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A NOTE ON STABILITY OF HARDY INEQUALITIES

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ABSTRACT. In this note, we formulate recent stability results for Hardy inequalities in the language of Folland and Stein's homogeneous groups. Consequently, we obtain remainder estimates for Rellich-type inequalities on homogeneous groups. Main differences from the Euclidean results are that the obtained stability estimates hold for any homogeneous quasinorm.

1. Introduction

Recall the L^p-Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla f|^p \, dx \geq \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f|^p}{|x|^p} \, dx \tag{1.1}$$

for every function $f \in C_0^\infty(\mathbb{R}^n)$, where $2 \leq p < n$.

Cianchi and Ferone [4] showed that for all $1 < p < n$ there exists a constant $C = C(p, n)$ such that

$$\int_{\mathbb{R}^n} |\nabla f|^p \, dx \geq \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f|^p}{|x|^p} \, dx \big(1 + C d_p(f)^{2p^*}\big)$$

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holds for all real-valued weakly differentiable functions f in $\mathbb{R}^n$ such that f and $|\nabla f| \in L^p(\mathbb{R}^n)$ go to zero at infinity. Here

$$d_p f = \inf_{c\in\mathbb{R}} \frac{\|f - c|x|^{-\frac{n-p}{p}}\|_{L^{p^*,\infty}(\mathbb{R}^n)}}{\|f\|_{L^{p^*,p}(\mathbb{R}^n)}}$$

with $p^* = \frac{np}{n-p}$, and $L^{\tau,\sigma}(\mathbb{R}^n)$ is the Lorentz space for $0 < \tau \leq \infty$ and $1 \leq \sigma \leq \infty$. Sometimes the improved versions of different inequalities, or remainder estimates, are called *stability of the inequality* if the estimates depend on certain distances (see, e.g., [1] for stability of trace theorems, [3] for stability of Sobolev inequalities, and so forth; for more general Lie group discussions of the above inequalities, we refer to our recent work [7]–[9] as well as the references therein).

Recently, Sano and Takahashi [11]–[14] obtained improved versions of Hardy inequalities. The aim of this note is to formulate their results on one of the largest classes of nilpotent Lie groups on $\mathbb{R}^n$, namely, homogeneous Lie groups, since obtained results give new insights even for the Abelian groups in terms of the arbitrariness of the homogeneous quasinorm.

2. Preliminaries

First, we briefly review some main concepts of homogeneous groups following Folland and Stein [6] (see also recent work [2] and [5] on this topic). We also recall a few other facts that will be used in the proofs. A connected simply connected Lie group $\mathbb{G}$ is called a *homogeneous group* if its Lie algebra $\mathfrak{g}$ is equipped with a family of dilations

$$D_\lambda = \mathrm{Exp}(A \ln \lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \big(\ln(\lambda) A\big)^k,$$

where A is a diagonalizable positive linear operator on $\mathfrak{g}$ and every D_λ is a morphism of $\mathfrak{g}$; that is,

$$\forall X, Y \in \mathfrak{g}, \lambda > 0, \quad [D_\lambda X, D_\lambda Y] = D_\lambda [X, Y]$$

holds. We recall that $Q := \mathrm{Tr}\, A$ is called the *homogeneous dimension* of $\mathbb{G}$. The Haar measure on a homogeneous group $\mathbb{G}$ is the standard Lebesgue measure for $\mathbb{R}^n$ (see, e.g., [5, Proposition 1.6.6]).

Let $|\cdot|$ be a homogeneous quasinorm on $\mathbb{G}$. Then the quasiball centered at $x \in \mathbb{G}$ with radius $R > 0$ is defined by

$$B(x, R) := \big\{y \in \mathbb{G} : |x^{-1}y| < R\big\}.$$

We refer the reader to [6] for the proof of the following important polar decomposition on homogeneous Lie groups, which can also be found in [5, Section 3.1.7]. There is a (unique) positive Borel measure σ on the unit quasisphere

$$\wp := \big\{x \in \mathbb{G} : |x| = 1\big\}, \tag{2.1}$$

so that for every $f \in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x)\, dx = \int_0^\infty \int_\wp f(ry) r^{Q-1}\, d\sigma(y)\, dr. \tag{2.2}$$

We use the notation

$$\mathcal{R}f(x) := \mathcal{R}_{|x|}f(x) = \frac{d}{d|x|}f(x) = \mathcal{R}f(x), \quad \forall x \in \mathbb{G}, \tag{2.3}$$

for any homogeneous quasinorm $|x|$ on $\mathbb{G}$. We will also use the following result.

Lemma 2.1 ([10, Theorem 3.1]). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be any homogeneous norm on $\mathbb{G}$. Then for $u \in C_0^\infty(\mathbb{G}\setminus\{0\})$ and $u_R = u(R\frac{x}{|x|})$, we have*

$$\Big\| \frac{u - u_R}{|x|^{\frac{Q}{p}} \log \frac{R}{|x|}} \Big\|_{L^p(\mathbb{G})} \leq \frac{p}{p-1} \Big\| |x|^{\frac{p-Q}{p}} \mathcal{R}u \Big\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \tag{2.4}$$

for all $R > 0$, and the constant $\frac{p}{p-1}$ is sharp.

We will also use the following known relations.

Lemma 2.2. *Let $a, b \in \mathbb{R}$. Then*

(i) *we have*

$$|a-b|^p - |a|^p \geq -p|a|^{p-2}ab, \quad p \geq 1;$$

(ii) *there exists a constant $C = C(p) > 0$ such that*

$$|a-b|^p - |a|^p \geq -p|a|^{p-2}ab + C|b|^p, \quad p \geq 2;$$

(iii) *if $a \geq 0$ and $a - b \geq 0$, then*

$$(a-b)^p + pa^{p-1}b - a^p \geq |b|^p, \quad p \geq 2.$$

3. Stability of L^p-Hardy inequalities

Let us set

$$d_H(u;R) := \Big(\int_{\mathbb{G}} \frac{|u(x) - R^{\frac{Q-p}{p}} u(R\frac{x}{|x|})|x|^{-\frac{Q-p}{p}}|^p}{|x|^p| \log \frac{R}{|x|}|^p} \, dx \Big)^{\frac{1}{p}}, \quad x \in \mathbb{G}, R > 0.$$

Theorem 3.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be any homogeneous quasinorm on $\mathbb{G}$. Then there exists a constant $C > 0$ for all real-valued functions $u \in C_0^\infty(\mathbb{G})$, and we have*

$$\int_{\mathbb{G}} |\mathcal{R}u|^p \, dx - \Big(\frac{Q-p}{p}\Big)^p \int_{\mathbb{G}} \frac{|u|^p}{|x|^p} \, dx \geq C \sup_{R>0} d_H^p(u;R), \quad 2 \leq p < Q, \tag{3.1}$$

where $\mathcal{R} := \frac{d}{d|x|}$ is the radial derivative.

Proof of Theorem 3.1. Let us introduce polar coordinates $x = (r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \wp$ on $\mathbb{G}$, where $\wp$ is the unit quasisphere

$$\wp := \big\{ x \in \mathbb{G} : |x| = 1 \big\}, \tag{3.2}$$

and

$$v(ry) := r^{\frac{Q-p}{p}} u(ry), \tag{3.3}$$

where $u \in C_0^\infty(\mathbb{G})$. It follows that $v(0) = 0$ and $\lim_{r\to\infty} v(ry) = 0$ for $y \in \wp$ since u is compactly supported. Using the polar decomposition on homogeneous groups (see (2.2)) and integrating by parts, we get

$$\begin{aligned}
D :&= \int_{\mathbb{G}} |\mathcal{R}u|^p \, dx - \Big(\frac{Q-p}{p}\Big)^p \int_{\mathbb{G}} \frac{|u|^p}{|x|^p} \, dx \\
&= \int_{\wp}\int_0^\infty \Big|-\frac{\partial}{\partial r} u(ry)\Big|^p r^{Q-1} - \Big(\frac{Q-p}{p}\Big)^p \big|u(ry)\big|^p r^{Q-p-1} \, dr \, dy \\
&= \int_{\wp}\int_0^\infty \Big|\frac{Q-p}{p} r^{-\frac{Q}{p}} v(ry) - r^{-\frac{Q-p}{p}} \frac{\partial}{\partial r} v(ry)\Big|^p r^{Q-1} \\
&\quad - \Big(\frac{Q-p}{p}\Big)^p \big|v(ry)\big|^p r^{-1} \, dr \, dy.
\end{aligned}$$

Now using the second relation in Lemma 2.2 with the choice $a = \frac{Q-p}{p} r^{-\frac{Q}{p}} v(ry)$ and $b = r^{-\frac{Q-p}{p}} \frac{\partial}{\partial r} v(ry)$, and making use of the fact that $\int_0^\infty |v|^{p-2} v(\frac{\partial}{\partial r} v) \, dr = 0$, we obtain

$$D \geq \int_{\wp}\int_0^\infty -p\Big(\frac{Q-p}{p}\Big)^{p-1} \big|v(ry)\big|^{p-2} v(ry) \frac{\partial}{\partial r} v(ry) \tag{3.4}$$

$$+ C\Big|\frac{\partial}{\partial r} v(ry)\Big|^p r^{p-1} \, dr \, dy \tag{3.5}$$

$$= C \int_{\mathbb{G}} |x|^{p-Q} |\mathcal{R}v|^p \, dx.$$

Finally, combining (3.4) and Lemma 2.1, we arrive at

$$D \geq C \int_{\mathbb{G}} \frac{|v(x) - v(R\frac{x}{|x|})|^p}{|x|^Q |\log\frac{R}{|x|}|^p} \, dx = C \int_{\wp}\int_0^\infty \frac{|v(ry) - v(Ry)|^p}{r|\log\frac{R}{r}|^p} \, dr \, dy \tag{3.6}$$

$$= C \int_{\wp}\int_0^\infty \frac{|u(ry) - R^{\frac{Q-p}{p}} u(Ry) r^{-\frac{Q-p}{p}}|^p}{r^{1+p-Q} |\log\frac{R}{r}|^p} \, dr \, dy$$

for any $R > 0$. This proves the desired result. $\square$

4. Stability of critical Hardy inequalities

In this section, we establish a stability estimate for the critical Hardy inequality involving the distance to the set of extremizers. Let us denote

$$f_{T,R}(x) = T^{\frac{Q-1}{Q}} u\Big(Re^{-\frac{1}{T}} \frac{x}{|x|}\Big)\Big(\log\frac{R}{|x|}\Big)^{\frac{Q-1}{Q}} \tag{4.1}$$

and the following "distance"

$$d_{cH}(u; T, R) := \Big(\int_{B(0,R)} \frac{|u(x) - f_{T,R}(x)|^Q}{|x|^Q |\log\frac{R}{|x|}|^Q |T\log\frac{R}{|x|}|^Q} \, dx\Big)^{\frac{1}{Q}}, \tag{4.2}$$

for some parameter $T > 0$ and functions u and $f_{T,R}$ for which the integral in (4.2) is finite.

Theorem 4.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 2$. Let $|\cdot|$ be any homogeneous quasinorm on $\mathbb{G}$. Then there exists a constant $C > 0$ for all real-valued functions $u \in C_0^\infty(B(0,R))$, and we have*

$$\int_{B(0,R)} \left|\mathcal{R}u(x)\right|^Q dx - \Big(\frac{Q-1}{Q}\Big)^Q \int_{B(0,R)} \frac{|u(x)|^Q}{|x|^Q(\log\frac{R}{|x|})^Q}\, dx \\ \geq C \sup_{T>0} d_{cH}^Q(u;T,R), \tag{4.3}$$

where $\mathcal{R} := \frac{d}{d|x|}$ is the radial derivative.

Proof of Theorem 4.1. Introducing coordinates $(r,y) = (|x|, \frac{x}{|x|}) \in (0,\infty)\wp$ on $\mathbb{G}$, where $\wp$ is the sphere as in (3.2), we have $u(x) = u(ry) \in C_0^\infty(B(0,R))$. In addition, let us set

$$v(sy) := \Big(\log\frac{R}{r}\Big)^{-\frac{Q-1}{Q}} u(ry), \quad y \in \wp, \tag{4.4}$$

where

$$s = s(r) := \Big(\log\frac{R}{r}\Big)^{-1}.$$

Since $u \in C_0^\infty(B(0,R))$, we have $v(0) = 0$ and v has a compact support. Moreover, it is straightforward that

$$\frac{\partial}{\partial r}u(ry) = -\Big(\frac{Q-1}{Q}\Big)\Big(\log\frac{R}{r}\Big)^{-\frac{1}{Q}}\frac{v(sy)}{r} + \Big(\log\frac{R}{r}\Big)^{\frac{Q-1}{Q}}\frac{\partial}{\partial s}v(sy)s'(r).$$

A direct calculation gives

$$\begin{aligned} S &:= \int_{B(0,R)} |\mathcal{R}u|^Q\, dx - \Big(\frac{Q-1}{Q}\Big)^Q \int_{B(0,R)} \frac{|u|^Q}{|x|^Q(\log\frac{R}{|x|})^Q}\, dx \\ &= \int_\wp \int_0^R \Big|\frac{\partial}{\partial r}u(ry)\Big|^Q r^{Q-1} - \Big(\frac{Q-1}{Q}\Big)^Q \frac{|u(ry)|^Q}{r(\log\frac{R}{r})^Q}\, dr\, dy \\ &= \int_\wp \int_0^R \Big|\Big(\frac{Q-1}{Q}\Big)\Big(r\log\frac{R}{r}\Big)^{-\frac{1}{Q}} v(sy) + \Big(r\log\frac{R}{r}\Big)^{\frac{Q-1}{Q}}\frac{\partial}{\partial s}v(sy)s'(r)\Big|^Q \\ &\quad - \Big(\frac{Q-1}{Q}\Big)^Q \frac{|v(sy)|^Q}{r\log\frac{R}{r}}\, dr\, dy. \end{aligned}$$

Now by applying the second relation in Lemma 2.2 with the choice

$$a = \frac{Q-1}{Q}\Big(r\log\frac{R}{r}\Big)^{-\frac{1}{Q}} v(sy) \qquad \text{and} \qquad b = \Big(r\log\frac{R}{r}\Big)^{\frac{Q-1}{Q}}\frac{\partial}{\partial s}v(sy)s'(r),$$

and by using the fact that $v(0) = 0$ and $\lim_{r\to\infty} v(ry) = 0$, we obtain

$$\begin{aligned} S \geq \int_\wp \int_0^R & -Q\Big(\frac{Q-1}{Q}\Big)^{Q-1}\left|v(sy)\right|^{Q-2} v(sy)\frac{\partial}{\partial s}v(sy)s'(r) \\ & + C\Big|\frac{\partial}{\partial s}v(sy)\Big|^Q \big(s'(r)\big)^Q \Big(r\log\frac{R}{r}\Big)^{Q-1}\, dr\, dy \end{aligned}$$

$$
\begin{aligned}
&= \int_{\wp}\int_0^R -Q\Big(\frac{Q-1}{Q}\Big)^{Q-1}\big|v(sy)\big|^{Q-2} v(sy)\frac{\partial}{\partial s}v(sy)s'(r)\\
&\quad + C\Big|\frac{\partial}{\partial s}v(sy)\Big|^{Q}\frac{1}{r^{Q}(\log\frac{R}{r})^{2Q}}\Big(r\log\frac{R}{r}\Big)^{Q-1}\,dr\,dy\\
&= \int_{\wp}\int_0^R -Q\Big(\frac{Q-1}{Q}\Big)^{Q-1}\big|v(sy)\big|^{Q-2} v(sy)\frac{\partial}{\partial s}v(sy)s'(r)\\
&\quad + C\Big|\frac{\partial}{\partial s}v(sy)\Big|^{Q}\frac{1}{(\log\frac{R}{r})^{Q-1}}s'(r)\,dr\,dy\\
&= \int_{\wp}\int_0^R -Q\Big(\frac{Q-1}{Q}\Big)^{Q-1}\big|v(sy)\big|^{Q-2} v(sy)\frac{\partial}{\partial s}v(s)\\
&\quad + C\Big|\frac{\partial}{\partial s}v(sy)\Big|^{Q}s^{Q-1}\,ds\,dy\\
&= C\int_{\mathbb{G}}|\mathcal{R}v|^{Q}\,dx,
\end{aligned}
$$

that is,

$$
S \geq C\int_{\mathbb{G}}|\mathcal{R}v|^{Q}\,dx. \tag{4.5}
$$

According to Lemma 2.1 with $v\in C_0^{\infty}(\mathbb{G}\backslash\{0\})$ with $p=Q$ and (4.5), it implies that

$$
\begin{aligned}
S &\geq C\int_{\mathbb{G}}\frac{|v(x)-v(T\frac{x}{|x|})|^{Q}}{|x|^{Q}|\log\frac{T}{|x|}|^{Q}}\,dx = C\int_{\wp}\int_0^{\infty}\frac{|v(sy)-v(Ty)|^{Q}}{s|\log\frac{T}{s}|^{Q}}\,ds\,dy\\
&= C\int_{\wp}\int_0^R\frac{|(\log\frac{R}{r})^{-\frac{Q-1}{Q}}u(ry)-T^{\frac{Q-1}{Q}}u(Re^{-\frac{1}{T}}y)|^{Q}}{r(\log\frac{R}{r})|\log(T\log\frac{R}{r})|^{Q}}\,dr\,dy\\
&= C\int_{\wp}\int_0^R\frac{|u(ry)-T^{\frac{Q-1}{Q}}u(Re^{-\frac{1}{T}}y)(\log\frac{R}{r})^{\frac{Q-1}{Q}}|^{Q}}{r(\log\frac{R}{r})^{Q}|\log(T\log\frac{R}{r})|^{Q}}\,dr\,dy.
\end{aligned}
$$

Thus, we arrive at

$$
S\geq C\int_{B(0,R)}\frac{|u(x)-T^{\frac{Q-1}{Q}}u(Re^{-\frac{1}{T}}\frac{x}{|x|})(\log\frac{R}{|x|})^{\frac{Q-1}{Q}}|^{Q}}{|x|^{Q}|\log\frac{R}{|x|}|^{Q}|\log(T\log\frac{R}{|x|})|^{Q}}\,dx
$$

for all $T>0$. The proof is complete. $\square$

5. Improved critical Hardy and Rellich inequalities for radial functions

Proposition 5.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q\geq 2$. Let $|\cdot|$ be a homogeneous quasinorm on $\mathbb{G}$. Let $q>0$ be such that*

$$
\alpha=\alpha(q,L):=\frac{Q-1}{Q}q+L+2\leq Q \tag{5.1}
$$

for $-1 < L < Q-2$. Then for all real-valued positive nonincreasing radial functions $u \in C_0^\infty(B(0,R))$, we have

$$\begin{aligned}\int_{B(0,R)} |\mathcal{R}u|^Q \, dx - \Big(\frac{Q-1}{Q}\Big)^Q \int_{B(0,R)} \frac{|u(x)|^Q}{|x|^Q (\log \frac{Re}{|x|})^Q} \, dx \\ \geq |\wp|^{1-\frac{Q}{q}} C^{\frac{Q}{q}} \Big(\int_{B(0,R)} \frac{|u(x)|^q}{|x|^Q (\log \frac{Re}{|x|})^\alpha} \, dx\Big)^{\frac{Q}{q}},\end{aligned} \tag{5.2}$$

where $|\wp|$ is the measure of the unit quasisphere in $\mathbb{G}$ and

$$\begin{aligned}C^{-1} = C(L,Q,q)^{-1} &:= \int_0^1 s^L \Big(\log \frac{1}{s}\Big)^{\frac{Q-1}{Q}q} \, ds \\ &= (L+1)^{-(\frac{Q-1}{Q}q+1)} \Gamma\Big(\frac{Q-1}{Q}q+1\Big);\end{aligned}$$

here $\Gamma(\cdot)$ is the gamma function.

Proof of Proposition 5.1. As in previous proofs, we set

$$v(s) = \Big(\log \frac{Re}{r}\Big)^{-\frac{Q-1}{Q}} u(r), \quad \text{where } r = |x|, s = s(r) = \Big(\log \frac{Re}{r}\Big)^{-1}, \tag{5.3}$$
$$s'(r) = \frac{s(r)}{r \log \frac{Re}{r}} \geq 0.$$

Simply, we have $v(0) = v(1) = 0$ since $u(R) = 0$; moreover,

$$\begin{aligned}u'(r) &= -\Big(\frac{Q-1}{Q}\Big)\Big(\log \frac{Re}{r}\Big)^{-\frac{1}{Q}} \frac{v(s(r))}{r} + \Big(\log \frac{Re}{r}\Big)^{\frac{Q-1}{Q}} v'\big(s(r)\big) s'(r) \\ &\leq 0.\end{aligned} \tag{5.4}$$

It is straightforward that

$$\begin{aligned}I &:= \int_{B(0,R)} |\mathcal{R}u|^Q \, dx - \Big(\frac{Q-1}{Q}\Big)^Q \int_{B(0,R)} \frac{|u|^Q}{|x|^Q (\log \frac{Re}{|x|})^Q} \, dx \\ &= |\wp| \int_0^R \big|u'(r)\big|^Q r^{Q-1} \, dr - \Big(\frac{Q-1}{Q}\Big)^Q |\wp| \int_0^R \frac{|u(r)|^Q}{r (\log \frac{Re}{r})^Q} \, dr \\ &= |\wp| \int_0^R \Big(\frac{Q-1}{Q}\Big(\log \frac{Re}{r}\Big)^{-\frac{1}{Q}} \frac{v(s(r))}{r} - \Big(\log \frac{Re}{r}\Big)^{\frac{Q-1}{Q}} v'\big(s(r)\big) s'(r)\Big)^Q r^{Q-1} \, dr \\ &\quad - \Big(\frac{Q-1}{Q}\Big)^Q |\wp| \int_0^R \frac{|u(r)|^Q}{r (\log \frac{Re}{r})^Q} \, dr.\end{aligned}$$

By applying the third relation in Lemma 2.2 with

$$a = \frac{Q-1}{Q}\Big(\log \frac{Re}{r}\Big)^{-\frac{1}{Q}} \frac{v(s(r))}{r} \qquad \text{and} \qquad b = \Big(\log \frac{Re}{r}\Big)^{\frac{Q-1}{Q}} v'\big(s(r)\big) s'(r),$$

and dropping $a^Q \geq 0$ as well as using the boundary conditions $v(0) = v(1) = 0$, we get

$$\begin{aligned}
I &\geq -|\wp| Q\Big(\frac{Q-1}{Q}\Big)^{Q-1} \int_0^R v\big(s(r)\big)^{Q-1} v'\big(s(r)\big) s'(r)\, dr \qquad (5.5)\\
&\quad + |\wp| \int_0^R \big|v'\big(s(r)\big)\big|^Q \big(s'(r)\big)^Q \Big(r \log \frac{Re}{r}\Big)^{Q-1} dr\\
&= -|\wp| Q\Big(\frac{Q-1}{Q}\Big)^{Q-1} \int_0^R v\big(s(r)\big)^{Q-1} v'\big(s(r)\big) s'(r)\, dr \qquad (5.6)\\
&\quad + |\wp| \int_0^R \big|v'\big(s(r)\big)\big|^Q \frac{1}{r^Q (\log \frac{Re}{r})^{2Q}} \Big(r \log \frac{Re}{r}\Big)^{Q-1} dr\\
&= -|\wp| Q\Big(\frac{Q-1}{Q}\Big)^{Q-1} \int_0^R v\big(s(r)\big)^{Q-1} v'\big(s(r)\big) s'(r)\, dr \qquad (5.7)\\
&\quad + |\wp| \int_0^R \big|v'\big(s(r)\big)\big|^Q s(r)^{Q-1} s'(r)\, dr\\
&= -|\wp| Q\Big(\frac{Q-1}{Q}\Big)^{Q-1} \int_0^1 v(s)^{Q-1} v'(s)\, ds \qquad (5.8)\\
&\quad + |\wp| \int_0^1 \big|v'(s)\big|^Q s^{Q-1}\, ds\\
&= |\wp| \int_0^1 \big|v'(s)\big|^Q s^{Q-1}\, ds.
\end{aligned}$$

Moreover, by using the inequality

$$\begin{aligned}
\big|v(s)\big| &= \Big|\int_s^1 v'(t)\, dt\Big| = \Big|\int_s^1 v'(t) t^{\frac{Q-1}{Q} - \frac{Q-1}{Q}}\, dt\Big|\\
&\leq \Big(\int_0^1 \big|v'(t)\big|^Q t^{Q-1}\, dt\Big)^{\frac{1}{Q}} \Big(\log \frac{1}{s}\Big)^{\frac{Q-1}{Q}},
\end{aligned}$$

we obtain

$$\int_0^1 \big|v(s)\big|^q s^L\, ds \leq \Big(\int_0^1 \big|v'(s)\big|^Q s^{Q-1}\, ds\Big)^{\frac{q}{Q}} \int_0^1 s^L \Big(\log \frac{1}{s}\Big)^{\frac{Q-1}{Q} q} ds$$

for $-1 < L < Q-2$. Thus, we have

$$\int_0^1 \big|v'(s)\big|^Q s^{Q-1}\, ds \geq C^{\frac{q}{Q}} \Big(\int_0^1 \big|v(s)\big|^q s^L\, ds\Big)^{\frac{Q}{q}}. \qquad (5.9)$$

Now it follows from (5.5) and (5.9) that

$$\begin{aligned}
I &\geq |\wp| C^{\frac{Q}{q}} \Big(\int_0^1 \big|v(s)\big|^q s^L\, ds\Big)^{\frac{Q}{q}} = |\wp| C^{\frac{Q}{q}} \Big(\int_0^R \frac{|u(r)|^q}{r(\log \frac{Re}{r})^{\alpha}}\, dr\Big)^{\frac{Q}{q}}\\
&= |\wp|^{1-\frac{Q}{q}} C^{\frac{Q}{q}} \Big(\int_0^R \frac{|u(x)|^q}{|x|^Q (\log \frac{Re}{|x|})^{\alpha}}\, dx\Big)^{\frac{Q}{q}},
\end{aligned}$$

where $\alpha = \alpha(q, L) = \frac{Q-1}{Q} q + L + 2$. The proof is complete. ∎

The method used in the previous section also allows one to obtain the following stability inequality for Rellich-type inequalities.

Proposition 5.2. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasinorm on $\mathbb{G}$, and let $p \geq 1$. Let $k \geq 2, k \in \mathbb{N}$, be such that $kp < Q$. Then for all real-valued radial functions $u \in C_0^\infty(\mathbb{G})$, we have*

$$\begin{aligned}\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}u|^p}{|x|^{(k-2)p}}\,dx - K_{k,p}^p \int_{\mathbb{G}} \frac{|u|^p}{|x|^{kp}}\,dx \\ \geq C \sup_{R>0} \int_{\mathbb{G}} \frac{||u(x)|^{\frac{p-2}{2}}u(x) - R^{\frac{Q-kp}{2}}|u(R)|^{\frac{p-2}{2}}u(R)|x|^{-\frac{Q-kp}{2}}|^2}{|x|^{kp}|\log\frac{R}{|x|}|^2}\,dx,\end{aligned} \tag{5.10}$$

where

$$\tilde{\mathcal{R}}f = \mathcal{R}^2 f + \frac{Q-1}{|x|}\mathcal{R}f$$

is the Rellich-type operator on $\mathbb{G}$, and $K_{k,p} = \frac{(Q-kp)[(k-2)p+(p-1)Q]}{p^2}$.

Proof of Proposition 5.2. For $k \geq 2, k \in \mathbb{N}$ and $kp < Q$, let us set

$$v(r) := r^{\frac{Q-kp}{p}}u(r), \quad \text{where } r \in [0, \infty). \tag{5.11}$$

Thus, $v(0) = 0$ and $v(\infty) = 0$.

We have

$$\begin{aligned}-\tilde{\mathcal{R}}u &= -\mathcal{R}^2\big(r^{\frac{kp-Q}{p}}v(r)\big) - \frac{Q-1}{r}\mathcal{R}\big(r^{\frac{kp-Q}{p}}v(r)\big) \\ &= -\mathcal{R}\Big(\frac{kp-Q}{p}r^{\frac{kp-Q}{p}-1}v(r) + r^{\frac{kp-Q}{p}}\mathcal{R}v(r)\Big) \\ &\quad - \frac{Q-1}{r}\frac{kp-Q}{p}r^{\frac{kp-Q}{p}-1}v(r) - \frac{Q-1}{r}r^{\frac{kp-Q}{p}}\mathcal{R}v(r) \\ &= -\frac{kp-Q}{p}\Big(\frac{kp-Q}{p}-1\Big)r^{\frac{kp-Q}{p}-2}v(r) - \frac{kp-Q}{p}r^{\frac{kp-Q}{p}-1}\mathcal{R}v(r) \\ &\quad - \frac{kp-Q}{p}r^{\frac{kp-Q}{p}-1}\mathcal{R}v(r) - r^{\frac{kp-Q}{p}}\mathcal{R}^2v(r) \\ &\quad - \frac{Q-1}{r}\frac{kp-Q}{p}r^{\frac{kp-Q}{p}-1}v(r) - \frac{Q-1}{r}r^{\frac{kp-Q}{p}}\mathcal{R}v(r) \\ &= -r^{\frac{kp-Q}{p}-2}\Big(\frac{(kp-Q)(kp-Q-p)}{p^2} + \frac{(Q-1)(kp-Q)}{p}\Big)v(r) \\ &\quad - r^{\frac{kp-Q}{p}-2}r^2\Big(\mathcal{R}^2v(r) + \frac{1}{r}\Big(\frac{2(kp-Q)}{p} + (Q-1)\Big)\mathcal{R}v(r)\Big) \\ &= r^{k-2-\frac{Q}{p}}\big(K_{k,p}v(r) - r^2\tilde{\mathcal{R}}_k v(r)\big),\end{aligned}$$

where

$$\tilde{\mathcal{R}}_k f = \mathcal{R}^2 f + \frac{2k + \frac{Q(p-2)}{p} - 1}{r}\mathcal{R}f$$

and $K_{k,p} = \frac{(Q-kp)[(k-2)p+(p-1)Q]}{p^2}$. By using the first inequality in Lemma 2.2 with $a = K_{k,p}v(r)$ and $b = r^2\tilde{\mathcal{R}}_k v(r)$, and the fact that $\int_0^\infty |v|^{p-2}vv'\,dr = 0$ since

$v(0)=0$ and $v(\infty)=0$, we obtain

$$
\begin{aligned}
J &:= \int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}u|^p}{|x|^{(k-2)p}}\,dx - K_{k,p}^p \int_{\mathbb{G}} \frac{|u|^p}{|x|^{kp}}\,dx \\
&= |\wp| \int_0^\infty \big|-\tilde{\mathcal{R}}u(r)\big|^p r^{Q-1-(k-2)p}\,dr - K_{k,p}^p |\wp| \int_0^\infty \big|u(r)\big|^p r^{Q-kp-1}\,dr \\
&= |\wp| \int_0^\infty \Big(\big|K_{k,p}v(r) - r^2\tilde{\mathcal{R}}_k v(r)\big|^p - \big(K_{k,p}v(r)\big)^p\Big) r^{-1}\,dr \\
&\geq -p|\wp| K_{k,p}^{p-1} \int_0^\infty |v|^{p-2} v \tilde{\mathcal{R}}_k v r\,dr \\
&= -p|\wp| K_{k,p}^{p-1} \int_0^\infty |v|^{p-2} v \Big(v'' + \frac{2k + \frac{Q(p-2)}{p} - 1}{r} v'\Big) r\,dr \\
&= -p|\wp| K_{k,p}^{p-1} \int_0^\infty |v|^{p-2} v v'' r\,dr.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
-\int_0^\infty |v|^{p-2} v v'' r\,dr &= (p-1)\int_0^\infty |v|^{p-2}(v')^2 r\,dr + \int_0^\infty |v|^{p-2} v v'\,dr \\
&= (p-1)\int_0^\infty |v|^{p-2}(v')^2 r\,dr \\
&= \frac{4(p-1)}{p^2} \int_0^\infty \Big(\frac{p-2}{2}\Big)^2 |v|^{p-2}(v')^2\,dr \\
&\quad + \frac{4(p-1)}{p^2} \int_0^\infty (p-2)|v|^{p-2}(v')^2 + |v|^{p-2}(v')^2 r\,dr \\
&= \frac{4(p-1)}{p^2} \int_0^\infty \Big(\big(|v|^{\frac{p-2}{2}}\big)' v + |v|^{\frac{p-2}{2}} v'\Big)^2 r\,dr \\
&= \frac{4(p-1)}{p^2} \int_0^\infty \Big|\big(|v|^{\frac{p-2}{2}} v\big)'\Big|^2 r\,dr \\
&= \frac{4(p-1)}{|\wp_2| p^2} \int_{\mathbb{G}_2} \Big|\mathcal{R}\big(|v|^{\frac{p-2}{2}} v\big)\Big|^2\,dx,
\end{aligned}
$$

where $\mathbb{G}_2$ is a homogeneous group of homogeneous degree 2 and $|\wp_2|$ is the measure of the corresponding unit 2-quasiball. By using Lemma 2.1 for $|v|^{\frac{p-2}{2}}v \in C_0^\infty(\mathbb{G}_2 \setminus \{0\})$ in the $p=Q=2$ case, and combining the above equalities, we obtain

$$
\begin{aligned}
J &\geq C_1 \int_{\mathbb{G}_2} \frac{\big||v(x)|^{\frac{p-2}{2}} v(x) - |v(R\frac{x}{|x|})|^{\frac{p-2}{2}} v(R\frac{x}{|x|})\big|^2}{|x|^2 |\log\frac{R}{|x|}|^2}\,dx \\
&= C_1 \int_0^\infty \frac{\big||v(r)|^{\frac{p-2}{2}} v(r) - |v(R)|^{\frac{p-2}{2}} v(R)\big|^2}{r|\log\frac{R}{r}|^2}\,dr \\
&= C_1 \int_0^\infty \frac{\big||u(r)|^{\frac{p-2}{2}} u(r) - R^{\frac{Q-kp}{2}} |u(R)|^{\frac{p-2}{2}} u(R) r^{-\frac{Q-kp}{2}}\big|^2}{r^{1-Q+kp}|\log\frac{R}{r}|^2}\,dr
\end{aligned}
$$

for any $R>0$. That is,

$$J \geq C \sup_{R>0} \int_{\mathbb{G}} \frac{||u(x)|^{\frac{p-2}{2}} u(x) - R^{\frac{Q-kp}{2}} |u(R)|^{\frac{p-2}{2}} u(R) |x|^{-\frac{Q-kp}{2}}|^2}{|x|^{kp}| \log \frac{R}{|x|}|^2} \, dx.$$

The proof is complete. ☐

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