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Pathwise solvability of stochastic integral equations with generalized drift and non-smooth dispersion functions

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Abstract. We study one-dimensional stochastic integral equations with non-smooth dispersion coëfficients, and with drift components that are not restricted to be absolutely continuous with respect to Lebesgue measure. In the spirit of Lamperti, Doss and Sussmann, we relate solutions of such equations to solutions of certain ordinary integral equations, indexed by a generic element of the underlying probability space. This relation allows us to solve the stochastic integral equations in a pathwise sense.

Résumé. Nous étudions des équations intégrales stochastiques unidimensionnelles avec coefficient de diffusion non-régulier, et avec termes de dérive non nécessairement absolument continus par rapport à la mesure de Lebesgue. En s'inspirant de Lamperti, Doss et Sussmann, la résolution de ces équations se ramène à la résolution de certaines équations intégrales ordinaires, paramétrées par un élément ω variant dans l'espace de probabilité de base. Ce lien nous permet de résoudre les équations intégrales stochastiques d'une façon trajectorielle.

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1. Introduction

Stochastic integral equations (SIEs) are powerful tools for modeling dynamical systems subject to random perturbations. Any such equation has two components: a stochastic integral with respect to a process that models the "underlying noise" of the system, and a drift term that models some "trend." In many applications, the drift term is assumed to be absolutely continuous with respect to Lebesgue measure on the real line. However, motivated by the pioneering work of Walsh [68] and Harrison and Shepp [38] on the "skew Brownian motion," several authors have studied SIEs without such a continuity assumption in quite some generality, beginning with Stroock and Yor [64], Le Gall [48], Barlow and Perkins [3], and Engelbert and Schmidt [23]. In the years since, Engelbert and Schmidt [24], Engelbert [22], Flandoli et al. [32,33], Bass and Chen [4], Russo and Trutnau [60], and Blei and Engelbert [11,12] have provided deep existence and uniqueness results about such equations.

In this present work we extend the *pathwise approach* taken by Lamperti [47], Doss [20] and Sussmann [65], who focused on the case of an absolutely continuous drift and of a smooth (C^2) dispersion function, to one-dimensional SIEs with generalized drifts and with strictly positive dispersions which, together with their reciprocals, are of finite first variation on compact subsets of the state space. This pathwise approach proceeds via a suitable transformation of the underlying SIE which replaces the stochastic integral component by the process that models the driving noise (a Brownian motion or, more generally, a continuous semimartingale); this noise process enters the transformed equation, now an *Ordinary* (that is, non-stochastic) *Integral Equation* (OIE), only parametrically through its coëfficients. Such a transformation emphasizes the pathwise character of the SIE, that is, highlights the representation of the solution process ("output") as a measurable and non-anticipative functional of the driving noise ("input"). The pathwise point of view allows the modeler, who tries to solve the SIE, to construct an input-output map without having to worry about stochastic integration, which notoriously obscures the dependence of the solution path on the Brownian (or semimartingale) path, due to the "$\mathbb{L}^2$-smearing" of stochastic integration.

We emphasize here also the reverse implication: If one can show, say via probabilistic methods, that a certain SIE has a solution, then this directly yields existence results for certain OIEs. Such OIEs often may have very irregular input functions, so that such existence results would be very hard to obtain via standard analytical arguments.

Overview

Section 2 provides the setup, and Section 3 links an SIE with generalized drift to a collection of related OIEs. While we rely on some rather weak assumptions on the dispersion function, such as time-homogeneity and finite variation on compact subsets of the state space, we make hardly any assumptions on the drift function; we allow it, for example, to depend explicitly on the input noise. We discuss also the Stratonovich version of the SIE with generalized drift and non-smooth dispersion function under consideration.

Section 4 provides several examples, primarily in the context of three specific setups: Section 4.1 discusses the case when the drift does not depend on the solution process of the SIE itself; Section 4.2 treats the situation when the drift is absolutely continuous with respect to Lebesgue measure; and Section 4.3 treats the case of time-homogeneous coëfficients when the input process is a Brownian motion. Finally, Section 4.4 provides an example related to skew Brownian motion.

Section 5 presents a comparison result in the spirit of Section VI.1 in Ikeda and Watanabe [40] but using entirely different methods and with quite broader scope. Finally, Section 6 establishes under appropriate conditions the continuity of the input–output map in the sense of Wong and Zakai [71], Wong and Zakai [72] for equations of the type studied in this paper. An Appendix summarizes aspects concerning the regularization of OIEs by means of additive noise.

2. Setup, notation, and examples

2.1. *Path space*

We place ourselves on the canonical path space $\Omega = C([0,\infty);\mathbb{R})$ of continuous functions $\omega : [0,\infty) \to \mathbb{R}$, equipped with the topology of uniform convergence on compact sets, denote by $W(\cdot) = \{W(t)\}_{0\le t<\infty}$ the coördinate mapping process $W(t,\omega) = \omega(t), 0 \le t < \infty$ for all $\omega \in \Omega$, and consider the filtration $\mathbb{F}^W = \{\mathcal{F}^W(t)\}_{0\le t<\infty}$ with $\mathcal{F}^W(t) = \sigma(W(s), 0 \le s \le t)$ generated by $W(\cdot)$; this filtration is left-continuous, but not right-continuous. We introduce its right-continuous version $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ by setting $\mathcal{F}(t) := \bigcap_{s>t} \mathcal{F}^W(s)$ for all $t \in [0,\infty)$, and define $\mathcal{F} \equiv \mathcal{F}^W(\infty) := \bigvee_{0\le t<\infty} \mathcal{F}^W(t)$.

We shall consider, on the measurable space $(\Omega, \mathcal{F})$, the collection $\mathfrak{P}$ of *semimartingale measures*, that is, of probability measures $\mathbb{P}$ under which the canonical process $W(\cdot)$ is a semimartingale in its own filtration $\mathbb{F}^W$. The Wiener measure $\mathbb{P}_*$ is the prototypical element of $\mathfrak{P}$; and under every $\mathbb{P} \in \mathfrak{P}$, the canonical process can be thought of as the "driving noise" of the system that we shall consider.

We fix an open interval $\mathcal{I} = (\ell, r)$ of the real line, along with some starting point $x_0 \in \mathcal{I}$; the interval $\mathcal{I}$ will be the state-space of the solutions to the equations which are the subject of this work. We shall denote by $\overline{\mathcal{I}}$ the one-point compactification of $\mathcal{I}$, that is, $\overline{\mathcal{I}} = \mathcal{I} \cup \{\Delta\}$ for some $\Delta \notin \mathcal{I}$. We shall consider also the space $\Xi = C_a([0,\infty);\overline{\mathcal{I}})$ of $\overline{\mathcal{I}}$-valued continuous functions that get absorbed when they hit the "cemetery point" Δ. We use for all $n \in \mathbb{N}$, $\mathrm{x} \in C_a([0,\infty);\overline{\mathcal{I}})$ the stop-rules

$$\mathcal{S}_n(\mathrm{x}) := \inf\big\{t \ge 0 \colon\ \mathrm{x}(t) \notin \big(\ell^{(n)}, r^{(n)}\big)\big\}. \tag{2.1}$$

Here $\{r_n\}_{n\in\mathbb{N}}$ (respectively, $\{\ell_n\}_{n\in\mathbb{N}}$) are some strictly increasing (respectively, decreasing) sequences satisfying $\ell < \ell_{n+1} < \ell_n < x_0 < r_n < r_{n+1} < r$ for all $n \in \mathbb{N}$, as well as $\lim_{n\uparrow\infty} \uparrow r_n = r$ and $\lim_{n\uparrow\infty} \downarrow \ell_n = \ell$. For later use we also introduce, for every path $\mathrm{x} \in \varXi$, the stop-rule

$$\mathcal{S}(\mathrm{x}) := \lim_{n\uparrow\infty} \uparrow \mathcal{S}_n(\mathrm{x}), \tag{2.2}$$

as well as the following quantities:

- the double sequence $\{\tau^{(i,n)}\}_{(i,n)\in\mathbb{N}_0^2}$ of stop-rules defined inductively by $\tau^{(0,n)}(\mathrm{x}) = 0$ and

  $$\tau^{(i+1,n)}(\mathrm{x}) = \inf\bigl\{t \geq \tau^{(i,n)}(\mathrm{x})\colon \bigl|\mathrm{x}(t) - \mathrm{x}\bigl(\tau^{(i,n)}(\mathrm{x})\bigr)\bigr| \geq 2^{-n}\bigr\}$$

  for all $(i,n) \in \mathbb{N}_0^2$; the quadratic variation $\langle \mathrm{x}\rangle(\cdot)$ of the path $\mathrm{x}(\cdot)$, defined as

  $$\langle \mathrm{x}\rangle(t) = \liminf_{n\uparrow\infty} \sum_{i\in\mathbb{N}} \bigl(\mathrm{x}\bigl(t \wedge \tau^{(i+1,n)}\bigr) - \mathrm{x}\bigl(t \wedge \tau^{(i,n)}\bigr)\bigr)^2, \quad 0 \leq t < \infty, \tag{2.3}$$

  formally with the convention $\Delta - \Delta = 0$; and
- the *right* local time $L^{\mathrm{x}}(\cdot, \xi)$ of the path $\mathrm{x}(\cdot)$ at the site $\xi \in \mathcal{I}$, defined as

  $$L^{\mathrm{x}}(T, \xi) = \limsup_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^T \mathbf{1}_{[\xi,\xi+\varepsilon)}\bigl(\mathrm{x}(t)\bigr)\, \mathrm{d}\langle \mathrm{x}\rangle(t), \quad 0 \leq T < \infty. \tag{2.4}$$

We denote by $X(\cdot) = \{X(t)\}_{0\leq t<\infty}$ the coordinate mapping process $X(t, \mathrm{x}) = \mathrm{x}(t)$, $0 \leq t < \infty$ for all $\mathrm{x} \in \varXi$, and introduce the filtration $\mathbb{F}^X = \{\mathcal{F}^X(t)\}_{0\leq t<\infty}$ with $\mathcal{F}^X(t) = \sigma(X(s), 0 \leq s \leq t)$ generated by this new canonical process $X(\cdot)$.

Let us recall from Definition 3.5.15 in Karatzas and Shreve [44] the notion of progressive measurability, and adapt it to the present circumstances. We say that a mapping $\mathfrak{M} : [0,\infty) \times C([0,\infty); \mathbb{R}) \times C_a([0,\infty); \overline{\mathcal{I}}) \to [-\infty, \infty]$ is *progressively measurable*, if for every $t \in [0,\infty)$ its restriction to $[0,t] \times C([0,\infty); \mathbb{R}) \times C_a([0,\infty); \overline{\mathcal{I}})$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}^W(t) \otimes \mathcal{F}^X(t)/\mathcal{B}([-\infty,\infty])$-measurable.

We observe that $(t, \mathrm{x}) \mapsto \langle \mathrm{x}\rangle(t)$ as in (2.3) is a progressively measurable functional on $[0,\infty) \times C_a([0,\infty); \overline{\mathcal{I}})$, and that so is the functional $(t, \mathrm{x}) \mapsto L^{\mathrm{x}}(t, \xi)$ in (2.4) for each given $\xi \in \mathcal{I}$. Finally, let us recall from Section 7.14 in Bichteler [10], Section II.a in Bertoin [9], and Corollary VI.1.9 in Revuz and Yor [59] that, for any continuous semimartingale $X(\cdot)$, its quadratic variation can be cast as $\langle X\rangle(\cdot) = X^2(\cdot) - X^2(0) - 2\int_0^\cdot X(t)\,\mathrm{d}X(t)$, and its *right* local time has the representation in (2.4).

2.2. *Ingredients of the stochastic integral equation*

In order to describe the stochastic integral equation under consideration, we place ourselves on the filtered measurable space $(\Omega, \mathcal{F})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\leq t<\infty}$. We shall fix throughout a measurable function $\mathfrak{s} : \mathcal{I} \to (0,\infty)$ with the property

$$\log \mathfrak{s}(\cdot) \text{ is left-continuous and of finite first variation on compact subsets of } \mathcal{I}. \tag{2.5}$$

We define

$$\widetilde{\ell}_n := -\int_{\ell_n}^{x_0} \frac{\mathrm{d}z}{\mathfrak{s}(z)}, \qquad \widetilde{r}_n := \int_{x_0}^{r_n} \frac{\mathrm{d}z}{\mathfrak{s}(z)}, \qquad \widetilde{\ell} = \lim_{n\uparrow\infty} \widetilde{\ell}_n, \qquad \widetilde{r} = \lim_{n\uparrow\infty} \widetilde{r}_n, \qquad \mathcal{J} := (\widetilde{\ell}, \widetilde{r}), \tag{2.6}$$

and the stop-rules

$$\widetilde{\mathcal{S}}_n(\mathrm{y}) := \inf\bigl\{t \geq 0\colon \mathrm{y}(t) \notin (\widetilde{\ell}_n, \widetilde{r}_n)\bigr\}, \qquad \widetilde{\mathcal{S}}(\mathrm{y}) := \lim_{n\uparrow\infty} \uparrow \widetilde{\mathcal{S}}_n(\mathrm{y}) \tag{2.7}$$

for all $n \in \mathbb{N}$ and $\mathrm{y} \in C_a([0,\infty); \overline{\mathcal{J}})$, where $C_a([0,\infty); \overline{\mathcal{J}})$ is defined in the same manner as $C_a([0,\infty); \overline{\mathcal{I}})$.

We shall also fix a progressively measurable mapping $\mathfrak{B}:[0,\infty)\times C([0,\infty);\mathbb{R})\times C_a([0,\infty);\overline{\mathcal{I}})\to\mathbb{R}$ with $\mathfrak{B}(0,\cdot,\cdot)=0$. For instance, $\mathfrak{B}(\cdot,\cdot,\cdot)$ can be of the form $\mathfrak{B}(T,\cdot,\mathrm{x})=\int_0^T\mathfrak{b}(\mathrm{x}(t))\,\mathrm{d}t$ for some bounded, measurable function $\mathfrak{b}:\overline{\mathcal{I}}\to\mathbb{R}$, for all $T\in[0,\infty)$ and $\mathrm{x}\in C_a([0,\infty);\overline{\mathcal{I}})$.

For any given semimartingale measure $\mathbb{P}\in\mathfrak{P}$, we shall be interested in the pathwise solvability of SIEs of the form

$$X(\cdot)=x_0+\int_0^{\cdot}\mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t)+\mathrm{d}\mathfrak{B}(t,W,X)\big]-\int_{\mathcal{I}}L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)} \tag{2.8}$$

on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P}),\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$, where the local time $L^X(\cdot,\cdot)$ is defined as in (2.4). From a systems-theoretic point of view, the process $X(\cdot)$ represents the "state" or "output" and the canonical process $W(\cdot)$ the "input" of the system with the dynamics of (2.8). The solution of this equation is defined in general only up until the *explosion time* $\mathcal{S}(X)\in(0,\infty]$.

More precisely, we have the following formal notions of solvability (Definitions 2.1 and 2.2 below).

Definition 2.1. *For any given semimartingale measure $\mathbb{P}\in\mathfrak{P}$, we shall call a process $X(\cdot)$ with values in $C_a([0,\infty);\overline{\mathcal{I}})$ a solution to the SIE* (2.8) *on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P}),\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$ up until a stopping time $\mathcal{T}$ with $0<\mathcal{T}\le\mathcal{S}(X)$, if the following conditions hold on the stochastic interval* $[0,\mathcal{T})$:

(i) *the process $X(\cdot)$ is a continuous $\mathbb{F}$-semimartingale*;
(ii) *the process $\mathfrak{B}(\cdot,W,X)$ is continuous and of finite first variation on compact subintervals*;
(iii) *the equation in* (2.8) *holds*.

Point (iii) in the above definition requires the notion of stochastic integral. We refer to Section I.4.d in Jacod and Shiryaev [41], and to Section 4.3 in Stroock and Varadhan [63], for the development of stochastic integration with respect to a right-continuous filtration which is not necessarily augmented by the null sets of the underlying probability measure.

Definition 2.2. *By* pathwise solvability *of the SIE* (2.8) *over a stochastic interval* $[0,\mathcal{T})$ *for some stopping time $0<\mathcal{T}\le\mathcal{S}(X)$, we mean the existence of a progressively measurable functional $\mathfrak{X}:[0,\infty)\times C([0,\infty);\mathbb{R})\to\overline{\mathcal{I}}$ such that*

(i) *the process $X(\cdot)=\mathfrak{X}(\cdot,W)$ solves on the interval* $[0,\mathcal{T})$ *the SIE* (2.8) *under any semimartingale measure* $\mathbb{P}\in\mathfrak{P}$; *and*
(ii) *the "input–output" mapping $(t,\omega)\mapsto\mathfrak{X}(t,\omega)$ is determined by solving, for each $\omega\in C([0,\infty);\mathbb{R})$, an appropriate Ordinary* (*or more generally, Functional*) *Integral Equation* (*OIE, or OFE*).

A solution as mandated by Definition 2.2 is obviously *strong*, in the sense that the random variable $X(t)=\mathfrak{X}(t,W)$ is measurable with respect to the sigma algebra $\mathcal{F}^W(t)$ for each $t\in[0,\infty)$; and no stochastic integration with respect to $W(\cdot)$ is necessary for computing the input–output mapping $\mathfrak{X}(\cdot,\cdot)$.

Remark 2.3. *The function $\mathfrak{s}(\cdot)$, under the requirements of* (2.5), *is bounded away from both zero and infinity over compact subsets of $\mathcal{I}$; this is because the condition* (2.5) *implies that the functions $\mathfrak{s}(\cdot)=\exp(\log(\mathfrak{s}(\cdot)))$ and $1/\mathfrak{s}(\cdot)=\exp(-\log(\mathfrak{s}(\cdot)))$ are left-continuous and of finite first variation on compact subsets of $\mathcal{I}$. It follows then from these considerations that*

$$\frac{1}{\mathfrak{s}(\cdot)}\ \text{is bounded on compact subsets of}\ \mathcal{I}. \tag{2.9}$$

If the function $\mathfrak{s}(\cdot)$ is bounded away from zero and of finite first variation on compact subsets of $\mathcal{I}$, then $\log\mathfrak{s}(\cdot)$ *is of finite first variation on compact subsets of $\mathcal{I}$. However, if $\mathfrak{s}(\cdot)$ is not bounded away from zero, this implication does not hold; for instance, with $\mathcal{I}=\mathbb{R}$, and $\mathfrak{s}(x)=1$ for all $x\le 0$, and $\mathfrak{s}(x)=x$ for all $x>0$, the function $\mathfrak{s}(\cdot)$ is of finite first variation on compact subsets of $\mathcal{I}$, but* $\log\mathfrak{s}(\cdot)$ *is not. Moreover, in the setting under consideration, the process $\mathfrak{s}(X(\cdot))$ is integrable with respect to both $\mathfrak{B}(\cdot,W,X)$ and the driving semimartingale $W(\cdot)$.*

Remark 2.4. *Using the property* $\int_{x_0}^{\cdot} \mathfrak{s}(\xi)\,\mathrm{d}(1/\mathfrak{s}(\xi)) + \int_{x_0}^{\cdot} (1/\mathfrak{s}(\xi+))\,\mathrm{d}\mathfrak{s}(\xi) \equiv 0$ *of Lebesgue–Stieltjes integration* (*e.g.*, *Proposition* 0.4.5 *in Revuz and Yor* [59]), *we see that the last term in* (2.8) *can be written equivalently as*

$$-\int_{\mathcal{I}} L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)} = \int_{\mathcal{I}} L^X(\cdot,\xi)\frac{\mathrm{d}\mathfrak{s}(\xi)}{\mathfrak{s}(\xi+)}. \tag{2.10}$$

Consider now the SIE (2.8) *with* $\mathfrak{B}(\cdot,\cdot,\cdot) \equiv 0$, *under a probability measure* $\mathbb{P} \in \mathfrak{P}$ *which renders the canonical process* $W(\cdot)$ *a local martingale. By virtue of* (2.10), *we have the expression*

$$L^X(T,\xi) - L^X(T,\xi-) = L^X(T,\xi)\frac{\mathfrak{s}(\xi+)-\mathfrak{s}(\xi)}{\mathfrak{s}(\xi+)}, \quad \mathbb{P}\text{-a.e. on the event } \{\mathcal{S}(X) > T\}$$

for the jump of the local time of $X(\cdot)$ *at the site* $\xi \in \mathcal{I}$ *and at time* $T \geq 0$; *here, we are using*

$$L^X(T,\xi) - L^X(T,\xi-) = \int_0^T \mathbf{1}_{\{X(t)=\xi\}}\,\mathrm{d}V(t),$$

a basic property of local time for a continuous semimartingale $X(\cdot) = X(0) + M(\cdot) + V(\cdot)$ (*for instance*, *Theorem* VI.1.7 *in Revuz and Yor* [59]). *This leads to the "balance equation"*

$$\mathfrak{s}(\xi+)L^X(T,\xi-) = \mathfrak{s}(\xi)L^X(T,\xi),$$

and expresses the symmetric local time $\widehat{L}^X(T,\xi) := (L^X(T,\xi) + L^X(T,\xi-))/2$ *as*

$$\widehat{L}^X(T,\xi) = \frac{1}{2}\left(1 + \frac{\mathfrak{s}(\xi)}{\mathfrak{s}(\xi+)}\right)L^X(T,\xi), \quad \mathbb{P}\text{-a.e. on the event } \{\mathcal{S}(X) > T\}$$

for each $T \geq 0$.

2.3. *Stratonovich interpretation*

The SIE (2.8) can be cast in the Stratonovich form

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big) \circ \mathrm{d}W(t) + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\,\mathrm{d}\mathfrak{B}(t,W,X) \tag{2.11}$$

when the dispersion fuction $\mathfrak{s}(\cdot)$ is the *difference of two convex functions*, i.e., can be written as the primitive of some real-valued function $\mathfrak{r}(\cdot)$ with finite first variation on compact subsets of $\mathcal{I}$: namely,

$$\mathfrak{s}(x) = \mathfrak{s}(c) + \int_c^x \mathfrak{r}(\xi)\,\mathrm{d}\xi, \quad x \in \mathcal{I}$$

for some $c \in \mathcal{I}$. For convenience we shall adopt the convention that the function $\mathfrak{r}(\cdot)$ be left-continuous.

Indeed, in this case the process $\mathfrak{s}(X(\cdot))$ is a continuous semimartingale with decomposition

$$\begin{aligned}\mathfrak{s}\big(X(\cdot)\big) - \mathfrak{s}(x_0) &= \int_0^{\cdot} \mathfrak{r}\big(X(t)\big)\,\mathrm{d}X(t) + \int_{\mathcal{I}} L^X(\cdot,\xi)\,\mathrm{d}\mathfrak{r}(\xi) \\ &= \int_0^{\cdot} \mathfrak{r}\big(X(t)\big)\mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t) + \mathrm{d}\mathfrak{B}(t,W,X)\big] + \int_{\mathcal{I}} L^X(\cdot,\xi)\left[\mathrm{d}\mathfrak{r}(\xi) - \mathfrak{r}(\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)}\right]\end{aligned}$$

by the generalized Itô–Tanaka formula. Therefore, the Stratonovich and Itô integrals are then related via

$$\begin{aligned}\int_0^{\cdot} \mathfrak{s}\big(X(t)\big) \circ \mathrm{d}W(t) - \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\,\mathrm{d}W(t) &= \frac{1}{2}\int_0^{\cdot} \mathfrak{r}\big(X(t)\big)\mathfrak{s}\big(X(t)\big)\,\mathrm{d}\langle W\rangle(t) = \frac{1}{2}\int_0^{\cdot} \frac{\mathfrak{r}(X(t))}{\mathfrak{s}(X(t))}\,\mathrm{d}\langle X\rangle(t) \\ &= \int_{\mathcal{I}} L^X(\cdot,\xi)\frac{\mathfrak{r}(\xi)}{\mathfrak{s}(\xi)}\,\mathrm{d}\xi = \int_{\mathcal{I}} L^X(\cdot,\xi)\frac{\mathrm{d}\mathfrak{s}(\xi)}{\mathfrak{s}(\xi)}\end{aligned}$$

$$= -\int_{\mathcal{I}} L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)},$$

and (2.11) follows from (2.8). We have used here the occupation-time-density property of semimartingale local time; see, for instance, pp. 224–225 in Karatzas and Shreve [44], as well as Definition 3.3.13 there. These considerations allow the interpretation of the last integral in (2.8) as a "singular Stratonovich-type correction term."

3. Pathwise solvability

The possibility that an SIE such as that of (2.8) might be solvable pathwise, is suggested by the following observation: If we work under the Wiener measure $\mathbb{P}_*$, if the function $\mathfrak{s}(\cdot)$ is continuous and continuously differentiable, and if $\mathfrak{B}(T,\cdot,\mathrm{x}) = \int_0^T \mathfrak{b}(t,\mathrm{x}(t))\,\mathrm{d}t$ for some bounded measurable function $\mathfrak{b}:[0,\infty)\times\overline{\mathcal{I}}\to\mathbb{R}$ for all $T\geq 0$ and $\mathrm{x}\in C_a([0,\infty),\overline{\mathcal{I}})$, then, on the strength of the occupation-time-density property of semimartingale local time, the corresponding SIE

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t) + \mathfrak{b}\big(t,X(t)\big)\,\mathrm{d}t\big] - \int_{\mathcal{I}} L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)} \tag{3.1}$$

of (2.8) takes the familiar form

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\left[\mathrm{d}W(t) + \left(\mathfrak{b}\big(t,X(t)\big) + \frac{1}{2}\mathfrak{s}'\big(X(t)\big)\right)\mathrm{d}t\right]; \tag{3.2}$$

whereas, if the function $\mathfrak{s}(\cdot)$ is twice continuously differentiable, this equation can be cast in terms of Stratonovich stochastic integration as

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\circ\mathrm{d}W(t) + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\mathfrak{b}\big(t,X(t)\big)\,\mathrm{d}t.$$

From the results of Doss [20] and Sussmann [65] we know that, at least in the case $\mathcal{I}=\mathbb{R}$, solving this latter SIE (3.2) amounts to solving pathwise an ordinary integral equation in which the source of randomness, that is, the $\mathbb{P}_*$-Brownian motion $W(\cdot)$, appears only parametrically through its coëfficients, *not in terms of stochastic integration*; see the OIE (3.12) below. This "classical" theory is also covered in books, for instance in Section III.2 of Ikeda and Watanabe [40], Section 5.2.D in Karatzas and Shreve [44], or Chapter 2 in Lyons and Qian [51].

3.1. *Basic properties of the space transformation*

For each $c\in\mathcal{I}$, we define the strictly increasing function $H_c:(\ell,r)\to(-\infty,\infty)$ by

$$H_c(x) := \int_c^x \frac{\mathrm{d}z}{\mathfrak{s}(z)},\quad x\in\mathcal{I}. \tag{3.3}$$

Here and in what follows, for the context of Lebesgue–Stieltjes integration, we define $\int_y^y f(z)\,\mathrm{d}z = 0$ as well as $\int_y^x f(z)\,\mathrm{d}z = -\int_x^y f(z)\,\mathrm{d}z$ for all $(x,y)\in\mathbb{R}^2$ with $x<y$ and an appropriate function f. We note that the function $H_c(\cdot)$ is indeed well-defined for all $c\in\mathcal{I}$, thanks to (2.9). Next, for each $c\in\mathcal{I}$, we set

$$\widetilde{\ell}(c) := H_c(\ell+) := \lim_{x\downarrow\ell} H_c(x)\in[-\infty,\infty),\qquad \widetilde{r}(c) := H_c(r-):=\lim_{x\uparrow r} H_c(x)\in(-\infty,\infty]. \tag{3.4}$$

We note $\mathcal{J} = (\widetilde{\ell}(x_0),\widetilde{r}(x_0))$ in the notation of (2.6), and consider the domain

$$\mathcal{D} := \big\{(c,w)\in\mathcal{I}\times\mathbb{R}:\ w\in\big(\widetilde{\ell}(c),\widetilde{r}(c)\big)\big\}. \tag{3.5}$$

For later use, we observe that

$$DH_c(x) = \frac{1}{\mathfrak{s}(x)}, \quad (c, x) \in \mathcal{I}^2, \tag{3.6}$$

where the symbol D stands for differentiation with respect to the second, parenthetical argument. The derivative is considered in its left-continuous version. The inverse function $\Theta_c : (\widetilde{\ell}(c), \widetilde{r}(c)) \to \mathcal{I}$ of $H_c(\cdot)$ is also well-defined for each $c \in \mathcal{I}$, thanks to the strict positivity of the function $\mathfrak{s}(\cdot)$, and satisfies $\Theta_c(0) = c$ as well as

$$D\Theta_c(w) = \mathfrak{s}\big(\Theta_c(w)\big), \quad (c, w) \in \mathcal{D}. \tag{3.7}$$

Once again, the derivative is considered in its left-continuous version. In particular, for each given $c \in \mathcal{I}$, the functions $x \mapsto H_c(x)$ and $w \mapsto \Theta_c(w)$ are strictly increasing.

Remark 3.1. *We note that, for each $c \in \mathcal{I}$, the function $\Theta_c(\cdot)$ solves the OIE*

$$\Theta_c(w) = c + \int_0^w \mathfrak{s}\big(\Theta_c(\zeta)\big)\,\mathrm{d}\zeta, \quad w \in \big(\widetilde{\ell}(c), \widetilde{r}(c)\big). \tag{3.8}$$

The function $\Theta_c(\cdot)$ is actually the only solution of the integral equation (3.8); *this is because any solution $\vartheta(\cdot)$ of the Equation* (3.8) *satisfies*

$$H_c\big(\vartheta(w)\big) = \int_{\vartheta(0)}^{\vartheta(w)} \frac{1}{\mathfrak{s}(\zeta)}\,\mathrm{d}\zeta = \int_0^w \frac{1}{\mathfrak{s}(\vartheta(u))}\,\mathrm{d}\vartheta(u) = w, \quad w \in \big(\widetilde{\ell}(c), \widetilde{r}(c)\big),$$

therefore $\vartheta(\cdot) \equiv \Theta_c(\cdot)$.

Let us also observe that the additivity property $H_c(\lambda) + H_\lambda(\xi) = H_c(\xi)$ for all $(c, \lambda, \xi) \in \mathcal{I}^3$, fairly evident from (3.3), translates into the *composition property*

$$\Theta_{\Theta_c(\gamma)}(w) = \Theta_c(\gamma + w), \quad c \in \mathcal{I}, \gamma \in \big(\widetilde{\ell}(c), \widetilde{r}(c)\big), \gamma + w \in \big(\widetilde{\ell}(c), \widetilde{r}(c)\big). \tag{3.9}$$

Finally we note that, thanks to the observations in Remark 2.3, both functions $H_c(\cdot)$ and $\Theta_c(\cdot)$ can be expressed as differences of two convex functions for each $c \in \mathcal{I}$.

3.2. *Preview of results*

The *Lamperti-type approach* reduces the problem of solving the SIE (2.8) under an arbitrary semimartingale measure $\mathbb{P} \in \mathfrak{P}$ to that of solving, for all "relevant" paths $\omega \in C([0, \infty); \mathbb{R})$, an Ordinary Functional Equation (OFE) of the form

$$\Gamma(t) = \mathfrak{B}\big(t, \omega, \Theta_{x_0}(\Gamma + \omega)\big), \quad 0 \le t < \widetilde{\mathcal{S}}(\Gamma + \omega) \tag{3.10}$$

in the notation of (2.7), and then produces a solution of the SIE (2.8) in the notation of (2.2), simply through the pointwise evaluation

$$X(t) := \Theta_{x_0}\big(\Gamma(t) + W(t)\big), \quad 0 \le t < \mathcal{S}(X) = \widetilde{\mathcal{S}}(\Gamma + W). \tag{3.11}$$

The *Doss–Sussmann-type approach* relies on the following observation. Given a function $\Gamma(\cdot)$ that satisfies the OFE (3.10), we can define the function $C(t) := \Theta_{x_0}(\Gamma(t))$, $0 \le t < \mathcal{S}(C) = \widetilde{\mathcal{S}}(\Gamma)$ and note that it satisfies an OIE of the form

$$C(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(C(t)\big)\,\mathrm{d}\mathfrak{B}\big(t, \omega, \Theta_{C(\cdot)}\big(\omega(\cdot)\big)\big), \tag{3.12}$$

thanks to the composition property in (3.9), at least up until the first time

$$\mathcal{R}(C,\omega) := \lim_{n\uparrow\infty} \uparrow \mathcal{R}_n(C,\omega) \tag{3.13}$$

the two-dimensional path $(C(\cdot), \omega(\cdot))$ exits the domain of (3.5); here we have denoted

$$\begin{aligned}\mathcal{R}_n(C,\omega) &:= \inf\big\{t \ge 0\colon\ C(t) \notin (\ell_n, r_n) \text{ or } \omega(t) \notin \big(H_{C(t)}(\ell_n), H_{C(t)}(r_n)\big)\big\}\\ &= \inf\big\{t \ge 0\colon\ C(t) \notin (\ell_n, r_n)\big\} \wedge \inf\big\{t \ge 0\colon\ \Theta_{C(t)}\big(\omega(t)\big) \notin (\ell_n, r_n)\big\}\\ &= \mathcal{S}_n(C) \wedge \mathcal{S}_n\big(\Theta_{C(\cdot)}\big(\omega(\cdot)\big)\big)\end{aligned} \tag{3.14}$$

in the manner of (2.1). Conversely, any $\mathbb{F}$-adapted solution $C(\cdot)$ to the OIE in (3.12) produces a solution of the SIE (2.8) simply through the pointwise evaluation

$$X(t) := \Theta_{C(t)}\big(W(t)\big), \quad 0 \le t < \mathcal{R}(C,W) = \mathcal{S}(X) \wedge \mathcal{S}(C); \tag{3.15}$$

the last equality here is obvious from (3.13), (3.14) and from the definition of the process $X(\cdot)$ in (3.15).

3.3. *Relating the SIE to a family of OIEs*

We are now ready to state and prove the first main result of this work.

Theorem 3.2 (A Lamperti-type result). *For any given semimartingale measure $\mathbb{P} \in \mathfrak{P}$, the following hold*:

(i) *Given any solution $X(\cdot)$ of the stochastic integral equation (2.8) on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ up until the explosion time $\mathcal{S}(X)$, the process*

$$Y(t) := H_{x_0}\big(X(t)\big), \quad 0 \le t < \mathcal{S}(X) \tag{3.16}$$

is well-defined up until its own explosion time $\widetilde{\mathcal{S}}(Y) = \mathcal{S}(X)$ as in (2.6), (2.2); and the process $\Gamma(\cdot) := Y(\cdot) - W(\cdot)$ is of finite first variation on compact intervals and solves the OFE

$$\Gamma(t) = \mathfrak{B}\big(t, W, \Theta_{x_0}(\Gamma + W)\big), \quad 0 \le t < \widetilde{\mathcal{S}}(\Gamma + W). \tag{3.17}$$

(ii) *Conversely, suppose we are given an $\mathbb{F}$-adapted process $\Gamma(\cdot)$ of finite first variation on compact intervals, defined up until the stopping time $\widetilde{\mathcal{S}}(\Gamma + W)$ and solving, for $\mathbb{P}$-almost every path $W(\cdot)$, the OFE of (3.17) up until $\widetilde{\mathcal{S}}(\Gamma + W)$.*

Then the process $X(\cdot) := \Theta_{x_0}(\Gamma(\cdot) + W(\cdot))$ is $\mathbb{F}$-adapted and solves the stochastic integral equation (2.8) on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$, up until the explosion time $\mathcal{S}(X) = \widetilde{\mathcal{S}}(\Gamma + W)$. In particular, if the process $\Gamma(\cdot)$ is $\mathbb{F}^W$-adapted, so is $X(\cdot)$.

Proof. We organize the proof in two steps. The first one is an analysis, showing the statement in part (i) of the theorem; the second one is a synthesis, proving the statement in part (ii).

Analysis: We start by assuming that such a solution $X(\cdot)$ to the SIE (2.8), as postulated in part (i) of the theorem, has been constructed on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ for the given semimartingale measure $\mathbb{P} \in \mathfrak{P}$, up until the explosion time $\mathcal{S}(X)$. Then the process $Y(\cdot) = H_{x_0}(X(\cdot))$ of (3.16) is well-defined up until the explosion time

$$\mathcal{S}(X) = \widetilde{\mathcal{S}}(Y) = \widetilde{\mathcal{S}}(\Gamma + W),$$

as follows from the definition of the function $H_{x_0}(\cdot)$ in (3.3). Finally, we recall that $H_{x_0}(x_0) = 0$ holds and that the function $1/\mathfrak{s}(\cdot)$ is of finite first variation on compact subsets of $\mathcal{I}$ (recall the discussion in Remark 2.3). Thus $H_{x_0}(\cdot)$

is the difference of two convex functions and the Itô–Tanaka rule (e.g., Theorem 3.7.1 in Karatzas and Shreve (1991)) gives

$$Y(T)=\int_0^T DH_{x_0}\big(X(t)\big)\,\mathrm{d}X(t)+\int_{\mathcal{I}} L^X(T,\xi)\,\mathrm{d}DH_{x_0}(\xi)\quad\text{on }\big\{\mathcal{S}(X)>T\big\}$$

for all $T\in[0,\infty)$. Now it is rather clear from (2.8) and (3.6) that the first of these integral terms is

$$\begin{aligned}&\int_0^T \frac{1}{\mathfrak{s}(X(t))}\cdot\mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t)+\mathrm{d}\mathfrak{B}(t,W,X)\big]-\int_0^T\frac{1}{\mathfrak{s}(X(t))}\int_{\mathcal{I}} L^X(\mathrm{d}t,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)}\\ &\quad=W(T)+\mathfrak{B}(T,W,X)-\int_{\mathcal{I}} L^X(T,\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)};\end{aligned}$$

and that the second integral term is $\int_{\mathcal{I}} L^X(T,\xi)\,\mathrm{d}(1/\mathfrak{s}(\xi))$. Combining these two terms we obtain

$$\Gamma(T)=Y(T)-W(T)=\mathfrak{B}(T,W,X)=\mathfrak{B}\big(T,W,\Theta_{x_0}(Y)\big)=\mathfrak{B}\big(T,W,\Theta_{x_0}(\Gamma+W)\big),$$

which yields all the claims in part (i) of the theorem.

Synthesis: We place ourselves on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P})$, $\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$ for the given semimartingale measure $\mathbb{P}\in\mathfrak{P}$. As postulated in part (ii) of the theorem, we assume that the OFE (3.17) has an $\mathbb{F}$-adapted solution $\Gamma(\cdot)$ up until the stopping time $\widetilde{\mathcal{S}}(\Gamma+W)$. With the process $X(\cdot)=\Theta_{x_0}(\Gamma(\cdot)+W(\cdot))$, it is clear that $\mathcal{S}(X)=\widetilde{\mathcal{S}}(\Gamma+W)$, and the Itô–Tanaka rule gives now

$$X(T)=x_0+\int_0^T D\Theta_{x_0}\big(\Gamma(t)+W(t)\big)\,\mathrm{d}\big(\Gamma(t)+W(t)\big)+\int_{\mathcal{J}} L^{\Gamma+W}(T,y)\,\mathrm{d}D\Theta_{x_0}(y)\tag{3.18}$$

on $\{\mathcal{S}(X)>T\}$, for all $T\in(0,\infty)$. On the strength of (3.7), the first integral in this expression is

$$\int_0^T \mathfrak{s}\big(\Theta_{x_0}\big(\Gamma(t)+W(t)\big)\big)\,\mathrm{d}\big[\mathfrak{B}(t,W,X)+W(t)\big]=\int_0^T\mathfrak{s}\big(X(t)\big)\,\mathrm{d}\big[\mathfrak{B}(t,W,X)+W(t)\big].$$

As for the second integral in (3.18), we recall the property

$$L^{\Gamma+W}(T,y)=\frac{1}{D^+\Theta_{x_0}(y)}L^X\big(T,\Theta_{x_0}(y)\big)$$

from Exercise 1.23 on p. 234 in Revuz and Yor [59]; we denote here by $D^+\Theta_{x_0}(\cdot)$ the *right*-derivative of the function $\Theta_{x_0}(\cdot)$, namely $D^+\Theta_{x_0}(y)=\mathfrak{s}(\Theta_{x_0}(y)+)$ from (3.7). These considerations allow us to cast the second integral in (3.18) as

$$\begin{aligned}\int_{\mathcal{J}} L^{\Gamma+W}(T,y)\,\mathrm{d}D\Theta_{x_0}(y)&=\int_{\mathcal{J}} L^X\big(T,\Theta_{x_0}(y)\big)\frac{\mathrm{d}D\Theta_{x_0}(y)}{D^+\Theta_{x_0}(y)}=\int_{\mathcal{J}} L^X\big(T,\Theta_{x_0}(y)\big)\frac{\mathrm{d}\mathfrak{s}(\Theta_{x_0}(y))}{\mathfrak{s}(\Theta_{x_0}(y)+)}\\&=\int_{\mathcal{I}} L^X(T,\xi)\frac{\mathrm{d}\mathfrak{s}(\xi)}{\mathfrak{s}(\xi+)}=-\int_{\mathcal{I}} L^X(T,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)},\end{aligned}$$

where the last equality follows from (2.10). All in all, we conclude that the process $X(\cdot)$ solves the SIE (2.8) on the stochastic interval $[0,\mathcal{S}(X))$. This completes the proof of part (ii) of the theorem. □

Remark 3.3 (Possible generalizations of Theorem 3.2). *We have assumed that the function* $\log\mathfrak{s}(\cdot)$ *is of finite first variation on compact subsets of* $\mathcal{I}$. *The question arises*: *How much of the pathwise approach of Theorem 3.2 goes through, if we only assume that the function* $1/\mathfrak{s}(\cdot)$ *is simply integrable on compact subsets of* $\mathcal{I}$? *As a first observation, the two functions* $H_c(\cdot)$ *and* $\Theta_c(\cdot)$ *will then not be expressible necessarily as differences of two convex functions*; *they will only be absolutely continuous with respect to Lebesgue measure. Therefore, by the arguments in Çinlar et al.* [15], *we cannot expect then the continuous process* $\Theta_{x_0}(\Gamma(\cdot)+W(\cdot))$ *to be a semimartingale.*

We also note that the second integral term in (2.8) *is not defined if the function* $\log \mathfrak{s}(\cdot)$ *is not of finite first variation. However, we might formally apply integration-by-parts to that integral and obtain an integral of Bouleau–Yor type, derived in Bouleau and Yor* [13]; *see, for example, Ghomrasni and Peskir* [37]. *The computations in Wolf* [70] *then will let us expect that* $\Theta_{x_0}(\Gamma(\cdot) + W(\cdot))$ *is a* local Dirichlet *process with a zero-quadratic variation term of Bouleau–Yor type. Dirichlet processes were introduced in Föllmer* [35] *and were studied by Bertoin* [8,9], *Fukushima et al.* [36], *among many others. For stochastic differential equations involving Dirichlet processes, we refer to Engelbert and Wolf* [25], *Flandoli et al.* [32,33] *and Coviello and Russo* [17].

The proof of Theorem 3.2 *relies on the Itô–Tanaka formula. Much work has been done on obtaining more general change-of-variable formulas that can accommodate Dirichlet processes as inputs and/or outputs. Here, we refer to Wolf* [69], *Dupoiron et al.* [21], *Bardina and Rovira* [2], *Lowther* [50], *and the many references therein. To the best of our knowledge, it is an open problem to connect these techniques and generalize our approach here to the situation when the function* $1/\mathfrak{s}(\cdot)$ *is only integrable on compact subsets of* $\mathcal{I}$. *A related open problem is to generalize the class of input processes* $W(\cdot)$ *from the class of all semimartingales to the larger class of all local Dirichlet processes. A first step in this direction, for smooth coefficients, was made in Errami et al.* [27], *and for so-called* weak Dirichlet processes *in Errami and Russo* [26].

Corollary 3.4. *We fix a semimartingale measure* $\mathbb{P} \in \mathfrak{P}$. *The SIE* (2.8) *has at most one solution, if and only if the OFE* (3.10) *has at most one* $\mathbb{F}$*-adapted solution for* $\mathbb{P}$*-almost every path* $W(\cdot)$. *Furthermore, the SIE* (2.8) *has a solution, if and only if the OFE* (3.10) *has an* $\mathbb{F}$*-adapted solution for* $\mathbb{P}$*-almost every path* $W(\cdot)$.

Corollary 3.5. *We assume that there exists an* $\mathbb{F}^W$*-adapted process* $\Gamma(\cdot)$, *defined up until the stopping time* $\widetilde{\mathcal{S}}(\Gamma + W)$ *of* (2.2), *of finite first variation on compact subintervals of* $[0, \widetilde{\mathcal{S}}(\Gamma + W))$ *and solving the OFE* (3.10) *for each* $\omega \in \Omega$. *Then Theorem* 3.2 *guarantees that the SIE* (2.8) *is pathwise solvable in the sense of Definition* 2.2, *with* $\mathfrak{X}(t, \omega) = \Theta_{x_0}(\Gamma(t) + \omega(t))$ *for all* $(t, \omega) \in [0, \infty) \times \Omega$.

Remark 3.6 (Pathwise stochastic integration). *In the setting of Corollary* 3.5, *the SIE* (2.8) *can be cast on the strength of Theorem* 3.2 *as*

$$\int_0^T \mathfrak{s}\big(\Theta_{x_0}\big(\Gamma(t) + W(t)\big)\big)\,\mathrm{d}W(t) = \Theta_{x_0}\big(\Gamma(T) + W(T)\big) - x_0 - \int_0^T \mathfrak{s}\big(\Theta_{x_0}\big(\Gamma(t) + W(t)\big)\big)\,\mathrm{d}\Gamma(t) + \int_{\mathcal{I}} L^{\Theta_{x_0}(\Gamma+W)}(T, \xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)}, \quad 0 \le T < \widetilde{\mathcal{S}}(\Gamma + W). \tag{3.19}$$

We note that the right-hand side of (3.19) *is defined path-by-path, and is an* $\mathbb{F}^W$*-adapted process. Moreover, these equalities hold under any semimartingale measure* $\mathbb{P} \in \mathfrak{P}$ (*at least* $\mathbb{P}$*-almost surely, as stochastic integrals are defined only thus*). *Consequently, the identification* (3.19) *corresponds to a pathwise definition of the stochastic integral on its left-hand side. This construction yields a version of the stochastic integral that is not only* $\mathbb{F}$*-adapted but also* $\mathbb{F}^W$*-adapted. We refer to Föllmer* [34], *Bichteler* [10], *Karandikar* [42], *Soner et al.* [61], *Nutz* [55], *and Perkowski and Prömel* [57] *for general results on pathwise stochastic integration.*

Corollary 3.7 (A Doss–Sussmann-type result). *For any given semimartingale measure* $\mathbb{P} \in \mathfrak{P}$, *the following hold*:

(i) *Given any solution* $X(\cdot)$ *of the stochastic integral equation* (2.8) *on the filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ *up until the explosion time* $\mathcal{S}(X)$, *the process*

$$C(t) := \Theta_{x_0}\big(\mathfrak{B}(t, W, X)\big), \quad 0 \le t < \mathcal{R}(C, W) = \mathcal{S}(X) \wedge \mathcal{S}(C)$$

is well-defined, and is the unique $\mathbb{F}$*-adapted solution of the ordinary integral equation*

$$C(t) = x_0 + \int_0^t \mathfrak{s}\big(C(u)\big)\,\mathrm{d}\mathfrak{B}(u, W, X), \quad 0 \le t < \mathcal{R}(C, W) = \mathcal{S}(X) \wedge \mathcal{S}(C). \tag{3.20}$$

Moreover, with this definition of the process $C(\cdot)$ *we have once again* (3.15), *namely*

$$X(t) = \Theta_{C(t)}\big(W(t)\big) = \Theta_{x_0}\big(\mathfrak{B}(t, W, X) + W(t)\big), \quad 0 \le t < \mathcal{R}(C, W) = \mathcal{S}(X) \wedge \mathcal{S}(C),$$

as well as the ordinary integral equation (3.12) *for* $\mathbb{P}$*-almost each path* $W(\cdot)$.

(ii) *Conversely, suppose we are given an* $\mathbb{F}$*-adapted process* $C(\cdot)$ *defined up until the explosion time* $\mathcal{S}(C)$ *as in* (2.2), *and solving the ordinary integral equation* (3.12) *for* $\mathbb{P}$*-almost every path* $W(\cdot)$ *up until the stopping time* $\mathcal{R}(C, W)$ *of* (3.13).

Then the process $X(\cdot) = \Theta_{C(\cdot)}(W(\cdot))$ *of* (3.15) *is* $\mathbb{F}$*-adapted and solves the stochastic integral equation* (2.8) *on the filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$, *up until the stopping time* $\mathcal{S}(C) \wedge \mathcal{S}(X) = \mathcal{R}(C, W)$. *In particular, if the process* $C(\cdot)$ *is* $\mathbb{F}^W$*-adapted, so is* $X(\cdot)$.

Proof. The corollary is a direct consequence of Theorem 3.2. More precisely, to establish part (ii), we define $\Gamma(\cdot) := H_{x_0}(C(\cdot))$ and note that $\Gamma(0) = 0$ and

$$\begin{aligned} \mathrm{d}\Gamma(t) &= \mathrm{d}H_{x_0}\big(C(t)\big) = \mathrm{d}\mathfrak{B}\big(t, W(\cdot), \Theta_{C(\cdot)}\big(W(\cdot)\big)\big) = \mathrm{d}\mathfrak{B}\big(t, W(\cdot), \Theta_{\Theta_{x_0}(\Gamma(\cdot))}\big(W(\cdot)\big)\big) \\ &= \mathrm{d}\mathfrak{B}\big(t, W(\cdot), \Theta_{x_0}\big(\Gamma(\cdot) + W(\cdot)\big)\big), \quad 0 \le t < \mathcal{R}(C, W). \end{aligned}$$

Here the second equality follows from (3.6) and (3.12), and the last equality from the composition property in (3.9). Therefore, the process $\Gamma(\cdot)$ satisfies the OFE in (3.17). Moreover, we note

$$X(\cdot) = \Theta_{C(\cdot)}\big(W(\cdot)\big) = \Theta_{\Theta_{x_0}(\Gamma(\cdot))}\big(W(\cdot)\big) = \Theta_{x_0}\big(\Gamma(\cdot) + W(\cdot)\big),$$

by the composition property (3.9), so Theorem 3.2(ii) applies. The identity $\mathcal{S}(C) \wedge \mathcal{S}(X) = \mathcal{R}(C, W)$ is clear from (3.13), (3.14), as already noted in Section 3.2.

For the statement in part (i) of the corollary, we appeal to part (i) in Theorem 3.2 and to the notation introduced there, and obtain the representations

$$\begin{aligned} &C(t) = \Theta_{x_0}\big(\mathfrak{B}(t, W, X)\big) = \Theta_{x_0}\big(\mathfrak{B}\big(t, W, \Theta_{x_0}(\Gamma + W)\big)\big) = \Theta_{x_0}\big(\Gamma(t)\big), \quad 0 \le t < \mathcal{R}(C, W); \\ &\Theta_{C(t)}\big(W(t)\big) = \Theta_{\Theta_{x_0}(\Gamma(t))}\big(W(t)\big) = \Theta_{x_0}\big(\Gamma(t) + W(t)\big) = X(t), \quad 0 \le t < \mathcal{R}(C, W), \end{aligned}$$

the latter on the strength of the composition property (3.9). These representations lead to the claims in part (i) of the corollary; the claimed uniqueness for the OIE (3.20) is argued as in Remark 3.1. □

Corollary 3.8 (Barrow–Osgood conditions). *We fix a semimartingale measure* $\mathbb{P} \in \mathfrak{P}$ *and impose the Barrow–Osgood conditions*

$$H_{x_0}(\ell+) = -\infty, \qquad H_{x_0}(r-) = \infty. \tag{3.21}$$

Then, in the notation of Theorem 3.2 *and Corollary* 3.7, $\mathcal{R}(C, W) = \mathcal{S}(X)$ *holds* $\mathbb{P}$*-almost surely. Moreover, we have*

$$\big\{\mathcal{S}(X) = \infty\big\} = \big\{[0, \infty) \ni t \longmapsto \mathfrak{B}(t, W, X) \text{ is real-valued}\big\}, \quad \textit{mod. } \mathbb{P}.$$

Proof. Under the conditions of (3.21) we have $\widetilde{\ell}(c) = -\infty$ and $\widetilde{r}(c) = \infty$ in (3.4) for every $c \in \mathcal{I}$; the function $\Theta_{x_0}(\cdot)$ is then defined on all of $\mathbb{R}$ and takes values in the interval $\mathcal{I} = (\ell, r)$; and the domain of (3.5) becomes the rectangle $\mathcal{D} = \{(c, w)\colon\ c \in \mathcal{I}, w \in \mathbb{R}\} = \mathcal{I} \times \mathbb{R}$. In particular, we then have $\mathcal{S}(X) = \mathcal{S}(C)$ and thus $\mathcal{R}(C, W) = \mathcal{S}(X)$ by the definition in (3.13). By Theorem 3.2(i) we have the representation $X(t) = \Theta_{x_0}(\mathfrak{B}(t, W, X) + W(t))$, which then yields the stated set equality. □

Remark 3.9. *One might wonder how the stopping times* $\mathcal{S}(C)$ *and* $\mathcal{S}(X)$ *of Corollary* 3.7 *relate to each other. In general, without the Barrow–Osgood conditions* (3.21), *anything is possible, as we illustrate here with a brief example where both events* $\{\mathcal{S}(X) < \mathcal{S}(C)\}$ *and* $\{\mathcal{S}(X) > \mathcal{S}(C)\}$ *have positive probabilities. We consider* $\mathcal{I} = (0, \infty)$, $\mathfrak{B}(t, \cdot, \cdot) = t$ *for all* $t \ge 0$, *and* $\mathfrak{s}(x) = x^2$ *for all* $x \in \mathcal{I}$. *Then*

$$H_c(x) = \frac{1}{c} - \frac{1}{x}, \quad (c, x) \in (0, \infty)^2 \quad \textit{and} \quad \Theta_c(w) = \left(\frac{1}{c} - w\right)^{-1}, \quad (c, w) \in \mathcal{D}$$

with $\mathcal{D} = \{(c,w) \in (0,\infty) \times \mathbb{R}\colon\ -\infty < w < 1/c\}$; *in particular,* $\mathcal{J} = (-\infty, 1/x_0)$, *and the second of the Barrow–Osgood conditions* (3.21) *fails. It follows that*

$$\Gamma(t) = t, \quad 0 \le t < \widetilde{\mathcal{S}}(\Gamma + W), \qquad C(t) = \left(\frac{1}{x_0} - t\right)^{-1}, \quad 0 \le t < \mathcal{S}(C) = \frac{1}{x_0},$$

$$X(t) = \Theta_{x_0}\big(\Gamma(t) + W(t)\big) = \left(\frac{1}{x_0} - t - W(t)\right)^{-1}, \quad 0 \le t < \mathcal{S}(X) = \widetilde{\mathcal{S}}(t + W).$$

Moreover, we have the representation $\mathcal{S}(X) = \inf\{t \ge 0\colon\ t + W(t) = 1/x_0\}$ *and it is clear that both events* $\{\mathcal{S}(X) < 1/x_0\}$ *and* $\{\mathcal{S}(X) > 1/x_0\}$ *have positive probabilities.*

4. Examples

We view the term corresponding to $\mathrm{d}\mathfrak{B}(\cdot, W, X)$ in (2.8) as a sort of generalized or "singular" drift that allows for both feedback effects (the dependence on the past and present of the "state" process $X(\cdot)$) and feed-forward effects (the dependence on the past and present of the "input" process $W(\cdot)$).

4.1. *The case of no dependence on the state process*

Let us consider mappings $\mathfrak{B}(\cdot,\cdot,\cdot)$ that do not depend on the state process $X(\cdot)$, namely

$$\mathfrak{B}(t,\omega,\mathrm{x}) = B(t,\omega), \quad (t,\omega,\mathrm{x}) \in [0,\infty) \times C\big([0,\infty);\mathbb{R}\big) \times C_a\big([0,\infty);\overline{\mathcal{I}}\big).$$

Here $B : [0,\infty) \times C([0,\infty);\mathbb{R}) \to \mathbb{R}$ is some progressively measurable mapping, such that $B(\cdot,\omega)$ is continuous and of finite first variation on compact intervals for all $\omega \in \Omega$. It should be stressed that $B(\cdot, W)$ *need not be absolutely continuous with respect to Lebesgue measure.*

We may define, for some bounded, measurable function $\beta : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, the progressively measurable functional

$$B(t,\omega) = \int_0^t \beta\big(s, \omega(s)\big)\,\mathrm{d}s, \quad 0 \le t < \infty, \omega \in C\big([0,\infty);\mathbb{R}\big).$$

We might be interested, for example, in a continuous semimartingale $X(\cdot)$ that is positively drifted whenever the driving noise is positive; in such a case, we might consider, for example, $\beta(t,\omega) = \mathbf{1}_{\{\omega(t)>0\}}$ for all $(t,\omega) \in [0,\infty) \times C([0,\infty);\mathbb{R})$. Alternatively, we may take $B(T,\omega) = \int_0^T \beta(t)\,\mathrm{d}t$, $(T,\omega) \in [0,\infty) \times \Omega$ for some integrable function $\beta : [0,\infty) \to \mathbb{R}$.

In this setting, the SIE (2.8) takes the form

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t) + \mathrm{d}B(t,W)\big] - \int_{\mathcal{I}} L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)}, \tag{4.1}$$

and the corresponding OFE (3.17) and OIE (3.20) become respectively

$$\Gamma(\cdot) = B(\cdot, W) \quad \text{and} \quad C(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(C(t)\big)\,\mathrm{d}B\big(t, W(t)\big).$$

In particular, under any semimartingale measure $\mathbb{P} \in \mathfrak{P}$, the solutions of (3.12) and (4.1) are then expressed as

$$\begin{aligned} &C(T) = \Theta_{x_0}\big(B(T,W)\big), \quad 0 \le T < \widetilde{\mathcal{S}}\big(B(\cdot,W)\big) \equiv \mathcal{S}(C), \\ &X(T) = \Theta_{x_0}\big(\mathfrak{B}(T,W) + W(T)\big), \quad 0 \le T < \widetilde{\mathcal{S}}\big(B(\cdot,W) + W\big) \equiv \mathcal{S}(X). \end{aligned} \tag{4.2}$$

Under the conditions (3.21), there are no explosions in the present context; i.e., $\mathcal{S}(X) = \mathcal{S}(C) = \infty$.

The state process $X(\cdot)$ in (4.2) is adapted not only to the right-continuous version $\mathbb{F}$ of the pure filtration $\mathbb{F}^W$, but also to this pure filtration itself. And if the function $B(\cdot,\cdot)$ does not depend on the second argument – i.e., if $B(\cdot,\omega) = B(\cdot)$ is equal to a given measurable function of finite first variation on compact subsets of $[0,\infty)$, for every $\omega \in \Omega$ – then for each $t \in [0,\infty)$ the random variables $X(t)$ and $W(t)$ are actually bijections of each other; to wit, $\sigma(X(t)) = \sigma(W(t))$ holds. Finally, we note that in the trivial case $B(\cdot,\cdot,\cdot) \equiv 0$ the solution in (4.2) simplifies further to $X(\cdot) = \Theta_{x_0}(W(\cdot))$.

The next example illustrates that the above arguments can be generalized somewhat.

Example 4.1. *In the notation of this subsection, let $A : \mathbb{R} \to (0,\infty)$ be a measurable function such that $1/A(\cdot)$ is integrable on compact subsets of $\mathbb{R}$. Moreover, we shall consider a continuous mapping $t \mapsto B(t,\omega)$ of finite first variation on compact intervals for all $\omega \in \Omega$. Let us fix*

$$\mathfrak{B}(T,\omega,\mathrm{x}) = \mathbf{1}_{\{\varrho(\omega,\mathrm{x})>T\}} \int_0^T A\big(H_{x_0}\big(\mathrm{x}(t)\big) - \omega(t)\big)\,\mathrm{d}B(t,\omega)$$

for all $(T,\omega,\mathrm{x}) \in [0,\infty) \times C([0,\infty);\mathbb{R}) \times C_a([0,\infty);\overline{\mathcal{I}})$, with

$$\varrho(\omega,\mathrm{x}) := \inf\left\{T \geq 0\colon \int_0^T A\big(H_{x_0}\big(\mathrm{x}(t)\big) - \omega(t)\big)\,\mathrm{d}|B|(t,\omega) = \infty\right\}.$$

Then the corresponding SIE (2.8) can be written as

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t) + A\big(H_{x_0}\big(\mathrm{x}(t)\big) - W(t)\big)\,\mathrm{d}B(t,W)\big] - \int_{\mathcal{I}} L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)} \tag{4.3}$$

and the corresponding OFE (3.10) as

$$\Gamma(\cdot) = \int_0^{\cdot} A\big(\Gamma(t)\big)\,\mathrm{d}B(t,W). \tag{4.4}$$

We define now the functions $\mathbf{H}_0(\cdot)$ and $\boldsymbol{\Theta}_0(\cdot)$ in the same way as $H_0(\cdot)$ and $\Theta_0(\cdot)$ but with $\mathfrak{s}(\cdot)$ replaced by $A(\cdot)$. The arguments in Remark 3.1 show that the unique solution of the Equation (4.4) is

$$\Gamma(t) = \boldsymbol{\Theta}_0\big(B(t,W)\big), \quad 0 \leq t < \widetilde{\mathbf{S}}\big(B(\cdot,W)\big)$$

with

$$\widetilde{\mathbf{S}}\big(B(T,\omega)\big) := \inf\left\{t \geq 0\colon\ B(t,\omega) \notin \left(-\int_{-\infty}^0 \frac{\mathrm{d}z}{A(z)}, \int_0^\infty \frac{\mathrm{d}z}{A(z)}\right)\right\}$$

in the manner of the stop-rule in (2.7); and that the SIE (4.3) has then a unique $\mathbb{F}^W$-adapted solution under each probability measure $\mathbb{P} \in \mathfrak{P}$, namely

$$X(t) = \Theta_{x_0}\big(\boldsymbol{\Theta}_0\big(B(t,W)\big) + W(t)\big), \quad 0 \leq t < \widetilde{\mathcal{S}}\big(\boldsymbol{\Theta}_0\big(B(\cdot,W)\big) + W\big) = \mathcal{S}(X). \tag{4.5}$$

For instance, let us consider the case $\mathcal{I} = (0,\infty)$, $x_0 = 1$, and $\mathfrak{s}(x) = x$ for all $x \in \mathcal{I}$. Then we have $H_{x_0}(x) =$ $\log(x)$ for all $x \in \mathcal{I}$, and the Equation of (4.3) simplifies to

$$X(\cdot) = 1 + \int_0^{\cdot} X(t)\left[\mathrm{d}W(t) + A\big(\log\big(X(t)\big) - W(t)\big)\,\mathrm{d}B(t,W) + \frac{1}{2}\,\mathrm{d}\langle W\rangle(t)\right]. \tag{4.6}$$

This SIE has then a unique $\mathbb{F}^W$-adapted solution under each probability measure $\mathbb{P} \in \mathfrak{P}$, given by (4.5) as

$$X(t) = \exp\big(\boldsymbol{\Theta}_0\big(B(t,W)\big) + W(t)\big), \quad 0 \leq t < \widetilde{\mathbf{S}}\big(B(\cdot,W)\big) = \mathcal{S}(X). \tag{4.7}$$

More specifically, let us consider the case $A(x) = \exp(-x)$ *for all* $x \in \mathbb{R}$. *Then we have* $\boldsymbol{\Theta}_0(y) = \log(1+y)$ *for all* $y \in (-1,\infty)$, *and the SIE* (4.6) *simplifies to*

$$X(\cdot) = 1 + \int_0^{\cdot} X(t)\,\mathrm{d}W(t) + \int_0^{\cdot} \exp\bigl(W(t)\bigr)\,\mathrm{d}B(t,W) + \frac{1}{2}\int_0^{\cdot} X(t)\,\mathrm{d}\langle W\rangle(t);$$

from (4.7), *the unique solution of this stochastic integral equation is* $X(t) = (1 + B(t,W))\exp(W(t))$, $0 \le t < \infty$, *and it is easy to check that this expression indeed solves the equation.*

4.2. *Absolutely continuous drifts*

Another very important example for the term $\mathrm{d}\mathfrak{B}(\cdot,W,X)$ involves a measurable function $\mathfrak{b}:[0,\infty)\times\mathbb{R}\times\mathcal{I}\to\mathbb{R}$ such that, for all $(T,K)\in(0,\infty)^2$, the functions

$$\overline{\mathfrak{b}}_K(\cdot) := \sup_{(t,w)\in[0,T]\times[-K,K]} \bigl|\mathfrak{b}(t,w,\cdot)\bigr| \text{ are integrable on compact subsets of } \mathcal{I};$$

see Engelbert [22]. For any given $(\omega,\mathrm{x}) \in C([0,\infty);\mathbb{R}) \times C_a([0,\infty);\overline{\mathcal{I}})$, we define

$$\mathfrak{B}(T,\omega,\mathrm{x}) := \mathbf{1}_{\{\varrho(\omega,\mathrm{x})>T\}} \int_0^T \mathfrak{b}\bigl(t,\omega(t),\mathrm{x}(t)\bigr)\,\mathrm{d}t$$

for all $T \ge 0$ along with the stop-rule

$$\varrho(\omega,\mathrm{x}) := \inf\left\{T \ge 0\colon \int_0^T \bigl|\mathfrak{b}\bigl(t,\omega(t),\mathrm{x}(t)\bigr)\bigr|\,\mathrm{d}t = \infty\right\}.$$

The SIE (2.8) takes then the form

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\bigl(X(t)\bigr)\bigl[\mathrm{d}W(t) + \mathfrak{b}\bigl(t,W(t),X(t)\bigr)\,\mathrm{d}t\bigr] - \int_{\mathcal{I}} L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)}; \tag{4.8}$$

and when $\mathfrak{b}(\cdot,\cdot,\cdot)$ does not depend on the second argument, this equation simplifies further to the SIE (3.1). In the context of this example, (3.17) becomes an OIE of the form

$$\Gamma(T) = \int_0^T \mathfrak{b}\bigl(t, W, \Theta_{x_0}\bigl(\Gamma(t)+W(t)\bigr)\bigr)\,\mathrm{d}t, \quad 0 \le T < \widetilde{\mathcal{S}}(\Gamma+W) = \mathcal{S}(X). \tag{4.9}$$

On the other hand, the OIE (3.12) corresponding to the SIE (4.8) takes the form

$$C(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\bigl(C(t)\bigr)\mathfrak{b}\bigl(t,\omega(t),\Theta_{C(t)}\bigl(\omega(t)\bigr)\bigr)\,\mathrm{d}t. \tag{4.10}$$

Remark 4.2. *Under the Barrow–Osgood conditions of* (3.21) *we have* $\mathcal{S}(C) = \mathcal{S}(X)$ *by Corollary* 3.8 *and also* $\mathcal{S}(X) = \varrho(W,X)$, *since those conditions imply*

$$\widetilde{\mathcal{S}}(\Gamma+W) = \widetilde{\mathcal{S}}(\Gamma) = \varrho(W,X).$$

In particular, if the drift function $\mathfrak{b}(\cdot,\cdot,\cdot)$ *is bounded, then all the stopping times in the above display are infinite.*

Example 4.3 (A counterexample). *We cannot expect the SIE* (2.8), *or for that matter the OIE* (3.12), *to admit* $\mathbb{F}$-*adapted solutions for a general progressively measurable functional* $\mathfrak{B}(\cdot,\cdot,\cdot)$. *For instance, take* $\mathbb{P} = \mathbb{P}_*$ *to be Wiener measure, take* $\mathfrak{s}(\cdot) \equiv 1$, *and consider*

$$\mathfrak{B}(T,\omega,\mathrm{x}) = \int_0^T \mathrm{b}(t,\mathrm{x})\,\mathrm{d}t, \quad (T,\omega,\mathrm{x}) \in [0,\infty) \times C\bigl([0,\infty);\mathbb{R}\bigr) \times C\bigl([0,\infty);\mathbb{R}\bigr)$$

for the bounded drift

$$\mathrm{b}(t,\mathrm{x}) = \left\{\frac{\mathrm{x}(t_k)-\mathrm{x}(t_{k-1})}{t_k - t_{k-1}}\right\}, \quad t_k < t \le t_{k+1}; \qquad \mathrm{b}(t,\mathrm{x}) = 0 \quad \textit{for } t=0, t>1$$

of Tsirel'son [66]. *Here* $\{\xi\}$ *stands for the fractional part of the number* $\xi \in \mathbb{R}$, *and* $(t_k)_{k\in-\mathbb{N}}$ *is a strictly increasing sequence of numbers with* $t_0 = 1$, *with* $0 < t_k < 1$ *for* $k < 0$, *and with* $\lim_{k\downarrow-\infty} t_k = 0$. *It was shown in the landmark paper of Tsirel'son* [66] (*see also pages* 195–197 *of Ikeda and Watanabe* [40] *or pages* 392–393 *of Revuz and Yor* [59]) *that the resulting SIE*

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathrm{b}(t,X)\,\mathrm{d}t + W(\cdot)$$

in (2.8), *driven by the* $\mathbb{P}_*$*-Brownian motion* $W(\cdot)$, *admits a weak solution which is unique in distribution*, *but no strong solution*; *see also the deep work of Beneš* [6,7] *for far-reaching generalizations and interpretations of Tsirelson's result*. *As a result*, *the OIE*

$$C(\cdot) = x_0 + \int_0^{\cdot} \mathrm{b}(t, W + C)\,\mathrm{d}t$$

of (3.12) *cannot possibly admit an* $\mathbb{F}$*-adapted solution in this case.*

4.3. *The time-homogenous case*

We consider a measurable function $\mathfrak{b} : \mathcal{I} \to \mathbb{R}$ which is integrable on compact subsets of $\mathcal{I}$, as well as a signed measure $\boldsymbol{\mu}$ on the Borel sigma algebra $\mathcal{B}(\mathcal{I})$ which is finite on compact subsets of $\mathcal{I}$. As in Section 4.2, we then introduce the progressively measurable mapping

$$\mathfrak{B}(T,\omega,\mathrm{x}) \equiv \mathfrak{B}(T,\mathrm{x}) := \mathbf{1}_{\{\boldsymbol{\varrho}(\mathrm{x})>T\}}\left(\int_0^T \mathfrak{b}\big(\mathrm{x}(t)\big)\,\mathrm{d}t + \int_{\mathcal{I}} L^{\mathrm{x}}(T,\xi)\frac{\boldsymbol{\mu}(\mathrm{d}\xi)}{\mathfrak{s}(\xi)}\right), \quad T\in[0,\infty)$$

as well as the stop-rule

$$\boldsymbol{\varrho}(\omega,\mathrm{x}) \equiv \boldsymbol{\varrho}(\mathrm{x}) := \inf\left\{T \ge 0\colon \int_0^T \big|\mathfrak{b}\big(\mathrm{x}(t)\big)\big|\,\mathrm{d}t + \int_{\mathcal{I}} L^{\mathrm{x}}(T,\xi)\frac{|\boldsymbol{\mu}|(\mathrm{d}\xi)}{\mathfrak{s}(\xi)} = \infty\right\}$$

for all $(\omega,\mathrm{x}) \in C([0,\infty);\mathbb{R}) \times C_a([0,\infty);\overline{\mathcal{I}})$. With this choice of drift, the SIE (2.8) can be written as

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t) + \mathfrak{b}\big(X(t)\big)\,\mathrm{d}t\big] + \int_{\mathcal{I}} L^X(\cdot,\xi)\left[\boldsymbol{\mu}(\mathrm{d}\xi) - \mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)}\right]; \tag{4.11}$$

whereas the corresponding OFE (3.17) and OIE (3.12) take respectively the form

$$\Gamma(\cdot) = \int_0^{\cdot} \mathfrak{b}\Big(\Theta_{x_0}\big(\Gamma(t)+W(t)\big)\Big)\,\mathrm{d}t + \int_{\mathcal{I}} L^{\Theta_{x_0}(\Gamma+W)}(\cdot,\xi)\frac{\boldsymbol{\mu}(\mathrm{d}(\xi)}{\mathfrak{s}(\xi)}; \tag{4.12}$$

$$C(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(C(t)\big)\left[\mathfrak{b}\Big(\Theta_{C(t)}\big(\omega(t)\big)\Big)\,\mathrm{d}t + \int_{\mathcal{I}} L^{\Theta_{C(\cdot)}(\omega(\cdot))}(\mathrm{d}t,\xi)\frac{\boldsymbol{\mu}(\mathrm{d}(\xi)}{\mathfrak{s}(\xi)}\right]. \tag{4.13}$$

We also note that the special case $\boldsymbol{\mu} \equiv 0$ leads to the time-homogeneous version

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t) + \mathfrak{b}\big(X(t)\big)\,\mathrm{d}t\big] - \int_{\mathcal{I}} L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)}$$

of (3.1). If we choose the measure $\boldsymbol{\mu}$ so that $\boldsymbol{\mu}([a,b)) = \int_{[a,b)} \mathfrak{s}(\xi)\,\mathrm{d}(1/\mathfrak{s}(\xi))$ holds for all $(a,b) \in \mathcal{I}^2$ with $a < b$, then the local time term in (4.11) disappears entirely.

The time-homogenous case under Wiener measure
Let us consider next *under the Wiener measure* $\mathbb{P}_*$ the SIE (4.11), now written in the more "canonical" form

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\,\mathrm{d}W(t) + \int_{\mathcal{I}} L^X(\cdot,\xi)\boldsymbol{\nu}(\mathrm{d}\xi);$$

here $\boldsymbol{\nu}$ is the measure on the Borel sigma algebra of $\mathcal{I}$, given by

$$\boldsymbol{\nu}\big([a,b)\big) = \boldsymbol{\mu}\big([a,b)\big) - \int_{[a,b)} \mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)} + 2\int_a^b \frac{\mathfrak{b}(\xi)}{\mathfrak{s}(\xi)}\,\mathrm{d}\xi, \quad \ell < a < b < r. \tag{4.14}$$

Theorem 4.4. *In the context of this subsection, suppose that the signed measure* $\boldsymbol{\nu}$ *of* (4.14) *satisfies*

$$\boldsymbol{\nu}\big(\{x\}\big) < 1, \quad x \in \mathcal{I}.$$

Suppose also that there exist an increasing function $f : \mathcal{I} \to \mathbb{R}$, *a nonnegative, measurable function* $g : \mathbb{R} \to [0,\infty)$, *and a real constant* $c > 0$, *such that we have*

$$\int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{d}y}{g(y)} = \infty, \quad \varepsilon > 0,$$

as well as

$$\big|\mathfrak{s}(\xi+y) - \mathfrak{s}(\xi)\big|^2 \le \frac{g(y)}{|y|}\big|f(\xi+y) - f(\xi)\big|$$

for all $\xi \in \mathcal{I}$ *and* $y \in (-c,c)\setminus\{0\}$ *with* $\xi + y \in \mathcal{I}$.

Then under the Wiener measure $\mathbb{P}_*$, *the SIE of* (4.11) *has a pathwise unique*, $\mathbb{F}$*-adapted solution* $X(\cdot)$. *Therefore, on account of Theorem* 3.2, *and again under the Wiener measure* $\mathbb{P}_*$, *the OFE* (4.12) *has also a unique* $\mathbb{F}$*-adapted solution* $\Gamma(\cdot)$; *and these solutions are related via the evaluation of* (3.11), *namely* $X(t) = \Theta_{x_0}(\Gamma(t) + W(t))$ *for* $0 \le t < \mathcal{S}(X) = \widetilde{\mathcal{S}}(\Gamma + W)$.

The first claim of Theorem 4.4 is proved as in Theorem 4.48 in Engelbert and Schmidt [24]; see also Le Gall [48], Barlow and Perkins [3], Engelbert and Schmidt [23], and Blei and Engelbert [11,12]. The argument proceeds by the familiar Zvonkin [74] method of *removal of drift*; Stroock and Yor [64], Le Gall [48], and Engelbert and Schmidt [23] contain early usage of this technique in the context of stochastic integral equations with generalized drifts. In these works the filtration is augmented by the $\mathbb{P}_*$-nullsets; however, there always exists a $\mathbb{P}_*$-indistinguishable modification $X(\cdot)$ of the solution process that is $\mathbb{F}$-adapted (see Remark I.1.37 in Jacod and Shiryaev [41]).

This reduction to a diffusion in natural scale, along with the classical Feller test, leads to necessary and sufficient conditions for the absence of explosions $\mathbb{P}_*(\mathcal{S}(X) = \infty) = 1$ in the spirit of Mijatović and Urusov [54], Karatzas and Ruf [43]; the straightforward details are left to the diligent reader.

4.4. *A close relative of the skew Brownian motion*

For two given real numbers $\rho > 0$, $\sigma > 0$, let us consider the SIE

$$X(\cdot) = \int_0^{\cdot} \big(\rho\mathbf{1}_{(-\infty,0]}\big(X(t)\big) + \sigma\mathbf{1}_{(0,\infty)}\big(X(t)\big)\big)\,\mathrm{d}W(t) + \frac{\sigma-\rho}{\sigma}L^X(\cdot,0). \tag{4.15}$$

This corresponds to the Equation (2.8) with $\mathfrak{B}(\cdot,\cdot,\cdot) \equiv 0$, state space $\mathcal{I} = \mathbb{R}$, initial condition $x_0 = 0$ and dispersion function $\mathfrak{s} = \rho\mathbf{1}_{(-\infty,0]} + \sigma\mathbf{1}_{(0,\infty)}$, thus

$$H_0(x) = \frac{x}{\rho}\mathbf{1}_{(-\infty,0]}(x) + \frac{x}{\sigma}\mathbf{1}_{(0,\infty)}(x), \qquad \Theta_0(w) = \rho w\mathbf{1}_{(-\infty,0]}(w) + \sigma w\mathbf{1}_{(0,\infty)}(w)$$

for the function of (3.3) and its inverse. The Barrow–Osgood conditions (3.21) are obviously satisfied here, explosions are non-existent, whereas Theorem 3.2 or Corollary 3.7 imply that

$$X(t) = \Theta_0\big(W(t)\big) = \sigma W^+(t) - \rho W^-(t), \quad 0 \le t < \infty \tag{4.16}$$

is the unique solution of (4.15). Indeed, it can be checked by fairly straightforward application of the Itô–Tanaka formula, that the process of (4.16) satisfies SIE (4.15) under any semimartingale measure $\mathbb{P} \in \mathfrak{P}$; and conversely, that every solution of this equation has to be given by the expression in (4.16).

Suppose now that the canonical process $W(\cdot)$ is skew Brownian motion with parameter $\alpha \in (0,1)$, to wit, that the process $V(\cdot) \equiv W(\cdot) - ((2\alpha - 1)/\alpha)L^W(\cdot, 0)$ is standard Brownian motion, under the probability measure $\mathbb{P} \in \mathfrak{P}$ (cf. Harrison and Shepp [38]). Then it can be checked, by considerations similar to those in Remark 2.4, that the SIE (4.15) takes the equivalent form

$$X(\cdot) = \int_0^{\cdot} \big(\rho \mathbf{1}_{(-\infty,0]}\big(X(t)\big) + \sigma \mathbf{1}_{(0,\infty)}\big(X(t)\big)\big)\,\mathrm{d}V(t) + \left(1 - \frac{(1-\alpha)\rho}{\alpha\sigma}\right)L^X(\cdot, 0).$$

Such equations have been studied before, for example by Ouknine [56] and Lejay and Martinez [49].

5. A comparison result

Let us place ourselves again in the context of Section 4.2 with the function $\mathfrak{b} : [0,\infty) \times \mathbb{R} \times \mathcal{I} \to \mathbb{R}$ *continuous*, and fix an arbitrary semimartingale measure $\mathbb{P} \in \mathfrak{P}$. Then, in terms of the continuous, real-valued function

$$G(t, w, \gamma) := \mathfrak{b}\big(t, w, \Theta_{x_0}(\gamma + w)\big); \quad t \in [0,\infty), (\gamma, w) \in \mathcal{E} := \big\{(\gamma, w) \in \mathbb{R}^2 \colon\ \gamma + w \in \mathcal{J}\big\} \tag{5.1}$$

we can write the OFE (4.9) in the slightly more compact form

$$\Gamma(\cdot) = \int_0^{\cdot} G\big(t, W(t), \Gamma(t)\big)\,\mathrm{d}t. \tag{5.2}$$

From Theorem III.2.1 in Hartman [39], we know that this equation has a *maximal* solution $\overline{\Gamma}(\cdot)$, defined up until the time $\widetilde{\mathcal{S}}(\overline{\Gamma} + W)$. Assuming that this solution $\overline{\Gamma}(\cdot)$ is $\mathbb{F}$-adapted, we observe – on the strength of Theorem 3.2 and of the strict increase of the mapping $\Theta_{x_0}(\cdot)$ (see also (3.7)) – that the corresponding $\mathbb{F}$-adapted process

$$\overline{X}(t) := \Theta_{x_0}\big(\overline{\Gamma}(t) + W(t)\big), \quad 0 \le t < \widetilde{\mathcal{S}}(\overline{\Gamma} + W) = \mathcal{S}(\overline{X})$$

from (3.11), (2.2) is the *maximal solution* on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ of the SIE (4.8), namely,

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t) + \mathfrak{b}\big(t, W(t), X(t)\big)\,\mathrm{d}t\big] - \int_{\mathcal{I}} L^X(\cdot, \xi)\mathfrak{s}(\xi)\,\mathrm{d}\frac{1}{\mathfrak{s}(\xi)}.$$

We fix now a number $\widehat{x}_0 \in (\ell, x_0]$ and consider yet another continuous function $\widehat{\mathfrak{b}} : [0,\infty) \times \mathbb{R} \times \mathcal{I} \to \mathbb{R}$ satisfying the pointwise comparison

$$\widehat{\mathfrak{b}}(t, w, x) \le \mathfrak{b}(t, w, x), \quad (t, w, x) \in [0,\infty) \times \mathbb{R} \times \mathcal{I}, \tag{5.3}$$

thus also the comparison

$$\widehat{\mathfrak{b}}\big(t, w, \Theta_{x_0}(\gamma + w)\big) =: \widehat{G}(t, w, \gamma) \le G(t, w, \gamma), \quad t \in [0,\infty), (\gamma, w) \in \mathcal{E}$$

with the notation of (5.1). Then we know from Theorem 3.2 that *any* $\mathbb{F}$-adapted process $\widehat{X}(\cdot)$ satisfying, on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$, the equation

$$\widehat{X}(\cdot) = \widehat{x}_0 + \int_0^{\cdot} \mathfrak{s}\big(\widehat{X}(t)\big)\big[\mathrm{d}W(t) + \widehat{\mathfrak{b}}\big(t, W(t), \widehat{X}(t)\big)\,\mathrm{d}t\big] - \int_{\mathcal{I}} L^{\widehat{X}}(\cdot, \xi)\mathfrak{s}(\xi)\,\mathrm{d}\big(1/\mathfrak{s}(\xi)\big), \tag{5.4}$$

can be cast in the manner of (3.11), (2.2) as

$$\widehat{X}(t) = \Theta_{x_0}\big(\widehat{\Gamma}(t) + W(t)\big), \quad 0 \le t < \mathcal{S}(\widehat{X}) = \widetilde{\mathcal{S}}(\widehat{\Gamma} + W).$$

Here the $\mathbb{F}$-adapted process $\widehat{\Gamma}(\cdot)$ satisfies, up until the stopping time $\mathcal{S}(\widehat{X}) = \widetilde{\mathcal{S}}(\widehat{\Gamma} + W)$, the analogue of the OIE (5.2), namely

$$\widehat{\Gamma}(\cdot) = \int_0^{\cdot} \widehat{G}\big(t, W(t), \widehat{\Gamma}(t)\big)\,\mathrm{d}t.$$

Corollary III.4.2 in Hartman [39] asserts now that the comparison $\widehat{\Gamma}(\cdot) \le \overline{\Gamma}(\cdot)$ holds on the interval $[0, \widetilde{\mathcal{S}}(\widehat{\Gamma} + W) \wedge \widetilde{\mathcal{S}}(\overline{\Gamma} + W))$ and, from the strict increase of the mapping $\Theta_{x_0}(\cdot)$ once again we deduce the "*comparison for SIEs*" result

$$\widehat{X}(t) \le \overline{X}(t), \quad 0 \le t < \mathcal{S}(\widehat{X}) \wedge \mathcal{S}(\overline{X}).$$

This compares the maximal solution $\overline{X}(\cdot)$ of the SIE (4.8) to an arbitrary solution $\widehat{X}(\cdot)$ of the SIE (5.4), under the conditions $\widehat{x}_0 \le x_0$ and (5.3).

6. Continuity of the input–output map

Once it has been established that the Equation (4.8) can be solved pathwise under appropriate conditions, it is important from the point of view of modeling and approximation to know whether the progressively measurable mapping $\mathfrak{X} : [0, \infty) \times C([0, \infty); \mathbb{R}) \to \overline{\mathcal{I}}$ that realizes its solution $X(\cdot) = \mathfrak{X}(\cdot, W)$ in terms of the canonical process $W(\cdot)$ (the "input" to this equation) is actually a *continuous* functional.

The first result of this type for classical SDEs was established by Wong and Zakai [71], Wong and Zakai [72]; similar results with simpler proofs were obtained by Doss [20] and Sussmann [65]. Wong–Zakai-type approximations have been the subject of intense investigation. Some pointers to the relevant literature are provided in McShane [53], Protter [58], Marcus [52], Ikeda and Watanabe [40], Kurtz et al. [46], Bass et al. [5], Aida and Sasaki [1], Da Pelo et al. [18], and Zhang [73].

We are now ready to state the main result of the present section, and two important corollaries. In order to simplify the exposition, we shall place ourselves in the context of Section 4.2, impose the Barrow–Osgood conditions (3.21), and assume that the drift function $\mathfrak{b}(\cdot, \cdot, \cdot)$ is bounded. In light of the Remark 4.2, the SIE (4.8) is then free of explosions.

Theorem 6.1 (Continuity of the input-output map). *In the context of Section* 4.2 *and under the Barrow–Osgood conditions* (3.21), *we assume that the drift function* $\mathfrak{b}(\cdot, \cdot, \cdot)$ *of the SIE* (4.8) *is bounded and satisfies, for each given* $n \in \mathbb{N}$, *the following conditions*:

(i) *the function* $\mathbb{R} \ni w \mapsto \mathfrak{b}(t, w, \xi)$ *is continuous for all* $(t, \xi) \in [0, \infty) \times \mathcal{I}$; *and*
(ii) *for all* $(t, w, \xi_1, \xi_2) \in [0, n] \times [-n, n] \times (\ell_n, r_n)^2$, *the local Lipschitz condition*

$$\big|\mathfrak{b}(t, w, \xi_1) - \mathfrak{b}(t, w, \xi_2)\big| \le L_n|\xi_1 - \xi_2|$$

is satisfied, where the constant $L_n < \infty$ *depends only on the integer* n.

Then the following statements hold:

1. *For each path* $\omega \in C([0, \infty); \mathbb{R})$, *the OIE* (4.9) *has a unique solution* $\Gamma_\omega(\cdot)$. *This solution is progressively measurable and satisfies* $\Gamma_\omega(\cdot) \in \mathbb{R}$.
2. *If* $\{\omega_k(\cdot)\}_{k\in\mathbb{N}}$ *is a sequence of continuous paths in* $C([0, \infty); \mathbb{R})$ *such that*

$$\lim_{k\uparrow\infty} \sup_{t\in[0,n]} \big|\omega(t) - \omega_k(t)\big| = 0, \quad n \in \mathbb{N} \tag{6.1}$$

holds for some $\omega(\cdot) \in C([0,\infty);\mathbb{R})$, *then with*

$$\mathbf{x}(\cdot) := \Theta_{x_0}\big(\varGamma_\omega(\cdot) + \omega(\cdot)\big) \quad \textit{and} \quad \mathbf{x}_k(\cdot) := \Theta_{x_0}\big(\varGamma_{\omega_k}(\cdot) + \omega_k(\cdot)\big),$$

we have

$$\lim_{k\uparrow\infty} \sup_{t\in[0,n]} \big|\mathbf{x}(t) - \mathbf{x}_k(t)\big| = 0, \quad n \in \mathbb{N}. \tag{6.2}$$

In particular, the "input-output mapping" $\omega \longmapsto \Theta_{x_0}(\varGamma_\omega(\cdot) + \omega(\cdot)) \equiv \mathfrak{X}(\cdot,\omega)$ *is continuous as a function from the canonical space* $C([0,\infty);\mathbb{R})$ *into the space* $C([0,\infty);\mathcal{I})$, *where both spaces are equipped with the topology of uniform convergence on compact subsets.*

The proof of Theorem 6.1 is provided at the end of this section. On the strength of Theorems 6.1 and 3.2, and under their conditions, the SIE (4.8) has a unique solution on the filtered space $(\varOmega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ for any given semimartingale measure $\mathbb{P} \in \mathfrak{P}$, with $\mathbb{P}(\mathcal{S}(X) = \infty) = 1$.

Corollary 6.2 (Wong–Zakai approximations). *Under the setting and assumptions of Theorem 6.1, and for an arbitrary but fixed semimartingale measure* $\mathbb{P} \in \mathfrak{P}$, *suppose that* $W_k(\cdot) = \int_0^\cdot \varphi_k(t)\,\mathrm{d}t$, $k \in \mathbb{N}$ *are absolutely continuous* $\mathbb{P}$*-almost sure approximations of the* $\mathbb{P}$*-semimartingale* $W(\cdot)$ *in the sense of* (6.1), *for some sequence* $\{\varphi_k(\cdot)\}_{k\in\mathbb{N}}$ *of* $\mathbb{F}$*-progressively measurable and locally integrable processes. Let* $\varGamma(\cdot)$ *and* $\{\varGamma_k(\cdot)\}_{k\in\mathbb{N}}$ *denote the solutions of the OIE* (4.9) *corresponding to* $W(\cdot)$ *and* $\{W_k(\cdot)\}_{k\in\mathbb{N}}$, *respectively.*

Then the processes $X_k(\cdot) \equiv \Theta_{x_0}(\varGamma_k(\cdot) + W_k(\cdot))$, $k \in \mathbb{N}$ *satisfy* $\mathbb{P}(\mathcal{S}(X_k) = \infty) = 1$ *and the analogues of* (4.8) *in the present context, namely, the OIEs*

$$X_k(\cdot) = x_0 + \int_0^\cdot \mathfrak{s}\big(X_k(t)\big)\big(\varphi_k(t) + \mathfrak{b}\big(t, W_k(t), X_k(t)\big)\big)\,\mathrm{d}t,$$

and converge almost surely to the solution $X(\cdot) = \varTheta_{x_0}(\varGamma(\cdot) + W(\cdot))$ *of the SIE* (4.8), *namely*

$$X(\cdot) = x_0 + \int_0^\cdot \mathfrak{s}\big(X(t)\big)\big[\mathrm{d}W(t) + \mathfrak{b}\big(t, W(t), X(t)\big)\,\mathrm{d}t\big] - \int_{\mathcal{I}} L^X(\cdot,\xi)\mathfrak{s}(\xi)\,\mathrm{d}\big(1/\mathfrak{s}(\xi)\big),$$

uniformly over compact intervals, in the manner of (6.2).

We formulate now a support theorem, which follows almost directly from Corollary 6.2. First, we introduce some necessary notation. For any given semimartingale measure $\mathbb{P} \in \mathfrak{P}$ and any initial position $x_0 \in \mathcal{I}$, we denote by $\mathfrak{U}^{\mathbb{P}}(x_0) \subseteq C([0,\infty);\mathcal{I})$ the support under $\mathbb{P}$ of the solution process $X(\cdot) = \Theta_{x_0}(\varGamma_W(\cdot) + W(\cdot))$ for the SIE (4.8); this is the smallest closed subset of $C([0,\infty);\mathcal{I})$, equipped with the topology of uniform convergence on compact sets, with the property $\mathbb{P}(X(\cdot) \in \mathfrak{U}^{\mathbb{P}}(x_0)) = 1$.

Moreover, we let C^{PL} and C^∞ denote, respectively, the spaces of piecewise linear and infinitely differentiable functions $\omega : [0,\infty) \to \mathbb{R}$. For any given subset A of $C([0,\infty);\mathcal{I})$, we denote by $\overline{A}$ its topological closure under the topology of uniform convergence on compact sets.

Corollary 6.3 (Support theorem). *Under the setting and assumptions of Theorem 6.1, and with a fixed semimartingale measure* $\mathbb{P} \in \mathfrak{P}$, *we have*

$$\mathfrak{U}^{\mathbb{P}}(x_0) \subseteq \overline{\big\{\varTheta_{x_0}\big(\varGamma_\omega(\cdot) + \omega(\cdot)\big) \colon \omega \in C^{\mathrm{PL}}\big\}} \quad \textit{and} \quad \mathfrak{U}^{\mathbb{P}}(x_0) \subseteq \overline{\big\{\varTheta_{x_0}\big(\varGamma_\omega(\cdot) + \omega(\cdot)\big) \colon \omega \in C^\infty\big\}}.$$

Moreover, if $\mathbb{P} = \mathbb{P}_*$ *is the Wiener measure, then the above set inclusions become equalities.*

Proof. The set inclusions follow from Corollary 6.2 and the fact that both spaces C^{PL} and C^∞ are dense in the space of continuous functions $C([0,\infty);\mathbb{R})$, equipped with the topology of uniform convergence on compact sets. Under the Wiener measure $\mathbb{P}_*$, the reverse implications follow from a change-of-measure argument similar to Lemma 3.1 in Stroock and Varadhan [62]. □

Proof of Theorem 6.1. To prove the first part of the theorem, fix a path $\omega(\cdot) \in C([0,\infty);\mathbb{R})$ and $n \in \mathbb{N}$. Next, recall the function $G(\cdot,\cdot,\cdot)$ of (5.1) and define the function

$$[0,\infty) \times \mathbb{R} \ni (t,\gamma) \longmapsto g_\omega(t,\gamma) = G\big(t,\omega(t),\gamma\big) \in \mathbb{R}.$$

If $\gamma \in [-n,n]$ we have $\omega(\cdot) + \gamma \in (\widetilde{\ell}_m, \widetilde{r}_m)$ on $[0,n]$ for some sufficiently large $m \in \mathbb{N}$ and thus, the local Lipschitz condition

$$\begin{aligned}\big|g_\omega(t,\gamma_1) - g_\omega(t,\gamma_2)\big| &\le L_m \big|\Theta_{x_0}\big(\omega(t)+\gamma_1\big) - \Theta_{x_0}\big(\omega(t)+\gamma_2\big)\big| \\ &\le \Bigg(L_m \sup_{\xi \in (\ell_m, r_m)} \mathfrak{s}(\xi)\Bigg)|\gamma_1 - \gamma_2|\end{aligned}$$

for all $t \in [0,n]$ and $(\gamma_1,\gamma_2) \in [-n,n]^2$. Since the function $g_\omega(\cdot,\cdot)$ is also bounded, Carathéodory's extension of the Peano existence theorem [see Coddington and Levinson [16], Theorems 2.1.1 and 2.1.3] guarantees the existence of a solution $\varGamma_\omega^{(n)}(\cdot)$ to the OIE

$$\varGamma_\omega^{(n)}(t) = x_0 + \int_0^t g_\omega\big(t, \varGamma_\omega^{(n)}(s)\big)\,\mathrm{d}s, \quad 0 \le t \le n$$

up to the first time that $\varGamma_\omega^{(n)}(\cdot)$ leaves the interval $(-n,n)$. Moreover, a Picard-Lindelöf-type argument yields the uniqueness of the solution, thus also the non-anticipativity of the function $(t,\omega) \mapsto \varGamma_\omega^{(n)}(t)$. Stitching those solutions for each $n \in \mathbb{N}$ together, yields then a unique non-anticipative mapping $[0,\infty) \times \Omega \ni (t,\omega) \mapsto \varGamma_\omega(t)$, as in the statement of the theorem. Since $g_\omega(\cdot,\cdot)$ is bounded, we have $\varGamma_\omega(\cdot) \in \mathbb{R}$.

In order to conclude the proof of the first claim, we need to show now that the mapping $\omega \mapsto \varGamma_\omega(\cdot)$ is measurable; however, while proving below the second claim of the theorem, we shall show that this mapping is actually continuous, thus, *a-fortiori*, measurable. Now measurability and $\mathbb{F}$-adaptivity – a consequence of non-anticipativity – lead to the progressive measurability of this mapping; see Propositions 1.1.12 and 1.1.13 of Karatzas and Shreve [44], in conjunction with the continuity of the mapping $t \mapsto \varGamma_\omega(t)$.

For the second claim of the theorem, let us fix paths $\{\omega_k(\cdot)\}_{k\in\mathbb{N}}$ and $\omega(\cdot)$ as in the statement, and an integer $n \in \mathbb{N}$. Let $\beta < \infty$ denote an upper bound on the function $|\mathfrak{b}(\cdot,\cdot,\cdot)|$, and fix $m \in \mathbb{N}$ so that

$$\sup_{t\le n, k\in\mathbb{N}} \big|\omega_k(t)\big| < m, \qquad x_0 + n\beta + \sup_{t\le n,k\in\mathbb{N}} \big|\omega_k(t)\big| < \widetilde{r}_m, \quad \text{and} \quad x_0 - n\beta - \sup_{t\le n,k\in\mathbb{N}} \big|\omega_k(t)\big| > \widetilde{\ell}_m.$$

Next, observe that we have

$$\begin{aligned}\big|\varGamma_\omega(\cdot) - \varGamma_{\omega_k}(\cdot)\big| \le &\int_0^n \big|G\big(t,\omega(t),\varGamma_\omega(t)\big) - G\big(t,\omega_k(t),\varGamma_\omega(t)\big)\big|\,\mathrm{d}t \\ &+ \int_0^\cdot \big|G\big(t,\omega_k(t),\varGamma_\omega(t)\big) - G\big(t,\omega_k(t),\varGamma_{\omega_k}(t)\big)\big|\,\mathrm{d}t\end{aligned}$$

on $[0,n]$ for all $k \in \mathbb{N}$. Since the function $G(\cdot,\cdot,\cdot)$ is bounded, the Dominated Convergence Theorem yields that the first term on the right-hand can be made arbitrarily small. For the second term, we can use the Lipschitz continuity of $G(\cdot,\cdot,\cdot)$ with Lipschitz constant L_m. An application of Gronwall's lemma then yields that $\lim_{k\uparrow\infty} \sup_{t\le n} |\varGamma_\omega(t) - \varGamma_{\omega_k}(t)| = 0$. Since the function $\Theta_{x_0}(\cdot)$ is locally Lipschitz continuous, the statement follows. □

Appendix: Regularization of OIEs

The implications of Theorem 3.2 or Corollary 3.7 can prove useful for obtaining existence and uniqueness statements of OFEs in the form of the OFE (3.10). For instance, Theorem 4.4 is a case in point.

Example A.1. *Let us look closer at the setup of the SIE* (3.1) *under the Wiener measure* $\mathbb{P}_*$ *and with* $\mathfrak{s}(\cdot) \equiv 1$, *for a bounded, measurable function* $\mathfrak{b} : [0, \infty) \times \mathbb{R} \to \mathbb{R}$. *It is well known* (*see*, *for example*, *Zvonkin* [74] *or Veretennikov* [67]) *that the resulting SIE*

$$X(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{b}\big(t, X(t)\big)\,\mathrm{d}t + W(\cdot) \tag{A.1}$$

has a unique $\mathbb{F}$*-adapted and non-exploding solution*. *In fact*, *Krylov and Röckner* [45] *show that the SIE* (A.1) *admits a pathwise unique*, *strong and non-explosive solution*, *under only very weak integrability conditions on the function* $\mathfrak{b}(\cdot, \cdot)$; *see also Fedrizzi and Flandoli* [28] *for a simpler argument*. *Theorem* 3.2 *now implies that the corresponding OIE* (4.9), *now in the form*

$$\Gamma(\cdot) = x_0 + \int_0^{\cdot} \mathfrak{b}\big(t, \Gamma(t) + W(t)\big)\,\mathrm{d}t, \tag{A.2}$$

also has a unique $\mathbb{F}$*-adapted solution* $\Gamma(\cdot)$; *a similar point is made by Davie* [19].

We do not know a theory of OIEs that can prove such existence and uniqueness statements of this type. *An explanation is given in Section* 1.6 *of Flandoli* [31]: *"...the intuitive reason behind these uniqueness results in spite of the singularities of the drift [is]...the regularity of the occupation measure. The measure distributed by single trajectories of diffusions...is ...very regular and diffused with respect to the occupation measure of solutions to deterministic ODEs. This regularity smooths out the singularities of the drift, opposite to the deterministic case in which the solution may persist on the singularities."*

The question answered affirmatively by Davie [19] (*see also Flandoli* [30] *for a simpler argument*), *is whether uniqueness holds for the OIE* (A.2) *also for almost all realizations of the Brownian paths* $W(\omega)$, *among all* (*possibly not* $\mathbb{F}$*-adapted*) *functions* $\Gamma(\cdot)$. *For a discussion of the subtle differences in those notions of uniqueness*, *we refer to the comments after Definition* 1.5 *in Flandoli* [29]. *Recently*, *Catellier and Gubinelli* [14] *further extended the regularization results to paths of fractional Brownian motion*.

Example A.2. *In the context of the SIE* (4.11) *in Section* 4.3, *under the Wiener measure* $\mathbb{P}_*$ *and with* $\mathfrak{s}(\cdot) \equiv 1$ *and* $\mathfrak{b}(\cdot) \equiv 0$, *the OFE* (4.12) *can be simplified to*

$$\Gamma(\cdot) = x_0 + \int_{\mathcal{I}} L^{\Gamma+W}(\cdot, \xi)\boldsymbol{\mu}(\mathrm{d}\xi)$$

up to an explosion time; *if additionally* $\mathcal{I} = \mathbb{R}$ *and* $\boldsymbol{\mu}(\mathrm{d}\xi) = \beta\boldsymbol{\delta}_0(\mathrm{d}\xi)$ *for some* $\beta \in \mathbb{R}$ *and the Dirac measure* $\boldsymbol{\delta}_0(\cdot)$ *at the origin*, *this expression simplifies further to the equation*

$$\Gamma(\cdot) = x_0 + \beta L^{\Gamma+W}(\cdot, 0). \tag{A.3}$$

Not much can be said directly about this equation; *we note*, *however*, *that the corresponding stochastic equation* (4.11) *simplifies to*

$$X(\cdot) = x_0 + W(\cdot) + \beta L^X(\cdot, 0), \tag{A.4}$$

the stochastic equation for the skew Brownian motion with skewness parameter $\alpha = 1/(2-\beta)$. *In terms of the symmetric local time at the origin* $\widehat{L}^X(\cdot, 0) = (1/2)(L^X(\cdot, 0) + L^{-X}(\cdot, 0))$, *the above equation can be written in the equivalent and perhaps more "canonical" form*

$$X(\cdot) = x_0 + W(\cdot) + \gamma\widehat{L}^X(\cdot, 0) \quad \text{with } \gamma = \frac{2\beta}{2-\beta} = 2(2\alpha - 1).$$

In accordance with Theorem 4.4, *the SIE* (A.4) *has a pathwise unique*, *strong solution for all* $\beta < 1$; *the theory of the Skorokhod reflection problem guarantees such a solution for* $\beta = 1$; *whereas it is shown in Harrison and Shepp* [38] *that there is no such solution for* $\beta > 1$. *From Theorem* 3.2, *analogous statements hold for* $\mathbb{F}$*-adapted solutions to the OFE* (A.3).

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