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On the asymptotic behavior of the density of the supremum of Lévy processes

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Abstract. Let us consider a real valued Lévy process X whose transition probabilities are absolutely continuous and have bounded densities. Then the law of the past supremum of X before any deterministic time t is absolutely continuous on $(0, \infty)$. We show that its density $f_t(x)$ is continuous on $(0, \infty)$ if and only if the potential density h' of the upward ladder height process is continuous on $(0, \infty)$. Then we prove that f_t behaves at 0 as h'. We also describe the asymptotic behaviour of f_t, when t tends to infinity. Then some related results are obtained for the density of the meander and this of the entrance law of the Lévy process conditioned to stay positive.

Résumé. Soit X un processus de Lévy réel dont les probabilités de transition sont absolument continues par rapport à la mesure de Lebesgue. Dans ce cas, il est connu que la loi du supremum passé avant un temps déterministe t est elle-même absolument continue sur $(0, \infty)$. En supposant de plus que les densités sont bornées, nous montrons que la densité $f_t(x)$ du supremum passé est continue en x sur $(0, \infty)$, si et seulement si la densité potentielle $h'(x)$ du subordinateur des hauteurs d'échelle ascendant est continue sur $(0, \infty)$. Nous montrons alors que f_t se comporte en 0 de la même manière que h'. Nous donnons également une description du comportement asymptotique de f_t, lorsque t tend vers l'infini. Enfin nous appliquons ces résultats pour étudier le comportement asymptotique de la densité du méandre des processus de Lévy et de la densité de la loi d'entrée des processus de Lévy conditionnés à rester positifs.

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1. Introduction

Since the work by Paul Lévy [16] for standard Brownian motion, the study of the law of the past supremum before a deterministic time of real valued Lévy processes has given rise to a significant literature. This is mainly justified by the important number of applications of this functional in various domains such as risk and queuing theories but properties of its law may also be useful for theoretical purposes. It is constantly involved in fluctuation theory, for instance.

Let us denote by $\overline{X}_t = \sup_{s \leq t} X_s$ the past supremum at time $t > 0$ of the real valued Lévy process X. Recently in [15] the asymptotic behaviour of the distribution function $\mathbb{P}(\overline{X}_t \leq x)$ was deeply investigated and in [2], necessary and sufficient conditions where given for the law of $\overline{X}_t$ to be absolutely continuous. A natural continuation of both these

works consists in a detailed study of the density f_t of this law, when it exists. For instance if the transition probabilities of the Lévy process are absolutely continuous, then the law of the past supremum is absolutely continuous on $(0, \infty)$. In this paper, under the additional assumption that the transition densities of the Lévy process are bounded, we show that f_t is continuous at $x \in (0, \infty)$ if and only if the potential density h' of the upward ladder height process is continuous at this point. Then, we describe the asymptotic behaviour of the density $f_t(x)$, when x tends to 0. This behaviour is the same as this of h', up to a constant which is given by the tail distribution of the lifetime of the generic excursion of the Lévy process reflected at its supremum. We also obtain some asymptotic results and estimates for f_t, when the time t tends to infinity. Most of the results displayed in this paper extend those obtained by Doney and Savov in [9] for stable Lévy processes.

In the next section we recall some elements of excursion and fluctuation theory for Lévy processes that are necessary for the proof of our main results. In Section 3, we state the main results and Section 4 is devoted to their proofs. The latter section as well as Section 3 also contain some intermediary results on bridges, meanders and Lévy processes conditioned to stay positive.

2. Preliminaries

We denote by $\mathcal{D}$ the space of càdlàg paths $\omega : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ with lifetime $\zeta(\omega) = \inf\{t \geq 0 \colon \omega_t = \infty\}$, with the usual convention that $\inf \varnothing = +\infty$. The space $\mathcal{D}$ is equipped with the Skorokhod topology, its Borel σ-algebra $\mathcal{F}$, and the usual completed filtration $(\mathcal{F}_s, s \geq 0)$ generated by the coordinate process $X = (X_t, t \geq 0)$ on the space $\mathcal{D}$. We write $\underline{X}$ and $\overline{X}$ for the infimum and supremum processes, that is

$$\underline{X}_t = \inf\{X_s \colon 0 \leq s \leq t\} \quad \text{and} \quad \overline{X}_t = \sup\{X_s \colon 0 \leq s \leq t\}.$$

We also define the first passage time by X in the open half line $(-\infty, 0)$ by:

$$\tau_0^- = \inf\{t > 0 \colon X_t < 0\}.$$

We denote by $\mathbb{P}_x$ the law on $(\mathcal{D}, \mathcal{F})$ of a Lévy process starting from $x \in \mathbb{R}$ and we will set $\mathbb{P} := \mathbb{P}_0$. Define $X^* := -X$, then the law of X^* under $\mathbb{P}_x$ will be denoted by $\mathbb{P}_x^*$, that is $(X^*, \mathbb{P}_x) = (X, \mathbb{P}_x^*)$. We recall that the process $(X, \mathbb{P}_x^*)$ is in weak duality with $(X, \mathbb{P})$, with respect to the Lebesgue measure. In this section, as well as in most of this paper, we make the following assumptions:

(H$_1$) The transition semigroup of $(X, \mathbb{P})$ is absolutely continuous and there is a version of its densities, denoted by $x \mapsto p_t(x)$, $x \in \mathbb{R}$, which are bounded for all $t > 0$.

(H$_2$) $(X, \mathbb{P})$ is not a compound Poisson process and for all $c \geq 0$, the process $((|X_t - ct|, t \geq 0), \mathbb{P})$ is not a subordinator.

Note that (H$_1$) is equivalent to the apparently stronger condition saying that the characteristic function $e^{-t\Psi(\xi)}$ of X is integrable for all $t > 0$. Here $\Psi(\xi)$ denotes the characteristic exponent of X given by the Lévy–Khintchine formula

$$\Psi(\xi) = -a\mathrm{i}\xi + \frac{1}{2}\sigma^2\xi^2 - \int_{\mathbb{R}\setminus\{0\}} \big(e^{\mathrm{i}\xi x} - 1 - \mathrm{i}\xi x \mathbb{1}_{|x|<1}\big) \Pi(\mathrm{d}x),$$

where (a, σ^2, Π) is a Lévy triplet. Indeed, boundedness of p_t implies that $p_t \in L^2(\mathbb{R})$ and consequently $e^{-t\Psi(\xi)} \in L^2(\mathbb{R})$, for all $t > 0$ which implies that $e^{-t\Psi(\xi)} \in L^1(\mathbb{R})$, for all $t > 0$. Conversely, if $e^{-t\Psi(\xi)} \in L^1(\mathbb{R})$, for all $t > 0$, then by the Riemann–Lebesgue lemma, $p_t \in \mathcal{C}_0(\mathbb{R})$, moreover the function $(t, x) \mapsto p_t(x)$ is jointly continuous on $(0, \infty) \times \mathbb{R}$. Regarding the latter equivalence, we emphasize that the weaker condition that X_t is absolutely continuous for all $t > 0$ does not have such a nice characterization in terms of Ψ, see Chapter 5 of [17].

Positivity of the density of the semigroup is ensured by conditions (H$_1$) and (H$_2$), that is,

$$p_t(x) > 0, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}. \tag{2.1}$$

Actually, from Theorem (3.3) in [19], under the assumption that X_t is absolutely continuous for all $t > 0$, condition (2.1) is equivalent to (H$_2$). The latter is an essential property for our purpose. Compound Poisson processes are

excluded here only because our setting is not adapted to their study. Note that assumptions (H_1) and (H_2) are satisfied in many classical cases, such as stable processes or subordinated Brownian motions.

Recall that the reflected process $X - \underline{X}$ is Markovian. Besides, 0 is regular for itself, for $X - \underline{X}$ if and only if 0 is regular for $(-\infty, 0)$, for the process X. When it is the case, we will simply write that $(-\infty, 0)$ is regular. Similarly, we will write that $(0, \infty)$ is regular when 0 is regular for $(0, \infty)$, for the process X. Then recall that in any case, at least one of the half lines $(-\infty, 0)$ or $(0, \infty)$ is regular. If $(-\infty, 0)$ is regular, then the local time at 0 of the process $X - \underline{X}$ is a continuous, increasing, additive functional which we will denote by L^*, satisfying $L_0^* = 0$, a.s., and such that the support of the measure dL_t^* is the set $\overline{\{t: X_t = \underline{X}_t\}}$. Moreover L^* is unique up to a multiplicative constant. We will normalize it by

$$\mathbb{E}\left(\int_0^\infty e^{-t} \, dL_t^* \right) = 1. \tag{2.2}$$

Then the Itô measure n^* of the excursions away from 0 of the process $X - \underline{X}$ is characterized by the *compensation formula*. More specifically, for any positive and predictable process F,

$$\mathbb{E}\left(\sum_{s \in G} F(s, \omega, \epsilon^s) \right) = \mathbb{E}\left(\int_0^\infty dL_s^* \left(\int_E F(s, \omega, \epsilon) n^*(d\epsilon) \right) \right), \tag{2.3}$$

where E is the set of excursions, G is the set of left end points of the excursions, and ϵ^s is the excursion which starts at $s \in G$. We refer to [1], Chapter IV, [14], Chapter 6 and [7] for more detailed definitions and some constructions of L^* and n^*.

When $(-\infty, 0)$ is not regular, the set $\{t: (X - \underline{X})_t = 0\}$ is discrete and following [1] and [14], we define the local time L^* of $X - \underline{X}$ at 0 by

$$L_t^* = \sum_{k=0}^{R_t} \mathbf{e}^{(k)}, \tag{2.4}$$

where for $t > 0$, $R_t = \text{Card}\{s \in (0, t]: X_s = \underline{X}_s\}$, $R_0 = 0$ and $\mathbf{e}^{(k)}$, $k = 0, 1, \ldots$ is a sequence of independent and exponentially distributed random variables with parameter

$$\gamma = \left(1 - \mathbb{E}(e^{-\tau_0^-})\right)^{-1}. \tag{2.5}$$

In this case, the measure n^* of the excursions away from 0 is proportional to the distribution of the process X under the law $\mathbb{P}$, killed at its first passage time in the negative half line. More formally, let us define $\epsilon^0 = (X_t \mathbb{1}_{\{t < \tau_0^-\}} + \infty \cdot \mathbb{1}_{\{t \geq \tau_0^-\}})$, then for any bounded Borel functional K on E,

$$\int_E K(\epsilon) n^*(d\epsilon) = \gamma \mathbb{E}\left[K(\epsilon^0) \right]. \tag{2.6}$$

From definitions (2.4), (2.6) and an application of the strong Markov property, we may check that the normalization (2.2) and the compensation formula (2.3) are still valid in this case.

In any case, n^* is a Markovian measure whose semigroup is this of the killed Lévy process when it enters in the negative half line. More specifically, for $x > 0$, let us denote by $\mathbb{Q}_x^*$ the law of the process $(X_t \mathbb{1}_{\{t < \tau_0^-\}} + \infty \cdot \mathbb{1}_{\{t \geq \tau_0^-\}}, t \geq 0)$ under $\mathbb{P}_x$, that is for $\Lambda \in \mathcal{F}_t$,

$$\mathbb{Q}_x^*(\Lambda, t < \zeta) = \mathbb{P}_x(\Lambda, t < \tau_0^-). \tag{2.7}$$

Then for all Borel positive functions f and g and for all $s, t > 0$,

$$n^*\left(f(X_t) g(X_{s+t}), s + t < \zeta \right) = n^*\left(f(X_t) \mathbb{E}_{X_t}^{\mathbb{Q}^*}\left(g(X_s) \right), s < \zeta \right), \tag{2.8}$$

where $\mathbb{E}_x^{\mathbb{Q}^*}$ means the expectation under $\mathbb{Q}_x^*$. Recall that $\mathbb{Q}_0^*$ is well defined when 0 is not regular for $(-\infty, 0)$, and in this case, from (2.6), we have $\mathbb{Q}_0^* = \gamma^{-1} n^*$. We define the probability measures $\mathbb{Q}_x$ in the same way as in (2.7) with respect to the dual process $(X, \mathbb{P}^*)$. Let us denote by $q_t^*(x, \mathrm{d}y)$ (resp. $q_t(x, \mathrm{d}y)$) the semigroup of the strong Markov process $(X, \mathbb{Q}_x^*)$ (resp. $(X, \mathbb{Q}_x)$). Note that from (H$_1$) and (2.7), the semigroups $q_t(x, \mathrm{d}y)$ and $q_t^*(x, \mathrm{d}y)$ are absolutely continuous. A slight extension of Lemma 2 in [21] actually leads to Lemma 1 below. Note also that the next two lemmas are closely related to Proposition 10.10 of [10], where similar results are obtained under slightly more restrictive hypothesis.

Lemma 1. *Under assumptions* (H$_1$) *and* (H$_2$), *for all* $t > 0$, *there are versions of the densities of the measures* $q_t(x, \mathrm{d}y)$ *and* $q_t^*(x, \mathrm{d}y)$ *which are strictly positive and continuous on* $(0, \infty)^2$. *We denote by* $q_t(x, y)$ *and* $q_t^*(x, y)$ *these densities. Both* q_t *and* q_t^* *satisfy Chapman–Kolmogorov equations and the duality relation,*

$$q_t^*(x, y) = q_t(y, x), \quad x, y > 0, t > 0. \tag{2.9}$$

Proof. It is obtained by following the proof of Lemma 2 in [21] along the lines. Indeed, the latter result is proved under the additional assumptions that both half lines $(-\infty, 0)$ and $(0, \infty)$ are regular. But we can see that these properties are actually not needed, although regularity of $(-\infty, 0)$ is argued at the beginning of this proof. □

Let us denote by $q_t^*(\mathrm{d}x)$, $t > 0$, the entrance law of n^*, that is for any positive Borel function f,

$$\int_{[0,\infty)} f(x) q_t^*(\mathrm{d}x) = n^* \big(f(X_t), t < \zeta \big). \tag{2.10}$$

The local time at 0 of the reflected process at its supremum $\overline{X} - X = X^* - \underline{X}^*$ and the measure of its excursions away from 0 are defined in the same way as for $X - \underline{X}$. They are respectively denoted by L and n. Then the entrance law $q_t(\mathrm{d}x)$ of n is defined in the same way as $q_t^*(\mathrm{d}x)$.

Lemma 2. *Under assumptions* (H$_1$) *and* (H$_2$) *the entrance laws* $q_t(\mathrm{d}x)$ *and* $q_t^*(\mathrm{d}x)$ *are absolutely continuous on* $[0, \infty)$ *and there are versions of their densities which are strictly positive and continuous on* $(0, \infty)$, *for all* $t > 0$. *We denote by* $q_t(x)$ *and* $q_t^*(x)$ *these densities. Then both* q_t *and* q_t^* *satisfy Chapman–Kolmogorov equations: for* $s, t > 0$ *and* $y > 0$,

$$q_{s+t}(y) = \int_0^\infty q_s(x) q_t(x, y) \, \mathrm{d}x \quad \text{and} \quad q_{s+t}^*(y) = \int_0^\infty q_s^*(x) q_t^*(x, y) \, \mathrm{d}x. \tag{2.11}$$

Proof. It suffices to prove the result for $q_t(\mathrm{d}x)$. It is proved in part 3 of Lemma 1 in [2], that under assumption (H$_1$), the measure $q_t(\mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure on $[0, \infty)$. Let h_t be any version of its density and for all $s > 0$ and $y > 0$, define

$$q_{s,t}(y) = \int_0^\infty h_t(x) q_s(x, y) \, \mathrm{d}x. \tag{2.12}$$

We derive from (H$_1$) and (2.7) (for the dual process) that $q_t(x, y)$ is uniformly bounded in $x, y \in (0, \infty)$. Moreover, from (2.10), (2.2) and (2.3), $\int_0^\infty h_t(x) \, \mathrm{d}x = n(t < \zeta) < \infty$. Then from the Lebesgue dominated convergence theorem and Lemma 1, relation (2.12) defines a continuous and strictly positive function on $(0, \infty)$. Moreover, from (2.8) we see that $q_{s,t}(x)$ is a density for $q_{t+s}(\mathrm{d}x)$. Hence it only depends on $t + s$. Let us set $q_{s,t}(x) = q_{t+s}(x)$.

Proceeding this way for all $s, t > 0$, we define a family of strictly positive and continuous densities $q_t(x)$, $t > 0$ of the entrance law of n which satisfies the Chapman–Kolmogorov equations $q_{t+s}(y) = \int_0^\infty q_t(x) q_s(x, y) \, \mathrm{d}x$, $x > 0$, $s, t > 0$. □

We end this section with the definition of the ladder processes. The ladder time processes τ and τ^*, and the ladder height processes H and H^* are the following (possibly killed) subordinators:

$$\tau_t = \inf\{s \colon L_s > t\}, \qquad \tau_t^* = \inf\{s \colon L_s^* > t\}, \qquad H_t = X_{\tau_t}, \qquad H_t^* = -X_{\tau_t^*}, \quad t \geq 0,$$

where $\tau_t = H_t = +\infty$, for $t \geq \zeta(\tau) = \zeta(H)$ and $\tau_t^* = H_t^* = +\infty$, for $t \geq \zeta(\tau^*) = \zeta(H^*)$. We denote by κ and κ^* the characteristic exponents of the ladder processes (τ, H) and (τ^*, H^*). Recall that the drifts $\mathtt{d}$ and $\mathtt{d}^*$ of the subordinators τ and τ^* satisfy

$$\int_0^t \mathbb{1}_{\{X_s = \overline{X}_s\}} \, ds = \mathtt{d} L_t, \qquad \int_0^t \mathbb{1}_{\{X_s = \underline{X}_s\}} \, ds = \mathtt{d}^* L_t^* \tag{2.13}$$

and that $\mathtt{d} > 0$ if and only if $(-\infty, 0)$ is not regular. In any case, we can check that $\mathtt{d} = \gamma^{-1}$, see [2]. We point out that $\mathtt{d} > 0$ if and only if 0 is not regular for $(-\infty, 0)$ (and $\mathtt{d}^* > 0$ if and only if 0 is not regular for $(0, \infty)$), so that $\mathtt{d}\mathtt{d}^* > 0$ always holds since 0 is necessarily regular for at least one of the half lines.

3. Main results

In all this section, $(X, \mathbb{P})$ is any Lévy process satisfying assumptions (H_1) and (H_2). Then from Corollary 3 of [2], the law of the past supremum $\overline{X}_t$ on $[0, \infty)$ takes the following form,

$$\mathbb{P}(\overline{X}_t \in dx) = \int_0^t n(t - s < \zeta) q_s^*(x) \, ds \, dx + \mathtt{d} q_t^*(x) \, dx + \mathtt{d}^* n(t < \zeta) \delta_{\{0\}}(dx). \tag{3.1}$$

Expression (3.1) shows that the law of $\overline{X}_t$ is absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$. Moreover, this law has an atom at 0 if and only if $(0, \infty)$ is not regular. Then we will denote by $f_t(x)$ the following version of the density of $\mathbb{P}(\overline{X}_t \in dx)$ on $(0, \infty)$,

$$f_t(x) = \int_0^t n(t - s < \zeta) q_s^*(x) \, ds + \mathtt{d} q_t^*(x), \quad x > 0. \tag{3.2}$$

Note that there are instances where the law of $\overline{X}_t$ is absolutely continuous whereas assumption (H_1) is not satisfied, see part 1 of Corollary 2 in [2]. Expression (3.2) will be the starting point of our study. The latter expression shows that certain properties of f_t, such as continuity or asymptotic behaviour at 0, are related to those of q_t^*. However, due to the "bad" behaviour of the function $(t, x) \mapsto q_t^*(x)$, when t and x are small, some features of the first term on the right-hand side of (3.2) cannot be directly derived from those of q_t^*. This study requires much sharper arguments which will be developed in the next section.

The next proposition extends Lemma 3 in [21]. It describes the asymptotic behaviour at 0 of the functions $x \mapsto q_t^*(x, y)$ and $x \mapsto q_t^*(x)$. The second assertion is to be compared with Propositions 6 and 7 in [8] where similar results are obtained in the case where the law of X is in the domain of attraction of a stable law.

Proposition 1. *For all $t > 0$,*

$$\lim_{x \to 0+} \frac{q_t^*(x, y)}{h^*(x)} = q_t^*(y), \quad \text{for all } y > 0 \quad \text{and} \quad \lim_{x \to 0+} \frac{q_t^*(x)}{h(x)} = \frac{p_t(0)}{t},$$

where h and h^* are the renewal functions of the ladder height processes H and H^*, that is $h(x) = \int_0^\infty \mathbb{P}(H_t \leq x) \, dt$ and $h^*(x) = \int_0^\infty \mathbb{P}(H_t^* \leq x) \, dt$, $x \geq 0$.

In general, the function h is finite, continuous, increasing and $h - h(0)$ is subadditive on $[0, \infty)$. Moreover, $h(0) = 0$ if $(-\infty, 0)$ is regular and $h(0) = \mathtt{d}$ if not. This function is known explicitly, for instance when X has no positive jumps. In this case, given our normalisation of the local time L, one has $H_t = ct$, where $c = \Phi(1)$ and Φ is the Laplace exponent of the subordinator $T_x = \inf\{t: X_t > x\}$, $x \geq 0$, so that $h(x) = c^{-1}x$. Also, when X is a stable process with index $\alpha \in (0, 2]$ and positivity coefficient $\mathbb{P}(X_1 > 0) = \rho$, then H is a stable subordinator with index $\alpha\rho$, and $h(x) = \mathbb{E}(H_1^{-\alpha\rho})x^{\alpha\rho}$. Lévy processes whose characteristic is of the form $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function ψ are also examples where h is explicit. The function $h(x)$ is then a Bernstein function and its integral representation in terms of $\psi(\xi)$ was given in Proposition 4.5 in [15].

Recall from (1.8) and (3.3) in [20], see also parts 2 and 3 of Lemma 1 in [2], that the renewal function h of the upward ladder process H is everywhere differentiable and that its derivative is given by

$$h'(x) = \int_0^\infty q_s^*(x)\,\mathrm{d}s, \quad \text{for all } x > 0. \tag{3.3}$$

Moreover Lemma 2 ensures that

$$h'(x) > 0, \quad \text{for all } x > 0.$$

Knowing that $x \mapsto q_t^*(x)$ is continuous on $(0, \infty)$ and considering the representation (3.2), it is natural to ask about continuity of f_t.

Proposition 2. *The following conditions are equivalent*:

(1) $x \to h'(x)$ *is continuous at* $x_0 > 0$,
(2) $x \to f_t(x)$ *is continuous at* $x_0 > 0$ *for every* $t > 0$,
(3) $x \to f_t(x)$ *is continuous at* $x_0 > 0$ *for some* $t > 0$.

The function h' is known to be continuous on $(0, \infty)$ in many instances. We have already seen that it is the case when X is a stable process. It is also continuous when the process has no positive jumps, but more generally if the ascending ladder height process H has a positive drift, then h' is continuous and bounded, see Theorem 19, Section VI.4 in [1]. In this case, a further study of the continuity of h' can be also deduced from Proposition 4.5 in [15] for a wide class of subordinated Brownian motions. Actually, this function is not always continuous, see for instance Lemma 2.4 in [13], where it is proved that if X has no negative jumps, bounded variations and a Lévy measure which admits atoms, then h' is not continuous.

Our next result deals with the asymptotic behaviour of f_t at 0.

Theorem 1. *The density of the law of the past supremum of* $(X, \mathbb{P})$ *fulfills the following asymptotic behaviour*,

$$\lim_{x \to 0^+} \frac{f_t(x)}{h'(x)} = n(t < \zeta),$$

uniformly on $[t_0, \infty)$ *for every fixed* $t_0 > 0$.

Note that the above-given result leads (by simple integration) to the estimates for the cumulative distribution function $\mathbf{P}(\overline{X}_t < x)$, which were previously studied in Theorem 3.1 in [15]. However, the result provided in [15] is valid for the whole range of time and space parameters and here we can recover the estimates only for x small enough and large t. We now state two results regarding the asymptotic behaviour of $f_t(x)$, when t tends to infinity. First recall the following equivalent forms of Spitzer's condition. Let $\rho \in (0, 1)$, and denote by $R_\rho(0)$ (resp. $R_{-\rho}(\infty)$) the set of regularly varying functions at $0+$ (resp. at $+\infty$) with index ρ (resp. $-\rho$), then

$$\lim_{t \to \infty} \mathbb{P}(X_t \geq 0) = \rho \quad \Leftrightarrow \quad \alpha \mapsto \kappa(\alpha, 0) \in R_\rho(0) \quad \Leftrightarrow \quad t \mapsto n(t < \zeta) \in R_{-\rho}(\infty). \tag{3.4}$$

The first equivalence can be found in Theorem 14, Section VI.3 in [1], see also the discussion after this theorem. The second equivalence follows from the discussion after Theorem 6, Section III.3 in [1] and the identity $n(t < \zeta) = \pi(t, \infty) + a$, where π is the Lévy measure of τ and a its killing rate. Then Theorem 2 provides a uniform limit in x on compact sets, under assumption (3.4). This result complements, and in some cases generalizes, the result of [11], where the same study was performed for the distribution function $\mathbb{P}(\overline{X}_t \leq x)$.

Theorem 2. *Assume that* (3.4) *is satisfied*, *then*

$$\lim_{t \to \infty} \frac{f_t(x)}{n(t < \zeta)} = h'(x),$$

uniformly in x *on every compact subset of* $(0, \infty)$.

Note that in the stable case, Theorem 2 is an immediate consequence of Theorem 1, the scaling property and the expressions for $h'(x)$ and $n(t < \zeta)$. The next theorem provides some general estimates for $f_t(x)$, when $t \geq t_0$ and $x \leq x_0$, for any given $t_0, x_0 > 0$. These estimates are sharp when (3.4) is satisfied.

Theorem 3. *For fixed $x_0, t_0 > 0$ there exist positive constants c_1 and c_2 such that*

$$c_1 n(t < \zeta) \leq \frac{f_t(x)}{h'(x)} \leq c_2 \frac{1}{t} \int_0^t n(s < \zeta) \, ds, \quad x \leq x_0, t \geq t_0.$$

If additionally (3.4) is satisfied, then there exists $c_3 > 0$ such that

$$c_1 h'(x) n(t < \zeta) \leq f_t(x) \leq c_3 h'(x) n(t < \zeta), \quad x \leq x_0, t \geq t_0.$$

The constants c_1 and c_2 depend on x_0 and t_0 by the relations described explicitly in (4.21) and (4.24).

Now we derive from Proposition 1 the asymptotics of the densities of the Lévy process $(X, \mathbb{P})$ conditioned to stay positive and this of its meander. Lévy processes conditioned to stay positive will also be involved in the proofs of Section 4. Let us briefly recall their definition which may be found in more details in [3] and [5]. The law of the Lévy process $(X, \mathbb{P})$ conditioned to stay positive is a Doob h-transform of the killed process $(X, \mathbb{Q}_x^*)$ defined in (2.7). It is obtained from the renewal function h^* of the downward ladder height process H^* which is excessive for $(X, \mathbb{Q}_x^*)$ and invariant if and only if $\limsup_{t \to \infty} X_t = +\infty$, a.s. The conditioned process is currently denoted by $(X, \mathbb{P}_x^\uparrow)$ and formally defined by

$$\mathbb{P}_x^\uparrow(\Lambda, t < \zeta) = \frac{1}{h^*(x)} \mathbb{E}_x^{\mathbb{Q}^*}\big(h^*(X_t) \mathbb{1}_{\{\Lambda, t < \zeta\}}\big), \quad x > 0, \Lambda \in \mathcal{F}_t. \tag{3.5}$$

We also recall from Theorem 2 in [3] that the family of measures $(\mathbb{P}_x^\uparrow)$ converges as $x \downarrow 0$, towards a probability measure $\mathbb{P}^\uparrow$ which is related to n^* by the following expression:

$$\mathbb{P}^\uparrow(\Lambda, t < \zeta) = n^*\big(h^*(X_t) \mathbb{1}_{\{\Lambda, t < \zeta\}}\big). \tag{3.6}$$

This convergence holds weakly on the Skohorod's space when $(0, \infty)$ is regular. In the nonregular case, we can only say that this convergence holds for the process $X \circ \theta_\varepsilon$, for all $\varepsilon > 0$, where $\theta_.$ is the shift operator. In any case, we derive from (3.6) that the density of the law $\mathbb{P}^\uparrow(X_t \in dx)$, for $t > 0$ is related to the entrance law q_t^* as follows:

$$p_t^\uparrow(x) = h^*(x) q_t^*(x). \tag{3.7}$$

The meander with length $t > 0$, is a process with the law of $(X_s, 0 \leq s \leq t)$ under the conditional distribution $\mathbb{P}(\cdot | \underline{X}_t \geq 0)$. This conditioning only makes sense when $(-\infty, 0)$ is not regular. When $(0, \infty)$ is regular, it corresponds to the law of $(X_s, 0 \leq s \leq t)$ under the limiting probability measure

$$M^{(t)} := \lim_{x \downarrow 0} \frac{1}{h^*(x)} \mathbb{P}_x(\cdot | \underline{X}_t \geq 0).$$

A general definition can be found in [5], see Section 4 and relation (4.5) therein. It implies in particular that on $\mathcal{F}_t$, the law $M^{(t)}$ of the meander of length t is absolutely continuous with respect to the process $(X, \mathbb{P}^\uparrow)$, with density $(h^*(X_t))^{-1}$. As a consequence, the density of the distribution $M^{(t)}(X_t \in dx)$ of the meander with length t at time t, which we denote by $m_t(x)$, is given by:

$$m_t(x) = n^*(t < \zeta)^{-1} q_t^*(x).$$

This relation together with (3.7) leads to the following straightforward consequence of Proposition 1.

Corollary 1. *The density $m_t(x)$ of the law of the meander with length t, at time t and the density $p_t^\uparrow(x)$ of the entrance law of the Lévy process conditioned to stay positive are continuous and strictly positive on $(0, \infty)$. Moreover they have the following asymptotic behaviour at 0:*

$$m_t(x) \sim \frac{p_t(0)}{tn^*(t < \zeta)} h(x) \quad and \quad p_t^\uparrow(x) \sim \frac{p_t(0)}{t} h(x) h^*(x), \quad as\ x \to 0.$$

4. Proofs

Before proceeding to the proofs of the theorems, we need a couple of additional preliminary results. We first extend Corollary 1 of [3] to the case where $(0, \infty)$ is not regular. Recall from (3.5) the definition of Lévy processes conditioned to stay positive.

Proposition 3. *Assume that $(X, \mathbb{P})$ is not a compound Poisson process and that $(|X|, \mathbb{P})$ is not a subordinator. Then for all bounded and continuous function f and for all $t > 0$,*

$$\lim_{x \to 0} \mathbb{E}_x^\uparrow \big(h^*(X_t)^{-1} f(X_t) \big) = n^* \big(f(X_t), t < \zeta \big).$$

Proof. When $(0, \infty)$ is regular for $(X, \mathbb{P}_x)$, the result is given by Corollary 1 of [3] whose proof can be found in [4].

Let us assume that $(0, \infty)$ is not regular for $(X, \mathbb{P}_x)$. Then from the second part of Theorem 2 of [3] and relation (3.2) in this article, we still have for all $t > 0$,

$$\lim_{x \to 0} \mathbb{E}_x^\uparrow \big(f(X_t) \big) = n^* \big(h^*(X_t) f(X_t), t < \zeta \big). \tag{4.1}$$

(Note that the constant k in (3.2) of [3] is equal to 1, according to the normalisation of the local time that is recalled in (2.2).) However, since $h^*(0) = 0$, the function $x \mapsto h^*(x)^{-1} f(x)$ is not necessarily bounded, so we cannot replace f by this function in (4.1) in order to get our result. But from the weak convergence stated in (4.1) with $f \equiv 1$ and using definitions (2.7) and (3.5), we derive that for all fixed $\delta > 0$ and $t > 0$ that

$$\lim_{x \to 0} \mathbb{E}_x^\uparrow \big(h^*(X_t)^{-1} \mathbb{1}_{\{X_t > \delta\}} \big) = \lim_{x \to 0} h^*(x)^{-1} \mathbb{P}_x \big(X_t > \delta, \tau_0^- > t \big)$$
$$= n^*(X_t > \delta, t < \zeta). \tag{4.2}$$

Then it remains to prove that

$$\lim_{x \to 0} \mathbb{E}_x^\uparrow \big(h^*(X_t)^{-1} \big) = \lim_{x \to 0} h^*(x)^{-1} \mathbb{P}_x \big(\tau_0^- > t \big)$$
$$= n^*(t < \zeta). \tag{4.3}$$

Indeed, it will follows from (4.2) and (4.3) that

$$\lim_{x \to 0} \frac{\mathbb{E}_x^\uparrow \big(h^*(X_t)^{-1} \mathbb{1}_{\{X_t \le \delta\}} \big)}{\mathbb{E}_x^\uparrow \big(h^*(X_t)^{-1} \big)} = n^*(X_t \le \delta | t < \zeta),$$

which proves that for all $t > 0$, the family of probability measures on $(0, \infty)$ with distribution function $\delta \mapsto \frac{\mathbb{E}_x^\uparrow (h^*(X_t)^{-1} \mathbb{1}_{\{X_t \le \delta\}})}{\mathbb{E}_x^\uparrow (h^*(X_t)^{-1})}$ converges weakly as $x \to 0$, toward the probability measure on $(0, \infty)$ with distribution function $\delta \mapsto n^*(X_t \le \delta | t < \zeta)$. The latter assertion is equivalent to the statement of our proposition.

Then let us prove (4.3). This point differs from the proof given in [4] in the regular case. First recall formula (1) in [4]:

$$\mathbb{P}_x \big(\tau_0^- > e/\varepsilon \big) = \mathbb{E} \bigg(\int_0^\infty e^{-\varepsilon s} \mathbb{1}_{\{\underline{X}_s \ge -x\}} \, dL_s^* \bigg) \big[d^* \varepsilon + n^*(e/\varepsilon < \zeta) \big], \tag{4.4}$$

which can be derived from the compensation formula (2.3) and (2.13). Set $h^{(\varepsilon)}(x) := \mathbb{E}(\int_0^\infty e^{-\varepsilon s} \mathbb{1}_{\{\underline{X}_s \geq -x\}} \, dL_s^*)$ and recall that $h^*(x) = \mathbb{E}(\int_0^\infty \mathbb{1}_{\{\underline{X}_s \geq -x\}} \, dL_s^*)$. Then we will first show that for all $\varepsilon > 0$,

$$h^{(\varepsilon)}(x) \sim h^*(x), \quad \text{as } x \to 0. \tag{4.5}$$

First note that for all $\varepsilon > 0$, $h^{(\varepsilon)}(x) \leq h^*(x)$. Then, for the lower bound, we can write for all $u > 0$, $h^{(\varepsilon)}(x) \geq e^{-\varepsilon u} \mathbb{E}(\int_0^u \mathbb{1}_{\{\underline{X}_s \geq -x\}} \, dL_s^*)$, so that

$$h^*(x) = \mathbb{E}\left(\int_0^u \mathbb{1}_{\{\underline{X}_s \geq -x\}} \, dL_s^* \right) + \mathbb{E}\left(\int_u^\infty \mathbb{1}_{\{\underline{X}_s \geq -x\}} \, dL_s^* \right)$$

$$\leq e^{\varepsilon u} h^{(\varepsilon)}(x) + \mathbb{E}\left(\int_u^\infty \mathbb{1}_{\{\underline{X}_s \geq -x\}} \, dL_s^* \right). \tag{4.6}$$

Then applying the Markov property at time u and using the monotonicity of h^*, we obtain that $\mathbb{E}(\int_u^\infty \mathbb{1}_{\{\underline{X}_s \geq -x\}} \, dL_s^*) \leq \mathbb{P}_x(\tau_0^- \geq u) h^*(x)$. Plunging this in (4.6), we get

$$h^*(x) \leq \frac{e^{\varepsilon u}}{1 - \mathbb{P}_x(\tau_0^- \geq u)} h^{(\varepsilon)}(x).$$

Observe that since $(-\infty, 0)$ is regular, for all $u > 0$, $\lim_{x \to 0} \mathbb{P}_x(\tau_0^- \geq u) = 0$. Let $\delta > 1$, then from the above inequality, for u sufficiently small, we can find $x_0 > 0$ such that for all $x \leq x_0$, $h^*(x) \leq \delta h^{(\varepsilon)}(x)$. So we have proved (4.5). Then let us rewrite (4.4) as follows:

$$\int_0^\infty e^{-\varepsilon s} \mathbb{P}_x(\tau_0^- > s) \, ds = h^{(\varepsilon)}(x) \left[d^* + \int_0^\infty e^{-\varepsilon s} n^*(s < \zeta) \, ds \right].$$

From (4.5), we obtain that for all $\varepsilon > 0$,

$$\lim_{x \to 0} \int_0^\infty e^{-\varepsilon s} \frac{\mathbb{P}_x(\tau_0^- > s)}{h^*(x)} \, ds = d^* + \int_0^\infty e^{-\varepsilon s} n^*(s < \zeta) \, ds,$$

which means that the measure with density $s \mapsto \mathbb{P}_x(\tau_0^- > s)/h^*(x)$ converges weakly toward the measure $d^* \delta_0(ds) + n^*(s < \zeta) \, ds$, as x tends to 0.

Then from this fact, we can derive (4.3) as it is done in the proof of Corollary 1 in [4]. Let $c \in (0, t)$, then

$$\liminf_{x \to 0} h^*(x)^{-1} \mathbb{P}_x(\tau_0^- > t) \geq c^{-1} \lim_{x \to 0} h^*(x)^{-1} \int_t^{t+c} \mathbb{P}_x(\tau_0^- > s) \, ds$$

$$= c^{-1} \int_t^{t+c} n^*(\zeta > s) \, ds \geq n^*(\zeta > t + c),$$

$$\limsup_{x \to 0} h^*(x)^{-1} \mathbb{P}_x(\tau_0^- > t) \leq c^{-1} \lim_{x \to 0} h^*(x)^{-1} \int_{t-c}^t \mathbb{P}_x(\tau_0^- > s) \, ds$$

$$= c^{-1} \int_{t-c}^t n^*(\zeta > s) \, ds \leq n^*(\zeta > t - c),$$

and the result follows, since c can be chosen arbitrarily small. $\qquad \square$

Let $\mathbb{P}_x^{*\uparrow}$, $x \geq 0$ be the law of the dual Lévy process $(X, \mathbb{P}_x^*)$ conditioned to stay positive. Then Proposition 3 is interpreted for the dual process as follows:

$$\lim_{x \to 0} \mathbb{E}_x^{*\uparrow} \big(h(X_t)^{-1} f(X_t) \big) = n \big(f(X_t), t < \zeta \big), \tag{4.7}$$

for all bounded and continuous function f and for all $t > 0$. We will actually use (4.7) in order to show Proposition 4 below.

In the next results, we will use some properties of the bridge of $(X, \mathbb{P})$. Let us now briefly recall its definition. We refer to [6] for a more complete account on the subject. Assume that (H_1) and (H_2) are satisfied, then the law $\mathbb{P}^t_{x,y}$ of the bridge from $x \in \mathbb{R}$ to $y \in \mathbb{R}$, with length $t > 0$ of the Lévy process $(X, \mathbb{P})$ is a regular version of the conditional law of $(X_s, 0 \le s \le t)$ given $X_t = y$, under $\mathbb{P}_x$. It satisfies $\mathbb{P}^t_{x,y}(X_0 = x, X_t = y) = 1$ and for all $s < t$, this law is absolutely continuous with respect to $\mathbb{P}_x$ on $\mathcal{F}_s$, with density $p_{t-s}(X_s - x)/p_t(y - x)$, i.e.,

$$\mathbb{P}^t_{x,y}(\Lambda) = \mathbb{E}\left(\mathbb{1}_\Lambda \frac{p_{t-s}(X_s - x)}{p_t(y - x)} \right), \quad \text{for all } \Lambda \in \mathcal{F}_s. \tag{4.8}$$

In the next proposition, we give the law of the time at which the bridge $(X, \mathbb{P}^t_{0,y})$, reaches its supremum over $[0, t]$.

Proposition 4. *Assume that* (H_1) *and* (H_2) *are satisfied and let us define*,

$$g_t := \sup\{u \le t \colon X_u = \overline{X}_u\}.$$

Let $y \in \mathbb{R}$, *then* $\mathbb{P}^t_{0,y}$-*a.s.*, g_t *is the unique instant such that* $X_{g_t} = \overline{X}_t$ *or* $X_{g_t-} = \overline{X}_t$.

Besides, the law of the time of the supremum of the bridge $(X, \mathbb{P}^t_{0,y})$ *is absolutely continuous on* $[0, t]$ *and its density is given by*:

$$\frac{\mathbb{P}^t_{0,y}(g_t \in \mathrm{d}s)}{\mathrm{d}s} = p_t(y)^{-1} \int_0^\infty q_s^*(x) q_{t-s}(x + y)\,\mathrm{d}x, \quad s \in [0, t].$$

Proof. It is well known that under our assumptions, $\mathbb{P}$-a.s., g_t is the unique instant such that $X_{g_t} = \overline{X}_t$ or $X_{g_t-} = \overline{X}_t$, see Section VI.2 in [1]. Let $s \in (0, t)$, then from the latter result applied to s and (4.8), we derive that $\mathbb{P}^t_{0,y}$-a.s., g_s is the unique instant in $[0, s]$ such that $X_{g_s} = \overline{X}_s$ or $X_{g_s-} = \overline{X}_s$. Now define $g_{s,t} := \sup\{u \in [s, t] \colon X_u = \sup_{v \in [s,t]} X_v\}$, then by a classical argument of time reversal, $\mathbb{P}^t_{0,y}$-a.s., $g_{s,t}$ is the unique instant in $[s, t]$ such that $X_{g_{s,t}} = \sup_{v \in [s,t]} X_v$ or $X_{g_{s,t}-} = \sup_{v \in [s,t]} X_v$. So it remains to prove that $\mathbb{P}^t_{0,y}(\overline{X}_s = \sup_{v \in [s,t]} X_v) = 0$. But it is a consequence of the following inequality,

$$\mathbb{P}^t_{0,y}\left(\overline{X}_s = \sup_{v \in [s,t]} X_v \right) \le \mathbb{P}^t_{0,y}\left(\overline{X}_s = \sup_{v \in [s,t]} X_v = y \right) + \sum_{n \ge 1} \mathbb{P}^t_{0,y}\left(\overline{X}_s = \sup_{v \in [s,t-1/n]} X_v \right)$$

$$\le \min\left\{ \mathbb{P}^t_{0,y}(\overline{X}_s = y), \mathbb{P}^t_{0,y}\left(\sup_{v \in [s,t]} X_v = y \right) \right\}$$

$$+ \sum_{n \ge 1} \mathbb{P}^t_{0,y}\left(\overline{X}_s = \sup_{v \in [s,t-1/n]} X_v \right).$$

Indeed, for all $u > 0$, the law of $\overline{X}_u$ under $\mathbb{P}$ has no atom whenever 0 is regular for $(0, \infty)$, see [2]. Then from (4.8) and a time reversal argument, we derive that at least one of the terms $\mathbb{P}^t_{0,y}(\overline{X}_s = y)$ and $\mathbb{P}^t_{0,y}(\sup_{v \in [s,t]} X_v = y)$ is equal to 0. Moreover, for all $n \ge 1$, $\mathbb{P}(\overline{X}_s = \sup_{v \in [s,t-1/n]} X_v) = 0$, so that from (4.8) the second term of right-hand side of the above inequality is equal to 0.

The second assertion is a direct consequence of Theorem 3 in [2] which asserts that

$$\mathbb{P}(g_t \in \mathrm{d}s, \overline{X}_t \in \mathrm{d}x, X_t - \overline{X}_t \in \mathrm{d}y)$$
$$= q_t^*(x) q_{t-s}(y) \mathbb{1}_{[0,t]}(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}\delta_{\{t\}}(\mathrm{d}s) q_t^*(x) \delta_{\{0\}}(\mathrm{d}y)\,\mathrm{d}x + \mathrm{d}^* \delta_{\{0\}}(\mathrm{d}s) \delta_{\{0\}}(\mathrm{d}x) q_t(y)\,\mathrm{d}y. \qquad \square$$

For $y = 0$, the time g_t of the supremum of the bridge $(X, \mathbb{P}^t_{0,y})$ is uniformly distributed over $[0, t]$, see [12]. Then as a consequence of this result and Proposition 4, we obtain the following equality:

$$\text{for all } s \in (0, t), \quad \int_0^\infty q_{t-s}^*(x) q_s(x)\,\mathrm{d}x = \frac{p_t(0)}{t}. \tag{4.9}$$

Proof of Proposition 1. When both half lines $(-\infty, 0)$ and $(0, \infty)$ are regular, the result follows directly from Lemma 3 of [21]. This lemma actually concerns the transition densities $p_t^{*\uparrow}(x, y)$ of the process $(X, \mathbb{P}_x^{*\uparrow})$, but it is easily interpreted in terms of the transition densities $q_t^*(x, y)$ and the entrance law $q_t^*(x)$, thanks to relations (3.5) and (3.7). Actually Lemma 3 of [21] yields

$$\lim_{y \to 0} \frac{q_t^*(y)}{h(y)} = \int_0^\infty q_{t-s}^*(x) q_s(x) \, dx, \tag{4.10}$$

and we conclude to the second assertion from identity (4.9).

Now let us consider the case where one of the half lines is not regular. Note that the main argument in the proof of Lemma 3 in [21] is the fact that for all $t > 0$, $\lim_{x \to 0} \mathbb{E}_x^{*\uparrow}(h(X_t)^{-1}) = \mathbb{E}^{*\uparrow}(h(X_t)^{-1}) = n^*(t < \zeta) < \infty$, which we have proved in Proposition 3, in the general case. (Here we actually use the result for the dual process, see (4.7).) Thanks to this result, we can follow the proof of Lemma 3 of [21] along the lines in order to check that it is still valid, when one of the half lines is not regular. Then we conclude as above.

A key point in the proof of our main result is the following proposition regarding integrability properties of $t \to p_t(0)/t$, both at zero and at infinity. $\qquad\qquad\square$

Proposition 5. *If there exists $t_0 > 0$ such that $x \to p_{t_0}(x)$ is bounded, then*

$$\int^\infty \frac{p_t(0)}{t} \, dt < \infty. \tag{4.11}$$

Moreover, if $t \to p_t(x)$ is bounded for every $t > 0$ (that is (H_1) holds) then

$$\int_{0+} \frac{p_t(0)}{t} \, dt = \infty. \tag{4.12}$$

Proof. Since boundedness of $x \to p_{t_0}(x)$ implies that $p_{t_0} \in L^2(\mathbb{R})$, its Fourier transform is also in $L^2(\mathbb{R})$ which means that $e^{-2t_0 \operatorname{Re}(\Psi(\cdot))} \in L^1(\mathbb{R})$. On the one hand it implies integrability of the characteristic function of X for $t \geq 2t_0$ and, by the Riemann–Lebesgue lemma, continuity of p_t for $t \geq 2t_0$. On the other hand, applying inverse Fourier transform together with Fubini–Tonelli theorem, we can write

$$\int_{3t_0}^\infty \frac{p_t(0)}{t} \, dt \leq \frac{1}{2\pi} \int_{3t_0}^\infty \frac{1}{t} \int_{\mathbb{R}} \left| e^{-t\Psi(\xi)} \right| d\xi \, dt = \frac{1}{\pi} \left(\int_0^1 + \int_1^\infty \right) \int_{t_0}^\infty \frac{1}{t} e^{-3t \operatorname{Re}(\Psi(\xi))} \, dt \, d\xi.$$

Recall that $2 \operatorname{Re} \Psi(\xi)$ is the Lévy–Khintchin exponent of the symmetrization of X and thus it is an increasing function (we have excluded the compound Poisson processes from our consideration). Consequently, for $t \geq t_0$ and $\xi \geq 1$, we can write

$$e^{-3t \operatorname{Re}(\Psi(\xi))} = e^{-2t \operatorname{Re}(\Psi(\xi))} e^{-t \operatorname{Re}(\Psi(\xi))} \leq e^{-2t_0 \operatorname{Re}(\Psi(\xi))} e^{-t \operatorname{Re}(\Psi(1))}.$$

By integrability of the characteristic function of X_{2t_0} and the fact that $\operatorname{Re} \Psi(1) > 0$, we obtain

$$\int_1^\infty \int_{t_0}^\infty \frac{1}{t} e^{-3t \operatorname{Re}(\Psi(\xi))} \, dt \, d\xi \leq \int_1^\infty e^{-2t_0 \operatorname{Re}(\Psi(\xi))} \, d\xi \cdot \int_{2t_0}^\infty \frac{1}{t} e^{-t \operatorname{Re}(\Psi(1))} \, dt < \infty,$$

hence it is enough to show the finiteness of the integral over $(0, 1)$. By Lévy–Khintchine formula, there exists a constant $c > 0$ such that $\operatorname{Re} \Psi(\xi) \geq c\xi^2$ whenever $\xi \in (0, 1)$. Moreover, we have

$$\int_{t_0}^\infty \frac{1}{t} e^{-3ct\xi^2} \, dt \approx -\ln \xi, \quad \xi \leq 1/2.$$

Here and below $f \approx g$ means that the ratio of the functions is bounded from below and above by positive constants for the indicated range of arguments. It finally gives

$$\int_0^1 \int_{t_0}^\infty \frac{1}{t} e^{-3t \operatorname{Re}(\Psi(\xi))} \, dt \, d\xi \leq \int_0^1 \int_{t_0}^\infty \frac{1}{t} e^{-3ct\xi^2} \, dt \, d\xi < \infty,$$

which ends the proof of (4.11).

To deal with (4.12) recall that (H$_1$) implies that the function $t \to p_t(0)$ is completely monotone (see [18], p. 118), so in particular it is decreasing. It entails

$$\int_0^{t_1} \frac{p_t(0)}{t} \, dt \geq p_{t_1}(0) \int_0^{t_1} \frac{dt}{t} = \infty.$$

This ends the proof. □

Finally, we introduce the following result providing the upper-bounds for q_t^* in terms of the transition probability density of the symmetrization of the process X and the renewal functions h.

Lemma 3. *If* (H$_1$) *holds then*

$$\frac{q_t^*(y)}{h(y)} \leq 6\left(\frac{e}{e-1}\right)^2 \frac{p_{t/6}^S(0)}{t}, \quad y > 0, t > 0,$$ (4.13)

where $p_t^S = p_t * \check{p}_t$ is the density of the semi-group of the symmetrization of X.

Proof. Note that under (H$_1$) by the inversion formula we have for every $t > 0$ that

$$q_t^*(x, y) \leq p_t(y - x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it(x-y)} e^{-t\Psi(\xi)} \, d\xi \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| e^{-t\Psi(\xi)} \right| \, d\xi = p_{t/2}^S(0),$$ (4.14)

where the first inequality follows from (2.7). By the upper-bounds given in Theorem 3.1 in [15] we have

$$\int_0^\infty q_t(x, y) \, dy = \mathbb{P}(\overline{X}_t \leq x) \leq \frac{e}{e-1} \kappa(1/t, 0) h(x).$$ (4.15)

Note that this bound is true for every Lévy process and that an analogous result holds for the reflected process $\overline{X} - X$. Then, applying the Chapman–Kolmogorov equation and using Inequalities (4.14) and (4.15), we obtain

$$
\begin{aligned}
q_{3t}^*(x, y) &= \int_0^\infty \int_0^\infty q_t^*(x, z) q_t^*(z, w) q_t^*(w, y) \, dz \, dw \\
&\leq p_{t/2}^S(0) \int_0^\infty q_t^*(x, z) \, dz \int_0^\infty q_t^*(w, y) \, dw \\
&= p_{t/2}^S(0) \int_0^\infty q_t^*(x, z) \, dz \int_0^\infty q_t(y, w) \, dw \\
&\leq \left(\frac{e}{e-1}\right)^2 p_{t/2}^S(0) h^*(x) h(y) \kappa(1/t, 0) \kappa^*(1/t, 0).
\end{aligned}
$$

This inequality together with the Wiener–Hopf factorization $\kappa(1/t, 0)\kappa^*(1/t, 0) = 1/t$ yields

$$\frac{q_{3t}^*(x, y)}{h^*(x) h(y)} \leq \left(\frac{e}{e-1}\right)^2 \frac{p_{t/2}^S(0)}{t}.$$ (4.16)

Taking the limit when $x \to 0$ and using Proposition 1 we can finally write

$$\frac{q_t^*(y)}{h(y)} \leq 6\left(\frac{e}{e-1}\right)^2 \frac{p_{t/6}^S(0)}{t}, \quad y > 0, t > 0.$$ (4.17)

□

We are now ready to proceed to the proofs of our main results.

Proof of Theorem 1. Let us first note that

$$\lim_{x \to 0^+} \frac{h(x)}{h'(x)} = 0. \tag{4.18}$$

Indeed, from (3.3), Proposition 1 and the Fatou lemma, we have

$$\liminf_{x \to 0^+} \frac{h'(x)}{h(x)} = \liminf_{x \to 0^+} \frac{1}{h(x)} \int_0^\infty q_s^*(x) \, ds \geq \int_0^\infty \liminf_{x \to 0^+} \frac{q_s^*(x)}{h(x)} \, ds = \int_0^\infty \frac{p_s(0)}{s} \, ds,$$

which is infinite from (4.12).

   Similarly as in the proof of Lemma 3, applying Chapman–Kolmogorov equation (2.11), we can write for $\delta \in (0, s)$

$$\frac{q_s^*(x)}{h(x)} = \int_0^\infty \int_0^\infty q_{s-\delta}^*(z) q_{\delta/2}^*(z, w) \frac{q_{\delta/2}^*(w, x)}{h(x)} \, dz \, dw$$

$$\leq p_{\delta/4}^S(0) \int_0^\infty q_{s-\delta}^*(z) \, dz \cdot \int_0^\infty \frac{q_{\delta/2}(x, w)}{h(x)} \, dw.$$

Consequently, using (4.15) together with the fact that $\int_0^\infty q_{s-\delta}^*(z) \, dz = n^*(s - \delta < \zeta)$, we get

$$\frac{q_s^*(x)}{h(x)} \leq c_\delta n^*(s - \delta < \zeta), \quad x > 0, \tag{4.19}$$

where

$$c_\delta = \frac{e}{e-1} p_{\delta/4}^S(0) \kappa(2/\delta, 0).$$

Since $t \to n(t < \zeta)$ is a continuous, nonnegative and decreasing function, it is uniformly continuous on $[t_0/2, \infty)$. For every $\varepsilon > 0$ we can choose $0 < \delta < t_0/2$ such that

$$n(t - \delta < \zeta) - n(t < \zeta) \leq \varepsilon, \quad t \geq t_0.$$

Then, using (3.2), the monotonicity of $t \to n(t < \zeta)$, (3.3) and (4.19), we can write for every $t \geq t_0$ that

$$\frac{f_t(x)}{h'(x)} = \int_0^\delta n(t - s < \zeta) \frac{q_s^*(x)}{h'(x)} \, ds + \frac{h(x)}{h'(x)} \int_\delta^t n(t - s < \zeta) \frac{q_s^*(x)}{h(x)} \, ds + d \frac{h(x)}{h'(x)} \frac{q_t^*(x)}{h(x)}$$

$$\leq n(t - \delta < \zeta) + \frac{h(x)}{h'(x)} \left[c_\delta \int_\delta^t n(t - s < \zeta) n^*(s - \delta < \zeta) \, ds + d c_{t_0/2} n^*(t - t_0/2 < \zeta) \right],$$

where the last term was estimated using (4.19) with $s := t$ and $\delta := t_0/2$. Moreover, we have the following simple inequality

$$\int_0^{t-\delta} n(t - \delta - s < \zeta) n^*(s < \zeta) \, ds \leq \int_{[0,\infty)} f_{t-\delta}(x) \, dx \leq 1,$$

which is a consequence of integration of the formula (3.2) with respect to x. The choice of δ and the monotonicity of $n(\cdot < \zeta)$ give

$$\frac{f_t(x)}{h'(x)} \leq n(t < \zeta) + \varepsilon + \frac{h(x)}{h'(x)} \left[c_\delta + c_{t_0/2} d n^*(t_0/2 < \zeta) \right],$$

for every $t > t_0$ and $x > 0$. Consequently, using (4.18) and the fact that ε was arbitrary, we have

$$\limsup_{x \to 0^+} \frac{f_t(x)}{h'(x)} \leq n(t < \zeta),$$

uniformly on $[t_0, \infty)$. For the lower bound, note that monotonicity of $t \to n(t < \zeta)$ and (3.2) give

$$\frac{f_t(x)}{h'(x)} \geq n(t < \zeta) \int_0^t \frac{q_s^*(x)}{h'(x)} \, ds \geq n(t < \zeta) - n(t < \zeta) \frac{h(x)}{h'(x)} \int_t^\infty \frac{q_s^*(x)}{h(x)} \, ds.$$

From (4.13) we have

$$n(t < \zeta) \int_t^\infty \frac{q_s^*(x)}{h(x)} \, ds \leq 6 \left(\frac{e}{e-1} \right)^2 n(t_0 < \zeta) \int_{6t_0}^\infty \frac{p_t^S(0)}{t} \, dt, \quad t \geq t_0.$$

Note that boundedness of $x \to p_t(x)$ implies boundedness of p_t^S (since p_t^S is a convolution of a function from $L^1(\mathbb{R})$ and a bounded function) and consequently, by (4.11) and (4.18) we finally obtain

$$\liminf_{x \to 0^+} \frac{f_t(x)}{h'(x)} \geq n(t < \zeta), \quad \text{uniformly for } t \geq t_0.$$

This ends the proof. □

Proof of Theorem 2. Let A be any compact subset of $(0, \infty)$. Since $t \mapsto n(t < \zeta)$ is regularly varying at infinity, we have

$$\frac{1}{t} \int_0^t n(s < \zeta) \, ds \approx n(t < \zeta), \quad t \to \infty. \tag{4.20}$$

Recall that $f(t) \approx g(t), t \to \infty$ means that there exists constant $c > 1$ such that $c^{-1} g(t) \leq f(t) \leq cg(t)$ for large t. Then let us split formula (3.2) into two parts by writing $f_t^1(x)$ for the integral component and $f_t^2(x) := \mathrm{d} q_t^*(x)$. Thus, for every fixed $\delta \in (0, 1)$, by monotonicity of $n(\cdot < \zeta)$ and (4.13) we have

$$f_t^1(x) = \left(\int_0^{(1-\delta)t} + \int_{(1-\delta)t}^t \right) n(s < \zeta) q_{t-s}^*(x) \, ds$$

$$\leq 6 \left(\frac{e}{e-1} \right)^2 h(x) \int_0^{(1-\delta)t} n(s < \zeta) \frac{p_{(t-s)/6}^S(0)}{t-s} \, ds + n\big((1-\delta)t < \zeta\big) \int_0^\infty q_s^*(x) \, ds.$$

Since $t \to p_t^S(0)$ is decreasing (by (H_1)), we can write

$$\frac{f_t^1(x)}{n(t < \zeta)} \leq \frac{n((1-\delta)t < \zeta)}{n(t < \zeta)} h'(x) + 6 \left(\frac{e}{e-1} \right)^2 \frac{h(x) p_{\delta t/6}^S(0)}{\delta} \frac{1}{tn(t < \zeta)} \int_0^t n(s < \zeta) \, ds.$$

Finally, using (4.20) and the facts that $\lim_{t \to \infty} p_t^S(0) = 0$ and $h(x)$ is bounded on $(0, x_0]$, we obtain

$$\limsup_{t \to \infty} \frac{f_t^1(x)}{n(t < \zeta)} = (1 - \delta)^{-\rho} h'(x).$$

Since δ was arbitrary and h' is bounded on A, we get

$$\limsup_{t \to \infty} \frac{f_t^1(x)}{n(t < \zeta)} = h'(x),$$

uniformly in $x \in A$. To deal with $f_t^2(x)$ we use (4.19) to get

$$\frac{f_t^2(x)}{n(t < \zeta)} \leq \mathrm{d} \frac{e}{e-1} p_{t/8}^S(0) \kappa(4/t, 0) h(x) \frac{n(t/2 < \zeta)}{n(t < \zeta)}.$$

Because $h(x)$ is bounded on A, and using the facts that

$$\lim_{t\to\infty} \frac{n(t/2 < \zeta)}{n(t < \zeta)} = 2^\rho$$

and $\lim_{t\to\infty} p_{t/8}^S(0)\kappa(4/t, 0) = 0$, we obtain

$$\limsup_{t\to\infty} \frac{f_t^2(x)}{n(t < \zeta)} = 0,$$

uniformly on A. Moreover, we have

$$\frac{f_t(x)}{n(t < \zeta)} \geq \int_0^t q_s^*(x)\,\mathrm{d}s = h'(x) - \int_t^\infty q_s^*(x)\,\mathrm{d}s,$$

where, for $x \in A$, we can write

$$\int_t^\infty q_s^*(x) \leq 6\left(\frac{e}{e-1}\right)^2 h(x) \int_t^\infty \frac{p_{s/6}^S(0)}{s}\,\mathrm{d}s < 6\left(\frac{e}{e-1}\right)^2 \sup_{x\in A} h(x) \int_{6t}^\infty \frac{p_s^S(0)}{s}\,\mathrm{d}s.$$

The last integral goes to zero, when t goes to infinity and consequently,

$$\liminf_{t\to\infty} \frac{f_t(x)}{n(t < \zeta)} = h'(x),$$

uniformly in $x \in A$. This ends the proof. $\square$

Proof of Theorem 3. Note that the function

$$g(x, t) := \frac{h(x)}{h'(x)} \int_t^\infty \frac{q_s^*(x)}{h(x)}\,\mathrm{d}s < 1$$

is a nonnegative function on $(0, x_0] \times [t_0, \infty]$ such that

$$c(x_0, t_0) := \sup_{x \leq x_0, t \geq t_0} g(x, t) < 1. \tag{4.21}$$

Since, by (4.11) and (4.18), the function $g(x, t)$ vanishes when $x \to 0$ or $t \to \infty$ and $g(x, t) \leq g(x, t_0)$, for $t \geq t_0$, it is enough to show that for every $0 < a < b$,

$$\sup_{x\in[a,b]} \frac{1}{h'(x)} \int_{t_0}^\infty q_s^*(x)\,\mathrm{d}s < 1.$$

If the above-given supremum were equal to 1, then we could choose a sequence of points $(x_n) \in [a, b]$ such that $\lim_n x_n = x_0$ and $\lim_n g(x_n, t_0) = 1$. Since, by continuity of $g_s^*(x)$ (see [2]) together with (4.13) and (4.11), the above-given integral is continuous in x we would get

$$\int_{t_0}^\infty q_s^*(x_0)\,\mathrm{d}s = \lim_n h'(x_n) = \lim_n \int_0^\infty q_s^*(x_n)\,\mathrm{d}s \geq \int_0^\infty \liminf_n q_s^*(x_n)\,\mathrm{d}s = \int_0^\infty q_s^*(x_0)\,\mathrm{d}s.$$

Here we have used the Fatou lemma and once again continuity of $q_s^*(x)$. Since $q_s^*(x_0)$ is strictly positive this is a contradiction.

By monotonicity of $n(\cdot < \zeta)$ we can write

$$f_t(x) \geq \int_0^t n(s < \zeta)q_{t-s}^*(x)\,\mathrm{d}s \geq n(t < \zeta) \int_0^t q_s^*(x)\,\mathrm{d}s$$

$$= n(t < \zeta)h'(x)\left(1 - \frac{h(x)}{h'(x)} \int_t^\infty \frac{q_s^*(x)}{h(x)}\right) \geq c_1 n(t < \zeta)h'(x)$$

whenever $x \leq x_0, t \geq t_0$. Here $c_1 = 1 - c(x_0, t_0) > 0$. To deal with the upper-bounds we use the fact that

$$n(t < \zeta) \leq \frac{1}{t} \int_0^t n(s < \zeta) \, ds.$$

This together with (4.13) enable us to write for every $t \geq t_0$ and $x \leq x_0$,

$$
\begin{aligned}
f_t^1(x) &= \left(\int_0^{t/2} + \int_{t/2}^t \right) n(s < \zeta) q_s^*(x) \, ds \\
&\leq 6 \frac{p_{t/12}(0)}{t} h(x) \int_0^{t/2} n(s < \zeta) \, ds + n(t/2 < \zeta) \int_{t/2}^t q_s^*(x) \, ds \\
&\leq 6 p_{t_0/12}(0) \frac{h(x)}{h'(x)} h'(x) \frac{1}{t} \int_0^t n(s < \zeta) \, ds + \frac{2h'(x)}{t} \int_0^t n(s < \zeta) \, ds.
\end{aligned}
\tag{4.22}
$$

We deal with the second part $f_t^2(x)$ similarly as in Theorem 2. We have

$$
\begin{aligned}
f_t^2(x) &\leq d h'(x) \frac{h(x)}{h'(x)} c_{t_0/2} n(t - t_0/2) \\
&\leq d h'(x) \sup_{x \leq x_0} \frac{h(x)}{h'(x)} c_{t_0/2} \frac{1}{t - t_0/2} \int_0^{t - t_0/2} n(s < \zeta) \, ds \\
&\leq d h'(x) \sup_{x \leq x_0} \frac{h(x)}{h'(x)} c_{t_0/2} \frac{2}{t} \int_0^t n(s < \zeta) \, ds.
\end{aligned}
\tag{4.23}
$$

Inequalities (4.22) and (4.23) prove the upper-bounds $f_t \leq c_2 n(t < \zeta) h'(x)$, with

$$c_2 = 6 p_{t_0/12}(0) \sup_{x \leq x_0} \frac{h(x)}{h'(x)} + 2 + 2 \sup_{x \leq x_0} \frac{h(x)}{h'(x)} c_{t_0/2}. \tag{4.24}$$

The second part of the thesis follows from the first one, (4.20) and the fact that $n(t < \zeta)$ is continuous and positive. Notice that positivity follows from monotonicity and the regular behaviour at infinity. $\qquad\square$

Proof of Proposition 2. (1) $\Rightarrow$ (2) Let x_0 be a point of continuity of h'. Fix $t > 0$ and take $\varepsilon > 0$. Since $q_s^*(x)$ is continuous, it is enough to show that the integral part $f_t^1(x) := \int_0^t n(t - s < \zeta) q_s^*(x) \, ds$ of (3.2) is continuous in x at x_0. Moreover, for every $t_0 < t$ we can write

$$f_t^1(x) = \left(\int_0^{t_0} + \int_{t_0}^t \right) n(t - s < \zeta) q_s^*(x) \, ds := k_{t_0}^1(x) + k_{t_0}^2(x).$$

Using the Lebesgue dominated convergence theorem, the fact that $n(t < \zeta)$ is integrable at zero and (4.13) we can easily show that $x \to k_{t_0}^2(x)$ is continuous on $(0, \infty)$ for every choice of $t_0 < t$. Moreover, the same arguments give continuity of the function $x \to \int_{t_0}^\infty q_s^*(x) \, ds$ for every positive t_0. We choose $t_0 < t/2$ such that

$$\int_0^{t_0} q_s^*(x_0) \, ds < \frac{\varepsilon}{4 n(t/2 < \zeta)},$$

where existence of such t_0 follows from integrability of $q_s^*(x_0)$ in s at 0. Since $x \to h'(x)$ is continuous at x_0 and the function $x \to \int_{t_0}^\infty q_s^*(x) \, ds$ is continuous on $(0, \infty)$, we can choose $\delta > 0$ such that for every $|x - x_0| < \delta$,

$$\int_0^{t_0} q_s^*(x) \, ds < \frac{\varepsilon}{2 n(t/2 < \zeta)}.$$

Writing for $|x - x_0| < \delta$,

$$\left| f_t^1(x) - f_t^1(x_0) \right| \le n(t - t_0 < \zeta)\left(\int_0^{t_0} q_s^*(x)\,\mathrm{d}s + \int_0^{t_0} q_s^*(x_0)\,\mathrm{d}s \right) + \left| k_{t_0}^2(x) - k_{t_0}^2(x_0) \right|$$

$$\le \varepsilon + \left| k_{t_0}^2(x) - k_{t_0}^2(x_0) \right|$$

and taking a limit, when $x \to x_0$ ends the proof in this case.

Since (3) follows directly from (2), it is enough to show (3) $\Rightarrow$ (1). Assume that for some $t > 0$ the function $x \to f_t(x)$ is continuous at x_0. We choose $t_0 > 0$ such that

$$\int_0^{t_0} n(t - s < \zeta)q_s^*(x_0)\,\mathrm{d}s < \varepsilon n(t < \zeta)/4,$$

for a given $\varepsilon > 0$. Our assumption implies that $x \to k_{t_0}^1(x)$ is continuous at x_0 and consequently, we can choose $\delta > 0$ such that

$$\int_0^{t_0} n(t - s < \zeta)q_s^*(x)\,\mathrm{d}s < \varepsilon n(t < \zeta)/2,$$

whenever $|x - x_0| < \delta$. Monotonicity of $n(\cdot < \zeta)$ entails,

$$\left| h'(x) - h'(x_0) \right| \le \left(\int_0^{t_0} q_s^*(x)\,\mathrm{d}s + \int_0^{t_0} q_s^*(x_0)\,\mathrm{d}s \right) + \left| \int_{t_0}^\infty q_s^*(x)\,\mathrm{d}s - \int_{t_0}^\infty q_s^*(x_0)\,\mathrm{d}s \right|$$

$$\le \varepsilon + \left| \int_{t_0}^\infty q_s^*(x)\,\mathrm{d}s - \int_{t_0}^\infty q_s^*(x_0)\,\mathrm{d}s \right|,$$

whenever $|x - x_0| < \delta$. Since the function $x \to \int_{t_0}^\infty q_s^*(x)\,\mathrm{d}s$ is continuous, the proof is complete. $\square$

Acknowledgements

References

[1] J. Bertoin. *Lévy Processes*. Cambridge Univ. Press, Cambridge, 1996. MR1406564

[2] L. Chaumont. On the law of the supremum of Lévy processes. *Ann. Probab.* **41** (3A) (2013) 1191–1217. MR3098676

[3] L. Chaumont and R. A. Doney. On Lévy processes conditionned to stay positive. *Electron. J. Probab.* **10** (28) (2005) 948–961. MR2164035

[4] L. Chaumont and R. A. Doney. Corrections to "On Lévy processes conditionned to stay positive." *Electron. J. Probab.* **13** (28) (2008) 1–4. MR2375597

[5] L. Chaumont and R. A. Doney. Invariance principles for local times at the maximum of random walks and Lévy processes. *Ann. Probab.* **38** (4) (2010) 1368–1389. MR2663630

[6] L. Chaumont and G. Uribe Bravo. Markovian bridges: Weak continuity and pathwise constructions. *Ann. Probab.* **39** (2) (2011) 609–647. MR2789508

[7] R. A. Doney. *Fluctuation Theory for Lévy Processes. Lectures from the 35th Summer School on Probability Theory Held in Saint-Flour, July 6–23, 2005. Lecture Notes in Math.* **1897**. Springer, Berlin, 2007. With a foreword by Jean Picard. MR2320889

[8] R. A. Doney and V. Rivero. Asymptotic behaviour of first passage time distributions for Lévy processes. *Probab. Theory Related Fields* **157** (2013) 1–45. MR3101839

[9] R. A. Doney and M. S. Savov. The asymptotic behavior of densities related to the supremum of a stable process. *Ann. Probab.* **38** (1) (2010) 316–326. MR2599201

[10] R. K. Getoor. Excursions of a Markov process. *Ann. Probab.* **7** (2) (1979) 244–266. MR0525052

[11] P. E. Greenwood and A. A. Novikov. One-sided boundary crossing for processes with independent increments. *Teor. Veroyatn. Primen.* **31** (2) (1986) 266–277. MR0850987

[12] F. B. Knight. The uniform law for exchangeable and Lévy process bridges. *Hommage à P. A. Meyer et J. Neveu. Astérisque* **236** (1996) 171–188. MR1417982

[13] A. Kuznetsov, A. E. Kyprianou and V. Rivero. The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II* 97–186. *Lecture Notes in Math.* **2061**. Springer, Heidelberg, 2012. MR3014147

[14] A. E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Universitext. Springer, Berlin, 2006. MR2250061

[15] M. Kwaśnicki, J. Małecki and M. Ryznar. Suprema of Lévy processes. *Ann. Probab.* **41** (3B) (2013) 2047–2065. MR3098066

[16] P. Lévy. *Processus Stochastiques et Mouvement Brownien*. Reprint of the second (1965) edition. Les Grands Classiques Gauthier-Villars, Paris. Éditions Jacques Gabay. Sceaux, 1992. MR0190953

[17] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge, 2013. MR3185174

[18] R. Schilling, R. Song and Z. Vondraček. *Bernstein Functions: Theory and Applications*. *Studies in Math.* **37**. De Gruyter, Berlin, 2010. MR2598208

[19] M. Sharpe. Supports of convolution semigroups and densities. In *Probability Measures on Groups and Related Structures XI* (*Oberwolfach*, *1994*) 364–369. World Sci. Publ., River Edge, NJ, 1995. MR1414946

[20] M. L. Silverstein. Classification of coharmonic and coinvariant functions for a Lévy process. *Ann. Probab.* **8** (1980) 539–575. MR0573292

[21] G. Uribe Bravo. Bridges of Lévy processes conditioned to stay positive. *Bernoulli* **20** (1) (2014) 190–206. MR3160578