Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2009, Vol. 45, No. 3, 876–886
DOI: 10.1214/08-AIHP194

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Maximal Brownian motions

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Received 13 December 2007; accepted 29 August 2008

Abstract. Let $Z = (X, Y)$ be a planar Brownian motion, $\mathcal{Z}$ the filtration it generates, and B a linear Brownian motion in the filtration $\mathcal{Z}$. One says that B (or its filtration) is maximal if no other linear $\mathcal{Z}$-Brownian motion has a filtration strictly bigger than that of B. For instance, it is shown in [In *Séminaire de Probabilités XLI* 265–278 (2008) Springer] that B is maximal if there exists a linear Brownian motion C independent of B and such that the planar Brownian motion (B, C) generates the same filtration $\mathcal{Z}$ as Z. We do not know if this sufficient condition for maximality is also necessary.

We give a necessary condition for B to be maximal, and a sufficient condition which may be weaker than the existence of such a C. This sufficient condition is used to prove that the linear Brownian motion $\int (X\,\mathrm{d}Y - Y\,\mathrm{d}X)/|Z|$, which governs the angular part of Z, is maximal.

Résumé. Soient $Z = (X, Y)$ un mouvement brownien plan de filtration naturelle $\mathcal{Z}$, et B un mouvement brownien linéaire de la filtration $\mathcal{Z}$. On dit que B est maximal, et que la filtration naturelle de B est maximale, lorsqu'aucun autre mouvement brownien linéaire de $\mathcal{Z}$ n'engendre une filtration strictement plus grosse que celle de B. Il est par exemple établi dans [In *Séminaire de Probabilités XLI* 265–278 (2008) Springer] que B est maximal dès qu'il existe dans $\mathcal{Z}$ un mouvement brownien linéaire C indépendant de B et tel que le mouvement brownien plan (B, C) engendre toute la filtration $\mathcal{Z}$; nous ne savons pas si cette condition suffisante de maximalité est aussi nécessaire.

Nous donnons une conditions nécessaire de maximalité, ainsi qu'une condition suffisante peut-être plus faible que l'existence d'un tel C. À l'aide de cette condition suffisante, nous démontrons que le mouvement brownien linéaire $\int (X\,\mathrm{d}Y - Y\,\mathrm{d}X)/|Z|$, qui régit la partie angulaire de Z, est maximal.

MSC: 60J65; 60G07; 60G44

Keywords: Brownian filtration; Maximal Brownian motion; Exchange property

1. Introduction

In the theory of filtered probability spaces, it is natural to study pairs of filtrations, one immersed in the other, and to attempt to understand possible different ways the smaller filtration can be immersed in the larger. A simple case is to consider various ways the filtration of a one-dimensional Brownian motion can be immersed in that of a two-dimensional Brownian motion. This study was initiated by two of us, who introduced in Section 2 of [2] the notions of complementability and maximality (Definition 1 below), and gave examples and counter-examples. The present work, which concentrates on maximality criteria, can be considered a sequel to [2].

Let $Z = (Z_t)_{t\geq 0}$ denote a planar Brownian motion with $Z_0 = 0$ (all Brownian motions will be started from the origin); call $\mathcal{Z} = (\mathcal{Z}_t)_{t\geq 0}$ the filtration generated by Z. *In the sequel*, Z *and* $\mathcal{Z}$ *are fixed*, and interest will focus on one-dimensional $\mathcal{Z}$-Brownian motions (shortly: $\mathcal{Z}$-BM1) and their filtrations. If B is a $\mathcal{Z}$-BM1, its natural filtration $\mathcal{F}^B$ is immersed in $\mathcal{Z}$, that is, every $\mathcal{F}^B$-martingale is also a $\mathcal{Z}$-martingale. Consequently, $\mathcal{F}^B_t = \mathcal{Z}_t \cap \mathcal{F}^B_\infty$ for every

$t \geq 0$, so $\mathcal{F}^B$ is characterized by its end σ-field $\mathcal{F}^B_\infty = \sigma(B)$. Many properties that $\mathcal{F}^B$ may enjoy can equivalently be stated either in terms of $\mathcal{F}^B_t$ for all t, or in terms of $\sigma(B)$ only.

An example of such equivalent statements is the following double definition, borrowed from [2].

Definition 1. *Let B be a $\mathcal{Z}$-BM1.*

One says that B is complementable if there exists a $\mathcal{Z}$-BM1 C independent of B, and such that the $\mathcal{Z}$-BM2 (B, C) generates the filtration $\mathcal{Z}$ (equivalently, the $\mathcal{Z}$-BM2 (B, C) generates the σ-field $\sigma(Z)$).

One says that B is maximal if for every $\mathcal{Z}$-BM1 D, one has

$$\big(\forall t \geq 0 \; \mathcal{F}^B_t \subset \mathcal{F}^D_t\big) \quad \Longrightarrow \quad \big(\forall t \geq 0 \; \mathcal{F}^B_t = \mathcal{F}^D_t\big)$$

(equivalently, $\sigma(B) \subset \sigma(D) \Longrightarrow \sigma(B) = \sigma(D)$).

These two properties (to be complementable and to be maximal) can also be considered properties of the filtration $\mathcal{F}^B$ generated by B; in the sequel, we shall sometimes say that a filtration is complementable or maximal when it is generated by a complementable or maximal $\mathcal{Z}$-BM1.

An interesting example is obtained by setting

$$U_t = \int_0^t \frac{X_s \, \mathrm{d}X_s + Y_s \, \mathrm{d}Y_s}{\sqrt{X_s^2 + Y_s^2}} \quad \text{and} \quad V_t = \int_0^t \frac{X_s \, \mathrm{d}Y_s - Y_s \, \mathrm{d}X_s}{\sqrt{X_s^2 + Y_s^2}}.$$

The processes U and V are independent $\mathcal{Z}$-BM1 which respectively govern the radial part and angular part of Z, but (U, V) does not generate the whole filtration $\mathcal{Z}$. Actually, as recalled in Proposition 5, it only generates the quotient of $\mathcal{Z}$ by SO_2, that is, the filtration generated by Z up to an arbitrary rotation. Yet, U is complementable: an independent complement is exhibited in [2].

Corollary 6 of [2] states that *every complementable $\mathcal{Z}$-BM1 is maximal.* Thus, U is maximal. This corollary also gives many examples of non-complementable BM1, for instance $B' = \int \operatorname{sgn}(B) \, \mathrm{d}B$, where B is any $\mathcal{Z}$-BM1. Indeed, the filtration generated by B' is also generated by $|B|$, so it is strictly included in that of B, and not maximal; by the corollary it cannot be complementable.

Conversely, *is every maximal $\mathcal{Z}$-BM1 complementable?* We do not know. This question is already open in the simpler case when B is complementable except for a germ property at time $0+$ (this will be made precise in Definition 3). In that particular case, we shall give a necessary and sufficient condition for maximality (Theorem 1). Interestingly, this condition is an exchange property: it says that supremum and intersection of certain σ-fields commute.

The necessary and sufficient condition of Theorem 1 is no longer sufficient for an arbitrary $\mathcal{Z}$-BM1 to be maximal, but it is still necessary (Corollary 1).

Using a variation on this condition (Proposition 4), we establish maximality for the $\mathcal{Z}$-BM1 V defined above. When the present paper was first submitted, the question whether V is complementable, a possibly stronger property, was open. We now know that the answer is positive (September 2008); the construction by one of us of an independent complement to V is rather involved and needs to be simplified before publication.

2. Notation and reminders

In a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, all sub-$\sigma$-fields are assumed to be $(\mathcal{A}, \mathbb{P})$-complete. Similarly, all filtrations are right-continuous and $(\mathcal{A}, \mathbb{P})$-complete; a *raw filtration* is a filtration which is not necessarily right-continuous. If $S = (S_t)_{t \geq 0}$ is any process, the filtration generated by S is denoted by $\mathcal{F}^S$. For $0 \leq a < b < \infty$, we define new processes as follows:

$$S^{[a,b]} = (S_t)_{t \in [a,b]}, \qquad S^{]a,b]} = (S_{a+t} - S_a)_{t \in [0,b-a]},$$
$$S^{[a,\infty[} = (S_t)_{t \geq a}, \qquad S^{]a,\infty[} = (S_{a+t} - S_a)_{t \geq 0}.$$

If $\mathcal{G}$ is a filtration and $\mathcal{S}$ a σ-field, the *exchange property*

$$\bigcap_{h>0} (\mathcal{G}_{t+h} \vee \mathcal{S}) = \mathcal{G}_t \vee \mathcal{S}$$

does not hold in general; but it does when $\mathcal{G}$ and $\mathcal{S}$ are independent (see [5]). In plain words, independent enlargements preserve right-continuity. A key ingredient in the proof of this fact is that independent enlargements preserve conditional expectations: if $\mathcal{T}$ and $\mathcal{U}$ are two independent σ-fields, if T is $\mathcal{T}$-measurable and if $\mathcal{S}$ is a sub-σ-field of $\mathcal{T}$, then $\mathbb{E}[T|\mathcal{S}] = \mathbb{E}[T|\mathcal{S} \vee \mathcal{U}]$.

All Brownian motions are started at the origin. A planar Brownian motion $Z = (X, Y)$ is fixed; its natural filtration is denoted by $\mathcal{Z} = (\mathcal{Z}_t)_{t\geq 0}$.

Recall the previsible representation property: every $\mathcal{Z}$-martingale has the form $c + \int H\,\mathrm{d}X + \int K\,\mathrm{d}Y$, where c is a constant and H and K are $\mathcal{Z}$-previsible.

3. A necessary condition for maximality

Definition 2. *Two $\mathcal{Z}$-BM1 B and C are said to be parallel if $C = \int H\,\mathrm{d}B$ for some $\mathcal{Z}$-previsible process H.*

Since B and C are linear Brownian motions, $H = \pm 1$ on a $(\mathrm{d}t \times \mathrm{d}\mathbb{P})$-full set; so one also has $B = \int H\,\mathrm{d}C$. Parallelism is an equivalence relation.

Any two $\mathcal{Z}$-BM1 generating the same filtration, or more generally any two $\mathcal{Z}$-BM1 B and C such that the filtration $\mathcal{F}^B$ is included in $\mathcal{F}^C$, are parallel. Indeed, B is a $\mathcal{Z}$-martingale adapted to the sub-filtration $\mathcal{F}^C$, hence also an $\mathcal{F}^C$-martingale; so it is a stochastic integral w.r.t. C.

Proposition 1. (1) *Given any $\mathcal{Z}$-BM1 B, there always exists some $\mathcal{Z}$-BM1 independent of B.*

(2) *Given two $\mathcal{Z}$-BM1 B and C, the following are equivalent*:

(i) *B and C are parallel*;

(ii) *there exists a $\mathcal{Z}$-BM1 independent of B and independent of C*;

(iii) *every $\mathcal{Z}$-BM1 independent of B is also independent of C.*

Proof. (1) B has the form $\int H\,\mathrm{d}X + \int K\,\mathrm{d}Y$, where H and K are $\mathcal{Z}$-previsible, and $H^2 + K^2 = 1$. The martingale $D = -\int K\,\mathrm{d}X + \int H\,\mathrm{d}Y$ is also a $\mathcal{Z}$-BM1 since $\langle D, D\rangle = K^2 + H^2 = 1$; and D is independent of B since $\langle D, B\rangle = -KH + HK = 0$.

(i) $\Rightarrow$ (iii): if B and C are parallel, $C = \int H\,\mathrm{d}B$ for some previsible H; so if D is any $\mathcal{Z}$-BM1 independent of B, one has $\mathrm{d}\langle C, D\rangle = H\,\mathrm{d}\langle B, D\rangle = 0$, showing that D is independent of C too.

(iii) $\Rightarrow$ (ii): immediate consequence of 1.

(ii) $\Rightarrow$ (i): if D is a $\mathcal{Z}$-BM1 independent of B and of C, one has

$$\mathrm{d}B = H^B\,\mathrm{d}X + K^B\,\mathrm{d}Y, \qquad \mathrm{d}C = H^C\,\mathrm{d}X + K^C\,\mathrm{d}Y \quad \text{and} \quad \mathrm{d}D = H^D\,\mathrm{d}X + K^D\,\mathrm{d}Y,$$

where the vector (H^D, K^D) is $(\mathrm{d}t \times \mathrm{d}\mathbb{P})$-a.e. orthogonal to (H^B, K^B) and $(\mathrm{d}t \times \mathrm{d}\mathbb{P})$-a.e. orthogonal to (H^C, K^C). Hence, (H^B, K^B) and (H^C, K^C) are $(\mathrm{d}t \times \mathrm{d}\mathbb{P})$-a.e. parallel; so $(H^B, K^B) = L(H^C, K^C)$ for some previsible L defined $(\mathrm{d}t \times \mathrm{d}\mathbb{P})$-a.e. Consequently $B = \int L\,\mathrm{d}C$, showing parallelism. □

Open questions. *Given an arbitrary $\mathcal{Z}$-BM1 B, does there always exist a C parallel to B and maximal? Does there always exist a C parallel to B and complementable? We do not know.*

Proposition 2. *Let B be a $\mathcal{Z}$-BM1.*

(i) *The σ-fields $\mathcal{B}'_t = \bigcap_{\varepsilon>0}(\mathcal{Z}_\varepsilon \vee \mathcal{F}^B_t)$ form a filtration (i.e., they are right-continuous in t).*

(ii) *The filtration $\mathcal{B}'$ contains $\mathcal{F}^B$ and is generated by some $\mathcal{Z}$-BM1 B'.*

(iii) *Any such B' is parallel to B.*

(iv) *Applying the same procedure to B' instead of B does not yield anything new*: *the filtration $\mathcal{B}''$ defined by $\mathcal{B}''_t = \bigcap_{\varepsilon>0}(\mathcal{Z}_\varepsilon \vee \mathcal{B}'_t)$ is equal to $\mathcal{B}'$.*

Proof. (i) The raw filtration $\mathcal{B}'$ is right-continuous at 0 because $\mathcal{B}'_t \subset \mathcal{Z}_t$ and $\bigcap_{t>0} \mathcal{Z}_t$ is degenerate. It is also right-continuous at $t > 0$ for the following reason. For $\varepsilon > 0$, the raw filtration $(\mathcal{Z}_\varepsilon \vee \mathcal{F}^B_t)_{t\in[\varepsilon,\infty)}$ is generated by the σ-field

$\mathcal{Z}_\varepsilon$ and the Brownian motion $B^{]\varepsilon,\infty[}$ which is independent of that σ-field; by the independent exchange property, that raw filtration is right-continuous. Consequently, for $0<\varepsilon<t$ one has

$$\bigcap_{h>0}\mathcal{B}'_{t+h}\subset\bigcap_{h>0}\big(\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_{t+h}\big)=\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_t,$$

and right-continuity of $\mathcal{B}'$ follows from $\bigcap_{h>0}\mathcal{B}'_{t+h}\subset\bigcap_{\varepsilon>0}(\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_t)=\mathcal{B}'_t$.

(ii) The filtration $\mathcal{B}'$ clearly contains $\mathcal{F}^B$; to show that $\mathcal{B}'$ is generated by some $\mathcal{Z}$-BM1, it suffices by Corollary 1 of [3] to verify that $\mathcal{B}'$ is immersed in $\mathcal{Z}$ and is "1-Brownian after 0."

The immersion property will be established by checking that if R is any bounded, $\mathcal{B}'_\infty$-measurable r.v., $\mathbb{E}[R|\mathcal{Z}_t]$ is $\mathcal{B}'_t$-measurable. This is trivial for $t=0$ since $\mathcal{Z}_0$ is degenerate, so we suppose $t>0$. For any $\varepsilon\in]0,t]$, $\mathcal{B}'_\infty$ is included in $\mathcal{Z}_\varepsilon\vee\sigma(B)$, so one can write $R=f(Z^{[0,\varepsilon]},B^{[0,t]},B^{]t,\infty[})$ for some bounded Borel functional f. Now, the first two arguments of f are $\mathcal{Z}_t$-measurable and the third one is independent of $\mathcal{Z}_t$; hence $\mathbb{E}[R|\mathcal{Z}_t]=\int f(Z^{[0,\varepsilon]},B^{[0,t]},b)w(\mathrm{d}b)$, where w denotes the law of $B^{]t,\infty[}$. So $\mathbb{E}[R|\mathcal{Z}_t]$ is measurable for $\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_t$; as this holds for any $\varepsilon\in]0,t]$, $\mathbb{E}[R|\mathcal{Z}_t]$ is $\mathcal{B}'_t$-measurable and $\mathcal{B}'$ is immersed in $\mathcal{Z}$.

It remains to see that $\mathcal{B}'$ is 1-Brownian after 0, that is, given any $s>0$, the filtration $(\mathcal{B}'_t)_{t\geq s}$ is generated by its initial σ-field $\mathcal{B}'_s$ and some 1-dimensional BM independent of $\mathcal{B}'_s$. Now, for all $t\geq s$, one has

$$\mathcal{B}'_t=\bigcap_{\varepsilon\in]0,s]}\big(\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_s\vee\sigma\big(B^{]s,t]}\big)\big)=\Big(\bigcap_{\varepsilon\in]0,s]}\big(\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_s\big)\Big)\vee\sigma\big(B^{]s,t]}\big)=\mathcal{B}'_s\vee\sigma\big(B^{]s,t]}\big),$$

where the second equality (exchanging a supremum with an intersection) is due to the process $B^{]s,t]}$ being independent of the σ-field $\mathcal{Z}_s$, which contains all the σ-fields $\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_s$. The claim is established by observing that $B^{]s,\infty[}$ is a BM1 independent of $\mathcal{Z}_s$ and a fortiori of $\mathcal{B}'_s$. So $\mathcal{B}'$ is generated by some $\mathcal{Z}$-BM1 B'.

(iii) B' must be parallel to B because its filtration contains that of B.

(iv) The filtration $\mathcal{B}''$ obtained by iterating this procedure satisfies for $0<\varepsilon\leq t$ the inclusion

$$\mathcal{B}''_t\subset\mathcal{Z}_\varepsilon\vee\mathcal{B}'_t\subset\mathcal{Z}_\varepsilon\vee\big(\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_t\big)=\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_t.$$

Taking the intersection over ε gives $\mathcal{B}''_t\subset\mathcal{B}'_t$, hence equality. This extends to $t=0$ by right-continuity, or by observing that $\mathcal{B}''_0$ is degenerate. □

With the notation of Proposition 2, B cannot be maximal unless $\mathcal{F}^B$ equals the (a priori larger) filtration $\mathcal{B}'$. Thus we get the following corollary.

Corollary 1. *Let B be a $\mathcal{Z}$-BM1. A necessary condition for B to be maximal is the exchange property*

$$\forall t\geq 0\quad\bigcap_{\varepsilon>0}\big(\mathcal{Z}_\varepsilon\vee\mathcal{F}^B_t\big)=\mathcal{F}^B_t$$

(*or equivalently, the equality $\bigcap_{t>0}(\mathcal{Z}_t\vee\sigma(B))=\sigma(B)$ between the final σ-fields of these filtrations*).

4. Sufficient conditions for maximality

The following lemma was used in the proof of Corollary 6 of [2] (although it was not explicitly stated there).

Lemma 1. *Given $(\Omega,\mathcal{A},\mathbb{P})$, let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ be three ($(\mathcal{A},\mathbb{P})$-complete) sub-σ-fields of $\mathcal{A}$ verifying $\mathcal{S}\subset\mathcal{T}$ and $\mathcal{S}\vee\mathcal{U}=\mathcal{T}\vee\mathcal{U}$. If $\mathcal{U}$ is independent of $\mathcal{T}$, then $\mathcal{S}=\mathcal{T}$.*

Proof. It suffices to use an independent enlargement in conditional expectations so as to write, for any bounded, $\mathcal{T}$-measurable r.v. T,

$$\mathbb{E}[T|\mathcal{S}]=\mathbb{E}[T|\mathcal{S}\vee\mathcal{U}]=\mathbb{E}[T|\mathcal{T}\vee\mathcal{U}]=T.$$

□

Definition 3. *Let B be a $\mathcal{Z}$-BM1. We shall say that B is complementable after 0 if there exists a $\mathcal{Z}$-BM1 C independent of B such that*

$$\forall t > 0, \forall \varepsilon \in]0, t] \quad \mathcal{Z}_\varepsilon \vee \mathcal{F}_t^{(B,C)} = \mathcal{Z}_t \tag{1}$$

(*or equivalently*,

$$\forall \varepsilon > 0 \quad \mathcal{Z}_\varepsilon \vee \sigma(B, C) = \mathcal{Z}_\infty, \tag{2}$$

since the filtration $(\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^{(B,C)})_{t\geq\varepsilon}$ is immersed in $(\mathcal{Z}_t)_{t\geq\varepsilon}$).

Clearly, if B is complementable, it is also complementable after 0. The converse is false; for instance, if B is complementable, any Brownian motion generating the Goswami–Rao filtration associated with B (see [1]) is complementable after 0, but not complementable.

Complementability after 0 is easily seen to be a property of the filtration: if $\bar{B}$ is any other $\mathcal{Z}$-BM1 generating $\mathcal{F}^B$, then $\bar{B}$ is complementable after 0 if and only if B is (with the same independent complement C). Thus, in that case we may say that the filtration $\mathcal{F}^B$ itself is complementable after 0.

Observe that (2) is equivalent to $\bigcap_{\varepsilon>0}(\mathcal{Z}_\varepsilon \vee \sigma(B, C)) = \mathcal{Z}_\infty$ since $\mathcal{Z}_\infty$ contains everything. Similarly, (1) is equivalent to

$$\forall t > 0 \quad \bigcap_{\varepsilon\in]0,t]} \big(\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^{(B,C)}\big) = \mathcal{Z}_t.$$

If C in Definition 3 is not a complement to B, that is, if $\sigma(B, C)$ is strictly included in $\mathcal{Z}_\infty$, one has

$$\Big(\bigcap_{\varepsilon>0} \mathcal{Z}_\varepsilon\Big) \vee \sigma(B, C) = \mathcal{Z}_0 \vee \sigma(B, C) = \sigma(B, C) \neq \mathcal{Z}_\infty = \bigcap_{\varepsilon>0}\big(\mathcal{Z}_\varepsilon \vee \sigma(B, C)\big),$$

so the exchange property fails for these σ-fields.

Observe also that B and C play the same role in Definition 3: when (1) or (2) is satisfied, both B and C are complementable after 0.

Remark 1 (Not used in the sequel). *Let B and $\bar{B}$ be two $\mathcal{Z}$-BM1 such that $\sigma(B) \subset \sigma(\bar{B})$. If B is complementable after 0, so is also $\bar{B}$. Indeed, let C be as in Definition 3. For $\varepsilon > 0$, one has*

$$\mathcal{Z}_\infty = \mathcal{Z}_\varepsilon \vee \sigma(B, C) \subset \mathcal{Z}_\varepsilon \vee \sigma(\bar{B}, C) \subset \mathcal{Z}_\infty,$$

wherefrom $\mathcal{Z}_\varepsilon \vee \sigma(\bar{B}, C) = \mathcal{Z}_\infty$. Moreover, C is independent of $\bar{B}$ since $\bar{B}$ is parallel to B (Propositions 1 and 2(iii)). Consequently, $\bar{B}$ is complementable after 0.

In particular, if B is complementable after 0, so is also the filtration $\mathcal{B}'$ defined in Proposition 2.

Our first sufficient condition for maximality will derive from the next lemma.

Lemma 2. *Let B and D be two $\mathcal{Z}$-BM1. If B is complementable after 0 and $\sigma(B) \subset \sigma(D)$, then $\mathcal{F}^D \subset \mathcal{B}'$, where $\mathcal{B}'$ is the filtration defined in Proposition 2.*

Proof. As B is complementable after 0, let C be as in Definition 3. Consider any $\mathcal{Z}$-BM1 D whose filtration contains $\mathcal{F}^B$; to establish $\mathcal{F}^D \subset \mathcal{B}'$, it suffices to show the inclusion $\sigma(D) \subset \mathcal{B}'_\infty = \bigcap_{t>0}(\mathcal{Z}_t \vee \sigma(B))$ of their terminal σ-fields.

Since $\mathcal{F}^D$ contains $\mathcal{F}^B$, D is parallel to B, and also independent of C by condition (iii) of Proposition 1; so (C, D) is a $\mathcal{Z}$-BM2. For $t > 0$, we shall apply Lemma 1 to the three σ-fields

$$\mathcal{S} = \mathcal{Z}_t \vee \sigma(B) = \mathcal{Z}_t \vee \sigma\big(B^{]t,\infty[}\big), \qquad \mathcal{T} = \mathcal{Z}_t \vee \sigma(D) = \mathcal{Z}_t \vee \sigma\big(D^{]t,\infty[}\big), \qquad \mathcal{U} = \sigma\big(C^{]t,\infty[}\big).$$

First, $\mathcal{S}$ is included in $\mathcal{T}$ because $\sigma(D)$ contains $\sigma(B)$. Second, owing to the definition of C and to (2),

$$\mathcal{Z}_\infty = \mathcal{Z}_t \vee \sigma(B,C) = \mathcal{Z}_t \vee \sigma(B) \vee \sigma\big(C^{]t,\infty[}\big) = \mathcal{S} \vee \mathcal{U} \subset \mathcal{T} \vee \mathcal{U} \subset \mathcal{Z}_\infty,$$

which implies $\mathcal{S} \vee \mathcal{U} = \mathcal{T} \vee \mathcal{U}$. Third, $\mathcal{U}$ is independent of $\mathcal{T}$ because the three σ-fields $\mathcal{Z}_t$, $\sigma(C^{]t,\infty[})$ and $\sigma(D^{]t,\infty[})$ are independent, for (C,D) is a $\mathcal{Z}$-BM2. So Lemma 1 applies, yielding $\mathcal{S} = \mathcal{T}$. Therefore $\sigma(D) \subset \mathcal{T} = \mathcal{S} = \mathcal{Z}_t \vee \sigma(B)$, and, as $t > 0$ was arbitrary, $\sigma(D) \subset \bigcap_{t>0}(\mathcal{Z}_t \vee \sigma(B))$. □

Proposition 3. *Let B be a $\mathcal{Z}$-BM1. If B is complementable after 0, then the filtration $\mathcal{B}'$ defined in Proposition 2 is maximal.*

Proof. If $\mathcal{F}^D$ contains $\mathcal{B}'$ for D some $\mathcal{Z}$-BM1, then $\mathcal{F}^D$ contains a fortiori $\mathcal{F}^B$ and Lemma 2 gives $\mathcal{F}^D \subset \mathcal{B}'$; so $\mathcal{F}^D$ cannot strictly contain $\mathcal{B}'$. □

Remark 2 (Not used in the sequel). *Define B to be maximal after 0 if for D any other $\mathcal{Z}$-BM1, one has the maximality property*

$$\bigcap_{\varepsilon>0}\big(\mathcal{Z}_\varepsilon \vee \sigma(B)\big) \subset \bigcap_{\varepsilon>0}\big(\mathcal{Z}_\varepsilon \vee \sigma(D)\big) \quad\Longrightarrow\quad \bigcap_{\varepsilon>0}\big(\mathcal{Z}_\varepsilon \vee \sigma(B)\big) = \bigcap_{\varepsilon>0}\big(\mathcal{Z}_\varepsilon \vee \sigma(D)\big)$$

(*equivalently, with obvious notation, $\mathcal{B}' \subset \mathcal{D}' \Rightarrow \mathcal{B}' = \mathcal{D}'$). This property is clearly necessary for $\mathcal{B}'$ to be maximal; it is also sufficient because, if $\mathcal{B}' \subset \mathcal{F}^D$, one also has $\mathcal{B}' \subset \mathcal{F}^D \subset \mathcal{D}'$, and maximality after 0 forces $\mathcal{B}' = \mathcal{D}'$, whence $\mathcal{B}' = \mathcal{F}^D$. So Proposition 3 can be thus rephrased: complementability after 0 implies maximality after 0.*

The next maximality criterion also has two equivalent statements, in terms of the filtrations $\mathcal{F}^B$ and $\mathcal{B}'$, or of their end σ-fields $\sigma(B)$ and $\mathcal{B}'_\infty$.

Theorem 1. *Assume that B is a $\mathcal{Z}$-BM1, complementable after 0. Then B is maximal if and only if the following exchange property holds:*

$$\sigma(B) = \bigcap_{\varepsilon>0}\big(\mathcal{Z}_\varepsilon \vee \sigma(B)\big)$$

(*equivalently, the filtrations $\mathcal{F}^B$ and $\mathcal{B}'$ are equal*).

Proof. Corollary 1 says that the exchange property is necessary for maximality (without supposing B to be complementable after 0).

Conversely, if the exchange property holds, the filtration $\mathcal{B}'$ from Proposition 2 has the same terminal σ-field as the filtration $\mathcal{F}^B$, so these filtrations are equal; and Proposition 3 says that $\mathcal{B}'$ is maximal. □

We shall now propose a variant of the sufficient condition in Theorem 1, obtained by modifying both the assumption that B is complementable after 0 and the exchange property.

Proposition 4. *Let $W = (B,C)$ be a $\mathcal{Z}$-BM2; call $\mathcal{B}'$ the filtration constructed from B as in Proposition 2. Assume*

(i) *for each $t > 0$, $\sigma(Z^{[t,\infty[}) \subset \sigma(Z_t) \vee \sigma(W^{]t,\infty[})$;*
(ii) *for each $t > 0$, the conditional laws $\mathcal{L}[Z_t|\mathcal{F}^B_t]$ and $\mathcal{L}[Z_t|\mathcal{B}'_t]$ are equal.*
Then $\mathcal{F}^B = \mathcal{B}'$ and B is maximal.

Observe that hypothesis (i) is stronger than supposing B to be complementable after 0, since it features $\sigma(Z_t)$ instead of the larger σ-field $\mathcal{Z}_t$ in the right-hand side. On the opposite, hypothesis (ii) is weaker than the exchange property $\mathcal{F}^B = \mathcal{B}'$ from Theorem 1.

Remark also that one always has $\mathcal{L}[Z_t|\mathcal{F}_t^B] = \mathcal{L}[Z_t|\mathcal{F}_\infty^B]$ and $\mathcal{L}[Z_t|\mathcal{B}'_t] = \mathcal{L}[Z_t|\mathcal{B}'_\infty]$ by independent enlargements; so hypothesis (ii) in Proposition 4 amounts to

$$\forall t > 0 \quad \mathcal{L}\bigl[Z_t|\sigma(B)\bigr] = \mathcal{L}\biggl[Z_t\biggm|\bigcap_{\varepsilon>0}\bigl(\mathcal{Z}_\varepsilon \vee \sigma(B)\bigr)\biggr]. \tag{3}$$

Proof of Proposition 4. It suffices to show that $\sigma(B) = \mathcal{B}'_\infty$; this implies equality of the filtrations $\mathcal{F}^B$ and $\mathcal{B}'$, and maximality of B follows by Theorem 1, because hypothesis (i) is stronger than complementability after 0. For an arbitrary bounded, $\mathcal{Z}_\infty$-measurable r.v. J, we have to show

$$\mathbb{E}\biggl[J\biggm|\bigcap_{\varepsilon>0}\bigl(\mathcal{Z}_\varepsilon \vee \sigma(B)\bigr)\biggr] = \mathbb{E}\bigl[J|\sigma(B)\bigr]. \tag{4}$$

By L^1-density, we may restrict ourselves to the case that J is measurable in $\sigma(Z^{[t,\infty[})$ for some $t > 0$. By hypothesis (i), J is some measurable function of Z_t and of the processes $B^{]t,\infty[}$ and $C^{]t,\infty[}$. By linearity and density, we may suppose $J = f(Z_t)g(B^{]t,\infty[})h(C^{]t,\infty[})$ with f, g and h bounded and measurable. Calling w the 1-dimensional Wiener measure, one has

$$\mathbb{E}\bigl[J|\mathcal{Z}_t \vee \sigma(B)\bigr] = f(Z_t)g\bigl(B^{]t,\infty[}\bigr)\mathbb{E}\bigl[h\bigl(C^{]t,\infty[}\bigr)\big|\mathcal{Z}_t \vee \sigma(B)\bigr] = f(Z_t)g\bigl(B^{]t,\infty[}\bigr)\int h\,\mathrm{d}w$$

since $C^{]t,\infty[}$ is a BM1 independent of $\mathcal{Z}_t \vee \sigma(B^{]t,\infty[}) = \mathcal{Z}_t \vee \sigma(B)$. Consequently

$$\mathbb{E}\biggl[J\biggm|\bigcap_{\varepsilon>0}\bigl(\mathcal{Z}_\varepsilon \vee \sigma(B)\bigr)\biggr] = \biggl(\int h\,\mathrm{d}w\biggr)g\bigl(B^{]t,\infty[}\bigr)\mathbb{E}\biggl[f(Z_t)\biggm|\bigcap_{\varepsilon>0}\bigl(\mathcal{Z}_\varepsilon \vee \sigma(B)\bigr)\biggr].$$

Using (3), the right-hand side becomes $(\int h\,\mathrm{d}w)g(B^{]t,\infty[})\mathbb{E}[f(Z_t)|\sigma(B)]$; as this is $\sigma(B)$-measurable, (4) is established. □

Remark 3. *The same proof also yields a slightly stronger result*: *Proposition* 4 *remains true if* (i) *and* (ii) *are no longer supposed to hold for each* $t > 0$, *but only for a sequence of times tending to* 0.

5. Maximality of the BM1 driving the angular part of Z

It will be convenient to consider $Z = (X, Y)$ as the complex BM $Z = X + \mathrm{i}Y$. Call R the modulus of Z. Almost surely Z does not come back to 0, so another complex BM W can be defined by

$$W_t = \int_0^t \frac{R_s}{Z_s}\,\mathrm{d}Z_s.$$

The real and imaginary parts of W are the BM1 given by

$$U_t = \int_0^t \frac{X_s\,\mathrm{d}X_s + Y_s\,\mathrm{d}Y_s}{R_s} \quad \text{and} \quad V_t = \int_0^t \frac{X_s\,\mathrm{d}Y_s - Y_s\,\mathrm{d}X_s}{R_s}.$$

The next proposition recalls some well-known facts concerning U and V (see, for instance, Proposition 3.1 and Theorem 3.4 in [4]).

Proposition 5 (Classical properties of U and V; see [4]). (i) *The linear Brownian motions U and V are independent*.

(ii) *The process R is a strong solution of the stochastic differential equation*

$$\mathrm{d}R_s = \mathrm{d}U_s + \frac{\mathrm{d}s}{2R_s};$$

in particular, R and U generate the same filtration.

(iii) *For* $0 < s \le t$,

$$\frac{Z_t}{R_t} = \frac{Z_s}{R_s} \exp\left(\mathrm{i} \int_s^t \frac{\mathrm{d}V_r}{R_r}\right).$$

(iv) *For each* $t > 0$, $\mathcal{Z}_t = \mathcal{F}_t^W \vee \sigma(Z_t/R_t)$; *moreover, the r.v.* Z_t/R_t *is independent of* W *and uniformly distributed on the unit circle.*

It is shown in [2] that the $\mathcal{Z}$-BM1 U is complementable, hence also maximal. Clearly, V is not an independent complement to U, because (U, V) does not generate $\mathcal{Z}$ (Proposition 5(iv)); but V (and also U) is complementable after 0, because if Z has been observed during some time-interval $[0, \varepsilon]$ (however small), then observing U and V suffices to recover Z.

We shall show that V is maximal as an application of Proposition 4.

Theorem 2. *The* $\mathcal{Z}$*-BM1* $V = \int \frac{X\,\mathrm{d}Y - Y\,\mathrm{d}X}{R}$ *is maximal.*

Proof. We shall check that V satisfies hypotheses (i) and (ii) of Proposition 4.

Hypothesis (i) is readily verified: knowing Z_t and the increments of U and V after t, one can easily recover the path of Z after t: the modulus $R = |Z|$ is obtained as the solution to the stochastic differential equation

$$\mathrm{d}R_s = \mathrm{d}U_s + \frac{\mathrm{d}s}{2R_s},$$

and the argument Θ of $Z = R\mathrm{e}^{\mathrm{i}\Theta}$ is then given by $\mathrm{d}\Theta_s = \mathrm{d}V_s/R_s$.

Hypothesis (ii), or equivalently (3), says that the conditional laws of Z_t given the σ-fields $\bigcap_{\varepsilon>0}(\mathcal{Z}_\varepsilon \vee \sigma(V))$ and $\sigma(V)$ are equal. The latter, $\mathcal{L}[Z_t|V]$, is simply the law of Z_t because Z_t is independent of V. This is a direct consequence of Proposition 5: by (ii), R_t is independent of V, and by (iv), Z_t/R_t is independent of the pair (U, V), and a fortiori of (V, R_t). Hence the triple $(V, R_t, Z_t/R_t)$ is independent, and $Z_t = (Z_t/R_t) \times R_t$ is independent of V.

So, to verify hypothesis (ii), we have to check that Z_t is independent of $\bigcap_{s>0}(\mathcal{Z}_s \vee \sigma(V))$ too. This property is less straightforward than the independence of Z_t and V; we state it as a lemma. Strictly speaking, the elementary proof, given above, that Z_t is independent of V, was not necessary, because it is also a consequence of that lemma, for $\bigcap_{\varepsilon>0}(\mathcal{Z}_\varepsilon \vee \sigma(V))$ contains $\sigma(V)$.

Lemma 3 (Key lemma). *For any* $t > 0$, *the r.v.* Z_t *is independent of the* σ*-field* $\bigcap_{s>0}(\mathcal{Z}_s \vee \sigma(V))$.

The end of this paper is devoted to the proof of Lemma 3, which consists of five steps; the first two, Lemmas 4 and 5, are general facts for later use. Lemma 4 is certainly well known.

Lemma 4. *If* $(T_p)_{p\ge1}$ *is any sequence of strictly positive, identically distributed r.v., then* $\limsup_p T_p$ *is a.s. strictly positive.*

Proof. For $\delta > 0$, observe that

$$\left\{\limsup_p T_p < \delta\right\} = \left\{\exists p \sup_{q\ge p} T_q < \delta\right\} \subset \liminf_p \{T_p < \delta\}.$$

By Fatou's lemma, the probability of these events is majorized by $\liminf_p \mathbb{P}[T_p < \delta]$, which equals $\mathbb{P}[T_1 < \delta]$ since the T_p have the same law. So $\mathbb{P}[\limsup T_p < \delta] \le \mathbb{P}[T_1 < \delta]$; the lemma follows by letting δ go to 0. □

Lemma 5. *On some probability space* $(\widehat{\Omega}, \widehat{\mathcal{A}}, \widehat{\mathbb{P}})$, *let* β *be a* BM1 *and* $(\alpha_t)_{[1,2]}$ *be a continuous process independent of* β, *with diffuse law. Then the conditional law of* $\int_1^2 \alpha_t \, \mathrm{d}\beta_t$ *given* β *is almost surely diffuse. A fortiori, for every real* $m \neq 0$,

$$\left|\widehat{\mathbb{E}}\left[\exp\left(\mathrm{i}m\int_1^2 \alpha_t \, \mathrm{d}\beta_t\right)\middle|\beta\right]\right| < 1 \quad a.s.$$

Proof. Let $\tilde{\alpha}$ be a process with the same law as α, independent of (β, α). The conditional law

$$\widehat{\mathcal{L}}\left[\int_1^2 (\alpha_t - \tilde{\alpha}_t) \, \mathrm{d}\beta_t \middle| \alpha, \tilde{\alpha}\right] = \mathcal{N}\left(0, \int_1^2 (\alpha_t - \tilde{\alpha}_t)^2 \, \mathrm{d}t\right)$$

is a.s. non-degenerate. Consequently, $\widehat{\mathbb{P}}[\int_1^2 (\alpha_t - \tilde{\alpha}_t) \, \mathrm{d}\beta_t = 0] = 0$, wherefrom

$$\widehat{\mathbb{P}}\left[\int_1^2 \alpha_t \, \mathrm{d}\beta_t = \int_1^2 \tilde{\alpha}_t \, \mathrm{d}\beta_t \middle| \beta\right] = 0 \quad \text{a.s.}$$

But, conditional on β, the r.v. $\int_1^2 \alpha_t \, \mathrm{d}\beta_t$ and $\int_1^2 \tilde{\alpha}_t \, \mathrm{d}\beta_t$ are independent and have the same law; so this conditional law must be diffuse. □

For the remaining three steps of the proof, we come back to the setting of Theorem 2, with R denoting $|Z|$.

Lemma 6. *Call* $\mathcal{R}$ *the* σ*-field* $\sigma((R_{2^k})_{k\in\mathbb{Z}})$. *Then for every* $k \in \mathbb{Z}$, *the* σ*-fields* $\mathcal{Z}_{2^k}$ *and* $\sigma(V, R^{[2^k,\infty[})$ *are conditionally independent given* $\sigma(V) \vee \mathcal{R}$.

Proof. Conditional on the σ-field $\mathcal{R}$, the process R is still Markov (it is a succession of independent Bessel bridges); hence the σ-fields $\mathcal{F}^R_{2^k}$ and $\sigma(R^{[2^k,\infty[})$ are conditionally independent given $\mathcal{R}$.

As V is independent of R, the σ-fields $\mathcal{F}^R_{2^k} \vee \sigma(V)$ and $\sigma(R^{[2^k,\infty[}) \vee \sigma(V)$ are conditionally independent given $\mathcal{R} \vee \sigma(V)$.

Last, Proposition 5(iv) gives the equality $\mathcal{Z}_{2^k} = \mathcal{F}^R_{2^k} \vee \mathcal{F}^V_{2^k} \vee \sigma(Z_{2^k}/R_{2^k})$, with Z_{2^k}/R_{2^k} independent of (R, V). The result follows. □

Lemma 7. *The r.v.* Z_1/R_1 *is independent of* $\bigcap_{s>0}(\mathcal{Z}_s \vee \sigma(V) \vee \mathcal{R})$ *and uniformly distributed on the unit circle.*

Proof. Owing to the reverse martingale convergence theorem, it suffices to prove that for every $m \in \mathbb{Z} \setminus \{0\}$,

$$\mathbb{E}\left[\left(\frac{Z_1}{R_1}\right)^m \middle| \mathcal{Z}_{2^n} \vee \sigma(V) \vee \mathcal{R}\right] \longrightarrow 0 \quad \text{a.s. when } n \to -\infty.$$

Call Q_n this conditional expectation, and Δ_k the angular variation of Z from time 2^k to 2^{k+1}:

$$\Delta_k = \int_{2^k}^{2^{k+1}} \frac{\mathrm{d}V_s}{R_s}.$$

As this stochastic integral can be computed in the filtration generated by the processes V_t and $R_t \mathbf{1}_{\{t \geq 2^k\}}$, Δ_k is measurable in $\sigma(V, R^{[2^k,\infty[})$; so Lemma 6 ensures that Δ_k is independent of $\mathcal{Z}_{2^k}$ conditionally on $\sigma(V) \vee \mathcal{R}$. An easy induction shows that, for $n \leq -1$, the r.v. $\Delta_n, \ldots, \Delta_{-1}$ are conditionally independent given $\mathcal{Z}_{2^n} \vee \sigma(V) \vee \mathcal{R}$. For $n \leq -1$, writing

$$\left(\frac{Z_1}{R_1}\right)^m = \left(\frac{Z_{2^n}}{R_{2^n}}\right)^m \prod_{k=n}^{-1} \mathrm{e}^{\mathrm{i}m\Delta_k}$$

one gets the product expansion

$$Q_n = \left(\frac{Z_{2^n}}{R_{2^n}}\right)^m \prod_{k=n}^{-1} C_k, \quad \text{where } C_k = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}m\Delta_k} | \mathcal{Z}_{2^n} \vee \sigma(V) \vee \mathcal{R}\big]. \tag{5}$$

Using Lemma 6 again, notice that

$$C_k = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}m\Delta_k} | \mathcal{Z}_{2^n} \vee \sigma(V) \vee \mathcal{R}\big] = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}m\Delta_k} | \sigma(V) \vee \mathcal{R}\big] = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}m\Delta_k} | \sigma(V, R_{2^k}, R_{2^{k+1}})\big].$$

From (5), one draws $|C_k| \leq 1$ and $|Q_n| = |C_n| \cdots |C_{-1}|$; to finish proving Lemma 7, it remains to see that this product tends to 0 when $n \to -\infty$. We shall in fact prove a stronger property, namely

$$\liminf_{k \to -\infty} |C_k| < 1.$$

This stems from Lemma 4 applied to the sequence of r.v. $1 - |C_k|$; we only have to check that both hypotheses of this lemma are satisfied.

First, all the $|C_k|$ are identically distributed, since, by scaling invariance of Z, the joint law of

$$\big(\Delta_k, \big(2^{-k/2} V_{2^k t}\big)_{t \geq 0}, 2^{-k/2} R_{2^k}, 2^{-k/2} R_{2^{k+1}}\big)$$

does not depend upon k.

The other hypothesis to be checked is $|C_0| < 1$ a.s. Write $C_0 = \mathbb{E}[\exp(\mathrm{i}m \int_1^2 R_s^{-1}\, \mathrm{d}V_s) | \sigma(V, R_1, R_2)]$, and observe that

$$\mathbb{E}\left[\exp\left(\mathrm{i}m \int_1^2 R_s^{-1}\, \mathrm{d}V_s\right) \Big| V = v, R_1 = r_1, R_2 = r_2\right] = \widehat{\mathbb{E}}\left[\exp\left(\mathrm{i}m \int_1^2 \alpha_t\, \mathrm{d}\beta_t\right) \Big| \beta = v\right],$$

where α and β are as in Lemma 5, with α the inverse of the 2-dimensional Bessel bridge from r_1 to r_2. Lemma 5 says that given r_1 and r_2, the modulus of the right-hand side is strictly less than 1 for almost all v; this proves the claim, and completes the proof of Lemma 7. □

The fifth step is the final one:

Proof of Lemma 3. By scaling, we may suppose $t = 1$; we have to establish that the r.v.

$$\mathbb{E}\left[f(Z_1) \Big| \bigcap_{s>0} \big(\mathcal{Z}_s \vee \sigma(V)\big)\right]$$

is constant, where f is any bounded, Borel function.

By linearity and L^1-density, it suffices to consider functions of the form $f(z) = g(|z|) h(z/|z|)$, with g and h bounded. Observing that R_1 is $\mathcal{R}$-measurable and calling on Lemma 7, one can write

$$\mathbb{E}\left[f(Z_1) \Big| \bigcap_{s>0} \big(\mathcal{Z}_s \vee \sigma(V) \vee \mathcal{R}\big)\right] = g(R_1) \mathbb{E}\left[h\left(\frac{Z_1}{R_1}\right) \Big| \bigcap_{s>0} \big(\mathcal{Z}_s \vee \sigma(V) \vee \mathcal{R}\big)\right] = \bar{h} g(R_1),$$

where $\bar{h}$ is the constant $\mathbb{E}[h(Z_1/R_1)]$. Therefore

$$\mathbb{E}\left[f(Z_1) \Big| \bigcap_{s>0} \big(\mathcal{Z}_s \vee \sigma(V)\big)\right] = \bar{h} \mathbb{E}\left[g(R_1) \Big| \bigcap_{s>0} \big(\mathcal{Z}_s \vee \sigma(V)\big)\right]. \tag{6}$$

Using $\mathcal{Z}_s \vee \sigma(V) = \mathcal{F}_s^R \vee \sigma(V) \vee \sigma(Z_s/R_s)$ and the independence of R, V and Z_s/R_s, one has for $0 < s \leq 1$

$$\mathbb{E}\big[g(R_1) | \mathcal{Z}_s \vee \sigma(V)\big] = \mathbb{E}\big[g(R_1) | \mathcal{F}_s^R \vee \sigma(V) \vee \sigma(Z_s/R_s)\big] = \mathbb{E}\big[g(R_1) | \mathcal{F}_s^R\big].$$

Let s go to zero. By reverse martingale convergence, and recalling that $\mathcal{F}^R_{0+} = \mathcal{F}^U_{0+}$ is degenerate, one ends up with

$$\mathbb{E}\Big[g(R_1)\Big|\bigcap_{s>0}\big(\mathcal{Z}_s \vee \sigma(V)\big)\Big] = \lim_{\substack{s\to 0\\ s>0}} \mathbb{E}\big[g(R_1)|\mathcal{F}^R_s\big] = \mathbb{E}\big[g(R_1)\big];$$

so the right-hand side of (6) is constant. This proves Lemma 3, and Theorem 2 by the same token. □

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