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Planar Ising magnetization field II. Properties of the critical and near-critical scaling limits

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Abstract. In (*Ann. Probab.* **43** (2015) 528–571), we proved that the renormalized critical Ising magnetization fields $\Phi^a := a^{15/8}\sum_{x\in a\mathbb{Z}^2}\sigma_x\delta_x$ converge as $a\to 0$ to a random distribution that we denoted by Φ^∞. The purpose of this paper is to establish some fundamental properties satisfied by this Φ^∞ and the near-critical fields $\Phi^{\infty,h}$. More precisely, we obtain the following results.

(i) If $A\subset\mathbb{C}$ is a smooth bounded domain and if $m=m_A:=\langle\Phi^\infty,1_A\rangle$ denotes the limiting rescaled magnetization in A, then there is a constant $c=c_A>0$ such that

$$\log\mathbb{P}[m>x]\underset{x\to\infty}{\sim}-cx^{16}.$$

In particular, this provides an alternative way of seeing that the field Φ^∞ is non-Gaussian (another proof of this fact would use the explicit n-point correlation functions established in (*Ann. Math.* **181** (2015) 1087–1138) which do not satisfy Wick's formula).

(ii) The random variable $m=m_A$ has a smooth *density* and one has more precisely the following bound on its Fourier transform: $|\mathbb{E}[e^{itm}]|\le e^{-\tilde{c}|t|^{16/15}}$.

(iii) There exists a one-parameter family $\Phi^{\infty,h}$ of near-critical scaling limits for the magnetization field in the plane with vanishingly small external magnetic field.

Résumé. Dans l'article (*Ann. Probab.* **43** (2015) 528–571), nous avons montré que le champ de magnétisation du modèle d'Ising critique $\Phi^a := a^{15/8}\sum_{x\in a\mathbb{Z}^2}\sigma_x\delta_x$ converge lorsque $a\to 0$ vers une distribution aléatoire limite Φ^∞. Le but de cet article est d'analyser certaines propriétés fondamentales de cet objet limite Φ^∞ ainsi que ses analogues presque-critiques $\Phi^{\infty,h}$. Plus précisément, nous obtenons les résultats suivants :

(i) Si $A\subset\mathbb{C}$ est un domaine borné régulier du plan et si $m=m_A:=\langle\Phi^\infty,1_A\rangle$, alors il existe une constante $c=c_A>0$ telle que

$$\log\mathbb{P}[m>x]\underset{x\to\infty}{\sim}-cx^{16}.$$

On obtient ainsi une preuve alternative du fait que Φ^∞ est non-Gaussian (une autre façon de voir le coté non-Gaussien utilise les fonctions de corrélations à n-points obtenues dans (*Ann. Math.* **181** (2015) 1087–1138) qui ne satisfont pas la formule de Wick).

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(ii) La variable aléatoire $m = m_A$ a une densité qui est analytique. Plus précisément, on obtient la borne suivante sur sa transformée de Fourier : $|\mathbb{E}[e^{itm}]| \le e^{-\tilde{c}|t|^{16/15}}$.
(iii) Il existe une famille à un paramètre $\Phi^{\infty,h}$ de limite d'échelle presque-critiques pour le champ de magnétisation dans le plan avec un champ magnétique extérieur infinitésimal.

MSC: 82B20; 82B27; 60K35

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1. Introduction

1.1. *Overview*

In [2], we considered the scaling limit of the appropriately renormalized magnetization field of a critical Ising model (i.e., at $\beta = \beta_c$) on the lattice $a\mathbb{Z}^2$ where the mesh a shrinks to zero. The natural object to consider is the following field:

$$\Phi^a := \sum_{x \in a\mathbb{Z}^2} a^{15/8} \sigma_x \delta_x, \tag{1.1}$$

where $\{\sigma_x\}_{x \in a\mathbb{Z}^2}$ is the realization of a critical Ising model on $a\mathbb{Z}^2$. Note that the renormalization of $a^{15/8}$ assumes Wu's computation [13,15]. If one does not want to rely on Wu's derivation, one should consider instead the following renormalization $\Phi^a := a^2 \varrho(a)^{-1/2} \sum_{x \in a\mathbb{Z}^2} \sigma_x \delta_x$, where the quantity $\varrho(a)$ was defined in [5]. See Remark 1.6 for more details. The following theorem is proved in [2] (see Theorem 1.2 and Appendix *A* in [2] for more details).

Theorem 1.1 ([2]). *As the mesh $a \searrow 0$, the random field Φ^a converges in law to a limiting random field Φ^∞ under the topology of the Sobolev space $\mathcal{H}^{-3}(\mathbb{C})$.*

In the case of a bounded smooth simply connected domain Ω equipped with $+, -$ or free boundary conditions along $\partial\Omega$, one also obtains a limiting magnetization field Φ^∞_Ω whose law depends on the choice of the prescribed boundary conditions $\xi \in \{+, -, \text{free}\}$. See Theorem 1.3 in [2].

Two proofs of these results are provided in [2]: the first one relies on the recent breakthrough results from [5] on the n-point correlation functions of the critical Ising model. The second proof is more conditional and relies for example on the recent work [6]. See Section 2 in [2].

The purpose of [2] was to identify a limit in law for these magnetization fields (i.e. Φ^∞ and Φ^∞_Ω). Beyond the conformal covariance nature of these fields (Theorem 1.8 in [2]), we did not investigate the fine properties of these fields. This is what we wish to address in this paper:

1. To start with, we will focus on the *tail behavior* of the field Φ^∞ (and its bounded domain analog). For any bounded smooth domain A, we obtain a precise tail estimate for the block magnetization $m = m_A = \langle \Phi^\infty, 1_A \rangle$ of $e^{-cx^{16}}$ where the constant $c > 0$ does not depend on the prescribed boundary conditions along ∂A. See Theorem 1.2.
2. Then, we investigate whether the random variable m_A defined above has a density function or not and if so what is its regularity. We answer this question by studying the tail of its characteristic function. Namely we prove that $|\mathbb{E}[e^{itm}]| \le e^{-\tilde{c}|t|^{16/15}}$. See Theorem 1.3.
3. Finally, we address a question of a different flavor: we prove in Theorem 1.4 that the magnetization field $\Phi^{a,h}$ for the near-critical Ising model with external field $ha^{15/8}$ has a scaling limit denoted by $\Phi^{\infty,h}$.

1.2. *Main statements*

In Section 2 we will prove the following result.

Theorem 1.2. *There exists a universal constant $c > 0$ such that for any prescribed boundary conditions $\xi \in \{+, -, \text{free}\}$ around the square $[0, 1]^2$, the (continuum) magnetization $m = m^\xi = \Phi^\xi([0, 1]^2)$ in $[0, 1]^2$ satisfies as $x \to \infty$:*

$$\log \mathbb{P}[m > x] \sim -cx^{16}.$$

This result extends to the case of the plane field Φ^∞ tested against a bounded smooth domain A, i.e., $\Phi^\infty(1_A)$ (in which case the constant c will depend on A), or to the case of the limiting field Φ^∞_Ω for a bounded smooth domain Ω tested against a smooth sub-domain $A \subset \Omega$.

In Section 3, we will prove (assuming Wu's formula or the forthcoming work [4]):

Theorem 1.3. *Let us consider the scaling limit $m = m^\xi$ of the magnetization in the square $[0, 1]^2$ with prescribed boundary conditions $\xi \in \{+, -, \text{free}\}$. There is a constant $\tilde{c} > 0$ such that for all $t \in \mathbb{R}$ one has*

$$\left|\mathbb{E}^\xi\left[e^{itm}\right]\right| \le e^{-\tilde{c}|t|^{16/15}}.$$

In particular, the density function $f = f^\xi$ of the random variable $m = m^\xi$ can be extended to an entire function on the whole complex plane $\mathbb{C}$.[4]

As in Theorem 1.2, the result extends to the whole-plane field Φ^∞ tested against domains A as well as to the fields Φ^∞_Ω for smooth bounded domains Ω.

As we shall see later in Remark 3.2, this theorem should also easily extend to more general boundary conditions ξ such as finite combinations of $+$, $-$, free-arcs. In this case, the constant $c = c([0, 1]^2) > 0$ would still be independent of the boundary condition ξ.

Finally, in Section 4 we will prove the following theorem concerning the near critical (as $h \to 0$) scaling limit. Two recent reviews that discuss the significance of such near-critical models are [1] and [12].

Theorem 1.4. *Let us fix some constant $h > 0$. Consider the Ising model on $a\mathbb{Z}^2$ at $\beta = \beta_c$ and with vanishingly small external magnetic field equal to $a^{15/8}h$. Let $\Phi^{a,h}$ be the near-critical magnetization field in the plane defined, as in [2] (where $h = 0$), by*

$$\Phi^{a,h} := \sum_{x \in a\mathbb{Z}^2} \delta_x \sigma_x a^{15/8},$$

where $\{\sigma_x\}_{x \in a\mathbb{Z}^2}$ is a realization of the above Ising model with external magnetic field equal to $ha^{15/8}$. Then, as the mesh $a \searrow 0$, the random distribution $\Phi^{a,h}$ converges in law to a near-critical field $\Phi^{\infty,h}$ under the topology of $\mathcal{H}^{-3}$ in the full plane defined in Section A.2 of [2].

The analogous statement in the case of a bounded smooth domain can be stated as follows.

Proposition 1.5. *Let Ω be a bounded smooth domain of the plane with boundary conditions either $+$, $-$ or free and let $h > 0$ be some positive constant. Then, with the obvious notation, $\Phi^{a,h}_\Omega$ converges in law to a field $\Phi^{\infty,h}_\Omega$ as $a \to 0$ under the topology of the Sobolev space $\mathcal{H}^{-3}(\Omega)$.*

This result is stated only as a proposition since as we shall see in Section 4, it follows almost readily from our previous work [2]. We will then prove Theorem 1.4 using this proposition by considering larger and larger domains Λ_L and by showing that the near-critical fields do stabilize as $L \to \infty$. The relation between $\Phi^{\infty,h}$ and $\Phi^{\infty,h}_\Omega$ to Φ^∞ and Φ^∞_Ω is discussed in Section 4.

[4]See for example Theorem IX.13 in [14].

1.3. *Brief outline of proofs*

- The proof of the tail behaviour given by Theorem 1.2 will be based on the study of the exponential moments of the magnetization m, i.e., of $\mathbb{E}[e^{tm}]$, with $t > 0$ large. Theorem 1.2 will then follow from a specific *Tauberian* theorem of Kasahara [11]. One issue in this program is to show that the random variable m indeed has exponential moments. This property was established in the first part of this series of papers, i.e. in [2] and the proof relied essentially on the *GHS* inequality. The other difficulty is to adapt the classical arguments which lead to the existence of *free energies* to our present continuum setting, where one cannot use the standard trick of fixing the spins along dyadic squares in order to use subadditivity. To overcome this, one relies on RSW within thin long tubes.
- In our study of $|\mathbb{E}^\xi[e^{itm}]|$, we rely on the FK representation and we prove that with very high probability (of order $1 - e^{-c|t|^{16/15}}$), one can find $O(1/\varepsilon^2)$ mesoscopic squares of well-chosen size $\varepsilon = \varepsilon_t$ which contain an FK cluster of "mass" about $1/t$.
- For the proof of Theorem 1.4, most of the non-trivial work is done in Lemma 4.1 whose purpose is to prove that the law of the full-plane near-critical field $\Phi^{a,h}$ is very close in $\mathcal{H}^{-3}$ to the law of a large domain near-critical field $\Phi^{a,h}_{\Lambda_L}$. The technique used here is a coupling argument similar to the one used in Section 2 in [2] and which relies on the RSW theorem from [7].

Remark 1.6. *Note that the renormalization we used in order to define Φ^a relies on Wu's derivation of the two-point function* ([13,15]). *See Remark* 1.5. *in* [2] *for a discussion on this. This renormalization is also the choice made in the last versions* (*v*2 *and v*3) *of the breakthrough work* [5]. *Furthermore a fully rigorous derivation of Wu's formula* [4] *by Chelkak and Hongler has been announced recently in* [5]. *Yet, in the meantime* (*i.e. before* [4] *appears*), *it turns out that if one does not wish to rely on Wu's formula, then our main results Theorems* 1.2 *and* 1.4 *still hold except that the discrete magnetization field Φ^a should no longer be renormalized by $a^{15/8}$* (*which does assume Wu*) *but instead by $a^2\varrho(a)^{-1/2}$, where the quantity $\varrho(a)$ was introduced in* [5] *and corresponds to the two-point function of the critical Ising model along the diagonal for points at distance $\sqrt{2}a^{-1}$. More precisely, without Wu and relying on* [5], *one still obtains a conformally covariant limiting field for $\Phi^a := a^2\varrho(a)^{-1/2}\sum_{x\in a\mathbb{Z}^2}\sigma_x\delta_x$. Indeed in* [5], *after adequate renormalization by $\varrho(a)$, the authors do obtain n-point limiting correlation functions which are conformally covariant with exponent* $1/8$ *without ever relying on Wu's result. This is all that we need in order to obtain a conformally covariant magnetization field Φ^∞ which can then be used to prove Theorems* 1.2 *and* 1.4 *in this paper. At the time of* [2], *we were not aware of this derivation of the exponent* $1/8$ *independent of Wu's computation in connection with continuum correlation functions. This is why we added a section there discussing the use of Wu's computation. Therefore Theorems* 1.2 *and* 1.4 *can be seen as unconditional* (*w.r.t. Wu's result*). *On the other hand, in* [3] *as well as in our present Theorem* 1.3, *we use different techniques which do rely more directly on the arm-exponent* $1/8$ *and thus will be fully unconditional once* [4] *is out.*

2. Tail behavior

In this section, we shall prove Theorem 1.2.

2.1. *Existence of exponential moments*

We will need the fact that the (continuum) magnetization m has all exponential moments. This property was proved in [2] and we provide below the corresponding statement.

Proposition 2.1 (Proposition 3.5 and Corollary 3.8 in [2]). *For any boundary condition ξ* (*either* $+$, $-$ *or* free *boundary conditions*) *around* $[0,1]^2$, *and for any $t \in \mathbb{R}$, if $m = m^\xi$ is the continuum* (*and m^a is the discrete*) *magnetization of the unit square*, *then one has*

(i) $\mathbb{E}[e^{tm}] < \infty$.
(ii) *Furthermore*, *as* $a \to 0$, $\mathbb{E}[e^{tm^a}] \to \mathbb{E}[e^{tm}]$.

See [2] for the proof of this proposition which relies essentially on the *GHS inequality* from [10].

2.2. *Asymptotic behavior of the moment generating function and scaling argument*

Since the exponential moments $\mathbb{E}^\xi[e^{tm}]$ are well-defined, our next step is to study the behavior for large t of the moment generating function $t \mapsto \mathbb{E}^\xi[e^{tm}]$. We will prove the following proposition.

Proposition 2.2. *There exists a* universal constant $b > 0$ *which does not depend on the boundary conditions* ξ *around* $[0, 1]^2$ *so that as* $t \to \infty$:

$$\log \mathbb{E}^\xi\left[e^{tm}\right] \sim bt^{16/15}.$$

Theorem 1.2 follows from the above proposition thanks to the following Tauberian theorem by Kasahara.

Theorem 2.3 (Corollary 1 in [11]). *For any random variable* X *which has all its exponential moments, if there is an exponent* $\alpha > 1$ *and a constant* $b > 0$ *such that*

$$\log \mathbb{E}\left[e^{tX}\right] \sim bt^\alpha,$$

as $t \to \infty$, *then the following holds for some explicit constant* $c = c(b, \alpha) > 0$:

$$\log \mathbb{P}[X > x] \sim -cx^{1/(1-1/\alpha)},$$

as $x \to \infty$.

Remark 2.4. *In fact, this result is stated only for positive random variables in* [11] *but it is very simple to extend it to any real-valued random variable* X. *Let us sketch a short argument here. Assume one has*

$$\log \mathbb{E}\left[e^{tX}\right] \sim bt^\alpha, \tag{2.1}$$

as $t \to \infty$ *for some* $b > 0$, *then necessarily,* $\mathbb{P}[X > 0]$ *has to be strictly positive. Now let* Y *be the random variable* X *conditioned to be positive. It is easy to check that as* $t \to \infty$, $\log \mathbb{E}[e^{tY}] \sim \log \mathbb{E}[e^{tX}]$. *One then concludes the argument by noticing that as* $x \to \infty$, $\log \mathbb{P}[X > x] \sim \log \mathbb{P}[X > x \mid X > 0] = \log \mathbb{P}[Y > x]$.

Remark 2.5. *Note that by a straightforward use of the exponential Chebyshev inequality, upper bounds on* $\mathbb{P}^\xi[m > x]$ *can be directly recovered from Proposition* 2.2.

Proof of Proposition 2.2. The main tools to prove the proposition will be the scaling covariance property of the total magnetization m which was proved in [2] (see Proposition 2.6 below) as well as Theorem 2.7 below which in some sense defines a *free energy* for our limiting magnetization field. Let us first state these two results.

Proposition 2.6 (Scaling covariance of m, Corollary 5.2 in [2]). *Let* $m = m^\xi$ *be the scaling limit of the renormalized magnetization in the square (i.e.,* $m = \langle \Phi^\infty, 1_{[0,1]^2}\rangle$), *with boundary conditions* ξ *being either* $+, -$ *or* free. *For any* $\lambda > 0$, *let* $m_\lambda = m_\lambda^\xi$ *be the scaling limit of the renormalized magnetization in the square* $[0, \lambda]^2$ *with the same boundary conditions* ξ. *Then one has the following identity in law*:

$$m_\lambda \overset{(d)}{=} \lambda^{15/8} m. \tag{2.2}$$

Theorem 2.7 (Existence of free energy). *For any* $L > 0$ *and any boundary conditions* ξ *(made of finitely many* $+$, $-$ *or* free *arcs) around* $[0, L]^2$, *let* $f_L^\xi(t) := \frac{1}{L^2} \log \mathbb{E}^\xi[e^{tm_L}]$.

There is a universal constant $b > 0$, *which does not depend on the boundary conditions* ξ, *such that for any* $t \in \mathbb{R}$

$$f_L^\xi(t) := \frac{1}{L^2} \log \mathbb{E}^\xi\left[e^{tm_L}\right] \underset{L\to\infty}{\longrightarrow} b|t|^{16/15}.$$

With these two ingredients, it is easy to conclude the proof of Proposition 2.2. Indeed if $\lambda_t := t^{8/15}$, then one has:

$$\log \mathbb{E}^{\pm}\big[e^{tm}\big] = \log \mathbb{E}^{\pm}\big[e^{m_{\lambda_t}}\big] \quad \text{using Proposition 2.6} \tag{2.3}$$

$$= t^{16/15}\left(\frac{1}{\lambda_t^2}\log \mathbb{E}^{\pm}\big[e^{m_{\lambda_t}}\big]\right) \tag{2.4}$$

$$\sim t^{16/15} b, \tag{2.5}$$

as $t \to \infty$. Other boundary conditions are handled by noting that $\mathbb{E}^{\xi}[e^{tm}]$ is squeezed between the $+$ and $-$ cases by the FKG inequalities. □

Remark 2.8. *Note that we did not need the full strength of Theorem 2.7, only the case $t = 1$. Nevertheless, since Theorem 2.7 is interesting in it own right, we prove it for all $t \in \mathbb{R}$ (which will result in a slight repetition of the above scaling argument in the proof of Lemma 2.13).*

Remark 2.9. *It is tempting to compare the above* free energy *with the classical one coming from the discrete system, i.e., defined as*

$$F(t) := \lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{E}^{+}\big[e^{t\sum_{x\in\Lambda_N}\sigma_x}\big], \tag{2.6}$$

but it is easy to see that they must be different, since clearly $F(t) \le |t|$ for any $t \in \mathbb{R}$. On the other hand, they behave essentially the same for small t as follows from the results of [3].

2.3. *Free energy estimates*

The purpose of this section is to prove Theorem 2.7 on the free energy of Φ^∞. The proof of this theorem will be divided into several steps as follows. First, we will show in Lemma 2.10 that for any $t \ge 0$, $f_L^+(t)$ and $f_L^-(t)$ have limits along dyadic scales $L_k = 2^k$, respectively denoted by $f^+(t)$ and $f^-(t)$. Then, in Lemma 2.11, we will show that

$$\begin{cases} \limsup_{L\to\infty} f_L^+(t) = f^+(t), \\ \liminf_{L\to\infty} f_L^-(t) = f^-(t). \end{cases}$$

In Lemma 2.12, we will prove that $f^+(t) = f^-(t) = f(t)$ for any $t \ge 0$. Finally Lemma 2.13 will identify the limit $f(t)$ to be exactly $b|t|^{16/15}$ for all $t \in \mathbb{R}$, thus concluding the proof of Theorem 2.7. The main difficulty in this last lemma will be to show that the constant b is positive.

We will first list these lemmas and then proceed with their proofs. Let us point out that some of the proofs below follow the standard arguments to prove that a *free energy* is well defined. Nevertheless, they turn out to be slightly more involved here since we are working with the continuum limit and therefore all the classical arguments based, for example, on counting the number of *lattice sites* on the boundary are no longer valid here. (Only the proof of Lemma 2.10 follows exactly the classical scheme.)

Lemma 2.10. *For any $t \ge 0$, and any $k \ge 1$,*

$$\begin{cases} f_{2^{k+1}}^+(t) \le f_{2^k}^+(t), \\ f_{2^{k+1}}^-(t) \ge f_{2^k}^-(t). \end{cases}$$

In particular, the sequences $f_{2^k}^{\pm}(t)$ converge as $k \to \infty$ and we will denote respectively their limits by $f^{\pm}(t)$.

Lemma 2.11. *For any $t \ge 0$, we have*

$$\begin{cases} \limsup_{L\to\infty} f_L^+(t) = f^+(t), \\ \liminf_{L\to\infty} f_L^-(t) = f^-(t). \end{cases}$$

Lemma 2.12. *For any* $t \geq 0$, *we have*

$$f^+(t) = f^-(t) = f(t).$$

Lemma 2.13. *There exists a universal constant* $b > 0$ *such that for any boundary conditions* ξ, *we have*

$$f(t) := \lim_{L\to\infty} \frac{1}{L^2}\mathbb{E}^\xi\big[\mathrm{e}^{tm_L}\big] = b|t|^{16/15},$$

for all $t \in \mathbb{R}$.

Proof of Lemma 2.10. Let us consider the case of + boundary conditions; the − case is similar. From Proposition 2.1(ii), we know that for any $L > 0$,

$$\mathbb{E}^+\big[\mathrm{e}^{tm_L}\big] = \lim_{a\to 0} \mathbb{E}^+\big[\mathrm{e}^{tm_L^a}\big].$$

Now, for any $k \in \mathbb{N}^+$, it is easy to check (by breaking the domain $[0, 2^{k+1}]^2$ into 4 squares with + boundary conditions and using FKG) that for suitable choices of the mesh size a (i.e. a such that $2^k \in a\mathbb{Z}$), then

$$\mathbb{E}^+\big[\mathrm{e}^{tm^a_{2^{k+1}}}\big] \leq \mathbb{E}^+\big[\mathrm{e}^{tm^a_{2^k}}\big]^4.$$

Taking the limit $a \to 0$, we get that

$$\mathbb{E}^+\big[\mathrm{e}^{tm_{2^{k+1}}}\big] \leq \mathbb{E}^+\big[\mathrm{e}^{tm_{2^k}}\big]^4,$$

which implies $f^+_{2^{k+1}}(t) \leq f^+_{2^k}(t)$. As pointed out above, this proof matches exactly the standard proof in the discrete setup. □

Proof of Lemma 2.11. We only consider the case of + boundary conditions and we will fix some $t \geq 0$ (the case of minus boundary conditions is handled in the same fashion). Let us also fix some integer $k_0 \geq 1$. We wish to show that $\limsup_{L\to\infty} f^+_L(t) \leq f^+_{2^{k_0}}(t)$.

For $L > 0$ large enough, let $M \geq 1$ be such that $L = M2^{k_0} + 2K$, with $K \in [2^{k_0-1}, 2^{k_0})$. Divide the domain $[0, L]^2$ into the inside square $Q := [K, L-K]^2$ and the annulus $A := [0, L]^2 \setminus Q$. Then, as in the proof of the above lemma, we have

$$\mathbb{E}^+\big[\mathrm{e}^{tm_L}\big] \leq \mathbb{E}^+\big[\mathrm{e}^{tm_{2^{k_0}}}\big]^{M^2}\mathbb{E}^+\big[\mathrm{e}^{tm_A}\big], \tag{2.7}$$

where m_A denotes the magnetization in the annulus A with + boundary conditions on its inner and outer boundaries. One can split this annulus into a number ($\leq 4L/K$) of squares of side-length K plus possibly 4 identical rectangles (up to a rotation) with one side of length K and the other side of shorter length $\tilde{K}$—see Figure 1. Call $R_1, \ldots, R_4$ those rectangles and let $\mathcal{R}$ be the family of possible shapes they can have. Then, we have

$$\begin{aligned}\mathbb{E}^+\big[\mathrm{e}^{tm_A}\big] &\leq \mathbb{E}^+\big[\mathrm{e}^{tm_K}\big]^{4L/K} \sup_{R\in\mathcal{R}} \mathbb{E}^+\big[\mathrm{e}^{tm_R}\big]^4 \\ &\leq \mathbb{E}^+\big[\mathrm{e}^{tm_K}\big]^{8(M+1)} \sup_{R\in\mathcal{R}} \mathbb{E}^+\big[\mathrm{e}^{tm_R}\big]^4. \end{aligned}\tag{2.8}$$

Now for any rectangle $R = [0, \tilde{K}] \times [0, K]$ with $0 < \tilde{K} \leq K$, one has

$$\begin{aligned}\mathbb{E}^+\big[\mathrm{e}^{tm_R}\big] &\leq \lim_{a\to 0} \mathbb{E}^+\big[\mathrm{e}^{tm^a_R}\big] \\ &\leq \lim_{a\to 0} \exp\Big(t\mathbb{E}^+\big[m^a_R\big] + \frac{t^2}{2}\mathbb{E}^+\big[\big(m^a_R - \langle m^a_R\rangle\big)^2\big]\Big),\end{aligned}$$

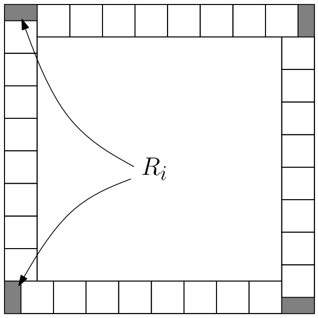

Fig. 1. The annulus A and the rectangles $R_1, \dots, R_4$.

using the GHS inequality (see Theorem 3.6 and Corollary 3.7 of [2]). As in the Appendix B of [2], it is easy to check that

$$\sup_{R\in\mathcal{R}}\Big(\limsup_{a\to 0}\big\{\mathbb{E}^+\big[m_R^a\big]+\mathbb{E}^+\big[\big(m_R^a\big)^2\big]\big\}\Big)<\infty,$$

which thus implies

$$\sup_{R\in\mathcal{R}}\mathbb{E}^+\big[\mathrm{e}^{tm_R}\big]<\infty. \tag{2.9}$$

In the same fashion, we have that

$$\sup_{K\in[2^{k_0-1},2^{k_0})}\mathbb{E}^+\big[\mathrm{e}^{tm_K}\big]<\infty. \tag{2.10}$$

Plugging the previous estimates into (2.7), we obtain

$$\frac{1}{L^2}\log\mathbb{E}^+\big[\mathrm{e}^{tm_L}\big]\le\frac{M^2}{L^2}\log\mathbb{E}^+\big[\mathrm{e}^{tm_{2^{k_0}}}\big]+\frac{8M+8}{L^2}\sup_K\log\mathbb{E}^+\big[\mathrm{e}^{tm_K}\big]+\frac{4}{L^2}\sup_{R\in\mathcal{R}}\log\mathbb{E}^+\big[\mathrm{e}^{tm_R}\big].$$

By letting $L, M\to\infty$, the last two terms tend to zero, while the first one converges to $f^+_{2^{k_0}}(t)$, which ends the proof of the lemma. □

Proof of Lemma 2.12. It is clear, by monotonicity, that for any $t\ge 0$, $f^-(t)\le f^+(t)$. Let us then show the reverse inequality. We will in fact compare the plus boundary conditions with free boundary conditions showing with the obvious notation that $f^{\mathrm{free}}(t)\ge f^+(t)$. Since the same proof allows us to show that $f^{\mathrm{free}}(t)\le f^-(t)$, this is enough to conclude the proof.

We wish to show that

$$f^{\mathrm{free}}(t):=\liminf_{k\to\infty}2^{-2k}\log\mathbb{E}^{\mathrm{free}}\big[\mathrm{e}^{tm_{2^k}}\big]\ge f^+(t)=\lim_{k\to\infty}2^{-2k}\log\mathbb{E}^+\big[\mathrm{e}^{tm_{2^k}}\big].$$

Note that we used lim inf to define f^{free} here since we have not proved (yet) that the limit exists in the case of free boundary conditions and lim inf is the worst possible case here.

Let us fix some small dyadic $\varepsilon=2^{-k_0}>0$. For any $L>10$, let $R=R_{L,\varepsilon}$ be the event that there is a $+$ cluster going around the annulus $a\mathbb{Z}^2\cap[0,L]^2\setminus[\varepsilon L,(1-\varepsilon)L]^2$. From the RSW Theorem in [7], we have that

$$H:=\inf_{L>10,a<1}\mathbb{P}^{\mathrm{free}}[R_{L,\varepsilon}]>0.$$

Recall furthermore that for any $L>10$:

$$\frac{1}{L^2}\log\mathbb{E}^{\mathrm{free}}\big[\mathrm{e}^{tm_L}\big]=\lim_{a\to 0}\frac{1}{L^2}\log\mathbb{E}^{\mathrm{free}}\big[\mathrm{e}^{tm_L^a}\big].$$

We have that

$$\begin{aligned}\liminf_{L\to\infty}\frac{1}{L^2}\log\mathbb{E}^{\text{free}}\big[e^{tm_L}\big] &\ge \liminf_{L\to\infty}\lim_{a\to 0}\frac{1}{L^2}\log\mathbb{E}^{\text{free}}\big[1_R e^{tm_L^a}\big]\\ &\ge \liminf_{L\to\infty}\lim_{a\to 0}\frac{1}{L^2}\log\mathbb{E}^{\text{free}}\big[e^{tm_L^a}\mid R\big]+\lim_{L\to\infty}\frac{1}{L^2}\log H\\ &= \liminf_{L\to\infty}\lim_{a\to 0}\frac{1}{L^2}\log\mathbb{E}^{\text{free}}\big[e^{tm_L^a}\mid R\big].\end{aligned}$$

For each dyadic $L = L_k = 2^k > 10$, let us divide the square $[0,L]^2$ into the annulus $A = A_L = [0,L]^2 \setminus [\varepsilon L,(1-\varepsilon)L]^2$ and the inside square $Q = Q_L = [\varepsilon L,(1-\varepsilon)L]^2$. As such and with the obvious notation, we will decompose the magnetization m_L^a into

$$m_L^a = m_A^a + m_Q^a. \tag{2.11}$$

Furthermore, we will denote by $\mathcal{F}_Q$ the filtration generated by the spins in $a\mathbb{Z}^2\cap Q$. By conditioning furthermore on $\mathcal{F}_Q$, we get

$$\liminf_{k\to\infty}\frac{1}{L_k^2}\log\mathbb{E}^{\text{free}}\big[e^{tm_{L_k}}\big]\ge\liminf_{k\to\infty}\lim_{a\to 0}\frac{1}{L_k^2}\log\mathbb{E}^{\text{free}}\big[e^{tm_Q^a}\mathbb{E}^{\text{free}}\big[e^{tm_A^a}\mid\mathcal{F}_Q,R\big]\bigm| R\big]. \tag{2.12}$$

Let us first show the following lemma.

Lemma 2.14. *There is a function $\alpha=\alpha(\varepsilon)>0$ such that uniformly in $L=2^k>10$ and on the configuration of spins σ_Q inside Q, one has*

$$\lim_{a\to 0}\mathbb{E}^{\text{free}}\big[e^{tm_A^a}\mid\mathcal{F}_Q,R\big]\ge\alpha(\varepsilon). \tag{2.13}$$

Proof. To prove the lemma, notice that by our choice of ε, the annulus A can be divided into $4(2^{k_0}-1)$ exact squares of side-length 2^{k-k_0} (as in Figure 1 except there are no thin rectangles there) and we have the bound

$$\lim_{a\to 0}\mathbb{E}^{\text{free}}\big[e^{tm_A^a}\mid\mathcal{F}_Q,R\big]\ge\lim_{a\to 0}\mathbb{E}^{-}\big[e^{tm^a_{2^{k-k_0}}}\big]^{4(2^{k_0}-1)}\quad\text{using FKG} \tag{2.14}$$

$$\ge\lim_{a\to 0}\mathbb{P}^{-}\big[m^a_{2^{k-k_0}}>0\big]^{4(2^{k_0}-1)} \tag{2.15}$$

$$\ge\mathbb{P}^{-}[m_{2^{k-k_0}}>0]^{4(2^{k_0}-1)} \tag{2.16}$$

$$=\mathbb{P}^{-}[m_{[0,1]^2}>0]^{4(2^{k_0}-1)}\ge\mathbb{P}^{-}[m_{[0,1]^2}>0]^{4/\varepsilon}, \tag{2.17}$$

where in the last line, we relied on the scaling covariance property given by Proposition 2.6. □

We conclude the proof of Lemma 2.14 by relying on the following easy lemma.

Lemma 2.15. *There is a constant $c>0$ such that*

$$\mathbb{P}^{-}[m_{[0,1]^2}>0]>c.$$

Proof (Sketch). Since $m^a_{[0,1]^2}$ converges in law to $m_{[0,1]^2}$, it is enough to prove that under $\mathbb{P}^-$, $m^a_{[0,1]^2}$ is bounded from below by some positive constant with uniform positive probability as $a\to 0$. To show this, use the FK representation of the spin-Ising model as in [2] and as in Section 3 below, in order to write $m^a_{[0,1]^2}$ as follows:

$$m^a_{[0,1]^2}=-\mathcal{A}_0+\sum_i\sigma_i\mathcal{A}_i,$$

where $\mathcal{A}_i = \mathcal{A}_i^a(C_i), i \geq 1$ stand for the renormalized areas of the FK clusters C_i which do not intersect $\partial[0,1]^2$, $\mathcal{A}_0$ stands for the renormalized area of the cluster C_0 intersecting $\partial[0,1]^2$, and $\{\sigma_i\}_i$ are independent ± 1 balanced Bernoulli variables. By a now standard second moment argument (exactly as in Appendix B in [2]), one can easily show (without using this FK representation) that there is some constant $M < \infty$ such that $\mathbb{P}[m_{[0,1]^2} > -M] > 1/2$. This means that the above sum is above $-M$ with probability more than $1/2$ uniformly in $a \to 0$. Now for each $\varepsilon > 0$, Lemma 3.1 below tells us that one can find $\Omega(1/\varepsilon^2)$ small FK clusters whose renormalized areas are of order $\varepsilon^{15/8}$. (Note that if one does not want to use Wu here, one still obtains $\Omega(1/\varepsilon^2)$ clusters whose renormalized areas are of order $\varepsilon^2 \varrho(\varepsilon)^{-1/2}$ which is enough.) One thus concludes the proof of the lemma by choosing ε small enough w.r.t. the value of M and forcing all these ε-clusters to have a + associated spin (which costs a uniform positive probability of order c^{1/ε^2}). □

Hence this ends the proof of Lemma 2.14 with $\alpha(\varepsilon) := c^{4/\varepsilon}$. □

Plugging (2.13) into (2.12) gives us

$$\liminf_{k\to\infty} \frac{1}{L_k^2} \log \mathbb{E}^{\text{free}}\big[e^{tm_{L_k}}\big] \geq \liminf_{k\to\infty} \lim_{a\to 0} \frac{1}{L_k^2} \log \mathbb{E}^{\text{free}}\big[e^{tm_Q^a} \bigm| R\big]. \tag{2.18}$$

Now, by FKG it is clear that

$$\begin{aligned}\lim_{a\to 0} \mathbb{E}^{\text{free}}\big[e^{tm_Q^a} \bigm| R\big] &\geq \lim_{a\to 0} \mathbb{E}^{+}\big[e^{tm_Q^a}\big] \\ &= \mathbb{E}^{+}\big[e^{tm_Q}\big],\end{aligned} \tag{2.19}$$

where in the latter expectations, the + boundary conditions are around $[0, L_k]^2$ and hence are further away from the domain $Q = Q_{L_k}$.

To conclude the proof of Lemma 2.12 we still need to compare $\mathbb{E}^+[e^{tm_Q}]$ with $\mathbb{E}^+[e^{tm_L}]$. This is done by the following lemma.

Lemma 2.16. *There is a function $\eta(x) > 0$ satisfying $\eta(x) \to 0$ as $x \to 0$, and such that for any $\varepsilon = 2^{-k_0}$, one has, with the above notation,*

$$\lim_{k\to\infty} \frac{1}{L_k^2} \log \mathbb{E}^{+}\big[e^{tm_Q}\big] \geq \lim_{k\to\infty} \frac{1}{L_k^2} \log \mathbb{E}^{+}\big[e^{tm_{L_k}}\big] - \eta(\varepsilon). \tag{2.20}$$

Proof. As in the proof of Lemma 2.11, and dividing $[0, L_k]^2$ as above, we have

$$\mathbb{E}^{+}\big[e^{tm_{L_k}}\big] = \mathbb{E}^{+}\big[e^{tm_Q + tm_A}\big] \tag{2.21}$$

$$\qquad \leq \mathbb{E}^{+}\big[e^{tm_Q}\big]\mathbb{E}^{+}\big[e^{tm_{2^{k-k_0}}}\big]^{4(2^{k_0}-1)}, \tag{2.22}$$

where, as above, the boundary conditions in the expectation $\mathbb{E}^+[e^{tm_Q}]$ are meant to be around the larger square $[0, L_k]^2$. Now, we have

$$\frac{1}{L_k^2} \log \mathbb{E}^{+}\big[e^{tm_{2^{k-k_0}}}\big]^{4(2^{k_0}-1)} = \frac{4(2^{k_0}-1)}{2^{2k_0}} \frac{1}{2^{2(k-k_0)}} \log \mathbb{E}^{+}\big[e^{tm_{2^{k-k_0}}}\big] \tag{2.23}$$

$$\qquad \leq 4\varepsilon f^{+}_{L_{k-k_0}}(t). \tag{2.24}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{1}{L_k^2} \log \mathbb{E}^{+}\big[e^{tm_Q}\big] \geq f^{+}(t) - 4\varepsilon f^{+}(t). \tag{2.25}$$

This ends the proof of Lemma 2.16. □

To conclude the proof of Lemma 2.12, we plug (2.25) into (2.18) and obtain, using (2.19),

$$\liminf_{k\to\infty} \frac{1}{L_k^2} \log \mathbb{E}^{\mathrm{free}}\big[\mathrm{e}^{tm_{L_k}}\big] \geq f^+(t) - 4\varepsilon f^+(t),$$

for any value of $\varepsilon = 2^{-k_0} > 0$. Hence, we have that

$$\liminf_{k\to\infty} \frac{1}{L_k^2} \log \mathbb{E}^{\mathrm{free}}\big[\mathrm{e}^{tm_{L_k}}\big] \geq f^+(t), \tag{2.26}$$

which thus implies

$$f^{\mathrm{free}}(t) = \liminf_{k\to\infty} \frac{1}{L_k^2} \log \mathbb{E}^{\mathrm{free}}\big[\mathrm{e}^{tm_{L_k}}\big] = f^+(t). \tag{2.27}$$

□

Proof of Lemma 2.13. As in the proof of Proposition 2.2, using the scaling covariance given by Proposition 2.6, we have that for any $L > 0$ and any $t > 0$ and for, say, $+$ boundary conditions,

$$tm_L \stackrel{(d)}{=} m_{Lt^{8/15}}.$$

This implies

$$\begin{aligned} f(t) = f^+(t) &= \lim_{L\to\infty} \frac{1}{L^2} \log \mathbb{E}^+\big[\mathrm{e}^{tm_L}\big] \\ &= \lim_{L\to\infty} \frac{1}{L^2} \log \mathbb{E}^+\big[\mathrm{e}^{m_{Lt^{8/15}}}\big] \\ &= t^{16/15} \lim_{\bar{L}\to\infty} \frac{1}{\bar{L}^2} \log \mathbb{E}^+\big[\mathrm{e}^{m_{\bar{L}}}\big] \\ &= f(1)t^{16/15}. \end{aligned}$$

To conclude the proof of the lemma when $t > 0$, it remains to show that the quantity (with $L_k = 2^k$)

$$f(1) = f^+(1) = \lim_{k\to\infty} \frac{1}{L_k^2} \log \mathbb{E}^+\big[\mathrm{e}^{m_{L_k}}\big]$$

is strictly positive.

To see this, let us first denote by M_L the magnetization $\Phi^\infty(1_{[0,L]^2})$ in $[0, L]^2$ of the full-plane field Φ^∞. By the results of [2], for any $L \in (0, \infty)$, M_L has zero mean and variance in $(0, \infty)$. Then by a few uses of the FKG inequalities, we have

$$\begin{aligned} \mathbb{E}^+\big[\mathrm{e}^{m_{2^k}}\big] &\geq \mathbb{E}\big[\mathrm{e}^{M_{2^k}}\big] \\ &\geq \big(\mathbb{E}\big[\mathrm{e}^{M_1}\big]\big)^{L_k^2} \\ &\geq \big(1 + \mathbb{E}\big[M_1^2\big]\big)^{L_k^2}, \end{aligned}$$

so that $f^+(1) \geq \log(1 + \mathbb{E}[M_1^2]) > 0$. □

3. Analyticity of the probability density function of *m*

In this section, we shall prove Theorem 1.3. First of all, by the convergence in law of m^a towards m, we have, as $a \to 0$:

$$\mathbb{E}^\xi\big[\mathrm{e}^{\mathrm{i}tm^a}\big] \to \mathbb{E}^\xi\big[\mathrm{e}^{\mathrm{i}tm}\big]. \tag{3.1}$$

It is thus sufficient to prove that there exists a constant $c > 0$ which is such that, for any $t \in \mathbb{R}$,

$$\limsup_{a\to 0}\left|\mathbb{E}^{\xi}\left[\mathrm{e}^{\mathrm{i}tm^a}\right]\right| \le \mathrm{e}^{-c|t|^{16/15}}.$$

To prove this, we will rely on the FK representation of the Ising model in $a\mathbb{Z}^2 \cap [0,1]^2$ endowed with its boundary conditions $\xi \in \{+, -, \text{free}\}$. Let us assume that $\xi = +$. (The case of free boundary conditions is even easier.) We can write

$$\begin{aligned}\left|\mathbb{E}^{+}\left[\mathrm{e}^{\mathrm{i}tm^a}\right]\right| &= \left|\mathbb{E}^{\mathrm{FK}}\left[\mathrm{e}^{\mathrm{i}t\mathcal{A}^+}\prod_{C_i}\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}t\mathcal{A}_i}+\mathrm{e}^{-\mathrm{i}t\mathcal{A}_i}\right)\right]\right| \\ &= \left|\mathbb{E}^{\mathrm{FK}}\left[\mathrm{e}^{\mathrm{i}t\mathcal{A}^+}\prod_{C_i}\cos t\mathcal{A}_i\right]\right|, \end{aligned} \tag{3.2}$$

where $\{C_i\}_i$ denotes the collection of clusters that do not intersect the boundary and C^+ is the cluster that intersects the boundary. Furthermore, we let $\mathcal{A}_i = \mathcal{A}_i^a = \mathcal{A}_i^a(C_i)$ stand for the renormalized areas of the cluster C_i, and $\mathcal{A}^+$ for the renormalized area of the cluster C^+. Our strategy, in order to obtain an upper bound for (3.2), is to show that with high probability, there are many clusters in $\{C_i\}_i$ with a renormalized area of order $1/t$.

We will rely on the following lemma:

Lemma 3.1. *There exist constants $c \in (0,1)$ and $M > 1$ such that for any $0 < \varepsilon < 1/10$ and any ε-square Q inside $[0,1]^2$, uniformly as $a \to 0$, and uniformly on the FK configuration outside of Q, with (conditional) probability at least $c > 0$, one can find at least one FK cluster C inside Q that does not intersect ∂Q and such that its renormalized area lies in the interval $[\varepsilon^{15/8}/M, M\varepsilon^{15/8}]$.*

Proof. Let Q be an ε-square inside $[0,1]^2$ and let ω_a be any FK-configuration outside Q. Let A_1 be the annulus $Q \setminus 7/8Q$, A_2 the annulus $7/8Q \setminus 3/4Q$ and A_3 the annulus $3/4Q \setminus 1/2Q$. Let us introduce the following events: let D_1 be the event that there is a *dual* circuit in the annulus A_1 and let O_2 and O_3 be the events that there is an *open* circuit around each annuli A_2 and A_3. Using the RSW Theorem from [7] for a free boundary condition, one has that there is a constant $b > 0$ such that uniformly on the outside configuration ω_a, one has

$$\mathbb{P}^{+}[D_1, O_2, O_3 \mid \omega_a] > b. \tag{3.3}$$

Now let $X = X_a$ be the number of points in $a\mathbb{Z}^2 \cap 1/2Q$ which are connected via an open-arm to $\partial(\frac{3}{4}Q)$. Then using similar computations as in Proposition B.2 in [2] or in Lemma 3.1 in [3], one can find a constant $C > 0$ such that

$$\begin{cases}\mathbb{E}^{+}[X_a \mid D_1, O_2, O_3, \omega_a] \ge (\varepsilon/a)^{15/8}/C, \\ \mathbb{E}^{+}[X_a^2 \mid D_1, O_2, O_3, \omega_a] \le C(\varepsilon/a)^{15/4}. \end{cases} \tag{3.4}$$

By a standard second-moment argument, and using the fact that all points counted in X_a belong to the same cluster (thanks to O_3), one obtains that with positive conditional probability, one can find a cluster C which does not intersect ∂Q and whose renormalized mass is larger than $1/M\varepsilon^{15/8}$. (Note that the event O_2 is there to ensure some positive information inside $3/4Q$.)

It remains to prove an upper bound. In the same way as X_a is smaller than the actual number of points in the open cluster we are interested in, one can also introduce $\tilde{X}_a$ to be the number of points inside the whole square Q which are connected to the boundary $\partial(3/4Q)$. This random variable dominates the size of the cluster we are interested in. It is enough to control its expectation and it is easy to see that, for a well-chosen constant $C > 0$, one has

$$\begin{aligned}\mathbb{E}^{+}[\tilde{X}_a \mid \omega_a] &\le \mathbb{E}^{+}[\tilde{X}_a \mid \text{wired } \partial Q] \\ &\le C(\varepsilon/a)^{15/8}. \end{aligned}$$

Since $\mathbb{P}^+[D_1, O_2, O_3 \mid \omega_a] > b$, this implies

$$\mathbb{P}^+\big[\tilde{X}_a \geq M(\varepsilon/a)^{15/8} \bigm| D_1, O_2, O_3, \omega_a\big] \leq \frac{1}{M}\frac{C}{b}.$$

By choosing M large enough (so that the conditional probabilities of lower bound and upper bound do not add up to something larger than one), one concludes the proof of the lemma. □

Proof of Theorem 1.3. For any $|t| > 100$, choose $\varepsilon = \varepsilon_t$ so that $M\varepsilon^{15/8} = \frac{1}{|t|}$ (we use the constants from Lemma 3.1). Use a tiling of the square $[0,1]^2$ so that one has $\frac{1}{2}\varepsilon^{-2}$ disjoint ε-squares Q. Recall from that lemma that for each such square Q, the probability that one has a cluster inside Q with renormalized area in $[(1/M)\varepsilon^{15/8}, M\varepsilon^{15/8}]$ is larger than $c > 0$ uniformly on what may happen outside of Q. We thus expect that at least about $\frac{c}{2}\varepsilon^{-2}$ squares Q will contain such a cluster. Let G be the event that at last $\frac{c}{4}\varepsilon^{-2}$ squares Q have a cluster with renormalized area in $[(1/M)\varepsilon^{15/8}, M\varepsilon^{15/8}]$. Then, by a classical Hoeffding inequality one has that

$$\mathbb{P}^+\big[G^c\big] \leq \mathrm{e}^{-d\varepsilon_t^{-2}} = \mathrm{e}^{-dM^{16/15}|t|^{16/15}}, \tag{3.5}$$

for some universal constant $d > 0$. Now, on the event G, we have

$$\begin{aligned}\Big|\mathrm{e}^{\mathrm{i}t\mathcal{A}^+}\prod_{C_i}\cos t\mathcal{A}_i\Big| &\leq \big[\cos 1/M^2\big]^{(c/4)\varepsilon_t^{-2}} \\ &= \big[\cos 1/M^2\big]^{(c/4)M^{16/15}|t|^{16/15}} \\ &\leq \mathrm{e}^{-\tilde{c}|t|^{16/15}},\end{aligned}$$

for some well-chosen constant $\tilde{c} > 0$. Combining the above estimate with equations (3.2) and (3.5), we thus end the proof of Theorem 1.3 with a possibly smaller value of $\tilde{c} > 0$ (due to $\mathbb{P}^+[G^c]$ as well as to the region $|t| \in [0, 100]$). □

Remark 3.2. *As suggested after Theorem* 1.3, *it should be possible to extend the above proof to basically any boundary conditions ξ (with the only constraint that one can prove a scaling limit result for m^a as in* [2]). *For example, if ξ is made of a finite combination of* $+, -,$ free *arcs, this is handled in* [2]. *In this latter case, one would rely on the following extension of* (3.2):

$$\begin{aligned}\big|\mathbb{E}^\xi\big[\mathrm{e}^{\mathrm{i}tm^a}\big]\big| &= \Big|\mathbb{E}^{\mathrm{FK},\xi}\Big[\prod_{C_i}\frac{1}{2}\big(\mathrm{e}^{\mathrm{i}t\mathcal{A}_i} - \mathrm{e}^{-\mathrm{i}t\mathcal{A}_i}\big)\prod_{C_k^+}\mathrm{e}^{\mathrm{i}t\mathcal{A}_k}\prod_{C_l^-}\mathrm{e}^{-\mathrm{i}t\mathcal{A}_l}\Big]\Big| \\ &\leq \mathbb{E}^{\mathrm{FK},\xi}\Big[\prod_{C_i}|\cos t\mathcal{A}_i|\Big].\end{aligned}$$

The additional difficulty when ξ is a general boundary condition lies in the FK-representation of the associated Ising model. Indeed, general boundary conditions induce negative information in the bulk (since the FK configuration is now conditioned to disconnect + and − arcs). But one can see from the above proof that negative information in fact makes Lemma 3.1 *even more likely. Indeed it makes the event D_1 of having a dual crossing in the annulus A_1 more likely.*

Remark 3.3. *We note that* $\log|\mathbb{E}^\xi[\mathrm{e}^{\mathrm{i}tm}]|$ *cannot behave like* $-|t|^{16/15}$ *as $t \to \infty$ because, by the Lee–Yang theorem,* $\mathbb{E}^\xi[\mathrm{e}^{\mathrm{i}tm}]$, *as a function of complex t, has infinitely many zeros $\{t_i\}_{i\geq 1}$, all purely real and s.t. $t_i \to \infty$. In particular,* $\log|\mathbb{E}^\xi[\mathrm{e}^{\mathrm{i}tm}]|$ *must diverge to $-\infty$ near each zero t_i, $i \geq 1$.*

4. Near-critical magnetization fields

We start by establishing Proposition 1.5.

Proof of Proposition 1.5. Let us assume that the boundary condition ξ is $+$ along $\partial\Omega$ (the case of free b.c. is treated in the same manner). The Ising model with an external field $h_a := ha^{15/8}$ can be thought of as a simple change of measure with respect to the Ising model without external field. In particular, one has for any field Φ:

$$\mathbb{P}\big[\Phi^{a,h} = \Phi\big] = \frac{\mathrm{e}^{h\langle\Phi,1_\Omega\rangle}}{\mathbb{E}[\mathrm{e}^{h\langle\Phi^a,1_\Omega\rangle}]}\mathbb{P}\big[\Phi^{a,h=0} = \Phi\big].$$

Or, written in terms of the Radon–Nikodym derivative, one has

$$\frac{\mathrm{d}\mu_\Omega^{a,h}}{\mathrm{d}\mu_\Omega^{a}}(\Phi) = \frac{\mathrm{e}^{h\langle\Phi,1_\Omega\rangle}}{\mathbb{E}[\mathrm{e}^{h\langle\Phi^a,1_\Omega\rangle}]} = \frac{\mathrm{e}^{h\langle\Phi,1_\Omega\rangle}}{\mu_\Omega^a[\mathrm{e}^{h\langle\Phi,1_\Omega\rangle}]},$$

where $\mu^{a,h}$ and μ^a denote respectively the laws of $\Phi_\Omega^{a,h}$ and Φ_Ω^a. Now it is not hard to check, using the fact that $\Phi^\infty \sim \mu_\Omega^\infty$ has exponential moments (Proposition 2.1), that $\mu_\Omega^{a,h}$ converges weakly for the topology of $\mathcal{H}^{-3}(\Omega)$ to the measure $\mu_\Omega^{\infty,h}$ which is absolutely continuous w.r.t. μ_Ω^∞ and whose Radon–Nikodym derivative is given by

$$\frac{\mathrm{d}\mu_\Omega^{\infty,h}}{\mathrm{d}\mu_\Omega^{\infty}}(\Phi) = \frac{\mathrm{e}^{h\langle\Phi,1_\Omega\rangle}}{\mathbb{E}[\mathrm{e}^{h\langle\Phi^\infty,1_\Omega\rangle}]}.$$

We refer to the Appendix A of [2] for details on the topological setup used here ($\mathcal{H}^{-3}$). □

We now wish to prove Theorem 1.4. It is based on the lemma below together with Proposition 1.5. In what follows, for each $L \in \mathbb{N}$, we will denote by Λ_L the domain $[-2^L, 2^L]^2$.

Lemma 4.1. *For any $\alpha > 0$, there exists $L = L(h,\alpha) \in \mathbb{N}$ sufficiently large so that, uniformly in $0 < a < \alpha$, one can find a coupling of $\Phi_{\Lambda_L}^{a,h}$ with $\Phi_{\mathbb{C}}^{a,h}$ satisfying*

$$\mathbb{E}\big[\big\|\Phi_{\Lambda_L}^{a,h} - \Phi_{\mathbb{C}}^{a,h}\big\|_{\mathcal{H}_{\mathbb{C}}^{-3}}\big] < \alpha,$$

where $\|\cdot\|_{\mathcal{H}_{\mathbb{C}}^{-3}}$ is defined by

$$\|h\|_{\mathcal{H}_{\mathbb{C}}^{-3}} := \sum_{k\geq 1}\frac{1}{2^k}\big(\|h_{|\Lambda_k}\|_{\mathcal{H}_{\Lambda_k}^{-3}} \wedge 1\big).$$

Remark 4.2. *It is easy to check that the distance defined in Lemma 4.1 induces the same topology on $\mathcal{H}_{\mathbb{C}}^{-3}$ as the one introduced in Appendix* A *of* [2].

Proof of Lemma 4.1. Let $L_1 \in \mathbb{N}$ be such that $\sum_{k\geq L_1} 2^{-k} < \alpha/2$ (its value will be fixed later, also depending on the value of $h > 0$). We wish to find some $L_2 \gg L_1$ such that one can couple the fields $\Phi_{\Lambda_{L_2}}^{a,h}$ and $\Phi_{\mathbb{C}}^{a,h}$ in such a way that with probability at least $1-\alpha$ they are identical once restricted to the sub-domain Λ_{L_1}. By the definition of $\|\cdot\|_{\mathcal{H}_{\mathbb{C}}^{-3}}$, this will clearly imply our result with $L = L_2$.

The coupling will be constructed similarly as in [8] and [2]. We will also use the FK representation with a ghost vertex used in [3] (see also for example [9]). We refer to [3] for more details on this representation. Since the proof below will follow very closely the proof of the lower bound given in Section 3 in [3], we will not give the full details here.

Following the notation of [3], let $\bar{\omega}_{\mathbb{C}}^{a,h} = (\omega_{\mathbb{C}}^{a,h}, \tau_{\mathbb{C}}^{a,h})$ and $\bar{\omega}_{L_2}^{a,h} = (\omega_{L_2}^{a,h}, \tau_{L_2}^{a,h})$ be respectively the FK representations of the Ising model with external field $h > 0$ on $a\mathbb{Z}^2$ and on $a\mathbb{Z}^2 \cap \Lambda_{L_2}$ with $+$ boundary conditions. These configurations are FK percolation configurations on the graph $a\mathbb{Z}^2 \cup \{\mathbf{g}\}$ and the notation $\bar{\omega} = (\omega, \tau)$ distinguishes between the nearest neighbor edges in $a\mathbb{Z}^2$ (ω) and the edges of the type $\langle x, \mathbf{g}\rangle$, with $x \in a\mathbb{Z}^2$ (τ). Furthermore, it is easy to check that $\omega_{L_2}^{a,h}$ stochastically dominates $\omega_{\mathbb{C}}^{a,h}$. Let us divide the annulus A_{L_1,L_2} into disjoint annuli of ratio 4: namely $A_1 := A_{4^{-1}L_2,L_2}$, $A_2 := A_{4^{-2}L_2,4^{-1}L_2}$ and so on. As such one has about $\log_4(L_2/L_1)$ annuli. As in [2], we will explore "inward" both configurations by preserving the monotonicity $\omega_{\mathbb{C}} \preccurlyeq \omega_{L_2}$ and by trying to find a matching circuit γ in each annulus with positive probability. As in [2], the main ingredient for the coupling is the RSW theorem from [7]. The difference in our present setting is that one also has to deal with the influence of the *ghost* vertex $\mathbf{g}$. In particular, finding a matching circuit γ is not enough if one wants to claim that the conditional laws "inside" γ are the same: one also has to make sure that the circuit γ is connected in both configurations to the ghost vertex $\mathbf{g}$.

We proceed as follows. Assume we did not succeed in coupling the two configurations in the first $i-1$ annuli $A_1, \dots, A_{i-1}$ and consider the ith annulus $A_i = A_{4^{-i}L_2,4^{-(i-1)}L_2}$. At this point, the configurations have been explored everywhere except inside the outer boundary of A_i, and $\omega_{L_2}^{a,h}$ dominates $\omega_{\mathbb{C}}^{a,h}$. Inside the annulus A_i, we will distinguish 3 sub-annuli: $B_1 = A_{3.4^{-i},4^{-i+1}}$, $B_2 = A_{2.4^{-i},3.4^{-i}}$ and $B_3 = A_{4^{-i},2.4^{-i}}$. From the RSW theorem of [7], there are open circuits in $\omega_{\mathbb{C}}^{a,h}$ (and thus $\omega_{L_2}^{a,h}$) with positive probability $c > 0$ in each of B_1, B_2, B_3. This is due to the fact that $\omega_{\mathbb{C}}^{a,h}$ dominates a critical FK configuration with zero magnetic field and with wired boundary conditions along $\partial_1 A_i$ (see Section 3 in [3]). Furthermore, due to the positive information inside B_2 (thanks to the open circuits in each B_1 and B_3), it is easy to extend the techniques used to prove Lemma 3.1 in [3] (i.e. an appropriate second moment argument) to show that with positive probability $c > 0$, there are at least $c(4^{-i}L_2/a)^{15/8}$ points inside B_2 which are connected in $\omega_{\mathbb{C}}^{a,h}$ to the "outermost" open circuit γ for the configuration $\omega_{\mathbb{C}}^{a,h}$ in the annulus B_3. Since $(4^{-i}L_2)^{15/8} \geq (L_1)^{15/8}$, the exact same proof as for Lemma 3.2 of [2] shows that if one chooses L_1 large enough (depending on h), then conditioned on the above event of having at least $c(4^{-i}L_2/a)^{15/8}$ points connected to γ, with conditional probability at least $1/2$, the cluster including γ will be connected to the ghost vertex $\mathbf{g}$ for the configuration $\omega_{\mathbb{C}}^{a,h}$ (and thus for $\omega_{L_2}^{a,h}$ as well). Once $\omega_{\mathbb{C}}^{a,h}$ and $\omega_{L_2}^{a,h}$ have a matching circuit γ connected to $\mathbf{g}$, one can sample the rest of the configurations so that they match "inside" the circuit γ. (As in [2], the exploration process is driven by $\omega_{\mathbb{C}}^{a,h}$.) To conclude, we choose L_1 so that it satisfies the two constraints discussed above (i.e., $\sum_{k\geq L_1} 2^{-k} < \alpha/2$ and the constraint relative to $h > 0$). This gives us a certain positive probability $c > 0$ to couple both configurations in any annulus $A_{4^{-1}L,L}$, $L \geq L_1$. The proof is then completed by choosing $L = L(h,\alpha) = L_2$ so that $c^{\log_4(L_2/L_1)} < \alpha/2$. □

Proof of Theorem 1.4. By Proposition 1.5, for any $L \in \mathbb{N}$, one has that $\Phi_{\Lambda_L}^{a,h}$ converges in law to $\Phi_{\Lambda_L}^{\infty,h}$ in $\mathcal{H}_{\Lambda_L}^{-3}$. It is easy to check that this convergence in law also holds in the space $(\mathcal{H}_{\mathbb{C}}^{-3}, \|\cdot\|_{\mathcal{H}_{\mathbb{C}}^{-3}})$. Since this latter space is Polish, for any $\alpha > 0$, there exists $a_0 = a_0(\alpha) > 0$ such that, for any $a < a_0$, one can couple $\Phi_{\Lambda_L}^{a,h}$ with $\Phi_{\Lambda_L}^{\infty,h}$ so that

$$\mathbb{E}\big[\big\|\Phi_{\Lambda_L}^{a,h} - \Phi_{\Lambda_L}^{\infty,h}\big\|_{\mathcal{H}_{\mathbb{C}}^{-3}}\big] < \alpha.$$

By using this fact together with Lemma 4.1 and the fact that $(\mathcal{H}_{\mathbb{C}}^{-3}, \|\cdot\|_{\mathcal{H}_{\mathbb{C}}^{-3}})$ is Polish, one easily obtains that $\{\Phi_{\Lambda_L}^{\infty,h}\}_{L\in\mathbb{N}}$ converges in law in $\mathcal{H}_{\mathbb{C}}^{-3}$ as $L \to \infty$ to a limiting field $\Phi_{\mathbb{C}}^{\infty,h}$. Now that our limiting random field is defined, to conclude about the convergence in law of $\Phi_{\mathbb{C}}^{a,h}$ to this limiting field, we proceed in the same manner: for any $\varepsilon > 0$, one can find $a_0 > 0$ sufficiently small so that for any $a < a_0$, there exists a joint coupling $(\Phi_{\mathbb{C}}^{a,h}, \Phi_{\Lambda_L}^{a,h}, \Phi_{\Lambda_L}^{\infty,h}, \Phi_{\mathbb{C}}^{\infty,h})$ such that all fields are ε-close to each other (for $\|\cdot\|_{\mathcal{H}_{\mathbb{C}}^{-3}}$) with probability at least $1-\varepsilon$. This proves the convergence in law of $\Phi_{\mathbb{C}}^{a,h}$ to $\Phi_{\mathbb{C}}^{\infty,h}$. □

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