omega^i \partial_\theta F_s^i + \partial_\theta F_s^i \sum_{k=-d}^{d} \lambda^k J * q_\delta^k \right\rangle. \tag{3.10}$$

Summing (3.8) over $i = -d, \dots, d$ and using (3.9) and (3.10), we obtain

$$\begin{aligned}\langle \nu_{N,t}, F_t\rangle_d = \langle \nu_{N,0}, F_0\rangle_d &+ \int_0^t \left\langle \nu_{N,s}, \partial_s F_s + \frac{1}{2}\partial_\theta^2 F_s + \delta\partial_\theta F_s \otimes w + \partial_\theta F_s \sum_{k=-d}^{d} \lambda_N^k J * q_\delta^k \right\rangle_d \mathrm{d}s \\ &+ \int_0^t \left\langle \nu_{N,s}, \partial_\theta F_s \sum_{k=-d}^{d} \lambda_N^k J * \nu_{N,s}^k \right\rangle_d \mathrm{d}s + \int_0^t \left\langle q_\delta, \partial_\theta F_s \sum_{k=-d}^{d} \lambda_N^k J * \nu_{N,s}^k \right\rangle_d \mathrm{d}s \\ &+ \int_0^t \left\langle q_\delta \sum_{k=-d}^{d} \left(\lambda_N^k - \lambda^k\right) J * q_\delta^k, \partial_\theta F_s \right\rangle_d \mathrm{d}s + M_{N,t}^F,\end{aligned} \tag{3.11}$$

where $M_{N,t}^F$ is defined in (3.4) and where we have used the notation $F \otimes \omega = (F^i\omega^i)_{i_1,\dots,d}$. Note that

$$\begin{aligned}\left\langle q_\delta, \partial_\theta F_s \sum_{k=-d}^{d} \lambda^k J * \nu_{N,s}^k \right\rangle_d &= \sum_{i=-d}^{d}\sum_{k=-d}^{d} \lambda^i\lambda^k \left\langle q_\delta^i, \partial_\theta F_s^i J * \nu_{N,s}^k \right\rangle \\ &= -\sum_{i=-d}^{d}\sum_{k=-d}^{d} \lambda^i\lambda^k \left\langle \nu_{N,s}^i, J * \left(q_\delta^i \partial_\theta F_s^i\right) \right\rangle \\ &= -\left\langle \nu_s, \sum_{k=-d}^{d} \lambda^k J * \left(q_\delta^k \partial_\theta F_s^k\right) \right\rangle_d.\end{aligned} \tag{3.12}$$

The result of Proposition 3.1 is a simple reformulation of (3.11) using (3.12) and the definition of L_δ^* in (3.5). □

In order to prove Proposition 2.8, we need to have some a priori control on the noise term in (2.22). This the purpose of Lemma 3.2:

Lemma 3.2. *The process $(Z_{N,t,t'})_{0<t<t'}$ defined in (2.25) satisfies the following estimate: for all $\varepsilon \in (0, \frac{1}{4})$ and all integer $m > 0$, there exists a positive constant $C_{m,\varepsilon}$ such that for all $0 < s' < s < t$, $s' < t' < t$,*

$$\mathbf{E}\left(\|Z_{N,t,t'} - Z_{N,s,s'}\|_{-1,d}^{2m}\right) \le \frac{C_{m,\varepsilon}}{N^m}\left((t-s)^{m(1/2-2\varepsilon)} + \left(t'-s'\right)^{m(1/2-\varepsilon)} + \left(t'-s'\right)^m\right). \tag{3.13}$$

Consequently, the almost-sure limit $Z_{N,t}$ of $Z_{N,t,t'}$ when $t' \nearrow t$ (recall (2.26)) exists in H_d^{-1} and $t \mapsto Z_{N,t}$ defines a continuous process in H_d^{-1}. Moreover, one has the estimate

$$\mathbf{E}\|Z_{N,t} - Z_{N,s}\|_{-1,d}^{2m} \le \frac{C_{m,\varepsilon}}{N^m}\left((t-s)^{m(1/2-2\varepsilon)} + (t-s)^m\right). \tag{3.14}$$

Remark 3.3. *Note that the estimates (3.13) and (3.14) are uniform in the choice of ψ, and in the starting configuration $(\varphi_j^i)_{i=-d,\dots,d;j=1,\dots,N^i}$.*

Proof of Lemma 3.2. We follow mostly here the ideas of [21]. Recall the definition of the process $Z_{N,t,t'}$ in (2.25): for $0 < t' < t$

$$Z_{N,t,t'}(h) = \sum_{i=-d}^{d} \frac{\lambda^i}{N^i} \sum_{j=1}^{N^i} \int_0^{t'} \partial_\theta\big[\big(e^{(t-s)L_\psi^*} h\big)^i\big]\big(\varphi_j^i(s)\big)\, \mathrm{d}B_j^i(s). \tag{3.15}$$

For $0 < s' < s < t$, $s' < t' < t$, we can decompose $Z_{N,t,t'} - Z_{N,s,s'}$ as follows:

$$Z_{N,t,t'} - Z_{N,s,s'} = M^1_{N,s',s,t} + M^2_{N,s',t',t}, \tag{3.16}$$

where

$$M^1_{N,s',s,t}(h) = \sum_{i=-d}^{d} \frac{\lambda^i}{N^i} \sum_{j=1}^{N^i} \int_0^{s'} \partial_\theta\big[\big(\big(e^{(t-u)L_\psi^*} - e^{(s-u)L_\psi^*}\big)h\big)^i\big]\big(\varphi_j^i(u)\big)\, \mathrm{d}B_j^i(u), \tag{3.17}$$

and

$$M^2_{N,s',t',t}(h) = \sum_{i=-d}^{d} \frac{\lambda^i}{N^i} \sum_{j=1}^{N^i} \int_{s'}^{t'} \partial_\theta\big[\big(e^{(t-u)L_\psi^*} h\big)^i\big]\big(\varphi_j^i(u)\big)\, \mathrm{d}B_j^i(u). \tag{3.18}$$

The processes $(M^1_{N,s',s,t}(h))_{s'\in[0,s)}$ and $(M^2_{N,s',t',t}(h))_{t'\in(s',t)}$ are martingales, with Itô brackets

$$\big[M^1_{N,\cdot,s,t}(h)\big]_{s'} = \sum_{i=-d}^{d} \sum_{j=1}^{N^i} \int_0^{s'} \big(U^{1,i,j}_{N,u,s,t}(h)\big)^2\, \mathrm{d}u, \tag{3.19}$$

and

$$\big[M^2_{N,s',\cdot,t}(h)\big]_{t'} = \sum_{i=-d}^{d} \sum_{j=1}^{N^i} \int_{s'}^{t'} \big(U^{2,i,j}_{N,u,t}(h)\big)^2\, \mathrm{d}u, \tag{3.20}$$

where we have used the notations

$$U^{1,i,j}_{N,u,s,t}(h) = \frac{\lambda^i}{N^i} \partial_\theta\big[\big(\big(e^{(t-u)L_\psi^*} - e^{(s-u)L_\psi^*}\big)h\big)^i\big]\big(\varphi_j^i(u)\big), \tag{3.21}$$

and

$$U^{2,i,j}_{N,u,t}(h) = \frac{\lambda^i}{N^i} \partial_\theta\big[\big(e^{(t-u)L_\psi^*} h\big)^i\big]\big(\varphi_j^i(u)\big). \tag{3.22}$$

Let $(h_l)_{l\geq 1}$ be a complete orthonormal basis in H^1_d. Using Parseval's identity, we obtain

$$\begin{aligned}
\mathbf{E}\|Z_{N,t,t'} - Z_{N,s,s'}\|^2_{-1,d} &= \sum_{l=1}^{\infty} \mathbf{E}\big|(Z_{N,t,t'} - Z_{N,s,s'})(h_l)\big|^2 \\
&\leq 2\sum_{l=1}^{\infty} \mathbf{E}\big|M^1_{N,s',s,t}(h_l)\big|^2 + 2\sum_{l=1}^{\infty} \mathbf{E}\big|M^2_{N,s',t',t}(h_l)\big|^2 \\
&\leq 2\sum_{l=1}^{\infty} \sum_{i=-d}^{d} \sum_{j=1}^{N^i} \int_0^{s'} \mathbf{E}\big(U^{1,i,j}_{N,u,s,t}(h_l)\big)^2\, \mathrm{d}u + 2\sum_{l=1}^{\infty} \sum_{i=-d}^{d} \sum_{j=1}^{N^i} \int_{s'}^{t'} \mathbf{E}\big(U^{2,i,j}_{N,u,t}(h_l)\big)^2\, \mathrm{d}u \\
&\leq 2\sum_{i=-d}^{d} \sum_{j=1}^{N^i} \int_0^{s'} \mathbf{E}\big\|U^{1,i,j}_{N,u,s,t}\big\|^2_{-1,d}\, \mathrm{d}u + 2\sum_{i=-d}^{d} \sum_{j=1}^{N^i} \int_{s'}^{t'} \mathbf{E}\big\|U^{2,i,j}_{N,u,t}\big\|^2_{-1,d}\, \mathrm{d}u.
\end{aligned} \tag{3.23}$$

For $m > 1$, we have

$$\begin{aligned}\mathbf{E}\|Z_{N,t,t'} - Z_{N,s,s'}\|_{-1,d}^{2m} &= \mathbf{E}\left(\sum_{l=1}^{\infty}\left|(Z_{N,t,t'} - Z_{N,s,s'})(h_l)\right|^2\right)^m \\ &\le m\mathbf{E}\left(\sum_{l=1}^{\infty}\left|M^1_{N,s',s,t}(h_l)\right|^2\right)^m + m\mathbf{E}\left(\sum_{l=1}^{\infty}\left|M^2_{N,s',t',t}(h_l)\right|^2\right)^m,\end{aligned} \tag{3.24}$$

and using Hölder and Burkholder–Davis–Gundy inequalities, we obtain for the terms involving M^1

$$\begin{aligned}\mathbf{E}\left(\sum_{l=1}^{\infty}\left|M^1_{N,s',s,t}(h_l)\right|^2\right)^m &= \sum_{l_1,l_2,\dots,l_m=1}^{\infty}\mathbf{E}\left|M^1_{N,s',s,t}(h_{l_1})\right|^2\cdots\left|M^1_{N,s',s,t}(h_{l_m})\right|^2 \\ &\le \sum_{l_1,l_2,\dots,l_m=1}^{\infty}\left(\mathbf{E}\left|M^1_{N,s',s,t}(h_{l_1})\right|^{2m}\right)^{1/m}\cdots\left(\mathbf{E}\left|M^1_{N,s',s,t}(h_{l_m})\right|^{2m}\right)^{1/m} \\ &\le C_m\sum_{l_1,l_2,\dots,l_m=1}^{\infty}\mathbf{E}\left[M^1_{N,\cdot,s,t}(h_{l_1})\right]_{s'}\cdots\mathbf{E}\left[M^1_{N,\cdot,s,t}(h_{l_m})\right]_{s'} \\ &\le C_m\sum_{l_1,l_2,\dots,l_m=1}^{\infty}\left(\sum_{i=-d}^{d}\sum_{j=1}^{N^i}\int_0^{s'}\mathbf{E}\left(U^{1,i,j}_{N,u,s,t}(h_{l_1})\right)^2\mathrm{d}u\right)\cdots \\ &\qquad\cdots\left(\sum_{i=-d}^{d}\sum_{j=1}^{N^i}\int_0^{s'}\mathbf{E}\left(U^{1,i,j}_{N,u,s,t}(h_{l_m})\right)^2\mathrm{d}u\right) \\ &= C_m\left(\sum_{l=1}^{\infty}\sum_{i=-d}^{d}\sum_{j=1}^{N^i}\int_0^{s'}\mathbf{E}\left(U^{1,i,j}_{N,u,s,t}(h_l)\right)^2\mathrm{d}u\right)^m \\ &= C_m\left(\sum_{i=-d}^{d}\sum_{j=1}^{N^i}\int_0^{s'}\mathbf{E}\left\|U^{1,i,j}_{N,u,s,t}\right\|^2_{-1,d}\mathrm{d}u\right)^m.\end{aligned} \tag{3.25}$$

The same work can be done for the terms involving M^2, which leads to

$$\begin{aligned}\mathbf{E}\|Z_{N,t,t'} - Z_{N,s,s'}\|_{-1,d}^{2m} &\le C'_m\left(\sum_{i=-d}^{d}\sum_{j=1}^{N^i}\int_0^{s'}\mathbf{E}\left\|U^{1,i,j}_{N,u,s,t}\right\|^2_{-1,d}\mathrm{d}u\right)^m \\ &\quad + C'_m\left(\sum_{i=-d}^{d}\sum_{j=1}^{N^i}\int_{s'}^{t'}\mathbf{E}\left\|U^{2,i,j}_{N,u,t}\right\|^2_{-1,d}\mathrm{d}u\right)^m.\end{aligned} \tag{3.26}$$

It remains now to find appropriate bounds for $\mathbf{E}\|U^{1,i,j}_{N,u,s,t}\|^2_{-1,d}$ and $\mathbf{E}\|U^{2,i,j}_{N,u,t}\|^2_{-1,d}$. On one hand, for $h \in H^1_d$, since $\delta_{\theta_0} \in H^{-1/2-\varepsilon}$ for all $\varepsilon > 0$, we have

$$\left|U^{2,i,j}_{N,u,t}(h)\right| \le \frac{C}{N^i}\left\|\partial_\theta\left[\left(e^{(t-u)L^*_\delta}h\right)^i\right]\right\|_{1/2+\varepsilon,d} \le \frac{C}{N^i}\left\|\left(e^{(t-u)L^*_\delta}h\right)^i\right\|_{3/2+\varepsilon,d}. \tag{3.27}$$

Applying Proposition B.6 with $\beta = 1/4 + \varepsilon/2$, we obtain, for any $0 < \gamma < \gamma_{L^*_\delta}$,

$$\left|U^{2,i,j}_{N,u,t}(h)\right| \le \frac{C}{N^i}\left(1 + e^{-\gamma(t-u)}(t-u)^{-1/4-\varepsilon/2}\right)\|h\|_{1,d}, \tag{3.28}$$

which means that $\|U^{2,i,j}_{N,u,t}\|_{-1,d} \le \frac{C}{N^i}(1+e^{-\gamma(t-u)}(t-u)^{-1/4-\varepsilon/2})$. On the other hand, proceeding as before, we get the bound:

$$\left|U^{1,i,j}_{N,u,s,t}(h)\right| \le \frac{C}{N^i}\left\|\left(\left[e^{(t-s)L^*_\delta}-1\right]e^{(s-u)L^*_\delta}h\right)^i\right\|_{3/2+\varepsilon,d}. \tag{3.29}$$

Applying Proposition B.6 with $\beta' = 1/4+\varepsilon/2$ and $\beta = 1/4-\varepsilon$, we get for all $\widetilde{h} \in H^{2-\varepsilon}_d$,

$$\left\|\left[e^{(t-s)L^*_\delta}-1\right]\widetilde{h}\right\|_{3/2+\varepsilon,d} \le C_\varepsilon (t-s)^{1/4-\varepsilon}\left\|\left(1-P^{0,*}\right)\widetilde{h}\right\|_{2-\varepsilon,d}. \tag{3.30}$$

For $\widetilde{h} = e^{(s-u)L^*_\delta}h$ and using again Proposition B.6 with this time $\beta = 1/2-\varepsilon/2$, this leads to

$$\left|U^{1,i,j}_{N,u,s,t}(h)\right| \le \frac{C}{N^i}(t-s)^{1/4-\varepsilon}(s-u)^{-1/2+\varepsilon/2}e^{-\gamma(s-u)}\|h\|_{1,d}, \tag{3.31}$$

which means that $\|U^{1,i,j}_{N,u,s,t}\|_{-1,d} \le \frac{C}{N^i}(t-s)^{1/4-\varepsilon}(s-u)^{-1/2+\varepsilon/2}e^{-\gamma(s-u)}$. We can now estimate (3.26): using that $N \le cN^i \le CN$, we obtain

$$\begin{aligned}
\mathbf{E}\|Z_{N,t,t'} - Z_{N,s,s'}\|^{2m}_{-1,d} &\le \frac{C''_m}{N^m}\left((t-s)^{1/2-2\varepsilon}\int_0^{s'}(s-u)^{-1+2\varepsilon}e^{-2\gamma(s-u)}\,\mathrm{d}u\right)^m \\
&\quad + \frac{C''_m}{N^m}\left(\int_{s'}^{t'}\left(1+e^{-2\gamma(t-u)}(t-u)^{-1/2-\varepsilon}\right)\mathrm{d}u\right)^m \\
&\le \frac{C'''_m}{N^m}\left((t-s)^{m(1/2-2\varepsilon)} + \left(t'-s'\right)^{m(1/2-\varepsilon)} + \left(t'-s'\right)^m\right)
\end{aligned}$$

which proves (3.13). One deduces from (3.13) and an application of the Kolmogorov Lemma that the almost-sure limit when $t' \nearrow t$ of $Z_{N,t,t'}$ defined in (2.26) exists in H^{-1}_d. Taking $t' \nearrow t$ and $s' \nearrow s$ and using Fatou Lemma, we deduce (3.14). The continuity of the limiting process $t \mapsto Z_{N,t}$ in H^{-1}_d is an easy consequence of (3.14). □

We are now in position to prove Proposition 2.8:

Proof of Proposition 2.8. Let us apply the identity (3.3) of Proposition 3.1 in the case of test functions F_s of the form

$$F_s = e^{(t-s)L^*_\delta}h,$$

for any test functions h of class $\mathcal{C}^2$ on $\mathbb{T}$. Then $\partial_s F_s = -L^*_\delta F_s$ and one obtains

$$\begin{aligned}
\langle \nu_{N,t}, h\rangle_d &= \left\langle \nu_{N,0}, e^{tL^*_\delta}h\right\rangle_d + \int_0^t \left\langle D_N, e^{(t-s)L^*_\delta}h\right\rangle_d \mathrm{d}s + \int_0^t \left\langle R_N(\nu_{N,s}), \partial_\theta e^{(t-s)L^*_\delta}h\right\rangle_d \mathrm{d}s \\
&\quad + M^F_{N,t}.
\end{aligned} \tag{3.32}$$

We aim at proving that one can write a mild version of this weak equation and that this mild formulation makes sense in H^{-1}_d. Consider a sequence $(v_l)_{l\ge 1}$ of elements of $\mathbf{L}^2_{0,d}$ converging as $l \to \infty$ in H^{-1}_d to $\nu_{N,0} \in H^{-1}_d$. Then, for h of class $\mathcal{C}^2$,

$$\left\langle v_l, e^{tL^*_\delta}h\right\rangle_d = \left\langle v_l, e^{tL^*_\delta}h\right\rangle_{2,d} = \left\langle e^{tL_\delta}v_l, h\right\rangle_{2,d} = \left\langle e^{tL_\delta}v_l, h\right\rangle_d. \tag{3.33}$$

By continuity of e^{tL_δ} on H^{-1}_d, $e^{tL_\delta}v_l$ converges in H^{-1}_d to $e^{tL_\delta}\nu_{N,0}$, as $l \to \infty$. In particular, for all $h \in H^1_d$,

$$\left|\left\langle e^{tL_\delta}\nu_{N,0}, h\right\rangle_d - \left\langle e^{tL_\delta}v_l, h\right\rangle_d\right| \le \|h\|_{1,d}\left\|e^{tL_\delta}\nu_{N,0} - e^{tL_\delta}v_l\right\|_{-1,d} \to_{l\to\infty} 0, \tag{3.34}$$

so that, at the limit for $l \to \infty$, for all $t \geq 0$,

$$\left\langle \nu_{N,0}, e^{tL_\delta^*} h \right\rangle_d = \left\langle e^{tL_\delta} \nu_{N,0}, h \right\rangle_d. \tag{3.35}$$

Since the function D_N defined in (2.23) is regular, it is straightforward to prove in the same way that

$$\left\langle D_N, e^{(t-s)L_\delta^*} h \right\rangle_d = \left\langle e^{(t-s)L_\delta} D_N, h \right\rangle_d. \tag{3.36}$$

The continuity of the mapping $t \mapsto e^{tL_\delta} \nu_{N,0}$ and $t \mapsto \int_0^t e^{(t-s)L_\delta} D_N \, \mathrm{d}s$ in H_d^{-1} is immediate from the continuity of the semigroup in H_d^{-1}.

We now focus on the term $R_N(\nu_{N,s})$ defined in (2.24). Note that since the function J is bounded and both μ_N and q are vectors of probability measures, $R_N(\nu_{N,s})$ naturally belongs to H_d^{-1} (Remark 2.3). Consider $(R_{s,l})_{l \geq 1}$ a sequence of elements of $\mathbf{L}^2_{0,d}$ converging in H_d^{-1} to $R_N(\nu_{N,s})$ (consider for example $w_{s,l} = \phi_l * R_N(\nu_{N,s})$ for a regular approximation of identity $(\phi_l)_{l \geq 1}$). For any $l \geq 1$, the following identity holds:

$$\left\langle R_{s,l}, \partial_\theta e^{(t-s)L_\delta^*} h \right\rangle_d = \left\langle R_{s,l}, \partial_\theta e^{(t-s)L_\delta^*} h \right\rangle_{2,d} = -\left\langle e^{(t-s)L_\delta} \partial_\theta R_{s,l}, h \right\rangle_{2,d}. \tag{3.37}$$

Since h is regular and $R_{s,l}$ converges in H_d^{-1} to $R_N(\nu_{N,s})$, the lefthand part of the previous identity converges as $l \to \infty$ to $\langle R_N(\nu_{N,s}), \partial_\theta e^{(t-s)L_\delta^*} h \rangle_d$. Moreover, for all h regular, using the estimate (B.22) on the regularity of the semigroup e^{tL_δ}, (note in particular that e^{tL_δ} can be extended to a continuous operator from H_d^{-2} to H_d^{-1}, see Proposition B.7 below),

$$\begin{aligned} \left| \left\langle e^{(t-s)L_\delta} \partial_\theta \left(R_{s,l} - R_N(\nu_{N,s}) \right), h \right\rangle_d \right| &\leq \|h\|_{1,d} \left\| e^{(t-s)L_\delta} \partial_\theta \left(R_{s,l} - R_N(\nu_{N,s}) \right) \right\|_{-1,d}, \\ &\leq C \|h\|_{1,d} \left(1 + \frac{1}{\sqrt{t-s}} \right) \left\| \partial_\theta \left(R_{s,l} - R_N(\nu_{N,s}) \right) \right\|_{-2,d}, \\ &\leq C \|h\|_{1,d} \left(1 + \frac{1}{\sqrt{t-s}} \right) \left\| R_{s,l} - R_N(\nu_{N,s}) \right\|_{-1,d}. \end{aligned}$$

Since the last estimate is true for all h regular, one obtains that

$$\left\| e^{(t-s)L_\delta} \partial_\theta \left(R_{s,l} - R_N(\nu_{N,s}) \right) \right\|_{-1,d} \leq C \left(1 + \frac{1}{\sqrt{t-s}} \right) \left\| R_{s,l} - R_N(\nu_{N,s}) \right\|_{-1,d}. \tag{3.38}$$

Since $R_{s,l}$ converges to $R_N(\nu_{N,s})$ in H_d^{-1}, one can take $l \to \infty$ in (3.37) and obtain:

$$\left\langle R_N(\nu_{N,s}), \partial_\theta e^{(t-s)L_\delta^*} h \right\rangle_d = -\left\langle e^{(t-s)L_\delta} \partial_\theta R_N(\nu_{N,s}), h \right\rangle_d.$$

The same argument as before shows also that

$$\begin{aligned} \left\| e^{(t-s)L_\delta} \partial_\theta R_N(\nu_{N,s}) \right\|_{-1,d} &\leq C \left(1 + \frac{1}{\sqrt{t-s}} \right) \left\| R_N(\nu_{N,s}) \right\|_{-1,d} \\ &\leq C \left(1 + \frac{1}{\sqrt{t-s}} \right) \|\nu_{N,s}\|_{-1,d} \leq C\pi \sqrt{\frac{2}{3} \sum_{k=-d}^{d} \left(\lambda^k\right)^{-1}} \left(1 + \frac{1}{\sqrt{t-s}} \right), \end{aligned} \tag{3.39}$$

where we used (2.4). The inequality (3.39) implies that the integral $\int_0^t \|e^{(t-s)L_\delta} \partial_\theta R_N(\nu_{N,s})\|_{-1,d} \, \mathrm{d}s$ is almost surely finite. Using [44], Theorem 1, p. 133, we deduce that $\int_0^t e^{(t-s)L_\delta} \partial_\theta R_N(\nu_{N,s}) \, \mathrm{d}s$ makes sense as a Bochner integral in H_d^{-1}. The continuity of $t \mapsto \int_0^t e^{(t-s)L_\delta} \partial_\theta R_N(\nu_{N,s}) \, \mathrm{d}s$ in H_d^{-1} is a direct consequence of the bounds found in Proposition B.7.

It remains to treat the noise term in (3.32). It is immediate to see from (3.4) and (2.26) that for all F regular, $\langle Z_{N,t}, F\rangle_d = M^F_{N,t}$, where we see the term $Z_{N,t}$ as a vector $(Z^{-d}_{N,t}, \dots, Z^{d}_{n,t})$, with $Z^k_{N,t}(h) = \frac{1}{\lambda^k} Z_{N,t}(\hat{h}_k)$ and $\hat{h}_k = (0, \dots, 0, h^k, 0, \dots, 0)$. One concludes from everything that we have done that, for all h regular that

$$\begin{aligned}\langle \nu_{N,t}, h\rangle_d = \left\langle e^{tL_\delta}\nu_{N,0}, h\right\rangle_d + \left\langle \int_0^t \left(e^{(t-s)L_\delta} D_N - e^{(t-s)L_\delta}\partial_\theta R_N(\nu_{N,s})\right) \mathrm{d}s, h\right\rangle_d \\ + \langle Z_{N,t}, h\rangle_d,\end{aligned} \tag{3.40}$$

where everything above makes sense as element of H_d^{-1}. Since this is true for all h regular, the identity (2.22) follows. Proposition 2.8 is proved. □

4. Controlling the noise

This section is devoted to control the noise term $Z_{n,t}$ defined in (2.39).

Remark 4.1. *It is important to note that since the definition of $Z_{n,t} = Z_{N,n,t}$ only differs from the one of $Z_{N,t}$ via a time translation, using Remark 2.11, the same estimates as (3.13) and (3.14) are also valid for the discretized process $Z_{n,t}$.*

More precisely, we prove the following proposition (recall the definition of $A^N = A^N(C_Z)$ given in (2.40)).

Proposition 4.2. *For all $\zeta > 0$, there exists a constant C_Z such that $\mathbf{P}(A^N) \to 1$, as $N \to \infty$.*

To prove Proposition 4.2, we rely on the following lemma:

Lemma 4.3 (Garsia–Rademich–Rumsey). *Let χ and Ψ be continuous, strictly increasing functions on $(0, \infty)$ such that $\chi(0) = \Psi(0) = 0$ and $\lim_{t\nearrow\infty}\Psi(t) = \infty$. Given $T > 0$ and ϕ continuous on $(0, T)$ and taking its values in a Banach space $(E, \|\cdot\|)$, if*

$$\int_0^T \int_0^T \Psi\left(\frac{\|\phi(t) - \phi(s)\|}{\chi(|t-s|)}\right) \mathrm{d}s\,\mathrm{d}t \le B < \infty, \tag{4.1}$$

then for $0 \le s \le t \le T$,

$$\left\|\phi(t) - \phi(s)\right\| \le 8\int_0^{t-s} \Psi^{-1}\left(\frac{4B}{u^2}\right)\chi(\mathrm{d}u). \tag{4.2}$$

Proof of Lemma 4.3 may be found in [42], Theorem 2.1.3. Let us now prove Proposition 4.2.

Proof of Proposition 4.2. Using Lemma 3.2 and Remark 4.1, we can apply Lemma 4.3 with the choices

$$\phi(t) = Z_{n,t}, \qquad \chi(u) = u^{\frac{2+\zeta}{2m}} \quad \text{and} \quad \Psi(u) = u^{2m}, \tag{4.3}$$

which implies that there exist a constant C (depending in m, ε and ζ) and a positive random variable B such that for every $0 \le s < t \le T$:

$$\|Z_{n,t} - Z_{n,s}\|^{2m}_{-1,d} \le C(t-s)^\zeta B, \tag{4.4}$$

where B satisfies

$$\mathbf{E}(B) \le \frac{C}{N^m}\int_0^T \int_0^T \left(|t-s|^{m(1/2-2\varepsilon)-2-\zeta} + |t-s|^{m-2-\zeta}\right) \mathrm{d}s\,\mathrm{d}t. \tag{4.5}$$

A simple integration shows that $\mathbf{E}(B) \leq \frac{C}{N^m}(T^{m(1/2-2\varepsilon)-\zeta} + T^{m-\zeta})$, whenever $m(1/2-2\varepsilon)-\zeta > 1$ and $m-\zeta > 1$, that is when $m > \frac{2(1+\zeta)}{1-4\varepsilon}$. We can fix for example $\varepsilon = 1/8$ and choose an integer m such that $m > 4(1+\zeta)$. Since $T \geq 1$, we have $\mathbf{E}(B) \leq C\frac{T^{m-\zeta}}{N^m}$ and we obtain:

$$\mathbf{E}\biggl(\sup_{0\leq s<t\leq T}\frac{\|Z_{n,t}-Z_{n,s}\|_{-1,d}^{2m}}{|t-s|^{\zeta}}\biggr) \leq C\frac{T^{m-\zeta}}{N^m}, \tag{4.6}$$

which implies

$$\begin{aligned}\mathbf{P}\biggl(\sup_{0\leq t\leq T}\|Z_{n,t}\|_{-1,d} \geq \sqrt{\frac{T}{N}}N^{\zeta}\biggr) &\leq \frac{N^m}{T^m}N^{-2m\zeta}\mathbf{E}\Bigl(\sup_{0\leq t\leq T}\|Z_{n,t}\|_{-1,d}^{2m}\Bigr) \\ &\leq \frac{N^m}{T^{m-\zeta}}N^{-2m\zeta}\mathbf{E}\biggl(\sup_{0\leq t\leq T}\frac{\|Z_{n,t}\|_{-1,d}^{2m}}{t^{\zeta}}\biggr) \leq CN^{-2m\zeta}.\end{aligned} \tag{4.7}$$

We deduce

$$\mathbf{P}\biggl(\sup_{1\leq n\leq n_f}\sup_{0\leq t\leq T}\|Z_{n,t}\|_{-1,d} \geq \sqrt{\frac{T}{N}}N^{\zeta}\biggr) \leq Cn_f N^{-2m\zeta}, \tag{4.8}$$

which tends to 0 as $N \to \infty$ if we choose $m > \frac{1}{4\zeta}$, since $n_f = O(N^{1/2})$. Proposition 4.2 is proved. □

5. Dynamics on the manifold *M*

The purpose of this section is to prove the results described in Section 2.6 concerning the process ν_n defined in (2.35).

Recall that the scheme defined in Section 2.6 starts at a time $t_0^N = O(N^{-1/2}\log N)$, such that there exists an event B^N with $\mathbf{P}(B^N) \to 1$ such that on B^N if we denote $\psi_0 = \mathrm{proj}_M(\mu_{N,N^{1/2}t_0^N})$ then $\|\mu_{N,N^{1/2}t_0^N} - q_{\psi_0}\|_{-1,d} \leq N^{-1/2+2\zeta}$. In other words, the initial condition of the scheme satisfies $\|\nu_{1,0}\|_{-1,d} \leq N^{-1/2+2\zeta}$ on B^N. The existence of these times t_0^N and event B^N will be proved in the Section 6. The first result proves estimate (2.41):

Proposition 5.1. *There exists an event Ω_1^N with $\mathbf{P}(\Omega_1^N) \to 1$ as $N \to \infty$ such that, on Ω_1^N,*

$$\sup_{1\leq n\leq n_f}\sup_{t\in[0,T]}\|\nu_{n,t}\|_{-1,d} = O\bigl(N^{-1/2+2\zeta}\bigr), \tag{5.1}$$

where the error $O(N^{-1/2+2\zeta})$ is uniform on Ω_1^N.

Proof. Recall the definition of the event A^N in (2.40) and define $\Omega_1^N := A^N \cap B^N$. Since the purpose of Section 4 was precisely to prove that $\mathbf{P}(A^N) \to 1$, we obviously have that $\mathbf{P}(\Omega_1^N) \to 1$, as $N \to \infty$.

Throughout this proof we work on the event Ω_1^N and proceed by induction. We already know that $\|\nu_{1,0}\|_{-1,d} \leq N^{-1/2+2\zeta}$. If we suppose that $\|\nu_{n,0}\|_{-1,d} \leq N^{-1/2+2\zeta}$, then from the mild formulation (2.38), from (2.19) and (2.20) and from the estimates on the noise term $Z_{n,t}$ on $\Omega_1^N \subset A^N$, we obtain

$$\begin{aligned}\|\nu_{n,t}\|_{-1,d} \leq{}& C_L e^{-\gamma_L t\wedge\tau^n}N^{-1/2+2\zeta} \\ &+ 2TC_L\|D_{\psi_{n-1}}\|_{-1,d} + C_L\bigl(T+2T^{1/2}\bigr)\sup_{0\leq s\leq t}\|R_{\psi_{n-1}}(\nu_{n,s})\|_{-1,d} \\ &+ T^{1/2}N^{-1/2+\zeta}.\end{aligned} \tag{5.2}$$

Since the sequence $(\omega_i)_{i\geq 1}$ is admissible (recall Definition 2.1), we have

$$\|D_{\psi_{n-1}}\|_{-1,d} \leq CN^{-1/2}\max_{k=-d,\dots,d}\bigl|\xi_N^k\bigr| \leq CN^{-1/2+\zeta}. \tag{5.3}$$

Define the time t^* as

$$t^* := \inf\{t \in [0, T] : \|\nu_{n,t}\|_{-1,d} \ge 2C_L N^{-1/2+2\zeta}\}. \tag{5.4}$$

Obviously $t^* > 0$ and if $t \le t^*$, one readily sees from (2.24) that

$$\begin{aligned}&\sup_{0\le s\le t} \big\| R_{\psi_{n-1}}(\nu_{n,s}) \big\|_{-1,d} \\ &\quad \le C\Big(\sup_{0\le s\le t} \|\nu_{n,s}\|^2_{-1,d} + N^{-1/2} \max_{k=-d,\dots,d} \big|\xi_N^k\big| \sup_{0\le s\le t} \|\nu_{n,s}\|_{-1,d}\Big) \le C N^{-1+4\zeta}.\end{aligned} \tag{5.5}$$

Putting together (5.2), (5.3) and (5.5) gives that $t^* = T$ if N is large enough. Consequently, by construction of the stopping time τ^n in (2.37), one has that $\tau^n = T$ and the choice of T (recall (2.30)) implies that

$$\|\nu_{n,T}\|_{-1,d} \le \frac{1}{2C_P} N^{-1/2+2\zeta}. \tag{5.6}$$

To conclude the recursion it remains to show that $\|\nu_{n+1,0}\|_{-1,d} \le N^{-1/2+2\zeta}$. To do this, let us write $\nu_{n+1,0}$ in terms of $\nu_{n,T}$:

$$\nu_{n+1,0} = q_{\psi_{n-1}} + \nu_{n,T} - q_{\psi_n}. \tag{5.7}$$

Since $P^s_{\psi_n}\nu_{n+1,0} = \nu_{n+1,0}$, where we recall that $P^s_{\psi_n}$ is the projection on the space N_{ψ_n}, we can rewrite it as

$$\begin{aligned}\nu_{n+1,0} &= P^s_{\psi_n}(q_{\psi_{n-1}} + \nu_{n,T} - q_{\psi_n}) \\ &= P^s_{\psi_n}(q_{\psi_{n-1}} - q_{\psi_n}) + \big(P^s_{\psi_n} - P^s_{\psi_{n-1}}\big)\nu_{n,T} + P^s_{\psi_{n-1}}\nu_{n,T}.\end{aligned} \tag{5.8}$$

Since $q_{\psi_{n-1}} - q_{\psi_n} = (\psi_{n-1} - \psi_n)q'_{\psi_n} + O((\psi_n - \psi_{n-1})^2)$ (and this estimate makes sense in H_d^{-1}) and $P^s_{\psi_n}\partial_\theta q_{\psi_n} = 0$, the first term of the second line of (5.8) is of order $O((\psi_n - \psi_{n-1})^2)$. Using the smoothness of the projection proj_M (Lemma 2.9),

$$\begin{aligned}|\psi_n - \psi_{n-1}| &= \big|\mathrm{proj}_M(\mu_{T_n\wedge\iota}) - \mathrm{proj}_M(\mu_{T_{n-1}\wedge\iota})\big| \\ &\le C\|\mu_{T_n\wedge\iota} - \mu_{T_{n-1}\wedge\iota}\|_{-1,d} \\ &\le C\|\nu_{n-1,T}\|_{-1,d} + C\|\nu_{n-1,0}\|_{-1,d} \le C N^{-1/2+2\zeta}.\end{aligned} \tag{5.9}$$

Combining the last two arguments, we obtain that the first term of the second line of (5.8) is of order $O(N^{-1+4\zeta})$. For the second term, the smoothness of the mapping $\psi \mapsto P^s_\psi$ gives

$$\big\|\big(P^s_{\psi_n} - P^s_{\psi_{n-1}}\big)\nu_{n,T}\big\|_{-1,d} \le C|\psi_n - \psi_{n-1}|\|\nu_{n,T}\|_{-1,d} \le C N^{-1+4\zeta}. \tag{5.10}$$

Taking the H_d^{-1} norm on the two sides in (5.8), we obtain

$$\|\nu_{n+1,0}\|_{-1,d} \le \|P^s_{\psi_{n-1}}\nu_{n,T}\|_{-1,d} + O\big(N^{-1+4\zeta}\big) \le \frac{1}{2}N^{-1/2+2\zeta} + O\big(N^{-1+4\zeta}\big), \tag{5.11}$$

which implies the result for N large enough. □

We are interested in the rescaled dynamics of the phase of the projection of the empirical measure on M and in particular use the rescaled discretization of this phase dynamics given by the process Ψ^N_t (recall (2.44)).

Proposition 5.2. *There exist a linear form $b : \mathbb{R}^{2d+1} \to \mathbb{R}$ and an event Ω_2^N satisfying $\mathbf{P}(\Omega_2^N) \to 1$ as $N \to \infty$ such that on the event Ω_2^N we have for $t \in [t_0^N, t_f]$:*

$$\Psi^N_t = \psi_0 + b(\xi_N)t + O\big(N^{-1/4+2\zeta}\big), \tag{5.12}$$

where the $O(N^{-1/4+2\zeta})$ is uniform on Ω_2^N.

Proof. We work for the moment on the event Ω_1^N defined in the proof of Proposition 5.1. Using Proposition 5.1, Lemma C.1 below and the fact that $\psi_n = \mathrm{proj}_M(q_{\psi_{n-1}} + \nu_{n,T})$, we have the following first order expansion of Ψ_t^N in (5.12) (recall the definition of $\mathtt{p}$ in (2.16) and note that there are $O(N^{1/2})$ terms in the sum):

$$\Psi_t^N := \psi_0 + \sum_{n=1}^{n_t} \mathtt{p}_{\psi_{n-1}}(\nu_{n,T}) + O\big(N^{-1/2+4\zeta}\big). \tag{5.13}$$

Let us now decompose the term $\mathtt{p}_{\psi_{n-1}}(\nu_{n,T})$, using the mild formulation (2.38). Remark that $\mathtt{p}_{\psi_{n-1}}(e^{tL_{\psi_{n-1}}}\nu_{n,0}) = \mathtt{p}_{\psi_{n-1}}(\nu_{n,0}) = 0$ and that $\mathtt{p}_{\psi_{n-1}}(e^{(t-s)L_{\psi_{n-1}}} D_{\psi_{n-1}}) = \mathtt{p}_{\psi_{n-1}}(D_{\psi_{n-1}})$. Note that Proposition 5.1 shows that $\tau^{n_f} = T$ on Ω_1^N, so that the time integration in the mild formulation (2.38) does not involve any stopping time. Hence it remains, since $D_{\psi_{n-1}}$ has no dependency in time,

$$\mathtt{p}_{\psi_{n-1}}(\nu_{n,T}) = T\mathtt{p}_{\psi_{n-1}}(D_{\psi_{n-1}}) - \int_0^T \mathtt{p}_{\psi_{n-1}}\big(e^{(t-s)L_{\psi_{n-1}}}\partial_\theta R_{\psi_{n-1}}(\nu_{n,s})\big)\,\mathrm{d}s + \mathtt{p}_{\psi_{n-1}}(Z_{n,T}). \tag{5.14}$$

Using (2.20) and (5.5)

$$\begin{aligned}
\left|\int_0^T \mathtt{p}_{\psi_{n-1}}\big(e^{(t-s)L_{\psi_{n-1}}}\partial_\theta R_{\psi_{n-1}}(\nu_{n,s})\big)\,\mathrm{d}s\right| &\le \int_0^T \big\|e^{(t-s)L_{\psi_{n-1}}}\partial_\theta R_{\psi_{n-1}}(\nu_{n,s})\big\|_{-1,d}\,\mathrm{d}s \\
&\le C\int_0^T \left(1+\frac{1}{\sqrt{t-s}}\right)\big\|R_{\psi_{n-1}}(\nu_{n,s})\big\|_{-1,d}\,\mathrm{d}s \\
&\le C(T+\sqrt{T})N^{-1+4\zeta},
\end{aligned} \tag{5.15}$$

which leads to

$$\mathtt{p}_{\psi_{n-1}}(\nu_{n,T}) = T\mathtt{p}_{\psi_{n-1}}(D_{\psi_{n-1}}) + \mathtt{p}_{\psi_{n-1}}(Z_{n,T}) + O\big(N^{-1+4\zeta}\big). \tag{5.16}$$

We would like to keep only $T\mathtt{p}_{\psi_{n-1}}(D_{\psi_{n-1}})$, since the sum of these terms produce the drift we are looking for, but unfortunately at each step $\mathtt{p}_{\psi_{n-1}}(Z_{n,T})$ has the same order as $T\mathtt{p}_{\psi_{n-1}}(D_{\psi_{n-1}})$. To get rid of this extra term $\mathtt{p}_{\psi_{n-1}}(Z_{n,T})$, we use the fact that it is an increment of a martingale and thus averages to 0 under summation. More precisely, denoting $z_n := \mathtt{p}_{\psi_{n-1}}(Z_{n,T\wedge\tau^n})$ and using Doob's inequality we obtain,

$$\mathbf{P}\left(\sup_{1\le m\le n_f}\left|\sum_{1\le n\le m} z_n\right| \ge N^{-1/4+2\zeta}\right) \le N^{1/2-4\zeta}\mathbf{E}\left(\left|\sum_{1\le n\le n_f} z_n\right|^2\right), \tag{5.17}$$

and we have the following decomposition:

$$\begin{aligned}
\mathbf{E}\left(\left|\sum_{1\le n\le n_f} z_n\right|^2\right) &\le \mathbf{E}\left(\sum_{1\le n\le n_f-1}\mathbf{E}\big[|z_{n+1}|^2|\mathcal{F}_{T_n}\big]\right) \\
&\le C\sum_{1\le n\le n_f-1}\mathbf{E}\big[\|Z_{n,T\wedge\tau^n}\|_{-1,d}^2\big] \le Cn_f T N^{-1},
\end{aligned} \tag{5.18}$$

where we have used (3.14). Since n_f is of order $N^{1/2}$, the probability in (5.17) tends to 0 when $N\to\infty$ and recalling (5.16), we deduce that there exists an event Ω_2^N satisfying $\mathbf{P}(\Omega_2^N) \to_{N\to\infty} 1$ such that on Ω_2^N

$$\Psi_t^N = \psi_0 + T\sum_{n=1}^{n_t}\mathtt{p}_{\psi_{n-1}}(D_{\psi_{n-1}}) + O\big(N^{-1/4+2\zeta}\big). \tag{5.19}$$

The quantity $\mathrm{p}_{\psi_{n-1}}(D_{\psi_{n-1}}) = N^{-1/2}\mathrm{p}_{\psi_{n-1}}(-\partial_\theta(\xi_N \cdot (J * q_{\psi_{n-1}})q_{\psi_{n-1}}))$ depends linearly in ξ_N and since the model is invariant by rotation, the projection does not depend on ψ_{n-1}. So we can write it as $N^{-1/2}b(\xi_N)$, where the linear form b is given by

$$b(\xi) := \mathrm{p}\big(-\partial_\theta\big(\xi \cdot (J * q)q\big)\big) = \mathrm{p}\left(-\partial_\theta\left(q \sum_{k=-d}^{d} \xi^k\big(J * q^k\big)\right)\right). \tag{5.20}$$

We can rewrite (5.19) as

$$\Psi_t^N = \psi_0 + \frac{T}{N^{1/2}}\left\lfloor \frac{N^{1/2}}{T}\big(t - t_0^N\big)\right\rfloor b(\xi_N) + O\big(N^{-1/4+2\zeta}\big). \tag{5.21}$$

Since $|t - t_0^N - \frac{T}{N^{1/2}}\lfloor\frac{N^{1/2}}{T}(t - t_0^N)\rfloor| \leq \frac{T}{N^{1/2}}$ and $b(\xi_N) = O(N^\zeta)$, we deduce

$$\Psi_t^N = \psi_0 + b(\xi_N)\big(t - t_0^N\big) + O\big(N^{-1/4+2\zeta}\big), \tag{5.22}$$

which implies the result, since $t_0^N = O(N^{-1/2}\log N)$. Proposition 5.2 is proved. □

We can now prove the following result, which together with Proposition 2.10 implies directly Theorem 2.4:

Proposition 5.3. *There exists N sufficiently large such that, on the event Ω_2^N,*

$$\sup_{t\in[t_0^N, t_f]} \|\mu_{N,N^{1/2}t} - q_{\psi_0+b(\xi_N)t}\|_{-1,d} = O\big(N^{-1/4+2\zeta}\big), \tag{5.23}$$

where the error $O(N^{-1/4+2\zeta})$ is uniform on Ω_2^N.

Proof. We place ourselves on the event Ω_2^N introduced in the proof of Proposition 5.2. For each t such that $N^{1/2}t \in [T_n, T_{n+1}]$ we can decompose $\mu_{N,N^{1/2}t}$ as

$$\mu_{N,N^{1/2}t} = q_{\psi_n} + \nu_{n+1,N^{1/2}t-T_n}. \tag{5.24}$$

But Proposition 5.1 implies that $\nu_{n+1,N^{1/2}t-T_n} = O(N^{-1/2+2\zeta})$ and for such time t we have

$$q_{\psi_n} = q_{\Psi_t^N} = q_{\psi_0+b(\xi_N)t} + O\big(N^{-1/4+2\zeta}\big), \tag{5.25}$$

where we have used Proposition 5.2. □

6. Approaching the manifold

The purpose of this section is to prove Proposition 2.10. We follow here the same ideas as in [7], Section 5. From now on, we fix $\varepsilon_0 > 0$ and $p_0 \in H_d^{-1}$ such that $\mathrm{dist}_{H_d^{-1}}(p_0, M) \leq \varepsilon_0$. The parameter ε_0 will be chosen sufficiently small in the following. We consider now a parameter $0 < \varepsilon < \varepsilon_0$ and proceed in three steps:

(1) We rely on the convergence in finite time of the empirical measure $\mu_{N,t}$ to the solution p_t of (1.7) starting from p_0 in order to show that $\mu_{N,t}$ approaches M (up to a distance of order ε). This step requires a time interval of order $\log\varepsilon$.
(2) We use the linear stability of M under (1.7) and control the noise terms of the dynamics to show that the empirical measure approaches M up to a distance of order $N^{-1/2+2\zeta}$. This step requires a time interval of order $\log N$.
(3) We show that the empirical measure stays at distance $N^{-1/2+2\zeta}$ from M up to the time t_0^N.

First step. As explained in Section 2.3, the stability of M implies that if ε_0 is small enough the deterministic solution p_t of the limit PDE (1.7) with initial condition p_0 converges to a $q_{\theta_0} \in M$. In particular, after a time s_1, p_t satisfies $\|p_{s_1} - q_{\theta_0}\|_{-1,d} \leq \varepsilon$. Due to the linear stability of M, this time s_1 is of order $-\frac{1}{\gamma_L}\log\varepsilon$.

In order to show that the empirical measure is close to the deterministic trajectory p_t when N is large, we use a mild formulation similar to the one obtained in Section 3, but this time relying on the $(2d+1)$-dimensional Laplacian operator Δ_d. More precisely using similar argument as in Section 3, one can obtain the following equality in H_d^{-1}:

$$\mu_{N,t} - p_t = e^{\frac{t}{2}\Delta_d}(\mu_{N,0} - p_0) - \int_0^t e^{\frac{t-s}{2}\Delta_d}\Bigg[\partial_\theta\Bigg(\mu_{N,s}\otimes\omega + \mu_{N,t}\sum_{k=-d}^{d}\lambda_N^k J * \mu_{N,s}^k\Bigg) - \partial_\theta\Bigg(p_s\otimes\omega + p_s\sum_{k=-d}^{d}\lambda^k J * p_s^k\Bigg)\Bigg]\mathrm{d}s + z_t, \tag{6.1}$$

where z_t satisfies, for all test function $f = (f^{-d}, \dots, f^d)$

$$z_t(f) = \sum_{i=-d}^{d}\frac{\lambda^i}{N^i}\sum_{j=1}^{N^i}\int_0^t \partial_\theta\big[\big(e^{\frac{t-s}{2}\Delta_d}f\big)^i\big]\big(\varphi_j^i(s)\big)\,\mathrm{d}B_j^i(s). \tag{6.2}$$

Since Δ_d is simply the classical one-dimensional Laplacian operator Δ on each coordinate, it is sectorial (in fact self-adjoint) with negative spectrum. Using the classical bound $\|e^{t\Delta}f\|_{-1} \leq \frac{C}{\sqrt{t}}\|f\|_{-2}$ for the one-dimensional Laplacian operator, we directly obtain

$$\big\|e^{t\Delta_d}f\big\|_{-1,d} \leq \frac{C}{\sqrt{t}}\|f\|_{-2,d}, \tag{6.3}$$

and with similar estimates as the one used in Section 4, one can show that the event B_1^N defined as

$$B_1^N := \Bigg\{\sup_{0\leq t\leq s_1}\|z_t\|_{-1,d} \leq \sqrt{\frac{t_1}{N}}N^\zeta\Bigg\} \tag{6.4}$$

satisfies $\mathbf{P}(B_1^N) \to 1$ as $N \to \infty$. Let us write the shortcut

$$U_{N,s,t} := e^{\frac{t-s}{2}\Delta_d}\Bigg[\partial_\theta\Bigg(\mu_{N,s}\otimes\omega + \mu_{N,t}\sum_{k=-d}^{d}\lambda_N^k J * \mu_{N,s}^k\Bigg) - \partial_\theta\Bigg(p_s\otimes\omega + p_s\sum_{k=-d}^{d}\lambda^k J * p_s^k\Bigg)\Bigg],$$

for the term within the integral in (6.1). Note that the mapping $(\mu, \nu) \mapsto \partial_\theta(\mu J * \nu)$ satisfies (see [7], Lemma A.3 for a proof)

$$\big\|\partial_\theta(\mu J * \nu)\big\|_{-2} \leq C\|\mu\|_{-1}\|\nu\|_{-1}. \tag{6.5}$$

Using (6.3) and (6.5), we obtain

$$\begin{aligned}\|U_{N,s,t}\|_{-1,d} &= \Bigg\|e^{\frac{t-s}{2}\Delta_d}\partial_\theta\Bigg(\mu_{N,s}\sum_{k=-d}^{d}\lambda_N^k J * \mu_{N,s}^k\Bigg) - \partial_\theta\Bigg(p_s\sum_{k=-d}^{d}\lambda^k J * p_s^k\Bigg)\Bigg\|_{-1,d} \\ &\leq \frac{C}{\sqrt{t-s}}\Bigg\|\partial_\theta\Bigg(\mu_{N,s}\sum_{k=-d}^{d}\lambda_N^k J * \mu_{N,s}^k\Bigg) - \partial_\theta\Bigg(p_s\sum_{k=-d}^{d}\lambda^k J * p_s^k\Bigg)\Bigg\|_{-2,d} \\ &\leq \frac{C}{\sqrt{t-s}}\sum_{i=-d}^{d}\sum_{k=-d}^{d}\lambda^k\big\|\partial_\theta\big(\mu_{N,s}^i J * \mu_{N,s}^k\big) - \partial_\theta\big(p_s^i J * p_s^k\big)\big\|_{-2}\end{aligned} \tag{6.6}$$

$$+\frac{C}{\sqrt{t-s}}\sum_{i=-d}^{d}\sum_{k=-d}^{d}\left|\lambda_N^k-\lambda^k\right|\left\|\partial_\theta\left(\mu_{N,s}^i J*\mu_{N,s}^k\right)\right\|_{-2} \tag{6.7}$$

$$\leq \frac{C}{\sqrt{t-s}}\left(\|p_s\|_{-1,d}+\|\mu_{N,s}\|_{-1,d}\right)\|\mu_{N,s}-p_s\|_{-1,d}+\frac{C}{\sqrt{t-s}}N^{-1/2+\zeta}\|\mu_{N,s}\|_{-1,d}^2 \tag{6.8}$$

$$\leq \frac{C'}{\sqrt{t-s}}\left(\|\mu_{N,s}-p_s\|_{-1,d}+N^{-1/2+\zeta}\right), \tag{6.9}$$

where we have used in particular (2.4), since both p_s and $\mu_{N,s}$ are probabilities. Let us place ourselves on the event

$$B_2^N := \left\{\|\mu_{N,0}-p_0\|_{-1,d}\leq\frac{\varepsilon}{2}\right\}\cap B_1^N, \tag{6.10}$$

which satisfies obviously $\mathbf{P}(B_2^N)\to 1$ as $N\to\infty$. Then, for all $t\leq s_1$, (6.3) and (6.6) imply that (6.1) can be rewritten on the event B_2^N as

$$\|\mu_{N,t}-p_t\|_{-1,d}\leq\frac{\varepsilon}{2}+C\sqrt{\frac{s_1}{N}}N^{\zeta}+C\int_0^t\frac{1}{\sqrt{t-s}}\|\mu_{N,s}-p_s\|_{-1,d}\,\mathrm{d}s, \tag{6.11}$$

so applying the Gronwall–Henry inequality ([24], Lemma 7.1.1 and Exercise 1), one obtains that for some $a>0$ (independent from N and ε), on the event B_2^N and for all $t\leq s_1$

$$\|\mu_{N,t}-p_t\|_{-1,d}\leq 2\left(\frac{\varepsilon}{2}+C\sqrt{\frac{s_1}{N}}N^{\zeta}\right)e^{as_1}. \tag{6.12}$$

We deduce that for N large enough, the projection

$$\psi_0^1 := \mathrm{proj}_M(\mu_{N,s_1})$$

is well defined and $\|\mu_{N,s_1}-p_{s_1}\|_{-1,d}\leq\varepsilon$ on B_2^N, which means that $|\psi_0^1-\theta_0|\leq C\varepsilon$ and $\|\mu_{N,s_1}-q_{\theta_0}\|\leq 2\varepsilon$.

Second step. Now that we know that $\mathrm{dist}(\mu_{N,s_1},M)\leq 2\varepsilon$ with increasing probability as $N\to\infty$, we can use a similar scheme as the one defined in Section 2.6 to show that the empirical measure approaches M up to a distance $N^{-1/2+2\zeta}$ with high probability. Since this part is very similar to the work done in Section 5, we do not specify all the details.

We consider the evolution of the dynamics on time intervals $[\widetilde{T}_n,\widetilde{T}_{n+1}]$ with $\widetilde{T}_n=s_1+n\widetilde{T}$ where $\widetilde{T}$ is such that $e^{-\gamma_L\widetilde{T}}\leq\frac{1}{4C_LC_P}$. We consider also a sequence of real numbers h_n satisfying $h_1=2\varepsilon$ and $h_{n+1}=\frac{h_n}{2}$ and take this time the number of step $\widetilde{n}_f$ of our scheme as

$$\widetilde{n}_f := \inf\left\{n: h_n\leq N^{-1/2+2\zeta}\right\}. \tag{6.13}$$

It is clear that $\widetilde{n}_f$ is bounded by $C\log N$ for some constant C independant from ε if ε is taken small. To ensure the existence of the projections of the process on M at each step, we introduce, as in Section 2.6, the stopping times

$$(\widetilde{n}_\tau,\widetilde{\tau})=\inf\left\{(n,t)\in\{1,\dots,\widetilde{n}_f\}\times[0,\widetilde{T}]:\mathrm{dist}_{H_d^{-1}}(\mu_{\widetilde{T}_{n-1}+t},M)\geq\sigma\right\}, \tag{6.14}$$

and

$$\widetilde{\tau}^n := \begin{cases}\widetilde{T} & \text{if } n<\widetilde{n}_\tau,\\ \widetilde{\tau} & \text{if } n=\widetilde{n}_\tau,\\ 0 & \text{if } n>\widetilde{n}_\tau.\end{cases} \tag{6.15}$$

This allows us to define for $n=0,\dots,\widetilde{n}_f$ the random phases $\widetilde{\psi}_n$ defined as (with $\widetilde{\iota}:=T_{\widetilde{n}_\tau-1}+\widetilde{\tau}$)

$$\widetilde{\psi}_n := \mathrm{proj}_M(\mu_{N,T_n\wedge\widetilde{\iota}}), \tag{6.16}$$

and the processes $\widetilde{\nu}_{n,t}$ defined for $n = 1, \dots, \widetilde{n}_f$ as

$$\widetilde{\nu}_{n,t} := \mu_{N,(T_{n-1}+t)\wedge\widetilde{\tau}} - q_{\widetilde{\psi}_{n-1}}. \tag{6.17}$$

This last process satisfies the mild equation

$$\widetilde{\nu}_{n,t} = e^{(t\wedge\widetilde{\tau}^n)L_{\widetilde{\psi}_{n-1}}}\widetilde{\nu}_{n,0} - \int_0^{t\wedge\widetilde{\tau}^n} e^{(t\wedge\widetilde{\tau}^n - s)L_{\widetilde{\psi}_{n-1}}}\big(D_{\widetilde{\psi}_{n-1}} + R_{\widetilde{\psi}_{n-1}}(\widetilde{\nu}_{n,s})\big)\,\mathrm{d}s + \widetilde{Z}_{n,t\wedge\widetilde{\tau}^n}, \tag{6.18}$$

where $\widetilde{Z}_{n,t}$ is defined as

$$\widetilde{Z}_{n,t}(f) = \sum_{i=-d}^{d}\frac{1}{N^i}\sum_{j=1}^{N^i}\int_0^t \partial_\theta\big[\big(e^{(t-s)L^*_{\widetilde{\psi}_{n-1}}}f\big)^i\big]\big(\varphi_j^i(\widetilde{T}_{n-1}+s)\big)\,\mathrm{d}B_j^i(\widetilde{T}_{n-1}+s). \tag{6.19}$$

Section 4 shows that the event

$$\widetilde{A}^N = \Big\{\sup_{1\le n\le\widetilde{n}_f}\sup_{t\in[0,\widetilde{T}]}\|\widetilde{Z}_{n,t}\|_{-1,d} \le \widetilde{T}^{1/2}N^{-1/2+\zeta}\Big\}, \tag{6.20}$$

satisfies $\mathbf{P}(\widetilde{A}^N) \to 1$ as $N \to \infty$.

In the first step of this proof we have shown, since $\widetilde{\psi}_0 = \psi_0^1$, that, on the event B_2^N (recall (6.10)), we have $\|\widetilde{\nu}_{1,0}\|_{-1,d} = \|\mu_{N,s_1} - q_{\psi_0^1}\|_{-1,d} \le h_1$. Our aim is to prove that on the event B_3^N defined as

$$B_3^N := \widetilde{A}^N \cap B_2^N, \tag{6.21}$$

we have $\|\widetilde{\nu}_{n,0}\|_{-1,d} \le h_n$ for all $n = 1, \dots, \widetilde{n}_f$. This would imply, using the notations $s_2 = \widetilde{T}_{n_f}$ and $\psi_0^2 = \mathrm{proj}_M(\mu_{N,s_2})$, that $\|\widetilde{\mu}_{N,s_2} - q_{\psi_0^2}\|_{-1,d} \le N^{-1/2+2\zeta}$. We place ourselves on the event B_3^N. From the mild formulation (6.18), if $n < \widetilde{n}_f$ and $\|\widetilde{\nu}_{n,0}\|_{-1,d} \le h_n$ we get

$$\begin{aligned}\|\widetilde{\nu}_{n,t}\|_{-1,d} \le{}& C_L e^{-\gamma_L t\wedge\widetilde{\tau}^n}h_n \\ &+ 2C_L\widetilde{T}\|D_{\widetilde{\psi}_{n-1}}\|_{-1,d} + C_L\big(\widetilde{T} + 2\widetilde{T}^{1/2}\big)\sup_{0\le s\le t}\big\|R_{\widetilde{\psi}_{n-1}}(\widetilde{\nu}_{n,s})\big\|_{-1,d} \\ &+ \widetilde{T}^{1/2}N^{-1/2+\zeta}.\end{aligned} \tag{6.22}$$

Consider the time $\widetilde{t}^*$ defined as

$$\widetilde{t}^* := \inf\big\{t \in [0, \widetilde{T}] : \|\widetilde{\nu}_{n,t}\|_{-1,d} \ge 2C_L h_n\big\}. \tag{6.23}$$

For all $t \le \widetilde{t}^*$ we have

$$\begin{aligned}&\sup_{0\le s\le t}\big\|R_{\widetilde{\psi}_{n-1}}(\widetilde{\nu}_{n,s})\big\|_{-1,d} \\ &\quad\le C\Big(\sup_{0\le s\le t}\|\widetilde{\nu}_{n,s}\|^2_{-1,d} + N^{-1/2}\max_{k=-d,\dots,d}\big|\xi_N^k\big|\sup_{0\le s\le t}\|\widetilde{\nu}_{n,s}\|_{-1,d}\Big) \\ &\quad\le C\big(C_L^2h_n^2 + C_L N^{-1/2+\zeta}h_n\big).\end{aligned} \tag{6.24}$$

The last quantity is smaller than $C(N, \varepsilon_0)h_n$, where $C(N, \varepsilon_0) \to 0$ as $N \to \infty$ and $\varepsilon_0 \to 0$. On the other hand we have shown in (5.3) that $\|D_{\widetilde{\psi}_{n-1}}\|_{-1,d} \le CN^{-1/2+\zeta}$. Since $n < \widetilde{n}_f$ we have $h_N > \frac{1}{2C_L}N^{-1/2+2\zeta}$, which means that $N^{-1/2+\zeta}$ is negligible with respect to h_n for N large enough. So for N large enough, $\widetilde{t}^* \ge \widetilde{T}$ and we have (recall that $e^{-\lambda\widetilde{T}} \le \frac{1}{4C_LC_P}$)

$$\|\widetilde{\nu}_{n,\widetilde{T}}\|_{-1,d} \le \frac{1}{4C_P}h_n + o(h_n) \le \frac{3}{8C_P}h_n, \tag{6.25}$$

when ε_0 is small enough. It remains to show that $\|\widetilde{\nu}_{n+1,0}\|_{-1,d} \le \frac{h_n}{2}$ to conclude the recursion. We do not prove it in details, since it can be done by proceeding exactly as in the proof of Proposition 5.1, decomposing $\|\widetilde{\nu}_{n,\widetilde{T}}\|_{-1,d}$ and showing that it can be written as

$$\|\widetilde{\nu}_{n+1,0}\|_{-1,d} \le \|P^{s}_{\widetilde{\psi}_{n-1}}\widetilde{\nu}_{n,\widetilde{T}}\|_{-1,d} + O\big(h_n^2\big), \tag{6.26}$$

which implies that $\|\widetilde{\nu}_{n+1,0}\|_{-1,d} \le \frac{3}{8}h_n + O(h_n^2) \le \frac{h_n}{2}$ on the event B_3^N when ε is small enough and concludes the recursion. Note that the estimate for $\widetilde{\psi}_n - \widetilde{\psi}_{n-1}$ obtained in (5.9) leads to

$$\big|\psi_0^2 - \psi_0^1\big| \le \sum_{n=1}^{n_f} |\widetilde{\psi}_n - \widetilde{\psi}_{n-1}| \le C\sum_{n=1}^{n_f} h_n \le 2Ch_1 \le 4C\varepsilon, \tag{6.27}$$

on the event B_2^N, which gives $|\psi_0^2 - \theta_0| \le C'\varepsilon$ for some C'.

Third step. In the previous step, we have constructed a time s_2 such that $s_2 \le -\frac{1}{\lambda}\log\varepsilon + C_1\log N$ for some constant C_1 and such that $\|\widetilde{\mu}_{N,s_2} - q_{\psi_0^2}\|_{-1,d} \le N^{-1/2+2\zeta}$ with high probability. We can now consider a time $s_3 = c\log N$ for $c = C_1 + 1$, which does not depend in ε. For N large enough and ε fixed, we obviously have $s_3 > s_2$. In order to prove that $\|\widetilde{\mu}_{N,s_3} - q_{\psi_0^3}\|_{-1,d} \le N^{-1/2+2\zeta}$ with high probability, where $\psi_0^3 = \mathrm{proj}_M(\mu_{N,s_3})$, it suffices to decompose the dynamics on the interval $[s_2, s_3]$ according to an iterative scheme with time step $\hat{T}$ satisfying $e^{-\gamma_L\hat{T}} \le \frac{1}{4C_LC_P}$ as does T and apply exactly the same procedure as in Proposition 5.1.

This last step induces a phase shift $|\psi_0^3 - \psi_0^2| \le CN^{-1/2+2\zeta}\log N \le C\varepsilon_0$, for N large enough. This concludes the proof, with $t_0^N = N^{-1/2}s_3$.

7. Estimates on the drift *b*

7.1. *The case of a symmetric disorder*

We prove here Proposition 2.5 and drop for simplicity the dependency in ψ and δ. We consider $\xi = (\xi^{-d}, \dots, \xi^d)$ such that $\xi^{-i} = \xi^i$ for all $i = 1, \dots, d$ and aim at proving that $b(\xi) = 0$, where the drift $b(\xi) = \mathtt{p}(-\partial_\theta(\{\sum_{k=-d}^d \xi^k(J * q^k)\}q))$ is given by (5.20).

The space of regular ($\mathcal{C}^2$, say) test functions $f = (f^{-d}, f^{-(d-1)}, \dots, f^{d-1}, f^d)$ can be naturally decomposed into the direct sum of the space $\mathcal{O}$ (resp. $\mathcal{E}$) of odd (resp. even) test function in both variables (θ, i), that is $f \in \mathcal{O}$ (resp. $f \in \mathcal{E}$) if and only if $f^{-i}(-\theta) = -f^i(\theta)$ (resp. $f^{-i}(-\theta) = f^i(\theta)$) for all $\theta \in \mathbb{T}$ and $i = 0, \dots, d$. One easily sees from the definition of $J(\cdot)$ in (1.4) and the definition of q in (1.8) that $q \in \mathcal{E}$ and $((J * q^{-d}), \dots, (J * q^d)) \in \mathcal{O}$. Let us denote $Q(\theta) := \sum_{k=-d}^d \xi^k(J * q^k)(\theta)$. Using that $\xi^{-i} = \xi^i$, one obtains that $Q(\theta) = \xi^0(J * q^0)(\theta) + \sum_{k=1}^d \xi^k((J * q^k)(\theta) + (J * q^{-k})(\theta))$, so that we deduce that Q is an odd function of θ and that $\theta \mapsto Q(\theta)q(\theta) \in \mathcal{O}$. Consequently $\partial_\theta(Q(\theta)q(\theta)) \in \mathcal{E}$. Hence, in order to prove Proposition 2.5, it suffices to prove that

$$\forall h \in \mathcal{E}, \quad \mathtt{p}(h) = 0. \tag{7.1}$$

This is indeed the case since one easily sees from the definition (2.14) of the operator $L = L_{\psi,\delta}$ that $L(\mathcal{E}) \subset \mathcal{E}$ and $L(\mathcal{O}) \subset \mathcal{O}$ and since $\mathtt{p}$ is the projection on the eigenfunction $\partial_\theta q \in \mathcal{O}$. Proposition 2.5 is proved.

7.2. *Small δ asymptotics of the drift*

Our aim here is to prove Proposition 2.6 that gives the first order expansion of the drift $b(\xi)$ defined in (5.20) as $\delta \to 0$. Due to the rotational invariance of the system, we can work with the stationary solution $q_{0,\delta}$ that we denote q_δ throughout this section. We denote $\mathtt{p}_\delta$ as $\mathtt{p}_{\psi=0,\delta}$ (recall (2.16)) and $D_\delta(\xi)$ as $D_{N,0,\delta}$, (recall (2.23)). With these notations the drift b is given by

$$b(\xi) = \mathtt{p}_\delta\big(D_\delta(\xi)\big). \tag{7.2}$$

When $\delta = 0$, it is straightforward to see that $q_\delta = (q_\delta^{-d}, \dots, q_\delta^d)$ is equal to $(q_0, \dots, q_0)$, where q_0 is the stationary solution of the nonlinear Fokker–Planck equation without disorder (2.13):

$$q_0(\theta) := \frac{e^{2Kr_0 \cos\theta}}{\mathcal{Z}_0(2Kr_0)}, \tag{7.3}$$

where

$$\mathcal{Z}_0(x) = \int_0^{2\pi} e^{x\cos(\theta)}\, \mathrm{d}\theta \tag{7.4}$$

and r_0 is the unique positive solution of the fixed-point problem

$$r_0 = \Psi_0(2Kr_0), \quad \text{with} \quad \Psi_0(x) := \frac{\int_0^{2\pi} \cos(\theta) e^{x\cos\theta}\, \mathrm{d}\theta}{\mathcal{Z}_0(x)}. \tag{7.5}$$

We refer to Section B.1 below for more details on the case $\delta = 0$. The following result (proved in Appendix D) provides the next order of the approximation of q_δ as $\delta \to 0$.

Lemma 7.1. *For $i = -d, \dots, d$ we have*

$$q_\delta^i(\theta) = q_0(\theta) + \delta\omega^i \kappa(\theta) q_0(\theta) + O(\delta^2), \tag{7.6}$$

where

$$\begin{aligned}\kappa(\theta) = {} & 2\theta + 4\pi \frac{\int_\theta^{2\pi} e^{-2Kr_0\cos u}\, \mathrm{d}u}{\mathcal{Z}_0(2Kr_0)} - 2\frac{\int_0^{2\pi} e^{2Kr_0\cos u} u\, \mathrm{d}u}{\mathcal{Z}_0(2Kr_0)} \\ & - 4\pi \frac{\int_0^{2\pi} e^{2Kr_0\cos u} \int_u^{2\pi} e^{-2Kr_0\cos v}\, \mathrm{d}v\, \mathrm{d}u}{\mathcal{Z}_0(2Kr_0)^2},\end{aligned} \tag{7.7}$$

and where the error $O(\delta^2)$ is uniform in $\theta \in \mathbb{T}$.

The projection p_δ also converges in some sense to the projection p_0 on the tangent space of the stable circle of stationary profiles of (2.13) at q_0. Moreover, the system given by (2.13) admits a nice Hilbertian structure, which allows to know p_0 explicitly. This allows us to obtain the following first order expansion of p_δ, whose proof is given in Appendix D.

Lemma 7.2. *For all componentwize primitive $(\mathcal{U}^{-d}, \dots, \mathcal{U}^d)$ of u smooth, we have*

$$\mathrm{p}_\delta(u) = \frac{\mathcal{Z}_0(2Kr_0)^2}{\mathcal{Z}_0(2Kr_0)^2 - 4\pi^2} \sum_{k=-d}^{d} \lambda^k \int_{\mathbb{T}} \left(1 - \frac{2\pi}{\mathcal{Z}_0(2Kr_0)^2 q_0}\right) \mathcal{U}^k + O(\delta \|u\|_{-1,d}). \tag{7.8}$$

We have now the tools required to obtain the first order expansion of the drift $b(\xi)$. The result we want to prove is

Proposition 7.3. *For all ξ such that $\sum_{k=-d}^d \xi^k = 0$ we have*

$$b(\xi) = \delta \sum_{k=-d}^{d} \xi^k \omega^k + O(\delta^2). \tag{7.9}$$

Proof. First remark that when $\delta = 0$, we obtain, using Lemma 7.1, that for all $i = -d, \dots, d$:

$$D_0^i(\xi) = \partial_\theta \left[q_0 \sum_{k=-d}^{d} \xi^k J * q_0 \right] = 0, \tag{7.10}$$

since $\sum_{k=-d}^{d} \xi^k = 0$. We deduce, using again Lemma 7.1, the following expansion for $D_\delta^i(\xi)$:

$$D_\delta^i(\xi) = \delta\omega^i \partial_\theta \left[\kappa q_0 \sum_{k=-d}^{d} \xi^k J * q_0\right] + \delta\partial_\theta \left[q_0 \sum_{k=-d}^{d} \xi^k \omega^k J * (\kappa q_0)\right] + O\big(\delta^2\big)$$
$$= \delta\partial_\theta \left[q_0 \sum_{k=-d}^{d} \xi^k \omega^k J * (\kappa q_0)\right] + O\big(\delta^2\big), \tag{7.11}$$

where we have used again the fact that $\sum_{k=-d}^{d} \xi^k = 0$. Applying Lemma 7.2, we deduce

$$b(\xi) = \delta\left[\frac{\mathcal{Z}_0(2Kr_0)^2}{\mathcal{Z}_0(2Kr_0)^2 - 4\pi^2} \int_{\mathbb{T}} \left(1 - \frac{2\pi}{\mathcal{Z}_0(2Kr_0)^2 q_0}\right) q_0 J * (\kappa q_0)\right] \sum_{i=-d}^{d} \sum_{k=-d}^{d} \lambda^i \xi^k \omega^k + O\big(\delta^2\big), \tag{7.12}$$

and recalling that $\sum_{i=-d}^{d} \lambda^i = 1$ and denoting

$$c_b := \frac{\mathcal{Z}_0(2Kr_0)^2}{\mathcal{Z}_0(2Kr_0)^2 - 4\pi^2} \int_{\mathbb{T}} \left(1 - \frac{2\pi}{\mathcal{Z}_0(2Kr_0)^2 q_0}\right) q_0 J * (\kappa q_0), \tag{7.13}$$

we simply obtain

$$b(\xi) = \delta c_b \sum_{k=-d}^{d} \xi^k \omega^k + O\big(\delta^2\big). \tag{7.14}$$

It remains to show that $c_b = 1$. Now using the fact that $J(\theta - \theta') = -K \sin\theta \cos\theta' + K \cos\theta \sin\theta'$, $\int_0^{2\pi} \sin(\theta) q_0(\theta) = 0$ and $\int_0^{2\pi} \cos(\theta) q_0(\theta) = r_0$, we obtain

$$\int_0^{2\pi} q_0(\theta) J * (\kappa q_0)(\theta)\, \mathrm{d}\theta = K r_0 \int_0^{2\pi} \sin\big(\theta'\big)\kappa\big(\theta'\big) q_0\big(\theta'\big)\, \mathrm{d}\theta', \tag{7.15}$$

and

$$\int_0^{2\pi} J * (\kappa q_0)(\theta)\, \mathrm{d}\theta = 0. \tag{7.16}$$

So the constant c_b can be simplified as follows

$$c_b = \frac{K r_0 \mathcal{Z}_0(2Kr_0)^2}{\mathcal{Z}_0(2Kr_0)^2 - 4\pi^2} \int_0^{2\pi} \sin(\theta)\kappa(\theta) q_0(\theta)\, \mathrm{d}\theta, \tag{7.17}$$

which leads to

$$c_b = \frac{2K r_0 \mathcal{Z}_0(2Kr_0)^2}{\mathcal{Z}_0(2Kr_0)^2 - 4\pi^2} \left[\int_0^{2\pi} \sin\theta \frac{e^{2Kr_0\cos\theta}}{\mathcal{Z}_0(2Kr_0)} \left(\theta + 2\pi \frac{\int_\theta^{2\pi} e^{-2Kr_0\cos u}\, \mathrm{d}u}{\mathcal{Z}_0(2Kr_0)}\right) \mathrm{d}\theta\right]. \tag{7.18}$$

Integrating by parts and using the fact that $\partial_\theta[e^{2Kr_0\cos(\theta)}] = -2Kr_0 \sin\theta e^{2Kr_0\cos\theta}$, we obtain

$$\int_0^{2\pi} \theta \sin\theta e^{2Kr_0\cos\theta}\, \mathrm{d}\theta = -\frac{\pi e^{2Kr_0}}{Kr_0} + \frac{\mathcal{Z}_0(2Kr_0)}{2Kr_0}, \tag{7.19}$$

and

$$\int_0^{2\pi} \sin\theta e^{2Kr_0\cos\theta} \int_\theta^{2\pi} e^{-2Kr_0\cos u}\, \mathrm{d}u = \frac{e^{2Kr_0}\mathcal{Z}_0(2Kr_0)}{2Kr_0} - \frac{\pi}{Kr_0}, \tag{7.20}$$

which implies that $c_b = \frac{2Kr_0\mathcal{Z}_0(2Kr_0)^2}{\mathcal{Z}_0(2Kr_0)^2-4\pi^2}(\frac{1}{2Kr_0} - \frac{4\pi^2}{2Kr_0\mathcal{Z}_0(2Kr_0)^2}) = 1$. Proposition 7.3 is proved. □

Using Proposition 7.3, we can now compute the first order of the variance v^2 of the limiting normal distribution of $b(\xi_N)$ when the disorder is i.i.d:

Proof of Proposition 2.6. From the Central Limit Theorem, we know that ξ_N converges as $N \to \infty$ to a Gaussian distribution with mean 0 and covariance matrix Σ satisfying

$$\begin{cases} \Sigma_{k,k} = \lambda^k(1-\lambda^k), & k \in \{-d,\dots,d\}, \\ \Sigma_{k,l} = -\lambda^k\lambda^l, & k,l \in \{-d,\dots,d\}, k \neq l. \end{cases} \tag{7.21}$$

Applying Proposition 7.3 we obtain

$$v^2 = \delta^2\left(\sum_{k\in\{-d,\dots,d\}} \lambda^k\big(1-\lambda^k\big)\big(\omega^k\big)^2 - \sum_{k,l\in\{-d,\dots,d\},k\neq l} \lambda^k\lambda^l\omega^k\omega^l\right) + O\big(\delta^3\big), \tag{7.22}$$

and since $\lambda^{-k} = \lambda^k$ and $\omega^{-k} = -\omega^k$ the terms with $l \neq -k$ cancel in the second sum, which gives the result. □

Appendix A: Construction of rigged-spaces

We specify here the construction of the Hilbert distributions spaces we work with in this paper. It is based on the notion of *rigged Hilbert spaces* (see [13], p. 81).

A.1. *Functional spaces on* $\mathbb{T}$

Consider $\mathbf{L}^2_0 := \{u \in \mathbf{L}^2, \int_{\mathbb{T}} u(\theta)\,\mathrm{d}\theta = 0\}$, the space of square integrable functions with zero mean value, endowed with the norm $\|u\|_2 := (\int_{\mathbb{T}} u(\theta)^2\,\mathrm{d}\theta)^{\frac{1}{2}}$. We call a *weight* any strictly positive function $\theta \mapsto w(\theta)$ on $\mathbb{T}$. For any weight w on $\mathbb{T}$, define H^1_w as the closure of $\{u \in \mathcal{C}^1(\mathbb{T}), \int_{\mathbb{T}} u(\theta)\,\mathrm{d}\theta = 0\}$ w.r.t. the norm

$$\|u\|_{1,w} := \left(\int_{\mathbb{T}} \big(\partial_\theta u(\theta)\big)^2 w(\theta)\,\mathrm{d}\theta\right)^{\frac{1}{2}}.$$

There is a continuous and dense injection of H^1_w into $\mathbf{L}^2_0$ and the corresponding dual space can be identified as $H^{-1}_{1/w}$, that is the closure of $\{u \in \mathcal{C}^1(\mathbb{T}), \int_{\mathbb{T}} u(\theta)\,\mathrm{d}\theta = 0\}$ under the norm

$$\|u\|_{-1,1/w} := \left(\int_{\mathbb{T}} \frac{\mathcal{U}(\theta)^2}{w(\theta)}\,\mathrm{d}\theta\right)^{\frac{1}{2}},$$

where $\mathcal{U}$ is the primitive of u such that $\int_{\mathbb{T}} \frac{\mathcal{U}}{w} = 0$.

A.2. *Functional spaces on* $\mathbb{T} \times \mathbb{R}$

The correct set-up of the paper is to consider test functions of both oscillators and frequencies, that is $(\theta,\omega) \mapsto u(\theta,\omega)$, where $\theta \in \mathbb{T}$ and $\omega \in \mathbb{R}$. Since the disorder is assumed to take a finite number of values $\{\omega^{-d},\dots,\omega^d\}$, it is equivalent to consider vector-valued test functions $\theta \mapsto (u^{-d}(\theta),\dots,u^d(\theta))$ and it is straightforward to define the counterparts of the norms defined in the last paragraph for these vector-valued functions: Consider $\mathbf{L}^2_{0,d} := (\mathbf{L}^2_0)^d$ endowed with the product norm

$$\|u\|_{2,d} := \left(\sum_{k=-d}^{d} \lambda^k \big\|u^k\big\|_2^2\right)^{\frac{1}{2}}.$$

In the same way, consider the space $H^1_{w,d}$, closure of $\{(u^{-d},\dots,u^d)\in\mathcal{C}^1(\mathbb{T}), \int_{\mathbb{T}} u^k(\theta)\,\mathrm{d}\theta=0\}$ under the norm

$$\|u\|_{1,w,d} := \left(\sum_{k=-d}^{d} \lambda^k \|u^k\|^2_{1,w}\right)^{\frac{1}{2}}, \tag{A.1}$$

as well as the space $H^{-1}_{1/w,d}$ endowed with the norm

$$\|u\|_{-1,1/w,d} := \left(\sum_{k=-d}^{d} \lambda^k \|u^k\|^2_{-1,1/w}\right)^{\frac{1}{2}}. \tag{A.2}$$

Note that if w_1 and w_2 are bounded weights, the norms $\|\cdot\|_{1,w_1}$ and $\|\cdot\|_{1,w_2}$ (resp. $\|\cdot\|_{-1,1/w_1}$ and $\|\cdot\|_{-1,1/w_2}$) are equivalent. The same holds for the $(2d+1)$-dimensional norms.

A.3. *Fractional spaces*

Define also the fractional norm $\|\cdot\|_{\alpha,d}$ (where $\alpha\geq 0$): consider Δ_d the Laplacian operator on each coordinate, $\|\cdot\|_0$ the L^2-norm on $\mathbb{T}$ and $\|u\|^2_{0,d}=\sum_k \lambda_k\|u^k\|^2_0$ and define

$$\|u\|^2_{\alpha,d} = \left\|(1-\Delta_d)^{\alpha/2}u\right\|^2_{0,d} = \sum_{k=-d}^{d} \lambda_k \left\|(1-\Delta)^{\alpha/2}u^k\right\|^2_0. \tag{A.3}$$

We denote as H^α_d the closure of regular functions with zero mean-value on $\mathbb{T}$ under the previous norm and $H^{-\alpha}_d$ the corresponding dual space.

Appendix B: Spectral estimates and regularity results on semigroups

The purpose of this paragraph is to establish spectral estimates on $L_{\psi,\delta}$ and its adjoint as well as regularity estimates on their semigroups $e^{tL_{\psi,\delta}}$ and $e^{tL^*_{\psi,\delta}}$.

B.1. *The case* $\delta=0$

The analysis of the dynamics of (1.5) and (1.7) is based on perturbations argument on the mean-field plane rotators system (2.12) and (2.13). The proof relies in particular strongly on the fact that (2.12) is reversible, with an explicit free energy [6,15]. However, one should note that the limit as $\delta\to 0$ of (1.5) or (1.7) is slightly different to the mean-field model (2.12)–(2.13). In particular, (1.7) becomes as $\delta\to 0$

$$\partial_t p^i_t(\theta) = \frac{1}{2}\partial^2_\theta p^i_t(\theta) - \partial_\theta\left(p^i_t(\theta)\left(\sum_{k=-d}^{d}\lambda^k J * p^k_t(\theta)\right)\right), \quad i=-d,\dots,d, \tag{B.1}$$

which corresponds to the situation where the disorder is no longer present but where the rotators have been (artificially) separated in different subpopulations. Following the terminology of [21] where (B.1) has been already encountered, we call this system the *non-disordered system*. It is shown in [21], Section 2.1, that the non-disordered system (B.1) presents most of the properties of the mean field plane rotators model (2.13). In particular, for all $K>1$, one can show that (B.1) admits a unique circle $M_{0,nd}$ of synchronized profiles, that is stable as $t\to\infty$. $M_{0,nd}$ is given by the translations of the profile $q_{0,nd}=(q_0,\dots,q_0)$, where q_0 is the profile generating the stable circle M_0 of non trivial solutions of (2.13) defined in (7.3) (recall also the definition of $\mathcal{Z}_0$ in (7.4) and r_0 in (7.5)).

The derivation of these stationary solutions is highly similar to the procedure described in Section 1.5 and we refer to the aforementioned references for more details. Note that one can draw a simple correspondance between the present definitions and the definitions of Section 1.5 in the case of $\delta=0$: namely, one readily sees that, for any $i=-d,\dots,d$, $S^i_0(\theta,x)=e^{x\cos(\theta)}\mathcal{Z}_0(x)$ (recall (1.9)) and $Z^i_0(x)=\mathcal{Z}_0(x)^2$, so that the definition of Ψ_δ when $\delta=0$ (recall (1.11)) coincides with Ψ_0 given in (7.5).

B.2. Spectral estimates when $\delta = 0$

Define the linearized operator around any stationary solution $q_{0,nd} \in M_{0,nd}$:

$$(Au)^i = \frac{1}{2}\partial_\theta^2 u^i - \partial_\theta\left((J * q_0)u^i + q_0 \sum_{k=-d}^{d} \lambda^k J * u^k\right), \quad i = -d, \dots, d, \tag{B.2}$$

with domain $\mathcal{D}(A) = \{(u^{-d}, \dots, u^d) \in \mathcal{C}^2(\mathbb{T})^{2d+1}, \int_{\mathbb{T}} u^k(\theta)\,\mathrm{d}\theta = 0, k = -d, \dots, d\}$. We recall the following result (see [21], Proposition 2.1):

Proposition B.1. *A is essentially self-adjoint with compact resolvent in $H^{-1}_{1/q_0,d}$. Its spectrum lies in $(-\infty, 0]$, 0 is a simple eigenvalue, with eigenspace spanned by $\partial_\theta q_{0,nd}$. The spectral gap between 0 and the rest of the spectrum is denoted as γ_A.*

One can deduce from Proposition B.1 similar spectral properties of its dual A^* in $\mathbf{L}^2_{0,d}$:

$$\left(A^* v\right)^i := \frac{1}{2}\partial_\theta^2 v^i + (J * q_0)\partial_\theta v^i - \int_{\mathbb{T}}\left((J * q_0)\partial_\theta v^i\right)\mathrm{d}\theta - \sum_{k=-d}^{d} \lambda^k J * \left(q_0 \partial_\theta v^k\right), \quad i = -d, \dots, d, \tag{B.3}$$

with domain $\mathcal{D}(A^*) = \mathcal{D}(A)$.

Proposition B.2. *A^* is essentially self-adjoint with compact resolvent in $H^1_{1/q_0,d}$. Its spectrum lies in $(-\infty, 0]$, and 0 is a simple eigenvalue and its spectral gap γ_{A^*} is equal to γ_A.*

Proof. Let us introduce the operator U defined from $H^1_{q_0,d}$ to $H^{-1}_{1/q_0,d}$ as

$$Uf(\theta) := -\partial_\theta\left(q_0(\theta)\partial_\theta f(\theta)\right).$$

U is an isometry between $H^1_{q_0,d}$ and $H^{-1}_{1/q_0,d}$: U realizes a bijection from $\{u \in \mathcal{C}^\infty(\mathbb{T})^d, \int_{\mathbb{T}} u^k(\theta)\,\mathrm{d}\theta = 0, k = -d, \dots, d\}$ into itself and for every $f, g \in H^1_{q_0,d}$,

$$\begin{aligned}\langle Uf, Ug\rangle_{-1,1/q_0,d} &= \sum_k \int_{\mathbb{T}} \frac{(q_0(\theta)\partial_\theta f^k(\theta))(q_0(\theta)\partial_\theta g^k(\theta))}{q_0(\theta)}\,\mathrm{d}\theta \\ &= \sum_k \int_{\mathbb{T}} q_0(\theta)\partial_\theta f^k(\theta)\partial_\theta g^k(\theta)\,\mathrm{d}\theta = \langle f, g\rangle_{1,q_0,d}.\end{aligned} \tag{B.4}$$

Moreover, the following identity holds:

$$A^* = U^{-1}AU, \tag{B.5}$$

so the operators A on $H^{-1}_{1/q_0,d}$ and A^* on $H^1_{1/q_0,d}$ have the same structural and spectral properties. □

B.3. Spectral estimates of $L_{\psi,\delta}$ and its adjoint

We are in position to deduce spectral estimates on the disordered operators L_δ and its adjoint L^*_δ in $\mathbf{L}^2_{0,d}$ (we drop the index ψ in this section for simplicity).

Proposition B.3. *The adjoint L_δ^* of L_δ in $\mathbf{L}^2_{0,d}$ is given by for all $i=-d,\dots,d$*

$$\left(L_\delta^* v\right)^i = \frac{1}{2}\partial_\theta^2 v^i + \delta\omega^i \partial_\theta v^i + \left(\partial_\theta v^i\right)\sum_{k=-d}^{d}\lambda^k J * q_\delta^k - \int_{\mathbb{T}}\left(\left(\partial_\theta v^i\right)\sum_{k=-d}^{d}\lambda^k J * q_\delta^k\right)\mathrm{d}\theta - \sum_{k=-d}^{d}\lambda^k J * \left(q_\delta^k \partial_\theta v^k\right), \tag{B.6}$$

with domain $D(L_\delta^*) = D(A)$.

Proof. For all regular u and v,

$$\begin{aligned}
\left\langle L_\delta^* v, u\right\rangle_{2,d} &= \langle v, L_\delta u\rangle_{2,d} \\
&= \sum_{i=-d}^{d}\lambda^i\left\langle v^i, \frac{1}{2}\partial_\theta^2 u^i - \delta\omega^i\partial_\theta u^i - \partial_\theta\left(u^i\sum_{k=-d}^{d}\lambda^k J * q_\delta^k + q_\delta^i\sum_{k=-d}^{d}\lambda^k J * u^k\right)\right\rangle_2 \\
&= \sum_{i=-d}^{d}\lambda^i\left\langle \frac{1}{2}\partial_\theta^2 v^i + \delta\omega^i\partial_\theta v^i + \partial_\theta v^i\sum_{k=-d}^{d}\lambda^k J * q_\delta^k - \int_{\mathbb{T}}\left(\partial_\theta v^i\sum_{k=-d}^{d} J * q_\delta^k\right)\mathrm{d}\theta, u^i\right\rangle_2 \\
&\quad + \sum_{i=-d}^{d}\sum_{k=-d}^{d}\lambda^i\lambda^k\left\langle q_\delta^i\partial_\theta v^i, J * u^k\right\rangle_2 \\
&= \left\langle \frac{1}{2}\partial_\theta^2 v^i + \delta\omega^i\partial_\theta v^i + \left(\partial_\theta v^i\right)\sum_{k=-d}^{d}\lambda^k J * q_\delta^k - \int_{\mathbb{T}}\left(\left(\partial_\theta v^i\right)\sum_{k=-d}^{d}\lambda^k J * q_\delta^k\right)\mathrm{d}\theta, u^i\right\rangle_2 \\
&\quad - \left\langle \sum_{k=-d}^{d}\lambda^k J * \left(q_\delta^k\partial_\theta v^k\right), u^i\right\rangle_2,
\end{aligned}$$

which precisely gives (3.5). □

The main result of this section is the following

Proposition B.4. *There exists $\delta_2 = \delta_2(K) > 0$ such that for all $\delta \le \delta_2$, everything that follows is true: the operator L_δ^* (resp. L_δ) is sectorial in $H^1_{q_0,d}$ (resp. $H^{-1}_{1/q_0,d}$), its spectrum lies in a sector of the type $\{\lambda \in \mathbb{C} : |\arg(\lambda)| > \pi/2 + \alpha\}$ for some $\alpha > 0$ and 0 is an isolated eigenvalue for L_δ^* (resp. L_δ), at a distance from the rest of the spectrum denoted by $\gamma_{L_\delta^*}$ (resp. γ_{L_δ}). Moreover, both L_δ and L_δ^* generate a $\mathcal{C}_0$-semigroup $t \mapsto e^{tL_\delta}$ (resp. $t \mapsto e^{tL_\delta^*}$) in $\mathbf{L}^2_{0,d}$ and $e^{tL_\delta^*} = (e^{tL_\delta})^*$.*

Proof. The result concerning the operator L_δ has been proved in [21], Th. 2.5. For the sake of completeness, we recall here the main arguments concerning L_δ^* in $H^1_{q_0,d}$ but we refer to [21], Section 6.2 for precise details. Note that we need a precise control of the spectrum of L_δ^* around the origin. In particular, one has to ensure that the spectrum of L_δ^* remains in the negative part of the complex plane. We write L_δ^* as a perturbation for small disorder of the non-disordered case:

$$L_\delta^* = A^* + B_\delta, \tag{B.7}$$

where A^* is given in (B.3) and B_δ is a small perturbation as $\delta \to 0$. More precisely, following the exact same strategy as in [21], Proposition 6.5, p. 356, one obtains that the operator B_δ is A^*-bounded: there exist constants a_δ and b_δ (only depending on δ and K) such that for all u in the domain of (the closure of) A^*

$$\|B_\delta u\|_{1,q_0,d} \le a_\delta\|u\|_{1,q_0,d} + b_\delta\left\|A^* u\right\|_{1,q_0,d}, \tag{B.8}$$

with $a_\delta = O(\delta)$ and $b_\delta = O(\delta)$, as $\delta \to 0$. Note that the only things that differs between this result and [21], Proposition 6.5 is that we work here with an H^1-norm whereas the result in [21] concerns an H^{-1}-norm.

Fix some $\varepsilon > 0$ (that will be specified later) and define $L^*_{\delta,\varepsilon} := L^*_\delta - \varepsilon$ and $A_\varepsilon := A - \varepsilon$, so that $L^*_{\delta,\varepsilon} = A^*_\varepsilon + B_\delta$. Fix $\alpha \in (0, \frac{\pi}{2})$ and introduce the following subset of the complex plane

$$\Sigma_\alpha := \left\{\lambda \in \mathbb{C}, \left|\arg(\lambda)\right| < \frac{\pi}{2} + \alpha\right\} \cup \{0\}.$$

The operator A_ε (as A itself) is self-adjoint in $H_{-1,1/q_0}$ and hence, sectorial. In particular, there exists $M > 0$ such that $\|R(\lambda, A_\varepsilon)\|_{H^{-1}_{1/q_0,d}} \le \frac{M}{|\lambda|}$, for all $\lambda \in \Sigma_\alpha$. Note that the constant M is indeed independent of $\varepsilon > 0$ and that the previous inequality is also true for A in place of A_ε (see [21], (6.12)). Using (B.5), one obtains that $\|R(\lambda, A^*_\varepsilon)\|_{H^1_{q_0,d}} \le \frac{M}{|\lambda|}$. For $\lambda \in \Sigma_\alpha$, $u \in H^1_{q_0,d}$,

$$\begin{aligned}\left\|B_\delta R\big(\lambda, A^*\big)u\right\|_{1,q_0,d} &\le a_\delta \left\|R\big(\lambda, A^*\big)u\right\|_{1,q_0,d} + b_\delta \left\|A^* R\big(\lambda, A^*\big)u\right\|_{1,q_0,d} \\ &\le \frac{M a_\delta}{|\lambda|}\|u\|_{1,q_0,d} + (M+1) b_\delta \|u\|_{1,q_0,d}.\end{aligned}$$

Choose δ sufficiently small so that $b_\delta(1 + M) \le \frac{1}{4}$ and $\frac{a_\delta M}{\varepsilon} \le \frac{1}{4}$. Then for $|\lambda| > \varepsilon \ge 4Ma_\delta$, we have $\|B_\delta R(\lambda, A^*)u\|_1 \le \frac{1}{2}\|u\|_1$ so that the operator $1 - B_\delta R(\lambda, A^*)$ is invertible from $H^1_{q_0,d}$ into itself, with norm smaller than 2. A simple computation shows that in this case

$$\big(\lambda - \big(A^* + B_\delta\big)\big)^{-1} = R\big(\lambda, A^*\big)\big(1 - B_\delta R\big(\lambda, A^*\big)\big)^{-1},$$

which gives that, for $\lambda \in \Sigma_\alpha$, $|\lambda| > \varepsilon$, $\|R(\lambda, L^*_\delta)\|_{H^1_{q_0,d}} \le \frac{2M}{|\lambda|}$. Consequently, the spectrum of L^*_δ is contained in

$$\Theta_{\alpha,\varepsilon} := \left\{\lambda \in \mathbb{C}, \frac{\pi}{2} + \alpha \le \arg(\lambda) \le \frac{3\pi}{2} - \alpha\right\} \cup \big\{\lambda \in \mathbb{C}, |\lambda| \le \varepsilon\big\}.$$

In particular, $0 \in \rho(L^*_{\delta,2\varepsilon})$ and for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ (hence $|\lambda| < |\lambda + 2\varepsilon|$), $\|R(\lambda, L^*_{\delta,2\varepsilon})\|_{H^1_{q_0,d}} \le \frac{M}{|\lambda+2\varepsilon|} \le \frac{M}{|\lambda|}$. The fact that this estimate can be extended to some $\Sigma_{\alpha'}$ for some α' is a consequence of a Taylor's expansion argument (see [21], Proposition 6.2), so that $L^*_{\delta,2\varepsilon}$ (and L^*_δ) is indeed sectorial.

At this point, we cannot rule out the possibility that some elements of the spectrum of L^*_δ may lie in $\Theta_{\varepsilon,\alpha} \cap \{\lambda \in \mathbb{C}, \Re(\lambda) > 0\}$. The last point of the proof is to show that one can choose ε and a smaller δ such that this situation does not hold: choose $\varepsilon = \frac{\gamma_A}{2} > 0$, where γ_A is the spectral gap of A. In particular, the circle centered in 0 with radius ε separates the eigenvalue 0 (of multiplicity 1) from the rest of the spectrum of A^*. An application of [26], Theorem IV-3.18, p. 214, shows that one can choose δ sufficiently small so that the spectrum of the perturbed operator L^*_δ is likewise separated by this circle: for such δ, there is a unique eigenvalue (with multiplicity 1) within the boundary of this circle. But we know already that 0 is an eigenvalue for the perturbed operator L^*_δ. By uniqueness, we conclude that there is no eigenvalue in the positive part of the complex plane. We leave the details of this argument to [21], Section 6.2.5.

Using [36], Corollary 10.6, p. 41, L^*_δ is the generator of the adjoint of $t \mapsto e^{tL_\delta}$ in $\mathbf{L}^2_{0,d}$, which is a C_0-semigroup. This concludes the proof of Proposition B.4. □

B.4. *Equivalence of norms*

For any $0 \le \beta \le 1$, consider the interpolation norm $\|\cdot\|_{V^\beta}$ associated to the sectorial operator $1 - L^*_\delta$ defined as

$$\|u\|_{V^\beta} = \left\|\big(1 - L^*_\delta\big)^\beta u\right\|_{1,q_0,d}. \tag{B.9}$$

Recall also the definition of the fractional norm in (A.3).

Lemma B.5. *Under the assumptions of Proposition B.4, for any $0 \le \beta \le 1$, there exists $c_1, C_1 > 0$ such that for all u,*

$$c_1 \|u\|_{1+2\beta,d} \le \|u\|_{V^\beta} \le C_1 \|u\|_{1+2\beta,d}. \tag{B.10}$$

Proof. We can decompose L_δ^* as follows:

$$L_\delta^* = \frac{1}{2}\Delta_d + R, \tag{B.11}$$

where, for all $i = -d, \dots, d$

$$\begin{aligned}(Rv)^i &= \delta\omega^i \partial_\theta v^i + \partial_\theta v^i \sum_{k=-d}^{d} \lambda^k J * q_0^k - \sum_{k=-d}^{d} \lambda^k J * \big(q_0^k(\partial_\theta v^k)\big) \\ &\quad - \int_{\mathbb{T}} \left(\partial_\theta v^i(\theta) \sum_{k=-d}^{d} \lambda^k J * q_0^k(\theta)\right) \mathrm{d}\theta. \end{aligned} \tag{B.12}$$

Since R only contains first order derivatives and J and q_0^k are smooth, it is easy to see that for all $u \in H_d^2$, we have

$$\|Ru\|_{1,d} \le C\|u\|_{2,d}. \tag{B.13}$$

One deduces immediately from this estimate that there exists a constant $C > 0$ such that for $u \in H_d^2$,

$$\big\|\big[2\big(1 - L_\delta^*\big) - (1 - \Delta_d)\big]u\big\|_{1,d} \le C\|u\|_{2,d}. \tag{B.14}$$

Consequently, the operator $[2(1 - L_\delta^*) - (1 - \Delta_d)](1 - \Delta_d)^{-1/2}$ is bounded in H_d^1. Since $1 - L_\delta^*$ is sectorial in H_d^1 with the same domain as Δ_d, an application of [24], Theorem 1.4.8 shows that the norms $\|(1 - L_\delta^*)^\beta \cdot \|_{1,d}$ and $\|(1 - \Delta_d)^\beta \cdot \|_{1,d}$ are equivalent. The norm equivalence (B.10) follows directly from the definitions (B.9) and (A.3). □

B.5. Regularity of semigroups

Recall here the definition of the projection $P^0_{\psi,\delta}$ on the kernel $\mathrm{Span}(\partial_\theta q_{\psi,\delta})$ of $L_{\psi,\delta}$ defined in Section 2.3. We drop here the dependance on ψ for simplicity. The corresponding projection on the kernel of L_δ^* is given by $P_\delta^{0,*}$. This kernel is one-dimensional, spanned by some $\theta \mapsto v_0(\theta)$ and there exists a linear form $\widetilde{p}$, bounded on H_d^1 such that, for all $u \in H_d^1$, $P_\delta^{0,*}u = \widetilde{p}(u)v_0$. Note that it is easy to see that v_0 is a regular ($\mathcal{C}^\infty$) function on $\mathbb{T}$.

Proposition B.6. *Suppose the assumptions of Proposition B.4 are true. For any $\gamma \in [0, \gamma_{L_\delta^*})$, any $\beta \in [0, 1]$ and all $t > 0$, $u \in H_d^1$,*

$$\big\|e^{tL_\delta^*}\big(1 - P_\delta^{0,*}\big)u\big\|_{1+2\beta,d} \le C\frac{e^{-\gamma t}}{t^\beta}\big\|\big(1 - P_\delta^{0,*}\big)u\big\|_{1,d}, \tag{B.15}$$

and

$$\big\|e^{tL_\delta^*}u\big\|_{1+2\beta,d} \le C\left(1 + \frac{e^{-\gamma t}}{t^\beta}\right)\|u\|_{1,d}, \tag{B.16}$$

and for all $\beta \ge 0$, $\beta' \ge 0$ such that $\beta + \beta' \le 1$ and all $h \in H_d^{1+2\beta+2\beta'}$,

$$\big\|\big(e^{tL_\delta^*} - 1\big)h\big\|_{1+2\beta',d} \le t^\beta \big\|\big(1 - P_\delta^{0,*}\big)h\big\|_{1+2\beta'+2\beta,d}. \tag{B.17}$$

Proof. Following Proposition B.4, $L_\delta^* P_\delta^{0,*} = 0$ and $L_\delta^*(1 - P_\delta^{0,*})$ is sectorial in H_d^1, with spectrum lying in $\{\lambda \in \mathbb{C} : |\arg(\lambda)| > \pi/2 + \varepsilon'\} - \gamma_{L_\delta^*}$ for some $\varepsilon' > 0$.

Let us first prove (B.15) and (B.16). Using [24], Theorem 1.4.3, (recall that $\gamma < \gamma_{L_\delta^*}$), we obtain that for all $t > 0$:

$$\big\|\big(-L_\delta^*\big)^\beta e^{tL_\delta^*}\big(1 - P_\delta^{0,*}\big)u\big\|_{1,d} \le C_\beta t^{-\beta} e^{-\gamma t}\|u\|_{1,d}. \tag{B.18}$$

Now as in the proof of Lemma B.5, we can apply [24], Theorem 1.4.8 to show that the norms induced by $(-L_\delta^*)^\beta$ and $(1 - L_\delta^*)^\beta$ are equivalent on the range of $(1 - P_\delta^{0,*})$ and we obtain for all $u \in H_d^1$

$$\big\|e^{tL_\delta^*}u\big\|_{1+2\beta,d} \le C\big\|\big(1 - L_\delta^*\big)^\beta e^{tL_\delta^*}\big(P_\delta^{0,*}u + \big(1 - P_\delta^{0,*}\big)u\big)\big\|_{1,d} \le C'_\beta\big(1 + e^{-\gamma t}t^{-\beta}\big)\|u\|_{1,d}. \tag{B.19}$$

We have used here in particular the fact that for all $u \in H_d^1$,

$$\big\|e^{tL_\delta^*}P_\delta^{0,*}u\big\|_{1+2\beta,d} \le \big|\widetilde{p}(u)\big|\big\|e^{tL_\delta^*}v_0\big\|_{1+2\beta,d} = \big|\widetilde{p}(u)\big|\|v_0\|_{1+2\beta,d} \le C\|u\|_{1,d},$$

since $\|v_0\|_{1+2\beta} < +\infty$. Concerning (B.17), remark that

$$e^{tL_\delta^*} - 1 = \big(e^{tL_\delta^*}\big(1 - P_\delta^{0,*}\big) - 1\big)\big(1 - P_\delta^{0,*}\big), \tag{B.20}$$

so applying Theorem 1.8.4 of [24] we have

$$\begin{aligned}\big\|\big(e^{tL_\delta^*} - 1\big)h\big\|_{1+2\beta',d} &\le C\big\|\big(1 - L_\delta^*\big)^{\beta'}\big(e^{tL_\delta^*}\big(1 - P_\delta^{0,*}\big) - 1\big)\big(1 - P_\delta^{0,*}\big)h\big\|_{1,d} \\ &\le C''_\beta t^\beta\big\|\big(1 - L_\delta^*\big)^{\beta'+\beta}\big(1 - P_\delta^{0,*}\big)h\big\|_{1,d} \le C'''_\beta t^\beta\big\|\big(1 - P_\delta^{0,*}\big)h\big\|_{1+2\beta'+2\beta,d}.\end{aligned} \tag{B.21}$$

This concludes the proof of Proposition B.6. □

One can deduce from Proposition B.6 a similar regularity result concerning the semigroup $t \mapsto e^{tL_\delta}$:

Proposition B.7. *For all $K \ge 1$, all $0 \le \delta < \delta(K)$, the semigroup $t \mapsto e^{tL_\delta}$ is continuous from H_d^{-2} to H_d^{-1}: for all $h \in H_d^{-2}$, $t > 0$,*

$$\big\|e^{tL_\delta}h\big\|_{-1,d} \le C\left(1 + \frac{1}{\sqrt{t}}\right)\|h\|_{-2,d}, \tag{B.22}$$

and for all $\varepsilon \in (0, 1/2)$, $t > 0$, $u \ge 0$,

$$\big\|e^{(t+u)L_\delta}h - e^{tL_\delta}h\big\|_{-1,d} \le Cu^\varepsilon\left(1 + \frac{1}{t^{1/2+\varepsilon}}\right)\|h\|_{-2,d}. \tag{B.23}$$

Proof. Let $\beta \in [0, 1]$, $t > 0$, $h \in H_d^{-1}$ and v a regular test function. Consider $(h_l)_{l\ge1}$ a sequence of elements of $\mathbf{L}_{0,d}^2$ converging to h in H_d^{-1}. For all $l \ge 1$,

$$\begin{aligned}\big|\langle e^{tL_\delta}h_l, v\rangle_d\big| &= \big|\langle e^{tL_\delta}h_l, v\rangle_{2,d}\big| = \big|\langle h_l, e^{tL_\delta^*}v\rangle_{2,d}\big| \\ &\le \|h_l\|_{-(1+2\beta),d}\big\|e^{tL_\delta^*}v\big\|_{1+2\beta,d} \le C\|h_l\|_{-(1+2\beta),d}\left(1 + \frac{1}{t^\beta}\right)\|v\|_{1,d},\end{aligned}$$

where we used (B.16) in the last inequality. Since h_l converges to h in H^{-1}, one can make $l \to \infty$ in the previous inequality and obtain $|\langle e^{tL_\delta}h, v\rangle_d| \le C\|h\|_{-(1+2\beta),d}(1 + \frac{1}{t^\beta})\|v\|_{1,d}$ and since this is true for all regular v, one deduces that

$$\big\|e^{tL_\delta}h\big\|_{-1,d} \le C\left(1 + \frac{1}{t^\beta}\right)\|h\|_{-(1+2\beta),d}, \tag{B.24}$$

which gives (B.15) when $\beta = \frac{1}{2}$. In the same way, an immediate corollary of (B.17) is that for all $\beta \geq 0$, $\beta' \geq 0$ such that $\beta + \beta' \leq 1$, for all $t > 0$

$$\left\| \left(e^{tL_\delta} - 1\right)h \right\|_{-(1+2\beta+2\beta'),d} \leq t^\beta \|h\|_{-(1+2\beta'),d}. \tag{B.25}$$

We now turn to the proof of (B.23). Fix $\varepsilon \in (0, 1/2)$ and apply (B.24) for $\beta = 1/2 + \varepsilon$ and (B.25) for $\beta = \varepsilon$ and $\beta' = \frac{1}{2}$,

$$\begin{aligned}\left\| e^{(t+u)L_\delta} h - e^{tL_\delta} h \right\|_{-1,d} &\leq C\left(1 + \frac{1}{t^{1/2+\varepsilon}}\right) \left\| \left(e^{uL_\delta} - 1\right)h \right\|_{-(2+2\varepsilon),d} \\ &\leq Cu^\varepsilon \left(1 + \frac{1}{t^{1/2+\varepsilon}}\right) \|h\|_{-2,d}.\end{aligned}$$

This concludes the proof of Proposition B.7. □

Appendix C: Projections

The purpose of this section is to prove several regularity results concerning the projection $P^0_{\psi,\delta} u = \mathrm{p}_{\psi,\delta}(u) \partial_\theta q_{\psi,\delta}$ (recall Section 2.3 and (2.16)) and the projection on the manifold M $\mathrm{proj}_M(\cdot)$ defined in Lemma 2.9.

Proof of Lemma 2.9. We first prove that $\psi \mapsto \mathrm{p}_\psi$ is smooth. This follows from the fact that the whole operator L_ψ is regular in $\psi \in \mathbb{T}$: we prove indeed that the mapping $\psi \mapsto L_\psi$ is in fact real holomorphic, in the sense of Kato [26], p. 375. Since the problem is invariant by rotation, it suffices to study the regularity of L_ψ is a neighborhood of $\psi = 0$. From the definition of the stationary solution q in (1.8), it is straightforward to see that one can expand q_ψ in series of ψ around $\psi = 0$:

$$q_\psi(\theta) = q_0(\theta) + \sum_{k \geq 1} \frac{\psi^k}{k!} \partial^k_\psi q_{\psi}{}_{|\psi=0}(\theta).$$

From this expansion, one deduces a similar expansion for L_ψ around $\psi = 0$: for all f regular

$$L_\psi f = L_0 f + \sum_{k \geq 1} \psi^k U_k f,$$

where each U_k is a differential operator of order 1, so that each U_k is relatively-bounded w.r.t. L_0. In particular the hypotheses of [26], Theorem 2.6, p. 377 are satisfied. In particular, $(L_\psi)_\psi$ forms a real-holomorphic family. In particular, the mapping $\psi \mapsto P^0_\psi$ is also regular ([26], Theorem 1.7, p. 368), and so is the mapping $\psi \mapsto \mathrm{p}_\psi$. Then the mapping $f(\psi, h) = \mathrm{p}_\psi(h - q_\psi)$ satisfies for each fixed ψ_0, $f(\psi_0, q_{\psi_0}) = 0$ and $\partial_\psi f(\psi_0, q_{\psi_0}) = -\mathrm{p}_{\psi_0} \partial_\psi q_{\psi_0} = -1$. So by the implicit function theorem, for all h in a certain neighborhood of q_{ψ_0}, there exists a unique $\psi =: \mathrm{proj}_M(h)$ such that $f(\psi, h) = 0$ and $h \mapsto \mathrm{proj}_M(h)$ is smooth. □

The next result states that the first order of the projection proj_M around q_ψ is given by the linear form p_ψ defined in (2.16).

Lemma C.1. *For $\psi \in \mathbb{T}$, $h \in H^{-1}_d$ such that $\mathrm{proj}_M(q_\psi + h)$ is well-defined, we have*

$$\mathrm{proj}_M(q_\psi + h) = \psi + \mathrm{p}_\psi(h) + O\left(\|h\|^2_{-1,d}\right). \tag{C.1}$$

Proof. Consider the real u such that $\mathrm{proj}_M(q_\psi + h) = \psi + u$. Due to the smoothness of proj_M, we have $u = O(\|h\|_{-1,d})$. The real number u satisfies

$$\mathrm{p}_{\psi+u}(q_\psi + h - q_{\psi+u}) = 0. \tag{C.2}$$

A first order expansion leads to

$$\mathtt{p}_\psi(h - u\partial_\psi q_\psi) = O(u^2), \tag{C.3}$$

which gives the result, since $\mathtt{p}_\psi(\partial_\psi q_\psi) = 1$. □

Appendix D: Expansions in δ

The aim of this section is to obtain first order asymptotic of the drift in Theorem 2.4 for small δ. We use the notations q_δ, $\mathtt{p}_\delta$ as in Section 7.2, putting the emphasis on the dependency of the different terms in δ. We denote also as $r_\delta > 0$ the unique positive solution to the fixed point relation $r_\delta = \Psi_\delta(2Kr_\delta)$ (recall (1.10)). We begin with a result concerning r_δ as $\delta \to 0$:

Lemma D.1. *The mapping $\delta \mapsto r_\delta$ is C^∞ and its derivative $r'(0)$ at $\delta = 0$ is zero, so that as $\delta \to 0$:*

$$r_\delta = r_0 + O(\delta^2), \tag{D.1}$$

where r_0 is the unique non-trivial solution of the fixed-point problem without disorder (7.5).

Proof. Consider the C^∞ mapping $g(r,\delta) = \Psi_\delta(2Kr) - r$. This mapping satisfies $\partial_r g(r_0, 0) = 2K\partial_x\Psi_0(2Kr_0) - 1$. The fixed-point function $r \mapsto \Psi_0(2Kr_0)$ is strictly convex when $K > 1$ ([37], Lemma 4), with derivative at the origin strictly greater than 1. One concludes that the derivative at the fixed point $r_0 > 0$ is strictly smaller than 1. Since this derivative is precisely equal to $2K\partial_x\Psi_0(2Kr_0)$, this shows that $\partial_r g(r_0, 0) < 0$. So the implicit function theorem implies that $\delta \mapsto r_\delta$ is C^∞. Using (1.10), one obtains that

$$r'(0) = \partial_\delta\Psi_\delta|_{\delta=0}(2Kr_0) + r'(0)2K\partial_x\Psi_0(2Kr_0). \tag{D.2}$$

Since $2K\partial_x\Psi_0(2Kr_0) < 1$, the proof of Lemma D.1 will be finished once we have proved that $\partial_\delta\Psi_\delta|_{\delta=0}(2Kr_0) = 0$. One has (recall the definition of $\mathcal{Z}_0$ in (7.4))

$$\begin{aligned}\partial_\delta\Psi_\delta|_{\delta=0}(2Kr_0) = \sum_{k=-d}^{d}\lambda^k\biggl(&\frac{\int_0^{2\pi}\cos(\theta)\partial_\delta S_\delta^k|_{\delta=0}(\theta, 2Kr_0)\,\mathrm{d}\theta}{\mathcal{Z}_0(2Kr_0)^2}\\ &-\frac{\int_0^{2\pi}\cos(\theta)S_0(\theta, 2Kr_0)}{\mathcal{Z}_0(2Kr_0)^4}\partial_\delta Z_\delta^k|_{\delta=0}(2Kr_0)\biggr).\end{aligned} \tag{D.3}$$

Some straightforward calculations show that, for all $k = -d, \dots, d$, $\theta \in \mathbb{T}$

$$\begin{aligned}\partial_\delta S_\delta^k|_{\delta=0}(\theta, 2Kr_0) = 2\omega^k e^{2Kr_0\cos(\theta)}\biggl(&\theta\int_0^{2\pi} e^{2Kr_0\cos(u)}\,\mathrm{d}u + 2\pi\int_\theta^{2\pi} e^{-2Kr_0\cos(u)}\,\mathrm{d}u\\ &-\int_0^{2\pi} ue^{-2Kr_0\cos(u)}\,\mathrm{d}u\biggr)\end{aligned} \tag{D.4}$$

and

$$\begin{aligned}\partial_\delta Z_\delta^k|_{\delta=0}(2Kr_0) &= 2\omega^k\biggl(2\pi\int_0^{2\pi} e^{2Kr_0\cos(\theta)}\int_\theta^{2\pi} e^{-2Kr_0\cos(u)}\,\mathrm{d}u\,\mathrm{d}\theta\\ &\qquad + \mathcal{Z}_0(2Kr_0)\int_0^{2\pi} u\bigl(e^{2Kr_0\cos(u)} - e^{-2Kr_0\cos(u)}\bigr)\,\mathrm{d}u\biggr)\\ &= 4\pi\omega^k\int_0^{2\pi} e^{2Kr_0\cos(\theta)}\int_\theta^{2\pi} e^{-2Kr_0\cos(u)}\,\mathrm{d}u\,\mathrm{d}\theta.\end{aligned} \tag{D.5}$$

Since $\sum_{k=-d}^{d} \lambda^k \omega^k = 0$, one obtains from (D.3), (D.4) and (D.5) that $\partial_\delta \Psi_\delta|_{\delta=0}(2Kr_0) = 0$. This concludes the proof of Lemma D.1. □

We now turn to the proof of Lemma 7.1:

Proof of Lemma 7.1. Obviously, for $\theta \in \mathbb{T}$,

$$q_\delta^i(\theta) = q_0(\theta) + \delta \partial_\delta q_\delta|_{\delta=0}(\theta) + O\big(\delta^2\big),$$

where the error $O(\delta^2)$ does not depend on $\theta \in \mathbb{T}$. The fact that $r'(0) = 0$ (Lemma D.1) implies that $\partial_\delta q_\delta|_{\delta=0}(\theta)$ only depends on the derivatives of S_δ and Z_δ w.r.t. δ, not w.r.t. x. Namely,

$$\partial_\delta q_\delta^i|_{\delta=0}(\theta) = \frac{\partial_\delta S_\delta^i|_{\delta=0}(\theta, 2Kr_0)}{\mathcal{Z}_0(2Kr_0)^2} - \frac{\partial_\delta Z_\delta^i|_{\delta=0}(2Kr_0) S_0^i(\theta, 2Kr_0)}{\mathcal{Z}_0(2Kr_0)^4}.$$

The expansion found in (7.6) is a simple consequence of (D.4), (D.5) and the expression of $\mathcal{Z}_0$ in (7.4). □

Proof of Lemma 7.2. In the case $\delta = 0$, the projection p_0 defined in (2.16) is given by $P_0^0(u) = \mathrm{p}_0(u)(\partial_\theta q_0, \dots, \partial_\theta q_0) = \mathrm{p}_0(u)\partial_\theta q_{0,nd}$. Since in this case, the operator $L_0 = A$ defined in (B.2) is essentially self-adjoint in $H^{-1}_{1/q_0,d}$ (Proposition B.1), the projection p_0 as a natural representation in terms of the scalar product $\langle \cdot, \cdot \rangle_{-1,1/q_0,d}$ associated to the norm defined in (A.2), namely

$$\mathrm{p}_0(u) = \frac{\langle \partial_\theta q_{0,nd}, u \rangle_{-1,1/q_0,d}}{\| \partial_\theta q_{0,nd} \|^2_{-1,1/q_0,d}}. \tag{D.6}$$

Using the notations of Appendix A, we deduce that

$$\| \partial_\theta q_{0,nd} \|^2_{-1,1/q_0,d} = \int_0^{2\pi} \frac{(q_0(\theta) - \frac{2\pi}{\mathcal{Z}_0^2})^2}{q_0} \, \mathrm{d}\theta = 1 - \frac{4\pi^2}{\mathcal{Z}_0^2},$$

and

$$\begin{aligned} \langle \partial_\theta q_{0,nd}, u \rangle_{-1,1/q_0,d} &= \sum_{k=-d}^{d} \lambda^k \int_0^{2\pi} \frac{\mathcal{U}^k(\theta)(q_0(\theta) - \frac{2\pi}{\mathcal{Z}_0^2})}{q_0(\theta)} \, \mathrm{d}\theta \\ &= \sum_{k=-d}^{d} \lambda^k \int_0^{2\pi} \mathcal{U}^k(\theta)\left(1 - \frac{2\pi}{\mathcal{Z}_0^2 q_0(\theta)}\right) \mathrm{d}\theta, \end{aligned}$$

which precisely gives the first order of (7.8). The validity of (7.8) comes from the definition of the projection P_δ^0 in (2.16) and the fact that L_δ is a relatively bounded perturbation of order δ of the operator $L_0 = A$. □

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