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Some examples of quenched self-averaging in models with Gaussian disorder

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Abstract. In this paper we give an elementary approach to several results of Chatterjee in (Disorder chaos and multiple valleys in spin glasses (2013) arXiv:0907.3381, *Comm. Math. Phys.* **337** (2015) 93–102), as well as some generalizations. First, we prove quenched disorder chaos for the bond overlap in the Edwards–Anderson type models with Gaussian disorder. The proof extends to systems at different temperatures and covers a number of other models, such as the mixed p-spin model, Sherrington–Kirkpatrick model with multi-dimensional spins and diluted p-spin model. Next, we adapt the same idea to prove quenched self-averaging of the bond magnetization for one system and use it to show quenched self-averaging of the site overlap for random field models with positively correlated spins. Finally, we show self-averaging for certain modifications of the random field itself.

Résumé. Dans cet article, nous présentons une approche élémentaire de plusieurs résultats de Chatterjee (Disorder chaos and multiple valleys in spin glasses (2013) arXiv:0907.3381, *Comm. Math. Phys.* **337** (2015) 93–102), et quelques généralisations. D'abord, nous prouvons, dans le cas d'un désordre quenched, un résultat de chaos pour le recouvrement par arêtes dans les modèles de type Edwards–Anderson avec désordre gaussien. La preuve s'étend à des systèmes à différentes températures et couvre d'autres modèles comme le modèle p-spins mixte, le modèle de Sherrington–Kirkpatrick avec des spins multi-dimensionnels et le modèle p-spin dilué. Ensuite, nous adaptons la même idée pour prouver la propriété d'auto-moyennisation du recouvrement par site et nous l'utilisons pour montrer le même résultat pour des modèles avec un champs aléatoire et des spins positivement corrélés. Enfin, nous montrons la propriété d'auto-moyennisation pour certaines modifications du champs aléatoire.

1. Introduction

The approach developed in this paper was motivated by several results of Chatterjee in [2,4]. One of the results in [2] described a quenched disorder chaos for the bond overlap in the setting of the Edwards–Anderson type spin glass models. Consider a finite undirected graph (V, E) and the Edwards–Anderson type Hamiltonian

$$H(\sigma) = \sum_{(i,j) \in E} g_{i,j} \sigma_i \sigma_j, \tag{1}$$

where $\sigma = (\sigma_i)_{i \in V} \in \{-1, +1\}^V$ and $g_{i,j}$ are i.i.d. standard Gaussian random variables. Given an inverse temperature parameter $\beta > 0$, the corresponding Gibbs measure is defined by

$$G(\sigma) = \frac{\exp \beta H(\sigma)}{Z}, \tag{2}$$

where $Z = \sum_\sigma \exp \beta H(\sigma)$ is called the partition function. Now, let us consider two copies of this system with different disorder parameters $(g^1_{i,j})$ and $(g^2_{i,j})$. We will denote the Hamiltonians and Gibbs measures of these systems by $H_1(\sigma)$, $H_2(\rho)$ and $G_1(\sigma)$, $G_2(\rho)$. Suppose that the disorder parameters of these two systems are correlated,

$$\mathbb{E} g^1_{i,j} g^2_{i,j} = t, \tag{3}$$

for some $t \in [0, 1]$. We still assume that $(g^1_{i,j}, g^2_{i,j})$ are independent for different $(i, j) \in E$. When $t = 1$, this gives us two copies of the same system, and the interesting case is when t is slightly smaller than one, so the interaction parameters of these two systems are slightly decoupled. Note that in [2] and [3] the correlation was written as e^{-2s} for $s \in [0, \infty)$, which is the same as our $t = e^{-2s}$. Consider i.i.d. samples $(\sigma^\ell)_{\ell \geq 1}$ from G_1 and $(\rho^\ell)_{\ell \geq 1}$ from G_2. The quantity

$$Q_{\ell, \ell'} = \frac{1}{|E|} \sum_{(i,j) \in E} \sigma^\ell_i \sigma^\ell_j \rho^{\ell'}_i \rho^{\ell'}_j \tag{4}$$

is called the bond overlap of configurations σ^ℓ and $\rho^{\ell'}$, which is a measure of similarity between bonds in these two configurations. Of course, one can similarly define the bond overlap of σ^ℓ and $\sigma^{\ell'}$, but here one is interested in the behavior of the bond overlap between two slightly decoupled systems. Up to a normalization factor $|E|$, the bond overlap is the covariance

$$\mathbb{E} H(\sigma^\ell) H(\rho^{\ell'}) = |E| Q_{\ell, \ell'}$$

of the Hamiltonian H in (1). Let us denote by $\langle \cdot \rangle$ the average with respect to $(G_1 \times G_2)^{\otimes \infty}$.

In Theorem 1.7 in [2] (see Theorem 11.5 in [3]), Chatterjee proved that, for any $0 < t < 1$,

$$\mathbb{E}\langle (Q_{1,1} - \langle Q_{1,1} \rangle)^2 \rangle \leq \frac{2\sqrt{2}}{\beta t^{1/4} \sqrt{|E| \log(1/t)}}. \tag{5}$$

This shows that for $t < 1$ and large $|E|$, the bond overlap $Q_{1,1}$ between replicas from these two systems concentrates around its Gibbs average $\langle Q_{1,1} \rangle$. The first goal of this paper will be to give an elementary proof of essentially the same inequality,

$$\mathbb{E}\langle (Q_{1,1} - \langle Q_{1,1} \rangle)^2 \rangle \leq \frac{8}{\beta \sqrt{|E|(1-t)}}, \tag{6}$$

as well as some generalizations. First of all, in addition to the proof being elementary, we get a better dependence on t when t approaches zero, which covers the case $t = 0$. In the case when t is close to 1, the dependence on t is the same, since $\log(1/t)$ is of order $1 - t$ in that case. Moreover, the same proof will give us quenched disorder chaos for two systems with different inverse temperature parameters β_1 and β_2, in which case (6) will be replaced by

$$\mathbb{E}\langle (Q_{1,1} - \langle Q_{1,1} \rangle)^2 \rangle \leq \frac{4(\beta_1 + \beta_2)}{\beta_1 \beta_2 \sqrt{|E|(1-t)}}. \tag{7}$$

It is not clear to us how to adapt Chatterjee's proof to this case because his approach heavily relies on the complete monotonicity of the quantity $\mathbb{E}\langle Q_{1,1} \rangle$ in $s := 2^{-1} \log t^{-1}$, which does not seem to hold when β_1 and β_2 are different. In Section 2, we will formulate a general disorder chaos result that will cover other examples in addition to the Edwards–Anderson type models, such as the mixed p-spin model, Sherrington–Kirkpatrick model with multi-dimensional spins, and diluted p-spin model.

In the second paper, [4], Chatterjee studied the random field Ising model on the d-dimensional lattice with the Hamiltonian

$$H(\sigma) = \beta \sum_{i \sim j} \sigma_i \sigma_j + h \sum_i g_i \sigma_i, \tag{8}$$

where $\sigma \in \{-1, +1\}^V$ for $V = \mathbb{Z}^d \cap [1, N]^d$, $i \sim j$ means that i and j are neighbors on this lattice, $\beta, h > 0$, and g_i are i.i.d. standard Gaussian random variables. The main goal in [4] was to show that for almost all values β and h, in the thermodynamic limit, the site overlap

$$R_{1,2} = \frac{1}{|V|} \sum_{i \in V} \sigma_i^1 \sigma_i^2$$

between two replicas σ^1 and σ^2 concentrates around a constant that depends only on β and h. We are not going to reproduce the entire proof, but will give simplified proofs of two key steps. The first key step was to show quenched self-averaging of the overlap,

$$\mathbb{E}\langle (R_{1,2} - \langle R_{1,2} \rangle)^2 \rangle \leq \frac{2\sqrt{2 + h^2}}{h\sqrt{|V|}}, \tag{9}$$

as a consequence of positive correlation of spins, which in this model follows from the FKG inequality [9]. Our approach in Section 4 will also remove the factor $\sqrt{2 + h^2}$. It will be based on some general result about quenched self-averaging of the bond magnetization for one system in Section 3.

Another key step in [4] was to show that the normalized random field

$$h(\sigma) = \frac{1}{|V|} \sum_{i \in V} g_i \sigma_i$$

concentrates around its quenched average $\langle h(\sigma) \rangle$,

$$\mathbb{E}\big(\langle h(\sigma)^2 \rangle - \langle h(\sigma) \rangle^2\big) \leq \frac{\sqrt{24}}{h\sqrt{|V|}} + \frac{1}{|V|}. \tag{10}$$

This step holds more generally and does not depend on the condition that the spins are positively correlated. Again, we will give a simplified proof of a slightly improved bound in Section 5 (see equations (40) and (46)), as well as certain generalizations (the most general statement appears in Theorem 6 in that section).

All the proofs of the results mentioned above are based on the same idea and follow the same simple pattern, which combines standard techniques of Gaussian interpolation and integration by parts with clever choices of observables and Gaussian functionals on which these observables are projected. The arguments are then reduced to simple applications of the Cauchy–Schwarz inequality, whereas the methods in [4] were based on the Poincaré inequality for the Gaussian measure. The simplicity of our approach results in better dependence of the bounds on the external parameters.

2. Quenched disorder chaos

We will formulate the main result of this section in a way that will cover a number of models as examples. We will consider two systems with the Hamiltonians

$$Y_1(\sigma) = \sum_{e \in E} g_e^1 f_e(\sigma), \tag{11}$$

$$Y_2(\rho) = \sum_{e \in E} g_e^2 f_e(\rho), \tag{12}$$

defined on the same measurable space $(\Sigma, \mathcal{F})$ (i.e. both $\sigma, \rho \in \Sigma$), which will usually be some finite set. Here the set E is some finite index set, $(f_e)_{e \in E}$ is a family of measurable functions $f_e : \Sigma \to [-1, 1]$, and (g_e^1, g_e^2) are independent Gaussian random pairs for $e \in E$ such that

$$\mathbb{E}(g_e^1)^2 = \mathbb{E}(g_e^2)^2 = 1 \quad \text{and} \quad \mathbb{E}g_e^1 g_e^2 = t \tag{13}$$

for some $t \in [0, 1]$. We can allow the functions $(f_e)_{e \in E}$ be random as long as their randomness is independent of the Gaussian random variables (g_e^1, g_e^2), but in all the examples below they will be non-random. In some models, such as diluted models, the cardinality of the index set E can be random as well and, in that case, we will also assume it to be independent of the Gaussian random variables (g_e^1, g_e^2). We will state our result for a fixed E, since one can average in $|E|$ later, as we will do, for example, in the diluted models.

Next, we consider the corresponding Gibbs measures G_1 and G_2 on $(\Sigma, \mathcal{F})$,

$$dG_1(\sigma) = \frac{\exp \gamma_1 Y_1(\sigma)}{Z_1} d\mu_1(\sigma), \tag{14}$$

$$dG_2(\rho) = \frac{\exp \gamma_2 Y_2(\rho)}{Z_2} d\mu_2(\rho), \tag{15}$$

where $\gamma_1, \gamma_2 > 0$ are some fixed constants, μ_1 and μ_2 are random finite measures on $(\Sigma, \mathcal{F})$ and Z_1, Z_2 are the partition functions. The randomness of μ_1 and μ_2 should be independent of the Gaussian random variables (g_e^1, g_e^2) but not necessarily of other random variables or each other. As above, we will consider i.i.d. replicas $(\sigma^\ell)_{\ell \geq 1}$ from G_1 and $(\rho^\ell)_{\ell \geq 1}$ from G_2, let $\langle \cdot \rangle$ denote the average with respect to $(G_1 \times G_2)^{\otimes \infty}$, and define the overlaps by

$$Q_{\ell, \ell'} = \frac{1}{|E|} \sum_{e \in E} f_e(\sigma^\ell) f_e(\rho^{\ell'}). \tag{16}$$

Then the following quenched disorder chaos for the overlap holds.

Theorem 1. *If $\gamma_1, \gamma_2 > 0$ and $t \in [0, 1)$ then*

$$\mathbb{E}\langle (Q_{1,1} - \langle Q_{1,1} \rangle)^2 \rangle \leq \frac{4(\gamma_1 + \gamma_2)}{\gamma_1 \gamma_2 \sqrt{|E|(1-t)}}. \tag{17}$$

Proof. The proof is based on a simple computation first used in the derivation of the (two-system) Ghirlanda–Guerra identities for the mixed p-spin model in Chen, Panchenko [6] and Chen [7] (for related results about disorder chaos, see also [5]). Because of the assumption (13), we can represent

$$Y_1(\sigma) = \sqrt{t} Z(\sigma) + \sqrt{1-t} Z_1(\sigma),$$
$$Y_2(\rho) = \sqrt{t} Z(\rho) + \sqrt{1-t} Z_2(\rho),$$

where, given i.i.d. standard Gaussian random variables z_e, z_e^1 and z_e^2 indexed by $e \in E$,

$$Z(\sigma) = \sum_{e \in E} z_e f_e(\sigma), \qquad Z(\rho) = \sum_{e \in E} z_e f_e(\rho),$$
$$Z_1(\sigma) = \sum_{e \in E} z_e^1 f_e(\sigma), \qquad Z_2(\rho) = \sum_{e \in E} z_e^2 f_e(\rho).$$

Let us consider the quantity

$$\mathbb{E}\left\langle Q_{1,1} \frac{Z_1(\rho^1)}{|E|} \right\rangle.$$

Notice that $Z_1(\rho^1)$ is a new object, with the randomness coming from the second term in the Hamiltonian Y_1 on the first system, and the argument ρ^1 that is a replica from the second system and is averaged with respect to G_2. As a result, if $\mathbb{E}'$ denotes the expectation in the Gaussian random variables z_e, z_e^1 and z_e^2, then

$$\mathbb{E}' Y_1(\sigma^\ell) Z_1(\rho^1) = \sqrt{1-t} |E| Q_{\ell,1}, \qquad \mathbb{E}' Y_2(\rho^\ell) Z_1(\rho^1) = 0,$$

and the usual Gaussian integration by parts (see e.g. [10], Appendix A.4) gives

$$\mathbb{E}\left\langle Q_{1,1} \frac{Z_1(\rho^1)}{|E|} \right\rangle = \gamma_1 \sqrt{1-t}\, \mathbb{E}\langle Q_{1,1}^2 - Q_{1,1} Q_{2,1}\rangle.$$

On the other hand, since $|Q_{1,1}| \le 1$,

$$\left| \mathbb{E}\left\langle Q_{1,1} \frac{Z_1(\rho^1)}{|E|} \right\rangle \right| \le \mathbb{E}\left\langle \frac{|Z_1(\rho^1)|}{|E|} \right\rangle.$$

The average on the right-hand side is with respect to $dG_2(\rho^1)$ only, which is independent of the Gaussian random variables z_e^1 that appear in $Z_1(\rho)$, so

$$\mathbb{E}\left\langle \frac{|Z_1(\rho^1)|}{|E|} \right\rangle = \mathbb{E}\left\langle \frac{\mathbb{E}_1 |Z_1(\rho^1)|}{|E|} \right\rangle,$$

where $\mathbb{E}_1$ is the expectation with respect to $(z_e^1)_{e \in E}$. Finally, since

$$\mathbb{E}_1 |Z_1(\rho^1)| \le \left(\mathbb{E}_1 Z_1(\rho^1)^2 \right)^{1/2} = \left(\sum_{e \in E} f_e(\rho^1)^2 \right)^{1/2} \le |E|^{1/2},$$

we prove that

$$\left| \gamma_1 \sqrt{1-t}\, \mathbb{E}\langle Q_{1,1}^2 - Q_{1,1} Q_{2,1}\rangle \right| = \left| \mathbb{E}\left\langle Q_{1,1} \frac{Z_1(\rho^1)}{|E|} \right\rangle \right| \le \frac{1}{\sqrt{|E|}}. \tag{18}$$

Next, by symmetry, $\langle Q_{2,1}^2 \rangle = \langle Q_{1,1}^2 \rangle$ and, therefore,

$$\mathbb{E}\langle (Q_{1,1} - Q_{2,1})^2 \rangle = 2\mathbb{E}\langle Q_{1,1}^2 - Q_{1,1} Q_{2,1} \rangle \le \frac{2}{\gamma_1 \sqrt{|E|(1-t)}},$$

where in the last inequality we used (18). Similarly, one can show that

$$\mathbb{E}\langle (Q_{2,2} - Q_{2,1})^2 \rangle \le \frac{2}{\gamma_2 \sqrt{|E|(1-t)}}.$$

Combining the above two inequalities and using Jensen's inequality,

$$\begin{aligned}
\mathbb{E}\left\langle \left(Q_{1,1} - \langle Q_{1,1}\rangle \right)^2 \right\rangle &\le \mathbb{E}\langle (Q_{1,1} - Q_{2,2})^2 \rangle \\
&\le 2\mathbb{E}\langle (Q_{1,1} - Q_{2,1})^2 \rangle + 2\mathbb{E}\langle (Q_{2,2} - Q_{2,1})^2 \rangle \\
&\le \frac{4}{\gamma_1 \sqrt{|E|(1-t)}} + \frac{4}{\gamma_2 \sqrt{|E|(1-t)}}.
\end{aligned}$$

This finishes the proof. $\qquad\square$

We will now give several examples of applications of Theorem 1. Since all the arguments are very similar, we will only give a detailed discussion of the mixed p-spin model.

Example 1 (Mixed p-spin model). *The Hamiltonian of the mixed p-spin model is given by*

$$H(\sigma) = \sum_{p \ge 1} \frac{\beta_p}{N^{(p-1)/2}} \sum_{1 \le i_1, \dots, i_p \le N} g_{i_1, \dots, i_p} \sigma_{i_1} \cdots \sigma_{i_p},$$

where $\sigma \in \Sigma_N := \{-1, +1\}^N$, $(\beta_p)_{p \geq 1}$ is a sequence of inverse temperature parameters such that $\beta_p \geq 0$ for all $p \geq 1$ and $\sum_{p \geq 1} 2^p \beta_p^2 < \infty$, and $g_{i_1,\ldots,i_p}$ are i.i.d. standard Gaussian for all $p \geq 1$ and all $1 \leq i_1, \ldots, i_p \leq N$. Let us now consider two such systems,

$$H_1(\sigma) = \sum_{p \geq 1} \frac{\beta_{1,p}}{N^{(p-1)/2}} \sum_{1 \leq i_1,\ldots,i_p \leq N} g^1_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p},$$

$$H_2(\rho) = \sum_{p \geq 1} \frac{\beta_{2,p}}{N^{(p-1)/2}} \sum_{1 \leq i_1,\ldots,i_p \leq N} g^2_{i_1,\ldots,i_p} \rho_{i_1} \cdots \rho_{i_p},$$

with the Gaussian interaction parameters coupled according to some sequence $(t_p)_{p \geq 1}$,

$$\mathbb{E}\big(g^1_{i_1,\ldots,i_p}\big)^2 = \mathbb{E}\big(g^2_{i_1,\ldots,i_p}\big)^2 = 1 \quad and \quad \mathbb{E}g^1_{i_1,\ldots,i_p} g^2_{i_1,\ldots,i_p} = t_p \in [0, 1].$$

Suppose that for some $p \geq 1$, $\beta_{1,p}, \beta_{2,p} > 0$ and $t_p < 1$. Let Y_1 and Y_2 be the p-spin terms in H_1 and H_2 correspondingly. This means that in (11)*, we should set $E = \{1, \ldots, N\}^p$, for $e = (i_1, \ldots, i_p) \in E$ define $f_e(\sigma) = \sigma_{i_1} \cdots \sigma_{i_p}$ for all $\sigma \in \Sigma_N$, and let*

$$\gamma_1 = \frac{\beta_{1,p}}{N^{(p-1)/2}} \quad and \quad \gamma_2 = \frac{\beta_{2,p}}{N^{(p-1)/2}}.$$

In this case, the bond overlap $Q_{1,1}$ will be equal to

$$Q_{1,1} = \frac{1}{N^p} \sum_{1 \leq i_1,\ldots,i_p \leq N} \sigma^1_{i_1} \cdots \sigma^2_{i_p} \rho^1_{i_1} \cdots \rho^1_{i_p} = (R_{1,1})^p,$$

where $R_{1,1} = N^{-1} \sum_{i=1}^N \sigma^1_i \rho^1_i$ is the usual site overlap. Finally, we can write the Gibbs measures corresponding to H_1 and H_2 as

$$G_1(\sigma) = \frac{\exp \gamma_1 Y_1(\sigma)}{Z_1} \mu_1(\sigma), \qquad G_2(\rho) = \frac{\exp \gamma_2 Y_2(\rho)}{Z_2} \mu_2(\rho),$$

where we denoted

$$\mu_1(\sigma) = \exp\big(H_1(\sigma) - \gamma_1 Y_1(\sigma)\big), \qquad \mu_2(\rho) = \exp\big(H_2(\rho) - \gamma_2 Y_2(\rho)\big).$$

By construction, these measures are independent of the Gaussian random variables in Y_1 and Y_2. Theorem 1 *implies that*

$$\mathbb{E}\big\langle\big((R_{1,1})^p - \big\langle(R_{1,1})^p\big\rangle\big)^2\big\rangle \leq \frac{4(\gamma_1 + \gamma_2)}{\gamma_1 \gamma_2 \sqrt{N^p(1 - t_p)}} = \frac{4(\beta_{1,p} + \beta_{2,p})}{\beta_{1,p}\beta_{2,p}\sqrt{N(1 - t_p)}}. \tag{19}$$

Clearly, for odd p this implies that $R_{1,1} \approx \langle R_{1,1}\rangle$ and for even p this implies that $|R_{1,1}| \approx \langle |R_{1,1}|\rangle$. This example was one of the main results in [6].

Example 2 (SK model with multidimensional spins). *Let S be a bounded Borel measurable subset of $\mathbb{R}^d$ and ν be a probability measure on $\mathcal{B}(S)$. Consider the configuration space*

$$\Sigma_N = \big\{(x_1, \ldots, x_N) : x_1 = (x_{1,u})_{1 \leq u \leq d}, \ldots, x_N = (x_{N,u})_{1 \leq u \leq d} \in S\big\}.$$

Consider the Hamiltonians and Gibbs measures of two SK type models with multidimensional spins on Σ_N,

$$H_1(\sigma) = \frac{\beta_1}{\sqrt{N}} \sum_{1 \leq i,j \leq N} g^1_{i,j}(\sigma_i, \sigma_j), \qquad dG_1(\sigma) = \frac{\exp H_1(\sigma)}{Z_1} d\nu(\sigma),$$

$$H_2(\rho) = \frac{\beta_2}{\sqrt{N}} \sum_{i \leq i,j \leq N} g^2_{i,j}(\rho_i, \rho_j), \qquad dG_2(\rho) = \frac{\exp H_2(\rho)}{Z_2} d\nu(\rho),$$

where (a, b) is the scalar product on $\mathbb{R}^d$, $\beta_1, \beta_2 > 0$, and $(g^1_{i,j}, g^2_{i,j})$ are independent Gaussian random vectors with covariance

$$\mathbb{E}(g^1_{i,j})^2 = \mathbb{E}(g^2_{i,j})^2 = 1 \quad and \quad \mathbb{E}g^1_{i,j}g^2_{i,j} = t \in [0, 1].$$

The bond overlap $Q_{1,1}$ will be defined in this case by

$$Q_{1,1} = \frac{1}{N^2} \sum_{1 \leq i, j \leq N} (\sigma^1_i, \sigma^1_j)(\rho^1_i, \rho^1_j) = \sum_{u,v=1}^{d} \left(\frac{1}{N} \sum_{i=1}^{N} \sigma^1_{i,u}\rho^1_{i,v} \right)^2,$$

and it is easy to see that Theorem 1 implies that

$$\mathbb{E}\langle (Q_{1,1} - \langle Q_{1,1} \rangle)^2 \rangle \leq \frac{4(\beta_1 + \beta_2)}{\beta_1\beta_2\sqrt{N(1-t)}} \tag{20}$$

for $t < 1$.

Example 3 (Diluted p-spin model). Let $\pi(\lambda N)$ be a Poisson random variable with mean λN and $(i_{j,k})_{j,k \geq 1}$ be i.i.d. uniform random variables on $\{1, \ldots, N\}$. Consider two diluted p-spin models,

$$H_1(\sigma) = \beta_1 \sum_{k \leq \pi(\lambda N)} g^1_k \sigma_{i_{1,k}} \cdots \sigma_{i_{p,k}}, \qquad G_1(\sigma) = \frac{\exp H_1(\sigma)}{Z_1},$$

$$H_2(\rho) = \beta_2 \sum_{k \leq \pi(\lambda N)} g^2_k \rho_{i_{1,k}} \cdots \rho_{i_{p,k}}, \qquad G_2(\rho) = \frac{\exp H_2(\rho)}{Z_2},$$

where $\beta_1, \beta_2 > 0$ and $(g^1_k, g^2_k)_{k \geq 1}$ are independent Gaussian random vectors with covariance

$$\mathbb{E}(g^1_k)^2 = \mathbb{E}(g^2_k)^2 = 1 \quad and \quad \mathbb{E}g^1_k g^2_k = t \in [0, 1].$$

If we define the bond overlap $Q_{1,1}$ by

$$Q_{1,1} = \frac{1}{\pi(\lambda N)} \sum_{k=1}^{\pi(\lambda N)} \sigma^1_{i_{1,k}} \cdots \sigma^1_{i_{p,k}} \rho^1_{i_{1,k}} \cdots \rho^1_{i_{p,k}}$$

when $\pi(\lambda N) \geq 1$, and $Q_{1,1} = 1$ (or any constant) when $\pi(\lambda N) = 0$, then applying Theorem 1 conditionally on $\pi(\lambda N)$ and then averaging in $\pi(\lambda N)$ implies that for $t < 1$,

$$\mathbb{E}\langle (Q_{1,1} - \langle Q_{1,1} \rangle)^2 \rangle \leq \frac{4(\beta_1 + \beta_2)}{\beta_1\beta_2\sqrt{1-t}} \mathbb{E} \frac{1}{\sqrt{\pi(\lambda N)}} I(\pi(\lambda N) \geq 1). \tag{21}$$

The last expectation is of order $1/\sqrt{\lambda N}$ and, in fact, it is easy to check that it is bounded by $1/(\sqrt{\lambda N} - \sqrt{2/(\lambda N)})$.

Example 4 (Edwards–Anderson model). Let (V, E) be an arbitrary undirected finite graph and let $\beta_1, \beta_2, h_1, h_2 \geq 0$. Consider two Edwards–Anderson models on $\{-1, +1\}^V$ with Gaussian random external fields,

$$H_1(\sigma) = \beta_1 \sum_{(i,j) \in E} g^1_{i,j}\sigma_i\sigma_j + h_1 \sum_{i \in V} g^1_i \sigma_i, \qquad G_1(\sigma) = \frac{\exp H_1(\sigma)}{Z_1},$$

$$H_2(\rho) = \beta_2 \sum_{(i,j) \in E} g^2_{i,j}\rho_i\rho_j + h_2 \sum_{i \in V} g^2_i \rho_i, \qquad G_2(\rho) = \frac{\exp H_2(\rho)}{Z_2},$$

where $(g_{i,j}^1, g_{i,j}^2)$ are independent Gaussian random vectors with covariance

$$\mathbb{E}\big(g_{i,j}^1\big)^2 = \mathbb{E}\big(g_{i,j}^2\big)^2 = 1 \quad and \quad \mathbb{E}g_{i,j}^1 g_{i,j}^2 = t_E \in [0, 1],$$

(g_i^1, g_i^2) *are independent Gaussian random vectors with covariance*

$$\mathbb{E}\big(g_i^1\big)^2 = \mathbb{E}\big(g_i^2\big)^2 = 1 \quad and \quad \mathbb{E}g_i^1 g_i^2 = t_V \in [0, 1],$$

and these two families of random vectors are independent of each other. From Theorem 1, *we can deduce two kinds of quenched disorder chaos. First, if $\beta_1, \beta_2 > 0$ and $t_E < 1$, we obtain*

$$\mathbb{E}\big\langle \big(Q_{1,1} - \langle Q_{1,1}\rangle\big)^2\big\rangle \leq \frac{4(\beta_1 + \beta_2)}{\beta_1 \beta_2 \sqrt{|E|(1 - t_E)}}, \tag{22}$$

where $Q_{1,1}$ is the bond overlap

$$Q_{1,1} = \frac{1}{|E|} \sum_{(i,j)\in E} \sigma_i^1 \sigma_j^1 \rho_i^1 \rho_j^1.$$

If $h_1, h_2 > 0$ and $t_V < 1$, then

$$\mathbb{E}\big\langle \big(R_{1,1} - \langle R_{1,1}\rangle\big)^2\big\rangle \leq \frac{4(h_1 + h_2)}{h_1 h_2 \sqrt{|V|(1 - t_V)}}, \tag{23}$$

where

$$R_{1,1} = \frac{1}{|V|} \sum_{i\in V} \sigma_i^1 \rho_i^1$$

is the usual site overlap. The bound in (22) *was the one discussed in the introduction.*

Remark. *In Theorem* 1.6 *of the same paper* [2], *Chatterjee also proved the following result. If d is the maximum degree of the graph (V, E) and*

$$q = \min\bigg(\beta^2, \frac{1}{4d^2}\bigg)$$

then for some choice of absolute constant C,

$$\mathbb{E}\langle Q_{1,1}\rangle \geq Cq t^{1/(Cq)}. \tag{24}$$

If d is fixed (for example, in the EA model on a finite dimensional lattice) and $t > 0$ then (24) *combined with* (5) *excludes the possibility that $Q_{1,1}$ concentrates near* 0 *for large $|E|$, since the quenched average $\langle Q_{1,1}\rangle$ must be strictly positive with positive probability. This seems to be in contrast with the predictions of Fisher, Huse* [8] *and Bray, Moore* [1] *for the site overlap*

$$R_{\ell,\ell'} = \frac{1}{|V|} \sum_{i\in V} \sigma_i^\ell \rho_i^{\ell'}, \tag{25}$$

which is expected to concentrate near zero when $t < 1$. One interpretation of (24) *is that there is no disorder chaos for the bond overlap. Another possible interpretation could be that the vectors $(\sigma_i^1 \sigma_j^1)$ and $(\rho_i^1 \rho_j^1)$ might have 'preferred directions' and the overlap $\langle Q_{1,1}\rangle$ of their Gibbs averages $(\langle \sigma_i^1 \sigma_j^1\rangle)$ and $(\langle \rho_i^1 \rho_j^1\rangle)$ could deviate from zero but, otherwise, they have no common structure, which is some sort of weak disorder chaos. To strengthen this statement, one could also try to show that $\langle Q_{1,1}\rangle$ concentrates around its expected value $\mathbb{E}\langle Q_{1,1}\rangle$.*

3. Self-averaging of the magnetization

From now on we will consider one system with the Hamiltonian as in (11),

$$Y(\sigma) = \sum_{e \in E} g_e f_e(\sigma),$$

and the Gibbs measure as in (14),

$$dG(\sigma) = \frac{\exp \gamma Y(\sigma)}{Z} d\mu(\sigma).$$

Consider a vector $a = (a_e)_{e \in E}$ of some arbitrary constants and denote

$$\|a\|_2 = \left(\sum_{e \in E} a_e^2\right)^{1/2} \quad \text{and} \quad \|a\|_1 = \sum_{e \in E} |a_e|.$$

We will define a weighted bond magnetization by

$$m(\sigma) = \sum_{e \in E} a_e f_e(\sigma). \tag{26}$$

The following holds.

Theorem 2. *If $\gamma > 0$ then*

$$\mathbb{E}\langle (m(\sigma) - \langle m(\sigma)\rangle)^2\rangle \leq \frac{1}{\gamma} \|a\|_2 \|a\|_1. \tag{27}$$

Proof. If we consider the random variable $g = \sum_e a_e g_e$ then Gaussian integration by parts gives

$$\mathbb{E}\langle m(\sigma^1) g\rangle = \gamma \mathbb{E}\langle m(\sigma^1)^2 - m(\sigma^1)m(\sigma^2)\rangle$$
$$= \gamma \mathbb{E}\langle (m(\sigma) - \langle m(\sigma)\rangle)^2\rangle.$$

On the other hand, since

$$|\langle m(\sigma^1)\rangle| \leq \sum_{e \in E} |a_e| |\langle f_e(\sigma^1)\rangle| \leq \|a\|_1, \tag{28}$$

we can write

$$|\mathbb{E}\langle m(\sigma^1) g\rangle| = |\mathbb{E}\langle m(\sigma^1)\rangle g| \leq \|a\|_1 \mathbb{E}|g| \leq \|a\|_1 (\mathbb{E}g^2)^{1/2} = \|a\|_2 \|a\|_1$$

and the proof follows. $\qquad \square$

Example 5. *Consider the mixed p-spin model as in the Example 1 above. Let us consider $b_1, \ldots, b_N$ such that $\sum_{i=1}^N |b_i| = 1$. If we denote $\gamma = \beta_p/N^{(p-1)/2}$, let*

$$f_e(\sigma) = \sigma_{i_1} \cdots \sigma_{i_p} \quad \text{and} \quad a_e = b_{i_1} \cdots b_{i_p} \quad \text{for } e = (i_1, \ldots, i_p) \in E = \{1, \ldots, N\}^p$$

then the bond magnetization is given by

$$m(\sigma) = \sum_{1 \leq i_1, \ldots, i_p \leq N} b_{i_1} \cdots b_{i_p} \sigma_{i_1} \cdots \sigma_{i_p} = \left(\sum_{i=1}^N b_i \sigma_i\right)^p$$

and (27) *implies that, for $\beta_p > 0$,*

$$\mathbb{E}\langle(m(\sigma) - \langle m(\sigma)\rangle)^2\rangle \leq \frac{\|a\|_2}{\gamma} = \frac{N^{(p-1)/2}\|b\|_2^p}{\beta_p}. \tag{29}$$

If we take $b_i = 1/N$, the bound becomes $(\beta_p\sqrt{N})^{-1}$ and $m(\sigma)$ is the pth power of the usual total site magnetization $N^{-1}\sum_{i\leq N}\sigma_i$. For odd p, this implies quenched self-averaging for the total site magnetization and, for even p, quenched self-averaging for its absolute value.

4. Self-averaging of the site overlap assuming positive spin correlation

Theorem 2 can be used to give a simplified proof of a slightly improved version of Lemma 2.6 in [4]. Consider a finite set V and consider any model with the Hamiltonian defined on $\sigma \in \{-1, +1\}^V$ that includes a Gaussian random field term,

$$H(\sigma) = H'(\sigma) + h\sum_{i\in V}g_i\sigma_i, \tag{30}$$

where (g_i) are i.i.d. standard Gaussian random variables, independent of $H'(\sigma)$. For the next result, let us assume that the spins are positively correlated under the Gibbs measure,

$$\langle\sigma_i\sigma_j\rangle \geq \langle\sigma_i\rangle\langle\sigma_j\rangle \quad \text{for all } i, j \in V. \tag{31}$$

For example, this was the case for the random field Ising model considered in [4] by the FKG inequality [9]. Let

$$R_{1,2} = \frac{1}{|V|}\sum_{i\in V}\sigma_i^1\sigma_i^2$$

denote the usual site overlap of two replicas.

Theorem 3. *If the inequalities* (31) *hold then*

$$\mathbb{E}\langle(R_{1,2} - \langle R_{1,2}\rangle)^2\rangle \leq \frac{2}{h\sqrt{|V|}}. \tag{32}$$

In particular, this removes the factor $\sqrt{2 + h^2}$ from the bound in Lemma 2.6 in [4].

Proof. To prove this, we start by copying the following equation from the proof of Lemma 2.6 in [4]:

$$\mathbb{E}(\langle R_{1,2}^2\rangle - \langle R_{1,2}\rangle^2) = \frac{1}{|V|^2}\sum_{i,j}\mathbb{E}(\langle\sigma_i\sigma_j\rangle^2 - \langle\sigma_i\rangle^2\langle\sigma_j\rangle^2)$$

$$= \frac{1}{|V|^2}\sum_{i,j}\mathbb{E}|\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle||\langle\sigma_i\sigma_j\rangle + \langle\sigma_i\rangle\langle\sigma_j\rangle|$$

$$\leq \frac{2}{|V|^2}\sum_{i,j}\mathbb{E}|\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_i\rangle|$$

$$= \frac{2}{|V|^2}\sum_{i,j}\mathbb{E}(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle),$$

where in the last step the positive correlation condition (31) was used. Next, if we consider the magnetization $m(\sigma) = |V|^{-1} \sum_i \sigma_i$ then

$$\mathbb{E}\langle (m(\sigma) - \langle m(\sigma) \rangle)^2 \rangle = \frac{1}{|V|^2} \sum_{i,j} \mathbb{E}\langle (\sigma_i - \langle \sigma_i \rangle)(\sigma_j - \langle \sigma_j \rangle) \rangle$$

$$= \frac{1}{|V|^2} \sum_{i,j} \mathbb{E}\big(\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \big).$$

Therefore, the inequalities (31) imply that

$$\mathbb{E}\big(\langle R_{1,2}^2 \rangle - \langle R_{1,2} \rangle^2 \big) \leq 2\mathbb{E}\langle (m(\sigma) - \langle m(\sigma) \rangle)^2 \rangle.$$

Finally, using Theorem 2 with $\gamma = h$ and $Y(\sigma) = \sum_{i \in V} g_i \sigma_i$ implies (32). $\qquad \square$

5. Self-averaging of random fields

Throughout this section, we will use the integration by parts formula

$$\mathbb{E}H_k(g)F(g) = \mathbb{E}H_{k-1}(g)F'(g) \tag{33}$$

for the Hermite polynomials

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}$$

of degree $k \geq 1$. In (33), g is a standard Gaussian random variable and F is a continuously differentiable function such that F' is of moderate growth. The case $k = 1$,

$$\mathbb{E}g F(g) = \mathbb{E}F'(g), \tag{34}$$

is often called the (usual) Gaussian integration by parts, and

$$\mathbb{E}(g^2 - 1)F(g) = \mathbb{E}g F'(g) \tag{35}$$

corresponds to the case $k = 2$.

Let $Y(\sigma)$ and $dG(\sigma)$ be as in Section 3. Consider a random field

$$W(\sigma) = \sum_{e \in E} a_e g_e f_e(\sigma) \tag{36}$$

for arbitrary constants a_e for $e \in E$. Denote

$$\|a\|_2 = \left(\sum_{e \in E} a_e^2 \right)^{1/2} \quad \text{and} \quad \|a\|_1 = \sum_{e \in E} |a_e|.$$

We will start with the following.

Theorem 4. *If $\gamma > 0$ then*

$$\mathbb{E}\big(\langle W(\sigma)^2 \rangle - \langle W(\sigma) \rangle^2 \big) \leq \frac{\sqrt{2}}{\gamma} \|a\|_2 \|a\|_1. \tag{37}$$

Proof. Using the integration by parts formula in (35), we can write

$$\mathbb{E}\big(g_e^2 - 1\big)\langle W(\sigma)\rangle = \mathbb{E}g_e\langle a_e f_e(\sigma)\rangle + \gamma\mathbb{E}g_e\langle W(\sigma)f_e(\sigma)\rangle - \gamma\mathbb{E}g_e\langle W(\sigma^1)f_e(\sigma^2)\rangle.$$

Multiplying both sides by a_e and summing over $e \in E$ gives

$$\mathbb{E}\sum_{e\in E}a_e\big(g_e^2 - 1\big)\langle W(\sigma)\rangle = \sum_{e\in E}a_e^2\mathbb{E}\langle g_e f_e(\sigma)\rangle + \gamma\mathbb{E}\langle W(\sigma)^2\rangle - \gamma\mathbb{E}\langle W(\sigma)\rangle^2$$

and, therefore,

$$\gamma\mathbb{E}\big(\langle W(\sigma)^2\rangle - \langle W(\sigma)\rangle^2\big) = \mathbb{E}\sum_{e\in E}a_e\big(g_e^2 - 1\big)\langle W(\sigma)\rangle - \sum_{e\in E}a_e^2\mathbb{E}\langle g_e f_e(\sigma)\rangle. \tag{38}$$

By the usual Gaussian integration by parts,

$$\mathbb{E}\langle g_e f_e(\sigma)\rangle = \gamma\mathbb{E}\big(\langle f_e(\sigma)^2\rangle - \langle f_e(\sigma)\rangle^2\big) \geq 0,$$

so omitting the last sum in (38) yields an upper bound

$$\gamma\mathbb{E}\big(\langle W(\sigma)^2\rangle - \langle W(\sigma)\rangle^2\big) \leq \mathbb{E}\sum_{e\in E}a_e\big(g_e^2 - 1\big)\langle W(\sigma)\rangle.$$

Let us note that

$$\mathbb{E}\bigg(\sum_{e\in E}a_e\big(g_e^2 - 1\big)\bigg)^2 = \sum_{e,e'\in E}a_e a_{e'}\mathbb{E}\big(g_e^2 - 1\big)\big(g_{e'}^2 - 1\big) = 2\|a\|_2^2,$$

since the terms for $e \neq e'$ are equal to 0 and $\mathbb{E}(g_e^2 - 1)^2 = 2$. By the Cauchy–Schwarz inequality,

$$\gamma\mathbb{E}\big(\langle W(\sigma)^2\rangle - \langle W(\sigma)\rangle^2\big) \leq \sqrt{2}\|a\|_2\big(\mathbb{E}\langle W(\sigma)\rangle^2\big)^{1/2}.$$

Finally, using that $|W(\sigma)| \leq \sum_{e\in E}|a_e g_e|$ and $\mathbb{E}(\sum_{e\in E}|a_e g_e|)^2 \leq \|a\|_1^2$ finishes the proof. □

Example 6. *If in* (41) *we take all $a_e = 1$, we get*

$$\mathbb{E}\big(\langle Y(\sigma)^2\rangle - \langle Y(\sigma)\rangle^2\big) \leq \frac{\sqrt{2}|E|^{3/2}}{\gamma}. \tag{39}$$

Applying this to the Hamiltonian (30) *with the Gaussian random field gives a new proof of Lemma* 2.9 *in* [4]. *If in Theorem* 5 *we take $E = V$, $\gamma = h$, for $i \in V$ take $f_i(\sigma) = \sigma_i$, and divide both sides of* (41) *by $|V|^2$, then the normalized random field*

$$h(\sigma) = \frac{1}{|V|}\sum_{i\in V}g_i\sigma_i$$

satisfies

$$\mathbb{E}\big(\langle h(\sigma)^2\rangle - \langle h(\sigma)\rangle^2\big) \leq \frac{\sqrt{2}}{h\sqrt{|V|}}. \tag{40}$$

This inequality was used in [4] *to establish 'half' of the Ghirlanda–Guerra identities for the first moment of the overlaps, with the other half following from the existence of the limit for the free energy.*

Next, we will show how one can push the above proof even further to improve the bound for small values of γ.

Theorem 5. *If $\gamma > 0$ then*

$$\mathbb{E}\big(\langle W(\sigma)^2\rangle - \langle W(\sigma)\rangle^2\big) \le \|a\|_2^2 + \sqrt{2}\|a\|_2\|a\|_1. \tag{41}$$

Remark. *When all $a_e = 1$, the bound becomes $|E| + \sqrt{2}|E|^{3/2}$, which is an improvement over (39) for small values of γ. In fact, for very small values of γ, if one simply integrates $\mathbb{E}\langle W(\sigma)^2\rangle$ by parts to obtain a trivial bound $\|a\|_2^2 + C\gamma^2\|a\|_1^2$, this gives further improvement for very small values of γ.*

Proof. To prove this inequality, let us look at the right-hand side of (38) more closely. First,

$$\mathbb{E}\sum_{e\in E} a_e\big(g_e^2 - 1\big)\langle W(\sigma)\rangle = \mathbb{E}\sum_{e,e'\in E} a_e a_{e'}\big(g_e^2 - 1\big)g_{e'}\langle f_{e'}(\sigma)\rangle.$$

It will be convenient to introduce the notation

$$F_e = \frac{1}{\gamma}\frac{\partial}{\partial g_e}\langle f_e(\sigma)\rangle = \langle f_e(\sigma)^2\rangle - \langle f_e(\sigma)\rangle^2. \tag{42}$$

For the terms $e = e'$, using the formula (35) for the factors $g_e^2 - 1$ gives

$$\sum_{e\in E} a_e^2 \mathbb{E}\big(g_e^2 - 1\big)g_e\langle f_e(\sigma)\rangle = \sum_{e\in E} a_e^2 \mathbb{E}g_e\langle f_e(\sigma)\rangle + \gamma\sum_{e\in E} a_e^2 \mathbb{E}g_e^2 F_e. \tag{43}$$

Since $0 \le F_e \le 1$, the second sum is bounded by $\gamma\|a\|_2^2$. The first sum cancels out the last sum in (38), so

$$\gamma\mathbb{E}\big(\langle W(\sigma)^2\rangle - \langle W(\sigma)\rangle^2\big) \le \mathbb{E}\sum_{e\ne e'} a_e a_{e'}\big(g_e^2 - 1\big)g_{e'}\langle f_{e'}(\sigma)\rangle + \gamma\|a\|_2^2. \tag{44}$$

If in the first term on the right-hand side we use the usual Gaussian integration by parts with respect to g_e', it can be rewritten as

$$\sum_{e'\in E} a_{e'}\mathbb{E}\bigg(\sum_{e\ne e'} a_e\big(g_e^2 - 1\big)\bigg)g_{e'}\langle f_{e'}(\sigma)\rangle = \gamma\sum_{e'\in E} a_{e'}\mathbb{E}\bigg(\sum_{e\ne e'} a_e\big(g_e^2 - 1\big)\bigg)F_{e'}. \tag{45}$$

Since $0 \le F_{e'} \le 1$ and

$$\mathbb{E}\bigg(\sum_{e\ne e'} a_e\big(g_e^2 - 1\big)\bigg)^2 \le 2\|a\|_2^2,$$

by the Cauchy–Schwarz inequality, we can bound the last sum by $\sqrt{2}\gamma\|a\|_2\|a\|_1$ and this finishes the proof. $\square$

Example 7. *Using the bound (41), one can supplement (40) in the Example 6 with*

$$\mathbb{E}\big(\langle h(\sigma)^2\rangle - \langle h(\sigma)\rangle^2\big) \le \frac{1}{|V|} + \frac{\sqrt{2}}{\sqrt{|V|}} \tag{46}$$

for small values of h.

There is a natural generalization of the previous results to the random field

$$W(\sigma) = \sum_{e\in E} a_e H_k(g_e) f_e(\sigma), \tag{47}$$

where, as above, H_k is the Hermite polynomial of degree $k \geq 0$. Let us denote

$$F_e^{(k)} = \frac{1}{\gamma^k} \frac{\partial^k}{\partial g_e^k} \langle f_e(\sigma) \rangle, \tag{48}$$

and let C_k be a constant such that $|F_e^{(k)}| \leq C_k$ with probability one. For example, $F_e^{(0)} = \langle f_e(\sigma) \rangle$ and $C_0 = 1$ (this was used in (28)) and $F_e^{(1)} = \langle f_e(\sigma)^2 \rangle - \langle f_e(\sigma) \rangle^2$ with $C_1 = 1$, which already appeared in (42). The following analogue of Theorems 2, 4 and 5 holds in this case.

Theorem 6. *We have that for $k \geq 0$,*

$$\mathbb{E}\big(\langle W(\sigma)^2 \rangle - \langle W(\sigma) \rangle^2\big) \leq \frac{\sqrt{k!(k+1)!}}{\gamma} \|a\|_1 \|a\|_2 \tag{49}$$

and for $k \geq 1$,

$$\mathbb{E}\big(\langle W(\sigma)^2 \rangle - \langle W(\sigma) \rangle^2\big) \leq C_k \sqrt{(k+1)!} \gamma^{k-1} \|a\|_1 \|a\|_2 + k! \|a\|_2^2. \tag{50}$$

Proof. Using the integration by parts formula (33),

$$\mathbb{E} H_{k+1}(g_e) \langle W(\sigma) \rangle = \mathbb{E} H_k(g_e) \frac{\partial}{\partial g_e} \langle W(\sigma) \rangle$$

$$= a_e \mathbb{E} H_k(g_e) H_k'(g_e) \langle f_e(\sigma) \rangle + \gamma \mathbb{E} H_k(g_e) \big(\langle W(\sigma) f_e(\sigma) \rangle - \langle W(\sigma^1) f_e(\sigma^2) \rangle\big).$$

Multiplying both sides by a_e and summing over $e \in E$ gives

$$\mathbb{E} \sum_{e \in E} a_e H_{k+1}(g_e) \langle W(\sigma) \rangle = \mathbb{E} \sum_{e \in E} a_e^2 H_k(g_e) H_k'(g_e) \langle f_e(\sigma) \rangle + \gamma \mathbb{E}\big(\langle W(\sigma)^2 \rangle - \langle W(\sigma) \rangle^2\big)$$

and, therefore,

$$\gamma \mathbb{E}\big(\langle W(\sigma)^2 \rangle - \langle W(\sigma) \rangle^2\big) = \mathbb{E} \sum_{e \in E} a_e H_{k+1}(g_e) \langle W(\sigma) \rangle$$

$$- \mathbb{E} \sum_{e \in E} a_e^2 H_k(g_e) H_k'(g_e) \langle f_e(\sigma) \rangle. \tag{51}$$

As above, the first term can be bounded as follows,

$$\mathbb{E} \sum_{e \in E} a_e H_{k+1}(g_e) \langle W(\sigma) \rangle \leq \left(\mathbb{E}\left(\sum_{e \in E} a_e H_{k+1}(g_e)\right)^2\right)^{1/2} \big(\mathbb{E} \langle W(\sigma) \rangle^2\big)^{1/2}$$

$$\leq \left(\sum_{e \in E} a_e^2 \mathbb{E} H_{k+1}(g_e)^2\right)^{1/2} \left(\mathbb{E}\left(\sum_{e \in E} |a_e| |H_k(g_e)|\right)^2\right)^{1/2}$$

$$\leq \sqrt{(k+1)!} \|a\|_2 \sqrt{k!} \|a\|_1,$$

where we used that $\mathbb{E} H_\ell(g)^2 = \ell!$ for $\ell = k, k+1$. This will finish the proof of (49) if we can show that the second term is negative. Using (33) for the factor $H_k(g_e)$ gives

$$\mathbb{E} H_k(g_e) H_k'(g_e) \langle f_e(\sigma) \rangle = \mathbb{E} H_{k-1}(g_e) H_k''(g_e) \langle f_e(\sigma) \rangle + \gamma \mathbb{E} H_{k-1}(g_e) H_k'(g_e) F_e^{(1)}.$$

Using a well-known relationship $H_k(x)' = k H_{k-1}(x)$, we can rewrite this as

$$\mathbb{E} H_k(g_e) H_k'(g_e) \langle f_e(\sigma) \rangle = k \mathbb{E} H_{k-1}(g_e) H_{k-1}'(g_e) \langle f_e(\sigma) \rangle + k \gamma \mathbb{E} H_{k-1}(g_e)^2 F_e^{(1)}.$$

Finally, using that $F_e^{(1)} \geq 0$ and proceeding by induction on k, we get

$$\mathbb{E} H_k(g_e) H_k'(g_e) \langle f_e(\sigma) \rangle \geq k \mathbb{E} H_{k-1}(g_e) H_{k-1}'(g_e) \langle f_e(\sigma) \rangle \geq 0.$$

To obtain (50), we need further calculations for the first term on the right-hand side of (51). Let us begin by writing

$$\mathbb{E} \sum_{e \in E} a_e H_{k+1}(g_e) \langle W(\sigma) \rangle = \mathbb{E} \sum_{e \neq e'} a_e a_{e'} H_{k+1}(g_e) H_k(g_{e'}) \langle f_{e'}(\sigma) \rangle$$
$$+ \mathbb{E} \sum_{e \in E} a_e^2 H_{k+1}(g_e) H_k(g_e) \langle f_e(\sigma) \rangle. \tag{52}$$

Using the integration by parts formula (33) repeatedly, for any $e \neq e'$ we get

$$\mathbb{E} H_{k+1}(g_e) H_k(g_{e'}) \langle f_{e'}(\sigma) \rangle = \mathbb{E} H_{k+1}(g_e) \frac{\partial^k}{\partial g_{e'}^k} \langle f_{e'}(\sigma) \rangle = \gamma^k \mathbb{E} H_{k+1}(g_e) F_{e'}^{(k)}.$$

Using (33) once for the factor $H_{k+1}(g_e)$, for $e = e'$ we get

$$\mathbb{E} H_{k+1}(g_e) H_k(g_e) \langle f_e(\sigma) \rangle = \mathbb{E} H_k(g_e) H_k'(g_e) \langle f_e(\sigma) \rangle + \gamma \mathbb{E} H_k(g_e)^2 F_e^{(1)}.$$

Therefore, we can rewrite (52) as

$$\mathbb{E} \sum_{e \in E} a_e H_{k+1}(g_e) \langle W(\sigma) \rangle = \gamma^k \mathbb{E} \sum_{e \neq e'} a_e a_{e'} H_{k+1}(g_e) F_{e'}^{(k)}$$
$$+ \mathbb{E} \sum_{e \in E} a_e^2 H_k(g_e) H_k'(g_e) \langle f_e(\sigma) \rangle + \gamma \mathbb{E} \sum_{e \in E} a_e^2 H_k(g_e)^2 F_e^{(1)}.$$

Plugging this into (51) and dividing both sides by γ,

$$\mathbb{E} \big(\langle W(\sigma)^2 \rangle - \langle W(\sigma) \rangle^2 \big) = \gamma^{k-1} \mathbb{E} \sum_{e \neq e'} a_e a_{e'} H_{k+1}(g_e) F_{e'}^{(k)} + \mathbb{E} \sum_{e \in E} a_e^2 H_k(g_e)^2 F_e^{(1)}.$$

Since $F_e^{(1)} \leq 1$, the second term can be bounded by

$$\mathbb{E} \sum_{e \in E} a_e^2 H_k(g_e)^2 F_e^{(1)} \leq k! \|a\|_2^2.$$

To bound the first term, let us rewrite it as

$$\gamma^{k-1} \mathbb{E} \sum_{e \neq e'} a_e a_{e'} H_{k+1}(g_e) F_{e'}^{(k)} = \gamma^{k-1} \sum_{e' \in E} a_{e'} \mathbb{E} F_{e'}^{(k)} \sum_{e \neq e'} a_e H_{k+1}(g_e).$$

Since, for any fixed $e' \in E$, $|F_{e'}^{(k)}| \leq C_k$ and

$$\mathbb{E} \bigg(\sum_{e \neq e'} a_e H_{k+1}(g_e) \bigg)^2 \leq (k+1)! \|a\|_2^2,$$

this can be bounded by $C_k \sqrt{(k+1)!} \gamma^{k-1} \|a\|_2 \|a\|_1$, which finishes the proof of (50). $\qquad \square$

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