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Higher moments of the natural parameterization for SLE curves

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Abstract. In this paper, we will show that the higher moments of the natural parametrization of *SLE* curves in any bounded domain in the upper half plane is finite. We prove this by estimating the probability that an *SLE* curve gets near n given points.

Résumé. Dans cet article, nous montrons que les grands moments de la paramétrisation naturelle d'une courbe *SLE* dans n'importe quel domaine borné du demi-plan supérieur sont finis. Nous prouvons ceci en estimant la probabilité qu'une courbe *SLE* soit proche d'un nombre n de points fixés.

MSC: 60J67

Keywords: SLE curve; Natural parametrization; Green's function

1. Introduction

A number of measures arise from statistical physics are believed to have conformally invariant scaling limits. In [13], a one-parameter family of measures on self-avoiding curves in the upper half plane, called (chordal) Schramm–Loewner evolution (SLE_κ) is defined. Here we only work with chordal version so we omit chordal. By conformal invariance, it is extended to other simply connected domains. Later, it was shown that *SLE* describes the limits of a number of models from physics so answering the question of conformal invariance for them. These models include loop-erased random walk for $\kappa = 2$ [9], Ising interfaces for $\kappa = 3$ and $\kappa = 16/3$ [16], harmonic explorer for $\kappa = 4$ [14], percolation interfaces for $\kappa = 6$ [15], and uniform spanning tree Peano curves for $\kappa = 8$ [9].

In order to define *SLE*, Schramm used *capacity parametrization*. We will see the definition of *SLE* as well as capacity parametrization in the next section. Capacity parametrization comes from Loewner evolution and makes it easy to analyze *SLE* curves by Ito's calculus. In all the physical models that we have above, in order to show the convergence, we have to first parametrize them with discrete version of the capacity and then prove the convergence to *SLE*. This parametrization is very different from the *natural* parametrization that we have for them which is just the length of the curve.

In order to prove the same results with the natural parametrization, we need to define a natural length for *SLE* curves. In [2], it is proved that the Hausdorff dimension of SLE_κ is $d = \min\{2, 1 + \frac{\kappa}{8}\}$. In [8], the authors conjectured that the Minkowski content of *SLE* should exist. They defined the natural parametrization in a different way using Doob–Meyer decomposition and proved the existence for $\kappa < 5.021\ldots$. Moreover, they conjectured that the natural length of *SLE* can be defined in terms of d-dimensional Minkowski content. Here is how it is defined (see [6] for more details). Let

$$\mathrm{Cont}_d\big(\gamma[0,t];r\big) = r^{d-2}\,\mathrm{Area}\big\{z : \mathrm{dist}\big(z,\gamma[0,t]\big) \le r\big\}.$$

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Then the d-dimensional content is

$$\mathrm{Cont}_d\big(\gamma[0,t]\big)=\lim_{r\to 0}\mathrm{Cont}_d\big(\gamma[0,t];r\big), \tag{1.1}$$

provided that the limit exists. If $\kappa>8$ the curve is space filling and $d=2$ so this is just the area and the problem is trivial. For $k<8$, the existence was shown in [6]. We assume for the purpose of this paper that $\kappa<8$. We call this parametrization, natural length or length from now on. Also a number of properties of the natural length were studied in [6]. For example the authors computed the first and second moments of the "natural length." Basically, this function is the appropriate scaled version of the probability that *SLE* hits given point(s). Precisely, the n-point Green's function at $z_1,\dots,z_n$ is

$$G(z_1,\dots,z_n)=\lim_{r_1,\dots,r_n\to 0}\prod_{k=1}^{n} r_k^{d-2}\mathbb{P}\left[\bigcap_{k=1}^{n}\big\{\mathrm{dist}(z_k,\gamma)\le r_k\big\}\right], \tag{1.2}$$

provided that the limit exists. The covariance rule of the Green's function is obvious, that is, if F maps $(\mathbb{H};0,\infty)$ conformally onto $(D;w_1,w_2)$, then

$$G_{(D;w_1,w_2)}(z_1,\dots,z_n)=\big|\big(F^{-1}\big)'(z)\big|^{2-d}G_{(\mathbb{H};0,\infty)}\big(F^{-1}(z_1),\dots,F^{-1}(z_n)\big), \tag{1.3}$$

if the Green's function at either side exists. Here we use $G_{(D;w_1,w_2)}$ to denote the Green's function for SLE_κ in D from w_1 to w_2.

It is proved in [10] that a modified version of 1-point and 2-point Green's function using conformal distance instead of distance exists. In [6], the authors prove the above limit exists for $n=1,2$. Lawler and Werness mentioned in [10] that the argument can be generalized to define higher order Green's function. So they conjectured the existence of multi-point Green's function. For $n=1$ the exact formula is given in [6] which is

$$G(z)=G_{(\mathbb{H};0,\infty)}(z)=C|z|^{d-2}\sin^{\kappa/8+8/\kappa-2}(\arg z)=C\,\mathrm{Im}(z)^{d-2}\sin^{8/\kappa-1}(\arg z), \tag{1.4}$$

where $C=C_\kappa>0$ is an unknown constant. In arbitrary domains the exact formula of the 1-point Green's function can be found by the covariance rule.

We now state the main theorems of this paper. Throughout, we fix $\kappa\in(0,8)$, the following constants depending on κ:

$$d=1+\frac{\kappa}{8},\qquad \alpha=\frac{8}{\kappa}-1.$$

We will use C to denote an arbitrary positive constant that depends only on κ, whose value may vary from one occurrence to another. If we allow C to depend on κ and another variable, say n, then we will use C_n. We introduce a family of functions. For $y\ge 0$, define P_y on $[0,\infty)$ by

$$P_y(x)=\begin{cases} y^{\alpha-(2-d)}x^{2-d}, & x\le y;\\ x^\alpha, & x\ge y.\end{cases}$$

Since $\alpha\ge 2-d>0$, if $0\le x_1<x_2$, then

$$\frac{x_1^\alpha}{x_2^\alpha}\le\frac{P_y(x_1)}{P_y(x_2)}\le\frac{x_1^{2-d}}{x_2^{2-d}}. \tag{1.5}$$

The first main theorem is:

Theorem 1.1. *Let $z_0,\dots,z_n$ be distinct points on $\overline{\mathbb{H}}$ such that $z_0=0$. Let $y_k=\mathrm{Im}\,z_k\ge 0$ and $l_k=\mathrm{dist}(z_k,\{z_j:0\le j<k\})$, $1\le k\le n$. Let $r_1,\dots,r_n>0$. Let γ be an SLE_κ curve in $\mathbb{H}$ from 0 to ∞. Then there is $C_n<\infty$ depending only on κ and n such that*

$$\mathbb{P}\big[\mathrm{dist}(\gamma,z_k)\le r_k,1\le k\le n\big]\le C_n\prod_{k=1}^{n}\frac{P_{y_k}(r_k\wedge l_k)}{P_{y_k}(l_k)}.$$

The second main theorem answers a question in [6].

Theorem 1.2. *If γ is an SLE curve from* 0 *to* ∞ *in* $\mathbb{H}$, *then for any bounded* $D \subset \mathbb{H}$, *we have*

$$\mathbb{E}\big[\mathrm{Cont}_d(\gamma \cap D)^n\big] < \infty, \quad n \in \mathbb{N}.$$

Remarks.

1. *The quantity on the right-hand side of the formula in Theorem 1.1 depends on the order of the points* $z_1, \dots, z_n$. *However, if* r_j*'s are sufficiently small, say,* $r_j < \mathrm{dist}(z_j, \{z_0, \dots, z_n\} \setminus \{z_j\})$, *then if we exchange any pair of consecutive points, i.e.,* z_k *and* z_{k+1}, *then the new quantity is no more than* C *times the old quantity, where* $C > 0$ *depends only on* κ. *Thus, if we permute those n points, the quantity will increase at most* C^{n^2} *times.*
2. *An immediate consequence of Theorem 1.1 is that the right-hand side of (1.2), with* lim *replaced by* lim sup, *is finite.*
3. *In fact, Theorem 1.1 implies an upper bound of the Green's function* $G(z_1, \dots, z_n)$ *for the above* γ, *if it exists. That is*

$$G(z_1, \dots, z_n) \le C_n \prod_{k=1}^{n} \frac{y_k^{\alpha-(2-d)}}{P_{y_k}(l_k)}.$$

A natural question to ask is whether the reverse inequality also holds (with smaller C_n). *The answer is yes if* $n \le 2$. *In the case* $n = 1$, *the right-hand side is* $C\frac{y^{\alpha-(2-d)}}{|z|^\alpha}$, *which agrees with the right-hand side of (1.4). In the case* $n = 2$, *the right-hand side is comparable to a sharp estimate of the* 2*-point Green's function given in [7] up to a constant. Thus, we expect that it holds for all* $n \in \mathbb{N}$.

4. *We guess that one can show* $\mathbb{E}[e^{\lambda\,\mathrm{Cont}_d(\gamma\cap D)}] < \infty$ *for some* $\lambda > 0$ *in any bounded domain* D. *This is nice because we can study natural length by its moment generating function. One way to prove it is to prove a similar bound for ordered multi-point Green's function but with* C^n *instead of* C_n. *See [10] for the definition of ordered Green's function.*
5. *If the Green's function* $G(z_1, \dots, z_n)$ *exits, the left-hand side of the displayed formula in Theorem 1.2 equals to* $\int_{D^n} G(z_1, \dots, z_n)\, dA(z_1) \cdots dA(z_n)$.
6. *Theorem 1.1 also provides an upper bound for the boundary Green's function, which is the scaled version of the probability that SLE hits given boundary point(s). The scaling exponent will be* α *instead of* $2 - d$ *so that the Green's function does not vanish. To be more precise, for the above* γ, *the boundary Green's function at* $x_1, \dots, x_n \in \mathbb{R} \setminus \{0\}$ *is*

$$\tilde{G}(x_1, \dots, x_n) = \lim_{r_1,\dots,r_n \to 0} \prod_{k=1}^{n} r_k^{-\alpha} \mathbb{P}\Bigg[\bigcap_{k=1}^{n} \{\mathrm{dist}(x_k, \gamma) \le r_k\}\Bigg], \tag{1.6}$$

provided that the limit exists. Lawler recently proved in [5] that the 1*-point and* 2*-point boundary Green's function exist, and gave good estimates of these functions. Using Theorem 1.1, we can derive the following conclusions. First, the right-hand side of (1.6), with* lim *replaced by* lim sup, *is finite. This result may help us to prove the existence of multi-point boundary Green's functions for SLE. Second, if* $\tilde{G}(x_1, \dots, x_n)$ *exits, then* $\tilde{G}(x_1, \dots, x_n) \le C_n \prod_{k=1}^{n} l_k^{-\alpha}$, *where* $l_k = \min_{0 \le j < k} |x_k - x_j|$ *with* $x_0 = 0$. *Similarly, we get upper bounds for mixed Green's functions, where some points lie on the boundary, and others lie in the interior.*

The organization of the rest of the paper goes as follows. In the next section we review the definition of *SLE* and some fundamental estimates for *SLE*. In the third section, we will prove two main lemmas. At the end, we will prove the two main theorems.

2. Preliminaries

2.1. *Definition of SLE*

In this subsection we review the definition of *SLE* and its basic properties. See [3,4,6,10] for more details.

A bounded set $K \subset \mathbb{H} = \{x + iy : y > 0\}$ is called an $\mathbb{H}$-hull if $\mathbb{H} \setminus K$ is a simply connected domain, and the complement $\mathbb{H} \setminus K$ is called an $\mathbb{H}$-domain. For every $\mathbb{H}$-hull K, there is a unique conformal map g_K from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ that satisfies

$$g_K(z) = z + \frac{c}{z} + O\big(|z|^{-2}\big), \quad |z| \to \infty$$

for some $c \ge 0$. The number c is called the half plane capacity of K, and is denoted by $\operatorname{hcap}(K)$.

Suppose that $\gamma : (0, \infty) \to \mathbb{H}$ is a simple curve with $\gamma(0+) \in \mathbb{R}$ and $\gamma(t) \to \infty$ as $t \to \infty$. Then for each t, $K_t := \gamma(0, t]$ is an $\mathbb{H}$-hull. Let $g_t = g_{K_t}$ and $a(t) = \operatorname{hcap}(K_t)$. We can reparameterize the curve such that $a(t) = 2t$. Then g_t satisfies the *(chordal) Loewner equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - V_t}, \quad g_0(z) = z, \tag{2.1}$$

where $V_t := \lim_{\mathbb{H}\setminus K_t \ni z \to \gamma(t)} g_t(z)$ is a continuous real-valued function.

Conversely, one can start with a continuous real-valued function V_t and define g_t by (2.1). For $z \in \mathbb{H} \setminus \{0\}$, the function $t \mapsto g_t(z)$ is well defined up to a blowup time T_z, which could be ∞. The evolution then generates an increasing family of $\mathbb{H}$-hulls defined by

$$K_t = \{z \in \mathbb{H} : T_z > t\}, \quad 0 \le t < \infty,$$

with $g_t = g_{K_t}$ and $\operatorname{hcap}(K_t) = 2t$ for each t. One may not always get a curve from the evolution.

The *(chordal) Schramm–Loewner evolution (SLE_κ) (from 0 to ∞ in $\mathbb{H}$)* is the solution to (2.1) where $V_t = \sqrt{\kappa} B_t$, where $\kappa > 0$ and $B(t)$ is a standard Brownian motion. It is shown in [9,12] that the limits

$$\gamma(t) = \lim_{\mathbb{H} \ni z \to V_t} g_t^{-1}(z), \quad 0 \le t < \infty,$$

exist, and give a continuous curve γ in $\overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\lim_{t\to\infty} \gamma(t) = \infty$. Only in the case $\kappa \le 4$, the curve is simple and stays in $\mathbb{H}$ for $t > 0$, and we recover the previous picture. For other cases, γ is not simple, and $H_t := \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$.

We can define SLE_κ in other simply connected domains using conformal maps. Roughly speaking, SLE_κ in a simply connected domain $D \subsetneqq \mathbb{C}$ is the image of the above γ under a conformal map F from $\mathbb{H}$ onto D. However, since γ in fact lies in $\overline{\mathbb{H}}$ instead of $\mathbb{H}$, the rigorous definition requires some regularity of D. For simplicity, we assume that ∂D is locally connected and call such domain D regular. This ensures that any conformal map F from $\mathbb{H}$ onto D has a continuous extension to $\overline{\mathbb{H}}$, and so $F \circ \gamma$ is a continuous curve in $\overline{D}$.

Now we state the definition. Let D be a regular simply connected domain, and w_0, w_∞ be distinct prime ends (cf. [3]) of D. Let $F : \mathbb{H} \to D$ be a conformal transformation of $\mathbb{H}$ onto D with $F(0) = w_0$, $F(\infty) = w_\infty$. Then $\tilde{\gamma} := F \circ \gamma$ is called an SLE_κ curve in D from w_0 to w_∞. Although such F is not unique, the definition is unique up to a linear time change.

Now we state the important *Domain Markov Property* (DMP) of *SLE*. Let D be a regular simply connected domain with prime ends $w_0 \ne w_\infty$, and γ an SLE_κ curve in D from w_0 to w_∞. For each $t_0 \ge 0$, let D_{t_0} be the connected component of $\mathbb{H} \setminus \gamma(0, t_0]$ which is a neighborhood of w_∞ in D, and $\gamma^{t_0}(t) = \gamma(t_0 + t)$, $0 \le t < \infty$. Let T be any stopping time w.r.t. γ. Then conditioned on $\gamma(0, T]$ and the event $\{T < \infty\}$, a.s. $\gamma(T) \in \partial D_T$ determines a prime end of D_T, and γ^T has the distribution of SLE_κ in D_T from (the prime end determined by) $\gamma(T)$ to w_∞.

2.2. *Crosscuts*

Let D be a simply connected domain. A simple curve $\rho : (a, b) \to D$ is called a crosscut in D if $\lim_{t\to a^+} \rho(t)$ and $\lim_{t\to b^-} \rho(t)$ both exist and lie on ∂D. We emphasize that by definition the end points of ρ do not belong to ρ, and so ρ completely lies in D. It is well known (cf. [11]) that as $t \to a^+$ or $t \to b^-$, $\rho(t)$ tends to a prime end of D. We say that these two prime ends are determined by ρ. Thus, if f maps D conformally onto $\mathbb{D}$, then $f(\rho)$ is a crosscut in $\mathbb{D}$. So we see that $D \setminus \rho$ has exactly two connected components.

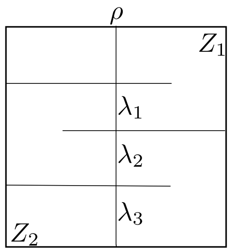

Fig. 1. This figure illustrates the situation of Lemma 2.1. Here $\widetilde{D}$ is the square and D is the comb domain. The λ_1, λ_2, λ_3 are sub-crosscuts of ρ in D that separate Z_1 from Z_2. Among them λ_1 is the crosscut given by the lemma.

For the ease of labeling the two components of $D \setminus \rho$, we introduce the following symbols. Let K be any subset of $\mathbb{C}$ such that $K \cap D$ is a relatively closed subset of D, and let S be a connected subset of $D \setminus K$. We use $D(K; S)$ to denote the connected component of $D \setminus K$ which is a neighborhood of S in D; and let $D^*(K; S) = D \setminus (K \cup D(K; S))$, which is the union of components of $D \setminus K$ other than $D(K; S)$. For example, $D(K; z_1) \neq D(K; z_2)$ means that z_1 and z_2 are separated in D by K. If ρ and η are disjoint crosscuts in D. Then $D \setminus \rho = D(\rho; \eta) \cup D^*(\rho; \eta)$ and $D \setminus \eta = D(\eta; \rho) \cup D^*(\eta; \rho)$; and we have $D^*(\rho; \eta) \subset D(\eta; \rho)$ and $D^*(\eta; \rho) \subset D(\rho; \eta)$.

The symbols $D(K; S)$ and $D^*(K; S)$ also make sense if S is a prime end of D such that $D \setminus K$ is a neighborhood of S in D. If D is an $\mathbb{H}$-domain, and S is the prime end ∞, then we omit the ∞ in $D(K; \infty)$ and $D^*(K; \infty)$. For example, for the SLE_κ curve γ in $\mathbb{H}$ from 0 to ∞, the corresponding $\mathbb{H}$-hull K_t satisfies that $\mathbb{H} \setminus K_t = \mathbb{H}(\gamma(0, t])$.

Lemma 2.1. *Let $D \subset \widetilde{D}$ be two simply connected domains. Let ρ be a Jordan curve in $\widetilde{D}$, which intersects ∂D, or a crosscut in $\widetilde{D}$. Let Z_1 and Z_2 be two connected subsets or prime ends of $\widetilde{D}$ such that $\widetilde{D}(\rho; Z_j)$, $j = 1, 2$, are well defined and not equal. In other words, $\widetilde{D} \setminus \rho$ is a neighborhood of both Z_1 and Z_2 in D, and Z_1 is disconnected from Z_2 in $\widetilde{D}$ by ρ. Suppose D is a neighborhood of both Z_1 and Z_2 in $\widetilde{D}$. Let Λ denote the set of connected components of $\rho \cap D$. Then there is a unique $\lambda_1 \in \Lambda$ such that $D(\lambda_1; Z_1) \neq D(\lambda_1; Z_2)$, and if $D(\lambda; Z_1) \neq D(\lambda; Z_2)$ for some $\lambda \in \Lambda$, then $D(\lambda_1; Z_1) \subset D(\lambda; Z_1)$ and $D(\lambda_1; Z_2) \supset D(\lambda; Z_2)$.*

Remark. *Every $\lambda \in \Lambda$ is a crosscut in D. We call the λ_1 given by the lemma the first sub-crosscut of ρ in D that disconnects Z_1 from Z_2. See Figure 1.*

Proof of Lemma 2.1. Let $\Lambda_0 = \{\lambda \in \Lambda : D(\lambda; Z_1) \neq D(\lambda; Z_2)\}$. We first show that Λ_0 is finite. Let γ be any curve in D connecting Z_1 with Z_2. Since $\gamma \cap \rho$ is a compact subset of $\bigcup_{\lambda \in \Lambda} \lambda$, and every $\lambda \in \Lambda$ is a relatively open subset of ρ, we see that γ intersects finitely many $\lambda \in \Lambda$. From the definition of Λ_0, γ intersects every $\lambda \in \Lambda_0$. Thus, Λ_0 is finite. We emphasize here that the above argument does not exclude the possibility that Λ_0 is empty.

Next, we show that Λ_0 is nonempty. We choose γ such that it minimizes the size of the set $\Lambda(\gamma) := \{\lambda \in \Lambda : \gamma \cap \lambda \neq \varnothing\}$, which can not be empty since $\bigcup_{\lambda \in \Lambda} \lambda = \rho \cap D$ disconnects Z_1 from Z_2 in D. Let $\lambda_0 \in \Lambda(\gamma)$. Let w_1 and w_2 be the first point and the last point on γ, which lies on λ_0, respectively. Let λ_0' be the sub curve of λ_0 with end points w_1 and w_2. There is $\varepsilon > 0$ such that $\operatorname{dist}(\lambda_0', \lambda) > \varepsilon$ for any $\lambda \in \Lambda \setminus \{\lambda_0\}$. Suppose $\lambda_0 \notin \Lambda_0$. Then $D(\lambda; Z_1) = D(\lambda; Z_2)$. We may choose for $j = 1, 2$, w_j' on the part of γ between Z_j and w_j, which is very close to w_j, such that there is a curve γ_ε connecting w_1' and w_2' in $D(\lambda_0; Z_1)$, which stays in the ε-neighborhood of λ_0'. Construct a new curve γ' in D connecting Z_1 and Z_2 by modifying γ such that the part of γ between w_1' and w_2' is replaced by γ_ε. Then we find that $\Lambda(\gamma') = \Lambda(\gamma) \setminus \{\lambda_0\}$, which contradicts the assumption on γ. Thus, $\Lambda_0 \supset \Lambda(\gamma)$ is nonempty.

Finally, we need to show that there is $\lambda_1 \in \Lambda_0$, which minimizes $\{D(\lambda; Z_1) : \lambda \in \Lambda_0\}$ and maximizes $\{D(\lambda; Z_2) : \lambda \in \Lambda_0\}$. This follows from the finiteness and nonemptyness of Λ_0 and the facts that for any $\lambda_1, \lambda_2 \in \Lambda_0$, one of $D(\lambda_1; Z_1)$ and $D(\lambda_2; Z_1)$ is a subset of the other, and the inclusion relation is reversed if Z_1 is replaced by Z_2. $\square$

Lemma 2.2. *Let D be a simply connected domain and ρ a crosscut in D. Let w_0, w_1 and w_∞ be connected subsets or prime ends of D such that $D \setminus \rho$ is a neighborhood of all of them in D. Suppose that ρ disconnects w_0 from w_∞ in D. Let $\gamma(t)$, $0 \le t < T$, be a continuous curve in $\overline{D}$ with $\gamma(0) \in \partial D$. Suppose for $0 \le t < T$, $D \setminus \gamma[0, t]$ is a neighborhood*

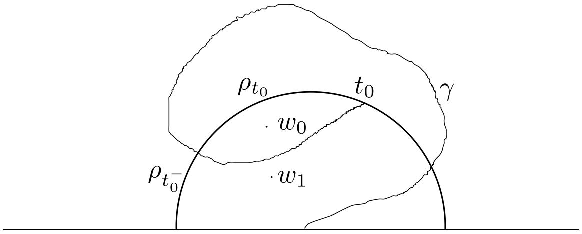

Fig. 2. This figure illustrates the situation of Lemma 2.2. Here D is the upper half plane, ρ is the semi-circle, and w_∞ is the prime end ∞. The curve γ is shown up to some time t_0. Since ρ_{t_0} does not disconnect w_1 from ∞, we have $f(t_0) = 1$. When $t < t_0$ and is close to t_0, ρ_t is the $\rho_{t_0^-}$ in the figure, which disconnects w_1 from ∞. This means that $f(t_0^-) = 0$, and f is not left-continuous at t_0.

of w_0, w_1 and w_∞ in D, and $w_0, w_1 \subset D_t := D(\gamma[0,t]; w_\infty)$. For $0 \le t < T$, let ρ_t be the first sub-crosscut of ρ in D_t that disconnects w_0 from w_∞ as given by Lemma 2.1. For $0 \le t < T$, let $f(t) = 1$ if $w_1 \in D_t(\rho_t; w_\infty)$; $= 0$ if $w_1 \in D_t^*(\rho_t; w_\infty)$. Then f is right-continuous on $[0,T)$, and left-continuous at those $t_0 \in (0,T)$ such that $\gamma(t_0)$ is not an end point of ρ_{t_0}.

Remark. *It is easy to see that $(D_t)_{0\le t<T}$ is a decreasing family of $\mathbb{H}$-domains. But $(\rho_t)_{0\le t<T}$ may not be a decreasing family. See Figure 2.*

Proof of Lemma 2.2. We first show that f is right-continuous. Fix $t_0 \in [0,T)$. From the definition of ρ_{t_0}, there exist a curve β_0 in D_{t_0}, which goes from w_0 to w_∞, crosses ρ_{t_0} for only once, and does not visit $\rho \setminus \rho_{t_0}$ before ρ_{t_0}. Let $S = w_\infty$ or w_0 depending on whether $f(t_0) = 1$ or 0. Then there is a curve β_1 in $D_{t_0} \setminus \rho_{t_0}$ that connects w_1 with S. Since $\gamma(t_0) \notin D_{t_0}$ and γ is continuous, there is $t_1 \in (t_0, T)$ such that $\gamma[t_0, t_1)$ is disjoint from β_0 and β_1. Fix $t \in (t_0, t_1)$. Then $\beta_0, \beta_1 \subset D_t$. From Lemma 2.1, there is the first sub-crosscut of ρ_{t_0}, denoted by $\rho_{t_0,t}$ in D_t that disconnects w_0 from w_∞. From the properties of β_0, $\rho_{t_0,t}$ is the connected component of $\rho_{t_0} \cap D_t$ that contains $\beta_0 \cap \rho_{t_0}$. Since β_0 does not intersect ρ before $\beta_0 \cap \rho_{t_0}$, we have $\rho_t = \rho_{t_0,t} \subset \rho_{t_0}$. Thus, β_1 is a curve in $D_t \setminus \rho_t$ connecting w_1 with S, which implies that f is constant on $[t_0, t_1)$.

Suppose $\gamma(t_0)$ is not an end point of ρ_{t_0} for some $t_0 \in (0,T)$. We now show that f is left-continuous at t_0. There exists $t_1 \in [0, t_0)$ such that $\gamma(t_1, t_0]$ does not intersect ρ_{t_0}. Fix $t \in (t_1, t_0]$. Then ρ_{t_0} is a crosscut in D_t. Let β_0, S, β_1 be as above. Then β_0 and β_1 are also curves in D_t. From the properties of β_0, we see that $\rho_t = \rho_{t_0}$. Thus, β_1 is a curve in $D_t \setminus \rho_t$ connecting w_1 with S, which implies that f is constant on $(t_1, t_0]$. □

2.3. *Estimates*

We give some important estimates for *SLE* in this subsection. The first one is the interior estimate. To begin with, we quote the following theorem proved in [2].

Theorem 2.1. *Suppose γ is an SLE_κ curve from w_1 to w_2 in a simply connected domain D. If $z \in D$, then*

$$\mathbb{P}\big[\operatorname{dist}(\gamma, z) \le r\big] \le C G_{(D;w_1,w_2)}(z) r^{2-d},$$

where $G_{(D;w_1,w_2)}$ is the 1-point Green's function for the γ.

A stronger estimate is obtained in [6]: $\mathbb{P}[\operatorname{dist}(\gamma, z) \le r] = r^{2-d} G_{(D;w_1,w_2)}(z)[1 + o(r^\alpha)]$, $\alpha > 0$. Using (1.4), (1.3) and Koebe's 1/4 theorem, we find that $G_{(D;w_1,w_2)}(z) \le C \operatorname{dist}(z, \partial D)^{d-2}$. So we have the following interior estimate which is a corollary of Theorem 2.1.

Lemma 2.3 (Interior estimate). *For any $z \in D$,*

$$\mathbb{P}\big[\operatorname{dist}(\gamma, z) \le r\big] \le C \left(\frac{r}{\operatorname{dist}(z, \partial D)}\right)^{2-d}.$$

We will state the boundary estimate for *SLE* in several different forms. The original one comes from [1], which is the following theorem.

Theorem 2.2 (Boundary estimate v.0). *Let γ be an SLE_κ curve in $\mathbb{H}$ from 0 to ∞. Then for any $x_0 \in \mathbb{R} \setminus \{0\}$ and $r > 0$,*

$$\mathbb{P}\big[\operatorname{dist}(\gamma, x_0) \le r\big] \le C\left(\frac{r}{|x_0|}\right)^{\alpha}.$$

We will express the above theorem in another form using the notation of extremal distance. The reader may refer to [11] for the definition and properties of extremal distance (length). We use $d_D(L_1, L_2)$ to denote the extremal distance between L_1 and L_2 in D. Suppose K is a nonempty $\mathbb{H}$-hull with $\overline{K} \cap \overline{\mathbb{R}}_- = \varnothing$. Let $x_K = \max\{\overline{K} \cap \mathbb{R}\}$ and $r_K = \max\{|z - x_K| : z \in \overline{K}\}$. It is well known that there are absolute constants C and M such that $\frac{r_K}{x_K} \le Ce^{-\pi d_{\mathbb{H}}(K,\mathbb{R}_-)}$ if $d_{\mathbb{H}}(K, \mathbb{R}_-) \ge M$. So the above theorem implies the following corollary.

Lemma 2.4 (Boundary estimate v.1). *Let γ be as above. Then for any $\mathbb{H}$-hull K with $\overline{K} \cap \overline{\mathbb{R}}_- = \varnothing$, we have*

$$\mathbb{P}[\gamma \cap K \neq \varnothing] \le Ce^{-\alpha\pi d_{\mathbb{H}}(K,\mathbb{R}_-)}.$$

The same is true if $\mathbb{R}_-$ is replaced with $\mathbb{R}_+$.

Using conformal invariance and comparison principle of extremal distance, we immediately get the following version of boundary estimate from the previous one.

Lemma 2.5 (Boundary estimate v.2). *Let D be a regular simply connected domain, and w_0 and w_∞ be two distinct prime ends of D. Let ρ and η be two disjoint crosscuts in D such that $D(\rho; \eta)$ is neither a neighborhood of w_0 nor a neighborhood of w_∞ in D. For w_0, the condition means that either $D \setminus \rho$ is a neighborhood of w_0 and $D(\rho; w_0) = D^*(\rho; \eta)$, or w_0 is a prime end determined by ρ; and likewise for w_∞. Let γ be an SLE_κ curve in D from w_0 to w_∞. Then*

$$\mathbb{P}\big[\gamma \cap \big(\eta \cup D^*(\eta; w_\infty)\big) \neq \varnothing\big] \le Ce^{-\alpha\pi d_D(\rho,\eta)}.$$

We now combine the interior estimate and the boundary estimate to get the following one-point estimate, which implies the case $n = 1$ in Theorem 1.1.

Lemma 2.6 (One-point estimate). *Let D be an $\mathbb{H}$-domain with a prime end $w_0 \neq \infty$. Let γ be an SLE_κ curve in D from w_0 to ∞. Let $z_0 \in \overline{\mathbb{H}}$, $y_0 = \operatorname{Im} z_0 \ge 0$, and $R > r > 0$. Let $\rho = \{z \in \mathbb{H} : |z - z_0| = R\}$ and $\eta = \{z \in \mathbb{H} : |z - z_0| = r\}$. Suppose $\{z \in \mathbb{H} : |z - z_0| \le R\} \subset D$ and $w_0 \notin \{x \in \mathbb{R} : |x - z_0| < R\}$. Then*

$$\mathbb{P}[\gamma \cap \eta \neq \varnothing] \le C\frac{P_{y_0}(r)}{P_{y_0}(R)}.$$

Proof. We consider different cases. Case 1: $y_0 \ge R$. The conclusion follows from the interior estimate because $\frac{P_{y_0}(r)}{P_{y_0}(R)} = (\frac{r}{R})^{2-d}$ and $\operatorname{dist}(z_0, \partial D) \ge R$. Case 2: $y_0 \le r$. We have $\frac{P_{y_0}(r)}{P_{y_0}(R)} = (\frac{r}{R})^{\alpha}$. By increasing the value of C, we may assume that $R > 4r$. The conclusion follows from the boundary estimate because ρ and η are separated in D by the two crosscuts $\{z \in \mathbb{H} : |z - \operatorname{Re} z_0| = 2r\}$ and $\{z \in \mathbb{H} : |z - \operatorname{Re} z_0| = R/2\}$, and the extremal distance between them in D is $\log(R/(4r))/\pi$. Case 3: $R > y_0 > r$. Let $\rho' = \{z \in \mathbb{H} : |z - z_0| = y_0\}$, which separates ρ from η in D. Let T be the first time that γ hits ρ', and $\gamma^T(t) = \gamma(T + t)$, $0 \le t < \infty$, if $T < \infty$. Then T is an stopping time, and $\{\gamma \cap \eta \neq \varnothing\} = \{\gamma^T \cap \eta \neq \varnothing\} \subset \{T < \infty\}$ almost surely. From the result of Case 2, $\mathbb{P}[T < \infty] \le C\frac{P_{y_0}(y_0)}{P_{y_0}(R)}$. From DMP, conditioned on $\gamma[0, T]$ and $\{T < \infty\}$, the γ^T is an SLE_κ curve in $D(\gamma[0, T])$ from $\gamma(T)$ to ∞. Since $\operatorname{dist}(z_0, \partial D_T) = y_0$, from the result of Case 1, we get $\mathbb{P}[\gamma^T \cap \eta \neq \varnothing | \gamma[0, T], T < \infty] \le C\frac{P_{y_0}(r)}{P_{y_0}(y_0)}$. Combining this with the estimate for $\mathbb{P}[T < \infty]$, we get the conclusion in Case 3. □

The following version of boundary estimate will be frequently used in this paper.

Lemma 2.7 (Boundary estimate v.3). *Let D be an $\mathbb{H}$-domain with a prime end $w_0 \neq \infty$. Let γ be an SLE_κ curve in D from w_0 to ∞. Let ρ be a crosscut in D such that $D^*(\rho)$ is not a neighborhood of w_0 in D, and $S \subset D^*(\rho)$. Let $\widetilde{D}$ be a domain that contains D, and $\widetilde{\rho}$ a subset of $\widetilde{D}$ that contains ρ. Let $\widetilde{\eta}$ be a Jordan curve in $\widetilde{D}$, which intersects ∂D, or a crosscut in $\widetilde{D}$. Suppose that $\widetilde{\eta}$ disconnects S from $\widetilde{\rho}$ in $\widetilde{D}$. Then*

$$\mathbb{P}[\gamma \cap S \neq \varnothing] \le C e^{-\pi\alpha d_{\widetilde{D}}(\widetilde{\rho},\widetilde{\eta})}.$$

Proof. From Lemma 2.1, $\widetilde{\eta}$ contains a sub-crosscut in E, denoted by η, which disconnects S from ρ. Since $S \subset D^*(\rho)$, we have $\eta \subset D^*(\rho)$ and $S \subset D^*(\eta)$. Thus, $D(\rho;\eta) = D^*(\rho)$ is not a neighborhood of either ∞ or w_0 in D. Using the boundary estimate v.2, we get

$$\mathbb{P}[\gamma \cap S \neq \varnothing] \le \mathbb{P}\big[\gamma \cap D^*(\eta) \neq \varnothing\big] \le C e^{-\pi\alpha d_D(\rho,\eta)} \le C e^{-\pi\alpha d_{\widetilde{D}}(\widetilde{\rho},\widetilde{\eta})}. \qquad \square$$

3. Main estimates

In this section, we let γ be an SLE_κ curve in $\mathbb{H}$ from 0 to ∞. Given any set S, let $\tau_S = \inf\{t \ge 0 : \gamma(t) \in S\}$; we set $\inf \varnothing = \infty$ by convention. Let $(\mathcal{F}_t)$ be the right-continuous filtration generated by γ. For $t_0 \ge 0$, let $\gamma^{t_0}(t) = \gamma(t_0 + t)$, $0 \le t < \infty$, and $H_{t_0} = \mathbb{H}(\gamma[0,t_0])$. Recall the DMP: if T is an $(\mathcal{F}_t)$-stopping time, then conditioned on $\mathcal{F}_T$ and $T < \infty$, γ^T is an SLE_κ curve in H_T from (the prime end of H_T determined by) $\gamma(T)$ to ∞.

Theorem 3.1. *Let $m \in \mathbb{N}$, $z_j \in \overline{\mathbb{H}}$ and $R_j \ge r_j > 0$, $0 \le j \le m$. Let $\widehat{\xi}_j = \{|z - z_j| = R_j\}$, $\xi_j = \{|z - z_j| = r_j\}$, and $\widehat{D}_j = \{|z - z_j| \le R_j\}$, $0 \le j \le m$. Suppose that $0 \notin \widehat{D}_j$, $0 \le j \le m$; and $\widehat{D}_0 \cap \widehat{D}_j = \varnothing$, $1 \le j \le m$. Let $r_0' \in (0, r_0)$ and $\xi_0' = \{|z - z_0| = r_0'\}$. Let*

$$E = \{\tau_{\xi_0} < \tau_{\widehat{\xi}_1} \le \tau_{\xi_1} < \cdots < \tau_{\widehat{\xi}_m} \le \tau_{\xi_m} < \tau_{\xi_0'} < \infty\}.$$

Let $y_j = \operatorname{Im} z_j$, $1 \le j \le m$. Then we have

$$\mathbb{P}[E|\mathcal{F}_{\tau_{\xi_0}}] \le C^m \left(\frac{r_0}{R_0}\right)^{\alpha/4} \prod_{j=1}^m \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

Discussion. *From the 1-point estimate, we see that, given γ up to hitting $\widehat{\xi}_j$, the probability that it reaches ξ_j is at most $C\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}$. The DMP allows us to put these estimates together to get the product on the righthand side of the above formula. The key point of the proof is to use the boundary estimate to derive the factor $(\frac{r_0}{R_0})^{\alpha/4}$. Recall that the boundary estimate can be applied when the SLE curve is required to cross a disjoint pair of crosscuts from the unbounded component to the bounded component determined by these crosscuts. But whether a given set lies in the bounded component may vary as the SLE curve grows. So we have to carefully keep track of the changes of the "topology" situations.*

Proof of Theorem 3.1. Let Ξ be the set of $\xi_j, \widehat{\xi}_j$, $0 \le j \le n$, and ξ_0'. By Theorem 2.2, for any $\xi \in \Xi$, γ almost surely does not visit $\xi \cap \mathbb{R}$. By discarding an event with probability zero, we may assume that γ does not visit $\xi \cap \mathbb{R}$ for any $\xi \in \Xi$. Then for any $\xi \in \Xi$, $\tau_\xi = \tau_{\xi \cap \mathbb{H}}$. Thus, it suffices to prove the lemma with each $\xi \in \Xi$ replaced by $\xi \cap \mathbb{H}$. This means that every $\xi \in \Xi$ is a Jordan curve or crosscut in $\mathbb{H}$. After that, we see that $\tau_\xi < \infty$ implies that $\gamma(\tau_\xi) \in \xi \cap \mathbb{H}$, and γ does not visit $\mathbb{H}^*(\xi)$ before ξ.

Let $\tau_0 = \tau_{\xi_0}$, $\widehat{\tau}_j = \tau_{\widehat{\xi}_j}$ and $\tau_j = \tau_{\xi_j}$, $1 \le j \le m$, and $\tau_{m+1} = \tau_{\xi_0'}$. From the DMP and one-point estimate (Lemma 2.6), we get

$$\mathbb{P}[\tau_j < \infty|\mathcal{F}_{\widehat{\tau}_j}] \le C\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}, \quad 1 \le j \le m. \tag{3.1}$$

Thus, $\mathbb{P}[E|\mathcal{F}_{\tau_0}] \le C^m \prod_{j=1}^m \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}$. If $R_0 = r_0$, the proof is finished.

Suppose $R_0 > r_0$. Let $\rho = \{z \in \mathbb{H} : |z - z_0| = \sqrt{R_0 r_0}\}$. Then ρ is a Jordan curve or crosscut in $\mathbb{H}$, which lies between $\widehat{\xi}_0$ and ξ_0, and

$$d_{\mathbb{H}}(\rho, \xi_0), d_{\mathbb{H}}(\rho, \widehat{\xi}_0) \ge \frac{\log(R_0/r_0)}{4\pi}. \tag{3.2}$$

Also note that ρ disconnects ξ_0' from ∞. Let $T = \inf\{t \ge 0 : \xi_0' \not\subset H_t\}$. For $\tau_0 \le t < T$, ξ_0' is a connected subset of H_t, and ρ intersects ∂H_t. Thus, we may use Lemma 2.1 to define ρ_t to be the first sub-crosscut of ρ in H_t that disconnects ξ_0' from ∞ for $\tau_0 \le t < T$. Note that every ρ_t is $\mathcal{F}_t$-measurable.

Let $I = \{(j, j+1) : 0 \le j \le m\} \cup \{(j, j) : 1 \le j \le m\}$, and define $(A_\iota)_{\iota \in I}$ by

$$\begin{aligned}
&A_{(0,1)} = \{T > \tau_0\} \cap \left\{\mathbb{H}^*(\xi_1) \subset H^*_{\tau_0}(\rho_{\tau_0})\right\} \in \mathcal{F}_{\tau_0};\\
&A_{(j,j)} = \{T > \tau_j\} \cap \left\{\mathbb{H}^*(\xi_j) \subset H_{\tau_{j-1}}(\rho_{\tau_{j-1}})\right\} \cap \left\{\mathbb{H}^*(\xi_j) \subset H^*_{\tau_j}(\rho_{\tau_j})\right\} \in \mathcal{F}_{\tau_j}, \quad 1 \le j \le m;\\
&A_{(j,j+1)} = \{T > \tau_j\} \cap \left\{\mathbb{H}^*(\xi_j) \subset H_{\tau_j}(\rho_{\tau_j})\right\} \cap \left\{\mathbb{H}^*(\xi_{j+1}) \subset H^*_{\tau_j}(\rho_{\tau_j})\right\} \in \mathcal{F}_{\tau_j}, \quad 1 \le j \le m-1;\\
&A_{(m,m+1)} = \{T > \tau_m\} \cap \left\{\mathbb{H}^*(\xi_m) \subset H_{\tau_m}(\rho_{\tau_m})\right\} \in \mathcal{F}_{\tau_m}.
\end{aligned}$$

Suppose E occurs. Then γ does not visit ξ_0' at any time $t \le \tau_m$. So ξ_0' is a connected subset of $\mathbb{H} \setminus \gamma[0, \tau_m]$. Then we must have $\xi_0' \subset H_{\tau_m}$ because γ^{τ_m} visits ξ_0', and $\gamma^{\tau_m} \subset \overline{H_{\tau_m}} \subset H_{\tau_m} \cup \gamma[0, \tau_m]$. Thus, $T > \tau_m > \tau_{m-1} > \cdots > \tau_1 > \tau_0$. Similarly, since $\mathbb{H}^*(\xi_j)$ is not visited by γ at any time $t \le \tau_j$, we conclude that $\mathbb{H}^*(\xi_j) \subset H_t$ for $t \le \tau_j$. Since $\mathbb{H}^*(\xi_j)$ is disjoint from $\rho \supset \rho_t$, we conclude that $\mathbb{H}^*(\xi_j)$ is contained in either $H_t(\rho_t)$ or $H_t^*(\rho_t)$ for any $t \le \tau_j$.

Define a strict total order on I such that $(0, 1) < (1, 1) < (1, 2) < (2, 2) < \cdots < (m-1, m) < (m, m) < (m, m+1)$. Define a family of events E_ι, $\iota \in I$, such that $E_\iota = E \setminus \bigcup_{\iota' : \iota' > \iota} A_{\iota'}$. Using induction, one can prove that

$$E_\iota \subset \left\{\mathbb{H}^*(\xi_{\iota_1}) \subset H^*_{\tau_{\iota_2}}(\rho_{\tau_{\iota_2}})\right\}, \quad \iota = (\iota_1, \iota_2) \in I \setminus \left\{(m, m+1)\right\}.$$

Especially, we get

$$E_{0,1} = E \setminus \bigcup_{\iota \in I \setminus \{(0,1)\}} A_\iota \subset \left\{\mathbb{H}^*(\xi_0) \subset H^*_{\tau_1}(\rho_{\tau_1})\right\} \subset A_{(0,1)}.$$

Thus, we have $E \subset \bigcup_{\iota \in I} A_\iota$. We will finish the proof by showing that

$$\mathbb{P}[E \cap A_\iota|\mathcal{F}_{\tau_0}] \le C^m \left(\frac{r_0}{R_0}\right)^{\alpha/4} \prod_{j=1}^m \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}, \quad \iota \in I. \tag{3.3}$$

Case 1. Suppose $A_{(0,1)}$ occurs and $\tau_0 < \widehat{\tau}_1$. Since $\widehat{\xi}_1$ and $\mathbb{H}^*(\xi_1)$ are subsets of $\mathbb{H}^*(\widehat{\xi}_1) \cup \widehat{\xi}_1$, which is a connected subset of $(\mathbb{H} \setminus \gamma[0, \tau_0]) \setminus \rho_{\tau_0}$, from $\mathbb{H}^*(\xi_1) \subset H^*_{\tau_0}(\rho_{\tau_0})$, we conclude that $\widehat{\xi}_1 \subset H^*_{\tau_0}(\rho_{\tau_0})$. Note that ρ disconnects $\widehat{\xi}_1$ from ξ_0' in $\mathbb{H}$, and intersects ∂H_{τ_0}. Applying Lemma 2.1, we get a sub-crosscut of ρ, denoted by ρ'_{τ_0}, that disconnects $\widehat{\xi}_1$ from ξ_0' in H_{τ_0}. Since both $\widehat{\xi}_1$ and ξ_0' lie in $H^*_{\tau_0}(\rho_{\tau_0})$, so does ρ'_{τ_0}. Thus, $H^*_{\tau_0}(\rho'_{\tau_0}) \subset H^*_{\tau_0}(\rho_{\tau_0})$. Since ρ_{τ_0} is the first sub-crosscut of ρ in H_{τ_0} that disconnects ξ_0' from ∞, we see that ρ'_{τ_0} does not disconnect ξ_0' from ∞. Thus, $\xi_0' \subset H_{\tau_0}(\rho'_{\tau_0})$, and $\widehat{\xi}_1 \subset H^*_{\tau_0}(\rho'_{\tau_0})$ as ρ'_{τ_0} disconnects $\widehat{\xi}_1$ from ξ_0' in H_{τ_0}. See Figure 3.

Since $\mathbb{H}^*(\xi_0)$ is a connected subset of $H_{\tau_0} \setminus \rho'_{\tau_0}$, and contains ξ_0' and a curve that approaches $\gamma(\tau_0) \in \xi_0$, we conclude that $H_{\tau_0}(\rho'_{\tau_0}; \gamma(\tau_0)) = H_{\tau_0}(\rho'_{\tau_0}; \xi_0') = H_{\tau_0}(\rho'_{\tau_0})$. Thus, $H_{\tau_0}(\rho'_{\tau_0}; \widehat{\xi}_1) = H^*_{\tau_0}(\rho'_{\tau_0})$ is not a neighborhood of $\gamma^{\tau_0}(0) = \gamma(\tau_0)$ in H_{τ_0}. Since $\tau_0 < \widehat{\tau}_1$, $\widehat{\tau}_1 < \infty$ implies that the SLE_κ curve γ^{τ_0} in H_{τ_0} (conditioned on $\mathcal{F}_{\tau_0}$) visits $\widehat{\xi}_1$. Since $\widehat{\xi}_0$ disconnects $\widehat{\xi}_1$ from $\rho \supset \rho'_{\tau_0}$ in $\mathbb{H}$, and intersects ∂H_{τ_0}, from the boundary estimate v.3 (Lemma 2.7) and (3.2), we get

$$\mathbb{P}[\widehat{\tau}_1 < \infty|\mathcal{F}_{\tau_0}, A_{(0,1)}, \tau_0 < \widehat{\tau}_1] \le C e^{-\alpha\pi d_{\mathbb{H}}(\rho, \widehat{\xi}_0)} \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4},$$

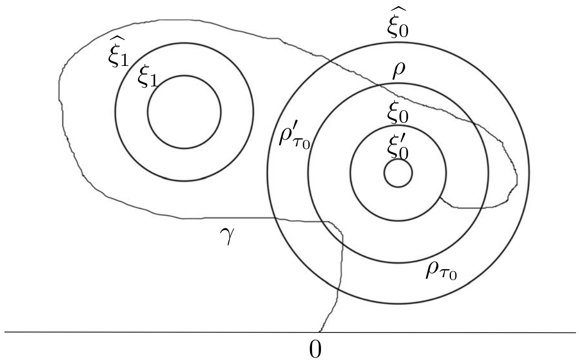

Fig. 3. This figure shows the event $A_{(0,1)}$ with γ stopped at $\tau_0 = \tau_{\xi_0}$.

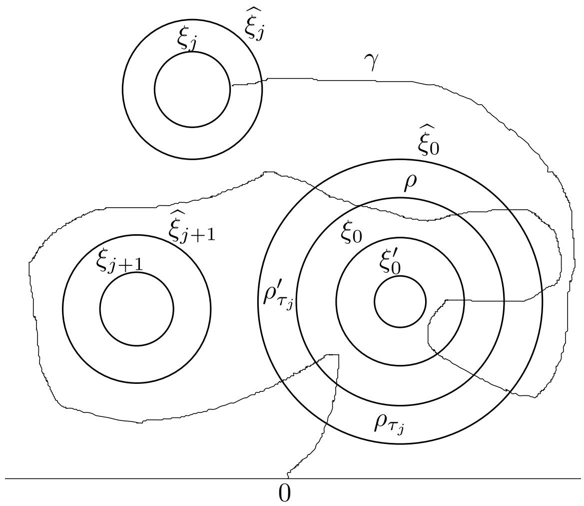

Fig. 4. This figure shows the event $A_{(j,j+1)}$ with γ stopped at $\tau_j = \tau_{\xi_j}$.

which together with (3.1) implies that (3.3) holds for $\iota = (0, 1)$.

Case 2. Suppose for some $1 \le j \le m-1$, $A_{(j,j+1)}$ occurs and $\tau_j < \widehat{\tau}_{j+1}$. Using the argument in the previous case with τ_0 and $\widehat{\xi}_1$ replaced by τ_j and $\widehat{\xi}_{j+1}$, respectively, we get a sub-crosscut of ρ, denoted by ρ'_{τ_j}, that disconnects $\widehat{\xi}_{j+1}$ from ξ'_0 in H_{τ_j}, and conclude that $H^*_{\tau_j}(\rho'_{\tau_j}) \subset H^*_{\tau_j}(\rho_{\tau_j})$, $\xi'_0 \subset H_{\tau_j}(\rho'_{\tau_j})$, and $\widehat{\xi}_{j+1} \subset H^*_{\tau_j}(\rho'_{\tau_j})$. See Figure 4.

Since $\mathbb{H}^*(\xi_j)$ is a connected subset of $H_{\tau_j} \setminus \rho_{\tau_j}$, and contains a curve that approaches $\gamma(\tau_j) \in \xi_j$, we conclude that $H_{\tau_j}(\rho_{\tau_j}; \gamma(\tau_j)) = H_{\tau_j}(\rho_{\tau_j}; \mathbb{H}^*(\xi_j)) = H_{\tau_j}(\rho_{\tau_j})$. Thus, $H^*_{\tau_j}(\rho'_{\tau_j}) \subset H^*_{\tau_j}(\rho_{\tau_j})$ is not a neighborhood of $\gamma^{\tau_j}(0) = \gamma(\tau_j)$ in H_{τ_j}. Since $\tau_j < \widehat{\tau}_{j+1}$, $\widehat{\tau}_{j+1} < \infty$ implies that the SLE_κ curve γ^{τ_j} in H_{τ_j} (conditioned on $\mathcal{F}_{\tau_j}$) visits $\widehat{\xi}_{j+1}$. Since $\widehat{\xi}_0$ disconnects $\widehat{\xi}_{j+1}$ from $\rho \supset \rho'_{\tau_j}$ in $\mathbb{H}$, from Lemma 2.7 and (3.2), we get

$$\mathbb{P}[\widehat{\tau}_{j+1} < \infty | \mathcal{F}_{\tau_j}, A_{(j,j+1)}, \tau_j < \widehat{\tau}_{j+1}] \le C e^{-\alpha\pi d_{\mathbb{H}}(\rho, \widehat{\xi}_0)} \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4},$$

which together with (3.1) implies that (3.3) holds for $\iota = (j, j+1)$, $1 \le j \le m-1$.

Case 3. Suppose $A_{(m,m+1)}$ and $\tau_m < \tau_{m+1}$ occur. Since $\mathbb{H}^*(\xi_m)$ is a connected subset of $H_{\tau_m} \setminus \rho_{\tau_m}$, and contains a curve that approaches $\gamma(\tau_m) \in \xi_m$, we conclude that $H_{\tau_m}(\rho_{\tau_m}; \gamma(\tau_m)) = H_{\tau_m}(\rho_{\tau_m}; \mathbb{H}^*(\xi_m)) = H_{\tau_m}(\rho_{\tau_m})$. Thus, $H^*_{\tau_m}(\rho_{\tau_m})$ is not a neighborhood of $\gamma^{\tau_m}(0) = \gamma(\tau_m)$ in H_{τ_m}. Since $\tau_m < \tau_{m+1}$, $\tau_{m+1} < \infty$ implies that the SLE_κ curve

γ^{τ_m} in H_{τ_m} (conditioned on $\mathcal{F}_{\tau_m}$) visits $\xi_0' \subset H^*_{\tau_m}(\rho_{\tau_m})$. Since ξ_0 disconnects ξ_0' from ρ in $\mathbb{H}$, and intersects ∂H_{τ_m}, we may apply Lemma 2.7 and (3.2) to get

$$\mathbb{P}[\tau_{m+1} < \infty | \mathcal{F}_{\tau_m}, A_{(m,m+1)}, \tau_m < \tau_{m+1}] \le C e^{-\pi d_{\mathbb{H}}(\xi_0,\rho)} \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4},$$

which together with (3.1) implies that (3.3) holds for $\iota = (m, m+1)$.

Case 4. Finally, we consider (3.3) in the case $\iota = (j, j)$. Fix $1 \le j \le m$ and define

$$\sigma_j = \inf\{t \ge \tau_{j-1} : \mathbb{H}^*(\xi_j) \subset H^*_t(\rho_t)\}.$$

From Lemma 2.2 and the right-continuity of $(\mathcal{F}_t)$, we have

1. Every σ_j is an $(\mathcal{F}_t)$-stopping time.
2. If $\sigma_j < \infty$, then $\mathbb{H}^*(\xi_j) \subset H^*_{\sigma_j}(\rho_{\sigma_j})$.
3. If $A_{(j,j)}$ occurs, then $\tau_{j-1} < \sigma_j < \tau_j$.
4. If $\tau_{j-1} < \sigma_j < \infty$, then $\gamma(\sigma_j)$ is an endpoint of ρ_{σ_j}.

Note that the last property implies that $H^*_{\sigma_j}(\rho_{\sigma_j})$ is not a neighborhood of either $\gamma(\sigma_j)$ or ∞ in H_{σ_j}. Let $F_< = \{\sigma_j < \widehat{\tau}_j\}$ and $F_\ge = \{\widehat{\tau}_j \le \sigma_j < \tau_j\}$. Then $A_{(j,j)} \subset F_< \cup F_\ge$.

Case 4.1. Suppose $F_\ge$ occurs. Let $N = \lceil \log(R_j/r_j) \rceil \in \mathbb{N}$. Let $\zeta_k = \{z \in \mathbb{H} : |z - z_j| = (R_j^{N-k} r_j^k)^{1/N}\}$, $0 \le k \le N$. Note that $\zeta_0 = \widehat{\xi}_j$ and $\zeta_N = \xi_j$. Then $F_\ge \subset \bigcup_{k=1}^N F_k$, where

$$F_k := \{\tau_{\zeta_{k-1}} \le \sigma_j < \tau_{\zeta_k}\}, \quad 1 \le k \le N.$$

If F_k occurs, then $\zeta_k \subset H^*_{\sigma_j}(\rho_{\sigma_j})$ because $\mathbb{H}^*(\zeta_k) \cup \zeta_k$ is a connected subset of $(\mathbb{H} \setminus \gamma[0,\sigma_j]) \setminus \rho$ that contains both ζ_k and $\mathbb{H}^*(\xi_j)$, and $\mathbb{H}^*(\xi_j) \subset H^*_{\sigma_j}(\rho_{\sigma_j})$. See Figure 5.

From Lemma 2.7 and (3.2), we get

$$\mathbb{P}[\tau_{\zeta_k} < \infty | \mathcal{F}_{\sigma_j}, F_k] \le C e^{-\alpha\pi d_{\mathbb{H}}(\rho,\zeta_{k-1})} \le C e^{-\alpha\pi (d_{\mathbb{H}}(\rho,\widehat{\xi}_0) + d_{\mathbb{H}}(\zeta_0,\zeta_{k-1}))} \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4}\left(\frac{r_j}{R_j}\right)^{(\alpha/2)((k-1)/N)}.$$

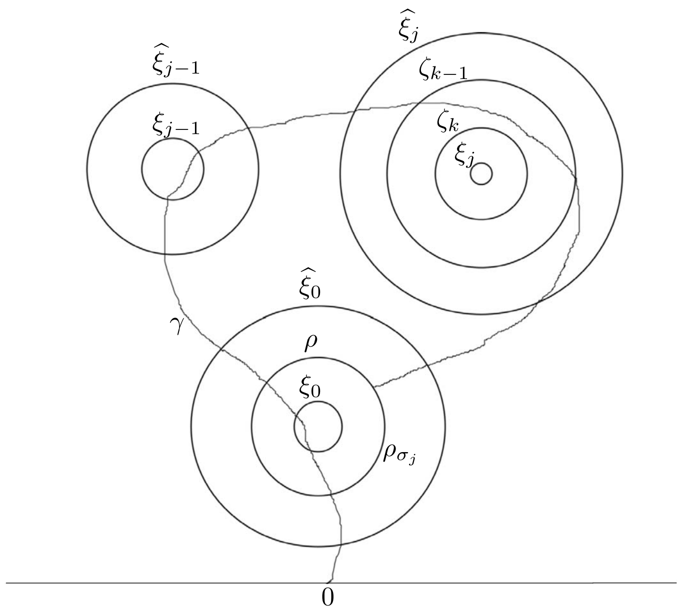

Fig. 5. This figure shows the event F_k, a sub event of $A_{(j,j)}$, with γ stopped at σ_j, the first time after $\tau_{j-1} = \tau_{\xi_{j-1}}$ that ξ_j lies in the bounded component of $H_t \setminus \rho_t$.

From Lemma 2.6, we get

$$\mathbb{P}[F_k|\mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \widehat{\tau}_j] \le C\frac{P_{y_j}((R_j^{N-k+1}r_j^{k-1})^{1/N})}{P_{y_j}(R_j)},$$

$$\mathbb{P}[\tau_j < \infty|\mathcal{F}_{\tau_{\zeta_k}}, F_k] \le C\frac{P_{y_j}(r_j)}{P_{y_j}((R_j^{N-k}r_j^k)^{1/N})}.$$

The above three displayed formulas together with (1.5) imply that

$$\mathbb{P}[\tau_j < \infty, F_k|\mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \widehat{\tau}_j] \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4}\left(\frac{r_j}{R_j}\right)^{(\alpha/2)((k-1)/N)}\left(\frac{r_j}{R_j}\right)^{-\alpha/N}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

Since $F_\ge \subset \bigcup_{k=1}^N F_k$, by summing up the above inequality over k, we get

$$\mathbb{P}[\tau_j < \infty, F_\ge|\mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \widehat{\tau}_j] \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}\left[\left(\frac{r_j}{R_j}\right)^{-\alpha/N}\frac{1-(r_j/R_j)^{\alpha/2}}{1-(r_j/R_j)^{\alpha/(2N)}}\right]. \tag{3.4}$$

By considering the cases $R_j/r_j \le e$ and $R_j/r_j > e$ separately, we see that the quantity inside the square bracket is bounded by the constant $\frac{e^\alpha}{1-e^{-\alpha/4}}$.

Case 4.2. Suppose $F_<$ occurs. Then $\mathbb{H}^*(\widehat{\xi}_j) \cup \widehat{\xi}_j$ is a connected subset of $(\mathbb{H} \setminus \gamma[0,\sigma_j]) \setminus \rho$ that contains $\mathbb{H}^*(\xi_j)$. So we get $\widehat{\xi}_j \subset H^*_{\sigma_j}(\rho_{\sigma_j}; \mathbb{H}^*(\xi_j) = H^*_{\sigma_j}(\rho_{\sigma_j})$. Since $\widehat{\xi}_0$ disconnects ρ from $\widehat{\xi}_j$ in $\mathbb{H}$, applying Lemma 2.7 and (3.2), we get

$$\mathbb{P}[\widehat{\tau}_j < \infty|\mathcal{F}_{\sigma_j}, F_<] \le Ce^{-\alpha\pi d_{\mathbb{H}}(\rho,\widehat{\xi}_0)} \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4},$$

which together with (3.1) implies that

$$\mathbb{P}[\tau_j < \infty, F_<|\mathcal{F}_{\tau_{j-1}}] \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\mathbb{P}[\tau_j < \infty, A_{(j,j)}|\mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \widehat{\tau}_j] \le C\left(\frac{r_0}{R_0}\right)^{\alpha/4}\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)},$$

which together with (3.1) implies that (3.3) holds for $\iota = (j, j)$, $1 \le j \le m$. □

Let Ξ be a family of mutually disjoint circles with center in $\overline{\mathbb{H}}$, each of which does not pass through or enclose 0. Define a partial order on Ξ such that $\xi_1 < \xi_2$ if ξ_2 is enclosed by ξ_1. One should keep in mind that a smaller element in Ξ has bigger radius, but will be visited earlier (if it happens) by a curve started from 0.

Suppose that Ξ has a partition $\{\Xi_e\}_{e\in\mathcal{E}}$ with the following properties:

1. For each $e \in \mathcal{E}$, the elements in Ξ_e are concentric circles with radii forming a geometric sequence with common ratio 1/4. We denote the common center z_e, the biggest radius R_e, and the smallest radius r_e.
2. Let $A_e = \{r_e \le |z - z_0| \le R_e\}$ be the closed annulus associated with Ξ_e, which is a single circle if $R_e = r_e$, i.e., $|\Xi_e| = 1$. Then the annuli A_e, $e \in \mathcal{E}$, are mutually disjoint.

Note that every Ξ_e is a totally ordered set w.r.t. the partial order on Ξ.

Theorem 3.2. *Let $y_e := \operatorname{Im} z_e \ge 0$, $e \in \mathcal{E}$. Then there is $C_{|\mathcal{E}|} < \infty$, which depends only on κ and $|\mathcal{E}|$, such that*

$$\mathbb{P}\left[\bigcap_{\xi\in\Xi}\{\gamma \cap \xi \ne \varnothing\}\right] \le C_{|\mathcal{E}|}\prod_{e\in\mathcal{E}}\frac{P_{y_e}(r_e)}{P_{y_e}(R_e)}.$$

Discussion. *Suppose γ visits all $\xi \in \Xi$. For $\xi_1, \xi_2 \in \Xi$, if $\xi_1 < \xi_2$, then γ will visit ξ_1 before ξ_2. Other than these constraints, γ can visit the elements in Ξ in any order. The simplest case is that γ does not jump back and forth between different groups $\{\Xi_e : e \in \mathcal{E}\}$. This means that γ first visits all circles in Ξ_{e_1} for some $e_1 \in \mathcal{E}$ before all other circles in Ξ, then visits all circles in Ξ_{e_2} for some $e_2 \in \mathcal{E} \setminus \{e_1\}$ before circles in $\Xi \setminus (\Xi_{e_1} \cup \Xi_{e_2})$, and so on. In this case, we can easily use the 1-point estimate and DMP to get the righthand side of the above formula. We use Theorem 3.1 to deal with the general cases. The key point is that γ has to pay a price to jump back and forth between different Ξ_e's due to the factor $(\frac{r_0}{R_0})^{\alpha/4}$ given in Theorem 3.1.*

Proof of Theorem 3.2. We write $\mathbb{N}_n$ for $\{k \in \mathbb{N} : k \le n\}$. Let S denote the set of bijections $\sigma : \mathbb{N}_{|\Xi|} \to \Xi$ such that $\xi_1 < \xi_2$ implies that $\sigma^{-1}(\xi_1) < \sigma^{-1}(\xi_2)$. Let $E = \bigcap_{\xi \in \Xi}\{\gamma \cap \xi \ne \varnothing\}$ and

$$E^\sigma = \{\tau_{\sigma(1)} < \tau_{\sigma(2)} < \cdots < \tau_{\sigma(|\Xi|)} < \infty\}, \quad \sigma \in S.$$

Then the above discussion gives

$$E = \bigcup_{\sigma \in S} E^\sigma. \tag{3.6}$$

We will derive an upper bound of $\mathbb{P}[E^\sigma]$ in (3.9).

Fix $\sigma \in S$. For $e \in \mathcal{E}$, we label the elements of Ξ_e by $\xi_0^e < \cdots < \xi_{N_e}^e$, where $N_e = |\Xi_e| - 1$. Let

$$J_e = \left\{1 \le n \le N_e : \sigma^{-1}\left(\xi_n^e\right) > \sigma^{-1}\left(\xi_{n-1}^e\right) + 1\right\} \cup \{0\}.$$

In plain words, $n \in J_e$ means that either $n = 0$ or after visiting ξ_{n-1}^e, γ does not immediately visit ξ_n^e without visiting other circles in Ξ that it has not visited before. In the latter case, after visiting ξ_{n-1}^e, γ visits the circles in $\bigcup_{e' \ne e} \Xi_{e'}$ before ξ_n^e.

Order the elements of J_e by $0 = s_e(0) < \cdots < s_e(M_e)$, where $M_e = |J_e| - 1$. Set $s_e(M_e + 1) = N_e + 1$. Every Ξ_e can be partitioned into $M_e + 1$ subsets:

$$\Xi_{(e,j)} = \left\{\xi_n^e : s_e(j) \le n \le s_e(j+1) - 1\right\}, \quad 0 \le j \le M_e.$$

The meaning of the partition is that, after γ visits the first element in $\Xi_{(e,j)}$, which must be $\xi_{s_e(j)}^e$, it then visits all elements in $\Xi_{(e,j)}$ without visiting any other circles in Ξ that it has not visited before. Let $I = \{(e, j) : e \in \mathcal{E}, 0 \le j \le M_e\}$. Then $\{\Xi_\iota : \iota \in I\}$ is another partition of Ξ, which is finer than $\{\Xi_e : e \in \mathcal{E}\}$. Note that every $\sigma^{-1}(\Xi_\iota)$, $\iota \in I$, is a connected subset of $\mathbb{Z}$.

For $\iota \in I$, let e_ι denote the first coordinate of ι, $z_\iota = z_{e_\iota}$ and $y_\iota = \operatorname{Im} z_\iota$. Let $P_\iota = \frac{P_{y_\iota}(R_{\max \Xi_\iota})}{P_{y_\iota}(R_{\min \Xi_\iota})}$. Recall that if $\iota = (e, j)$, $\min \Xi_\iota = \xi_{s_e(j)}^e$ and $\max \Xi_\iota = \xi_{s_e(j+1)-1}^e$. From Lemma 2.6 we get

$$\mathbb{P}[\tau_{\max \Xi_\iota} < \infty | \mathcal{F}_{\min \Xi_\iota}] \le C P_\iota, \quad \iota \in I. \tag{3.7}$$

Let $P_e = \frac{P_{y_e}(r_e)}{P_{y_e}(R_e)}$, $e \in \mathcal{E}$. From (1.5) we get

$$\prod_{j=0}^{M_e} P_{(e,j)} \le 4^{\alpha M_e} P_e, \quad e \in \mathcal{E}. \tag{3.8}$$

We have $|I| = \sum_{e \in \mathcal{E}}(M_e + 1)$. Considering the order that γ visits Ξ_ι, $\iota \in I$, we get a bijection map $\widehat{\sigma} : \mathbb{N}_{|I|} \to I$ such that $n_1 < n_2$ implies that $\max \sigma^{-1}(\Xi_{\widehat{\sigma}(n_1)}) < \min \sigma^{-1}(\Xi_{\widehat{\sigma}(n_2)})$, and $n_1 = n_2 - 1$ implies that $\max \sigma^{-1}(\Xi_{\widehat{\sigma}(n_1)}) = \min \sigma^{-1}(\Xi_{\widehat{\sigma}(n_2)}) - 1$. We may now express E^σ as

$$E^\sigma = \{\tau_{\min \Xi_{\widehat{\sigma}(1)}} < \tau_{\max \Xi_{\widehat{\sigma}(1)}} < \tau_{\min \Xi_{\widehat{\sigma}(2)}} < \tau_{\max \Xi_{\widehat{\sigma}(2)}} < \cdots < \tau_{\min \Xi_{\widehat{\sigma}(|I|)}} < \tau_{\max \Xi_{\widehat{\sigma}(|I|)}} < \infty\}.$$

Fix $e_0 \in \mathcal{E}$. Let $n_j = \widehat{\sigma}^{-1}((e_0, j))$, $0 \le j \le M_{e_0}$. Then $n_{j+1} \ge n_j + 2$, $0 \le j \le M_{e_0} - 1$. Fix $0 \le j \le M_{e_0} - 1$. Let $m = n_{j+1} - n_j - 1$. Applying Theorem 3.1 to $\widehat{\xi}_0 = \min \Xi_{e_0}$, $\xi_0 = \max \Xi_{(e_0,j)} = \max \Xi_{\widehat{\sigma}(n_j)}$, $\xi_0' = \min \Xi_{(e_0,j+1)} = \min \Xi_{\widehat{\sigma}(n_{j+1})}$, $\widehat{\xi}_k = \min \Xi_{\widehat{\sigma}(n_j+k)}$ and $\xi_k = \max \Xi_{\widehat{\sigma}(n_j+k)}$, $1 \le k \le m$, we get

$$\mathbb{P}\big[E^\sigma_{[\max \Xi_{\widehat{\sigma}(n_j)}, \min \Xi_{\widehat{\sigma}(n_{j+1})}]} | \mathcal{F}_{\tau_{\max \Xi_{\widehat{\sigma}(n_j)}}}\big] \le C^m 4^{-\alpha/4(s_{e_0}(j+1)-1)} \prod_{n=n_j+1}^{n_{j+1}-1} P_{\widehat{\sigma}(n)},$$

where $E^\sigma_{[\max \Xi_{\widehat{\sigma}(n_j)}, \min \Xi_{\widehat{\sigma}(n_{j+1})}]}$ is the $\mathcal{F}_{\tau_{\min \Xi_{\widehat{\sigma}(n_{j+1})}}}$-measurable event

$$\{\tau_{\max \Xi_{\widehat{\sigma}(n_j)}} < \tau_{\min \Xi_{\widehat{\sigma}(n_j+1)}} < \tau_{\max \Xi_{\widehat{\sigma}(n_j+1)}} < \cdots < \tau_{\max \Xi_{\widehat{\sigma}(n_j+m)}} < \tau_{\min \Xi_{\widehat{\sigma}(n_{j+1})}} < \infty\}.$$

Letting j vary between 0 and $M_{e_0} - 1$ and using (3.7) and we get

$$\mathbb{P}\big[E^\sigma\big] \le C^{|I|} 4^{-\alpha/4 \sum_{j=1}^{M_{e_0}} (s_{e_0}(j)-1)} \prod_{\iota \in I} P_\iota.$$

Using (3.8) and $|I| = \sum_e (M_e + 1)$, we find that the right-hand side is bounded by

$$C^{|\mathcal{E}|} C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\alpha/4 \sum_{j=1}^{M_{e_0}} s_{e_0}(j)} \prod_{e\in\mathcal{E}} P_e.$$

Taking a geometric average over $e_0 \in \mathcal{E}$, we get

$$\mathbb{P}\big[E^\sigma\big] \le C^{|\mathcal{E}|} C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\alpha/(4|\mathcal{E}|) \sum_{e\in\mathcal{E}} \sum_{j=1}^{M_e} s_e(j)} \prod_{e\in\mathcal{E}} P_e. \tag{3.9}$$

So far we have omitted the σ on I, M_e, $s_e(j)$ and etc.; we will put σ on the superscript if we want to emphasize the dependence on σ. From (3.6) and the above result, it follows that

$$\mathbb{P}[E] \le C^{|\mathcal{E}|} \sum_{(M_e; (s_e(j))_{j=0}^{M_e})_{e\in\mathcal{E}}} |S_{(M_e,(s_e(j)))}| C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\alpha/(4|\mathcal{E}|) \sum_{e\in\mathcal{E}} \sum_{j=1}^{M_e} s_e(j)} \prod_{e\in\mathcal{E}} P_e, \tag{3.10}$$

where

$$S_{(M_e,(s_e(j)))} := \big\{\sigma \in S : M_e^\sigma = M_e, s_e^\sigma(j) = s_e(j), 0 \le j \le M_e, e \in \mathcal{M}\big\},$$

and the first summation in (3.10) is over all possible $(M_e; (s_e(j))_{j=0}^{M_e})_{e\in\mathcal{E}}$, namely, $M_e \ge 0$ and $0 = s_e(0) < s_e(1) < \cdots < s_e(M_e) \le N_e$ for every $e \in \mathcal{E}$. It now suffices to show that

$$\sum_{(M_e; (s_e(j))_{j=1}^{M_e})_{e\in\mathcal{E}}} |S_{(M_e,(s_e(j)))}| C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\alpha/(4|\mathcal{E}|) \sum_{e\in\mathcal{E}} \sum_{j=1}^{M_e} s_e(j)} \le C_{|\mathcal{E}|}, \tag{3.11}$$

for some $C_{|\mathcal{E}|} < \infty$ depending only on $|\mathcal{E}|$ and κ.

We now bound the size of $S_{(M_e,(s_e(j)))}$. Note that M_e^σ and $s_e^\sigma(j)$, $0 \le j \le M_e^\sigma$, $e \in \mathcal{E}$, determine the partition Ξ_ι, $\iota \in I^\sigma$, of Ξ. When the partition is given, σ is then determined by $\widehat{\sigma} : \mathbb{N}_{|I^\sigma|} \to I^\sigma$, which is in turn determined by $e_{\widehat{\sigma}(n)}$, $1 \le n \le |I^\sigma| = \sum_{e\in\mathcal{E}} (M_e^\sigma + 1)$, because if $e_{\widehat{\sigma}(n)} = e_0$, then $\widehat{\sigma}(n) = (e_0, j_0)$, where $j_0 = \min\{0 \le j \le M_{e_0} : (e_0, j) \notin \widehat{\sigma}(m), m < n\}$. Since each $e_{\widehat{\sigma}(n)}$ has at most $|\mathcal{E}|$ possibilities, we have $|S_{(M_e,(s_e(j)))}| \le |\mathcal{E}|^{\sum_{e\in\mathcal{E}} (M_e+1)}$. Thus, the left-hand side of (3.11) is bounded by

$$
\begin{aligned}
&|\mathcal{E}|^{|\mathcal{E}|} \sum_{(M_e;(s_e(j))_{j=0}^{M_e})_{e\in\mathcal{E}}} \prod_{e\in\mathcal{E}} \big(C|\mathcal{E}|\big)^{M_e} 4^{-\alpha/(4|\mathcal{E}|)\sum_{j=1}^{M_e} s_e(j)} \\
&= |\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M_e=0}^{N_e} \big(C|\mathcal{E}|\big)^{M_e} \sum_{0=s_e(0)<\cdots<s_e(M_e)\le N_e} 4^{-\alpha/(4|\mathcal{E}|)\sum_{j=1}^{M_e} s_e(j)} \\
&\le |\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M=0}^{\infty} \big(C|\mathcal{E}|\big)^{M} \sum_{s(1)=1}^{\infty} \cdots \sum_{s(M)=M}^{\infty} 4^{-\alpha/(4|\mathcal{E}|)\sum_{j=1}^{M} s(j)} \\
&\le |\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M=0}^{\infty} \big(C|\mathcal{E}|\big)^{M} \prod_{j=1}^{M} \sum_{s(j)=j}^{\infty} 4^{-(\alpha/(4|\mathcal{E}|))s(j)} = \left[|\mathcal{E}| \sum_{M=0}^{\infty} \left(\frac{C|\mathcal{E}|}{1-4^{-\alpha/(4|\mathcal{E}|)}}\right)^{M} 4^{-(\alpha/(8|\mathcal{E}|))M(M+1)}\right]^{|\mathcal{E}|}.
\end{aligned}
$$

The conclusion now follows since the summation inside the square bracket equals to a finite number depending only on κ and $|\mathcal{E}|$. □

4. Proofs of the main theorems

First, we are going to use Theorem 3.2 to prove Theorem 1.1. What we need to do in the proof is to use the radii r_j's and the distances l_j's to construct a group of circles Ξ and a partition Ξ_e, $e\in\mathcal{E}$, that satisfy the conditions in Section 3, and prove that the upper bound given by Theorem 3.2 is comparable to the upper bound in Theorem 1.1.

Proof of Theorem 1.1. We assume that any r_j is of the form $\frac{l_j}{4^{h_j}}$ for some integer h_j. If not, it is between two of them and by changing C_n in the theorem and using (1.5) we can get the result easily. Also we can assume $h_j\ge 1$ for every j because otherwise the corresponding term on right-hand side i.e. $\frac{P_{y_j}(r_j\wedge l_j)}{P_{y_j}(l_j)}$ is 1 so we can just ignore it. We want to deduce this theorem from Theorem 3.2, so we want to construct a family Ξ. Consider

$$
\xi_j^s = \left\{|z-z_j| = \frac{l_j}{4^s}\right\}, \quad 1\le j\le n, 1\le s\le h_j.
$$

The family $\{\xi_j^s : 1\le j\le n, 1\le s\le h_j\}$ may not be mutually disjoint. To solve this issue, we will remove some circles as follows. For $1\le j<k\le n$, let $D_k=\{|z-z_k|\le l_k/4\}$, which contains every ξ_k^r, $1\le r\le h_k$, and

$$
I_{j,k} = \big\{\xi_j^s : 1\le s\le h_j, \xi_j^s\cap D_k\ne\varnothing\big\}. \tag{4.1}
$$

Then $\Xi := \{\xi_j^s : 1\le j\le n, 1\le s\le h_j\}\setminus\bigcup_{1\le j<k\le n} I_{j,k}$ is mutually disjoint. If $\operatorname{dist}(\gamma,z_j)\le r_j$, then γ intersects every ξ_j^s, $1\le s\le h_j$. So we get

$$
\mathbb{P}\big[\operatorname{dist}(\gamma,z_j)\le r_j, 1\le j\le n\big] \le \mathbb{P}\left[\bigcap_{j=1}^{n}\bigcap_{s=1}^{h_j}\{\gamma\cap\xi_j^s\ne\varnothing\}\right] \le \mathbb{P}\left[\bigcap_{\xi\in\Xi}\{\gamma\cap\xi\ne\varnothing\}\right]. \tag{4.2}
$$

Next, we construct a partition $\{\Xi_e : e\in\mathcal{E}\}$ of Ξ. First, Ξ has a natural partition Ξ_j, $1\le j\le n$, such that Ξ_j is composed of circles centered at z_j. For each j, we construct a graph G_j, whose vertex set is Ξ_j, and $\xi_1\ne\xi_2\in\Xi_j$ are connected by an edge iff the bigger radius is 4 times the smaller one, and the open annulus between them does not contain any other circle in Ξ. Let $\mathcal{E}_j$ denote the set of connected components of G_j. Then we partition Ξ_j into Ξ_e, $e\in\mathcal{E}_j$, such that every Ξ_e is the vertex set of $e\in\mathcal{E}_j$. Then the circles in every Ξ_e are concentric circles with radii forming a geometric sequence with common ratio $1/4$, and the closed annuli A_e associated with Ξ_e, $e\in\mathcal{E}_j$, are mutually disjoint. From the construction we also see that for any $j<k$, and $e\in\mathcal{E}_j$, A_e does not intersect D_k,

which contains every A_e with $e\in\mathcal{E}_k$. Let $\mathcal{E}=\bigcup_{j=1}^n\mathcal{E}_j$. Then A_e, $e\in\mathcal{E}$, are mutually disjoint. Thus, $\{\Xi_e:e\in\mathcal{E}\}$ is a partition of Ξ that satisfies the properties before Theorem 3.2. So we get

$$\mathbb{P}\Big[\bigcap_{\xi\in\Xi}\{\gamma\cap\xi\neq\varnothing\}\Big]\le C_{|\mathcal{E}|}\prod_{e\in\mathcal{E}}\frac{P_{y_e}(r_e)}{P_{y_e}(R_e)}=C_{|\mathcal{E}|}\prod_{j=1}^n\prod_{e\in\mathcal{E}_j}\frac{P_{y_j}(r_e)}{P_{y_j}(R_e)}. \tag{4.3}$$

Here we set $\prod_{e\in\mathcal{E}_j}=1$ if $\mathcal{E}_j=\varnothing$. We will finish the proof by comparing $|\mathcal{E}|$ with n and the product $\prod_{e\in\mathcal{E}_j}\frac{P_{y_j}(r_e)}{P_{y_j}(R_e)}$ with $\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}$.

Here is a useful fact: every $I_{j,k}$ defined in (4.1) contains at most one element. The reason is

$$\frac{\max_{z\in D_k}\{|z-z_j|\}}{\min_{z\in D_k}\{|z-z_j|\}}=\frac{|z_j-z_k|+l_k/4}{|z_j-z_k|-l_k/4}\le\frac{l_k+l_k/4}{l_k-l_k/4}<4.$$

The above formula also implies that, for $j<k$, $\bigcup_{\xi\in\Xi_k}\xi\subset D_k$ intersects at most 2 annuli from $\{l_j/4^r\le|z-z_j|\le l_j/4^{r-1}\}$, $2\le r\le h_j$. If $j>k$, by construction, $\bigcup_{\xi\in\Xi_k}\xi$ is disjoint from the annuli $\{l_j/4^r\le|z-z_j|\le l_j/4^{r-1}\}$, $2\le r\le h_j$, which are contained in D_j.

We now bound $|\mathcal{E}_j|$. We may obtain G by removing vertices and edges from a path graph $\widehat{G}_j$, whose vertex set is $\{\xi_j^s:1\le s\le h_j\}$, and two vertices are connected by an edge iff the bigger radius is 4 times the smaller one. Every edge e of $\widehat{G}_j$ determines an annulus, denoted by A_e. The vertices removed are the elements in $I_{j,k}$, $k>j$; and the edges removed are those e such that A_e intersects some $\xi\in\Xi_k$ with $k\neq j$, which may happen only if $k>j$. Thus, the total number of vertices or edges removed is not bigger than $\sum_{k>j}(1+2)=3(n-j)$. So we get $|\mathcal{E}_j|\le 1+3(n-j)$. Thus, $|\mathcal{E}|\le n+\frac{3n(n-1)}{2}$. This means that $C_{|\mathcal{E}|}$ may be written as C_n.

Finally we compare $\prod_{e\in\mathcal{E}_j}\frac{P_{y_j}(r_e)}{P_{y_j}(R_e)}$ with $\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}$. If A is an annulus $\{r\le|z-z_0|\le R\}$ for some $z_0\in\overline{\mathbb{H}}$ with $y_0\in\operatorname{Im}z_0\ge 0$ and $R\ge r>0$, we define $P_A=\frac{P_{y_0}(r)}{P_{y_0}(R)}$. Let $A_{j,s}=\{l_j/4^s\le|z-z_j|\le l_j/4^{s-1}\}$, $1\le s\le h_j$, and $\mathcal{S}_j=\{s\in\mathbb{N}_{h_j}:A_{j,s}\subset\bigcup_{e\in\Xi_j}A_e\}$. Then

$$\frac{P_{y_j}(r_j)}{P_{y_j}(l_j)}=\prod_{s=1}^{h_j}P_{A_{j,s}},\qquad \prod_{e\in\mathcal{E}_j}\frac{P_{y_j}(r_e)}{P_{y_j}(R_e)}=\prod_{s\in\mathcal{S}_j}P_{A_{j,s}}.$$

Using (1.5), we get

$$\prod_{e\in\mathcal{E}_j}\frac{P_{y_j}(r_e)}{P_{y_j}(R_e)}\le 4^{\alpha|\mathbb{N}_{h_j}\setminus\mathcal{S}_j|}\frac{P_{y_j}(r_j)}{P_{y_j}(l_j)}.$$

Now $s\in\mathbb{N}_{h_j}\setminus\mathcal{S}_j$ only if $s=1$ or there is some $k>j$ with $D_k\cap A_{j,s}\neq\varnothing$. Since for $k>j$, D_k intersects at most two $A_{j,s}$, we find that $|\mathbb{N}_{h_j}\setminus\mathcal{S}_j|\le 1+2(n-j)$. Thus,

$$\prod_{j=1}^n\prod_{e\in\mathcal{E}_j}\frac{P_{y_j}(r_e)}{P_{y_j}(R_e)}\le 4^{\alpha n^2}\prod_{j=1}^n\frac{P_{y_j}(r_e)}{P_{y_j}(R_e)}.$$

Combining the above formula with (4.2) and (4.3), we complete the proof. □

Proof of Theorem 1.2. As we mentioned before we can define natural length of *SLE* in a domain by Minkowski content. See equation (1.1). Similarly if D is a bounded subset of the upper half plane we can define $\mathrm{Cont}_d(\gamma\cap D)$ as the natural length of *SLE* in the domain D in the obvious way.

The main theorem of [6] becomes

$$\lim_{r\to 0}\mathrm{Cont}_d(\gamma\cap D;r)=\mathrm{Cont}_d(\gamma\cap D),$$

with probability 1. Now we compute

$$\begin{aligned}\mathbb{E}\big[\mathrm{Cont}_d(\gamma\cap D;r)^n\big]&=\mathbb{E}\big[r^{n(d-2)}\big(\mathrm{Area}\big(z\in D|\,\mathrm{dist}(z,\gamma)<r\big)^n\big)\big]\\&=r^{n(d-2)}\mathbb{E}\bigg[\bigg(\int_D 1_{\mathrm{dist}(z,\gamma)<r}\,dA(z)\bigg)^n\bigg]\\&=\int_{D^n} r^{n(d-2)}\mathbb{P}\big(\mathrm{dist}(z_1,\gamma)<r,\dots,\mathrm{dist}(z_n,\gamma)<r\big)\,dA(z_1)\cdots dA(z_n).\end{aligned}$$

For the above equality, we changed expectation and integral which is allowed because the integrand is always positive. We will find an upper bound for

$$\sup\big\{r^{n(d-2)}\mathbb{P}\big(\mathrm{dist}(z_1,\gamma)<r,\dots,\mathrm{dist}(z_n,\gamma)<r\big)\big\},$$

which is integrable over D^n. By Theorem 1.1 we know that this is bounded above by

$$r^{n(d-2)}C_n\prod_{k=1}^n\frac{P_{y_k}(r\wedge l_k)}{P_{y_k}(l_k)}.$$

Now assume that r is smaller than $l_{i_1},\dots,l_{i_k}$ and bigger than the rest. Then by equation (1.5) and the definition of P_y we get that the above quantity is bounded by

$$C_n r^{n(d-2)}\prod_{j=1}^k\frac{r^{2-d}}{l_{i_j}^{2-d}}\le C_n\prod_{s=1}^n l_s^{d-2}.$$

We have the last inequality because if $r>l$ then $r^{d-2}<l^{d-2}$. So now we should show

$$f(z_1,\dots,z_n)=\prod_{k=1}^n l_k^{d-2}=\prod_{k=1}^n\min\big\{|z_k-z_0|,|z_k-z_1|,\dots,|z_k-z_{k-1}|\big\}^{d-2}$$

is integrable over D^n. This is true because for every $1\le k\le n$,

$$\begin{aligned}&\int_D\min\big\{|z_k-z_0|,|z_k-z_1|,\dots,|z_k-z_{k-1}|\big\}^{d-2}\,dA(z_k)\\&\quad\le\sum_{j=0}^{k-1}\int_D|z_k-z_j|^{d-2}\,dA(z_k)\\&\quad\le k\int_{|z|\le\mathrm{diam}(D\cup\{0\})}|z|^{d-2}\,dA(z)=2\pi k\int_0^{\mathrm{diam}(D\cup\{0\})}r^{d-1}\,dr<\infty,\end{aligned}$$

as $d>0$. Finally, we may apply Fatou's lemma to conclude that

$$\mathbb{E}\big[\mathrm{Cont}_d(\gamma\cap D)^n\big]\le\int_D\cdots\int_D\prod_{k=1}^n l_k(z_1,\dots,z_n)\,dA(z_1)\cdots dA(z_n)<\infty.$$

□

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