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COMPACT AND "COMPACT" OPERATORS ON STANDARD HILBERT MODULES OVER W^*-ALGEBRAS

DRAGOLJUB J. KEČKIĆ* and ZLATKO LAZOVIĆ

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ABSTRACT. We construct a topology on the standard Hilbert module $l^2(\mathcal{A})$ over a unital W^*-algebra $\mathcal{A}$ such that any "compact" operator (i.e., any operator in the norm closure of the linear span of the operators of the form $x \mapsto z\langle y, x\rangle$) maps bounded sets into totally bounded sets.

1. INTRODUCTION

Given a unital W^*-algebra $\mathcal{A}$, we consider the standard Hilbert module denoted by $l^2(\mathcal{A})$ (the notation $\mathcal{H}_\mathcal{A}$ is also widespread)

$$l^2(\mathcal{A}) = \left\{ x = (\xi_1, \xi_2, \dots) \;\middle|\; \xi_j \in \mathcal{A}, \sum_{j=1}^{+\infty} \xi_j^* \xi_j \text{ converges in the norm topology} \right\},$$

equipped with the $\mathcal{A}$-valued inner product

$$l^2(\mathcal{A}) \times l^2(\mathcal{A}) \ni (x, y) \mapsto \sum_{j=1}^{+\infty} \xi_j^* \eta_j \in \mathcal{A}, \quad x = (\xi_1, \xi_2, \dots), y = (\eta_1, \eta_2, \dots).$$

Since an arbitrary $\mathcal{A}$-linear bounded operator on $l^2(\mathcal{A})$ does not need to have an adjoint, the natural algebra of operators is $B^a(l^2(\mathcal{A}))$—the algebra of all $\mathcal{A}$-linear bounded operators on $l^2(\mathcal{A})$ having an adjoint. It is known that $B^a(l^2(\mathcal{A}))$ is a

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*Corresponding author.
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C^*-algebra. Also, $B^a(\mathcal{M})$ is a W^*-algebra whenever $\mathcal{M}$ is a self-dual module over a W^*-algebra.

Among all operators in $B^a(l^2(\mathcal{A}))$, those that belong to the linear span of the operators of the form $x \mapsto \Theta_{y,z}(x) = z\langle y, x\rangle$ $(y, z \in l^2(\mathcal{A}))$ are called *finite-rank operators*. The norm closure of finite-rank operators is known as the algebra of all "compact" operators. The quotation marks are usually written in order to emphasize the fact that "compact" operators do not map bounded sets into relatively compact sets, as is the case in the framework of Hilbert (and also Banach) spaces, though they share many properties of proper compact operators on a Hilbert space (see [6], [7]). (For general literature concerning Hilbert modules over more general C^*-algebras, including the standard Hilbert module, the reader is referred to [5] or [8].)

The aim of this article is to introduce a locally convex topology on $l^2(\mathcal{A})$, where $\mathcal{A}$ is a unital W^*-algebra, such that any "compact" operator maps bounded sets (in the norm) into totally bounded sets in the introduced topology. In a very special case, where $\mathcal{A} \cong B(H)$ denotes the algebra of all bounded operators on a Hilbert space, the converse is also true. Namely, any operator $T \in B^a(l^2(\mathcal{A}))$ that maps bounded sets into totally bounded sets is "compact." Therefore, speaking freely, we can omit the quotation marks.

2. Preliminaries

Let us recall some basic definitions and facts concerning uniform spaces (for more details, see [1] or [4]). Uniform spaces are those topological spaces in which one can deal with notions such as Cauchy sequence, Cauchy net, or uniform continuity. Although it is usual to define them as spaces endowed with a family of sets in $X \times X$ given as some kind of neighborhoods of the diagonal, or so-called *entourages*, for our purposes it is more convenient to give an equivalent definition via a family of semimetrics.

Definition 2.1. A nonempty set endowed with a family of semimetrics, functions $d_\alpha : X \times X \to [0, +\infty)$ satisfying (i) $d_\alpha(x, y) \geq 0$, (ii) $d_\alpha(x, y) = d_\alpha(y, x)$, and (iii) $d_\alpha(x, z) \leq d_\alpha(x, y) + d_\alpha(y, z)$ is called a *uniform space*. All d_α's are metrics, except they do not need to distinguish points, that is, there might be $d_\alpha(x, y) = 0$ for some $x \neq y$. However, it is provided that for all $x \neq y$ there is an α such that $d_\alpha(x, y) > 0$.

The family of sets $B_{d_\alpha}(x; \varepsilon) = \{y \in X \mid d_\alpha(x, y) < \varepsilon\}$ makes a basis for some topology. It is well known that a topological space X is a uniform space if and only if it is completely regular.

Let X be a uniform space. We say that a net $x_i \in X$ is a *Cauchy net* if it is a Cauchy net with respect to all d_α's; that is, if for all α's and for all $\varepsilon > 0$ there is i_0 such that, for all i, $j > i_0$, we have $d_\alpha(x_i, x_j) < \varepsilon$. The notion of a complete uniform space is defined in an obvious way.

A set $A \subseteq X$ is called *totally bounded* if for all $\varepsilon > 0$ and all α's there is a finite set $c_1, c_2, \ldots, c_m \in X$ such that sets $B_\alpha(c_j; \varepsilon) = \{y \in X \mid d_\alpha(c_j, y) < \varepsilon\}$ cover A. It is well known that any relatively compact set is totally bounded and that the converse is true provided that X is complete. If X is not complete, then there are totally bounded sets that are not relatively compact, for instance, $\mathbb{Q}\cap[0,1]$ as a subset of $\mathbb{Q}$. (See also [1, Remark 4.2.2].)

Any locally convex topological vector space is a uniform space. Indeed, there is a family of seminorms generating its topology. This family can be obtained by Minkowski functionals of basic neighborhoods of zero. And an arbitrary seminorm defines a semimetric in a natural way. Conversely, any family of seminorms that distinguishes points leads to a locally convex Hausdorff topological vector space. Hence, a family of seminorms allows us to deal with notions such as totally bounded set, complete space, Cauchy net, and so on.

3. Topology

For an arbitrary Hilbert W^*-module $\mathcal{M}$, Paschke in his initial works on Hilbert C^*-modules [9], [10] and Frank in [2] introduced two topologies, τ_1 and τ_2, the first of them generated by functionals $x\mapsto\varphi(\langle y,x\rangle)$, $y\in\mathcal{M}$, φ normal state, and the second by seminorms $p(x)=\varphi(\langle x,x\rangle)^{1/2}$, φ normal state. Frank proved that $\mathcal{M}$ is self-dual if and only if the unit ball in $\mathcal{M}$ is complete in τ_1 (and this is equivalent to the completeness in τ_2). Therefore, if $\mathcal{M}=l^2(\mathcal{A})$ is a standard Hilbert module, it is not complete either in τ_1 or in τ_2, since $l^2(\mathcal{A})$ is never self-dual, except in the case where $\mathcal{A}$ is a finite-dimensional algebra. We will refer to τ_1 and τ_2 as *weak Paschke–Frank* (*PF*) and *strong PF* topologies, since obviously $\tau_1\subset\tau_2$.

However, we need a topology which is between a weak and a strong PF topology. Namely, on a standard Hilbert module $l^2(\mathcal{A})$, where $\mathcal{A}$ is a unital W^*-algebra, we define a locally convex topology τ by the family of seminorms

$$p_{\varphi,y}(x)=\sqrt{\sum_{j=1}^{+\infty}\left|\varphi(\eta_j^*\xi_j)\right|^2},\tag{3.1}$$

where φ is a normal state, $x=(\xi_1,\xi_2,\dots)\in l^2(\mathcal{A})$, and $y=(\eta_1,\eta_2,\dots)$ is a sequence of elements in $\mathcal{A}$ such that

$$\sup_{j\ge1}\varphi(\eta_j^*\eta_j)=1.\tag{3.2}$$

(Note that y does not need to belong to $l^2(\mathcal{A})$.)

Proposition 3.1. *Seminorms (3.1) are well defined; that is, the series is convergent. Also, $\tau_1\subset\tau\subset\tau_2$.*

Proof. Since $(\xi,\eta)\mapsto\varphi(\eta^*\xi)$ is a semi-inner product, we have $|\varphi(\eta_j^*\xi_j)|^2\le\varphi(\xi_j^*\xi_j)\varphi(\eta_j^*\eta_j)$. By this and by (3.2), we have

$$p_{\varphi,y}(x)^2=\sum_{j=1}^{+\infty}\left|\varphi(\eta_j^*\xi_j)\right|^2\le\sum_{j=1}^{+\infty}\varphi(\xi_j^*\xi_j)\varphi(\eta_j^*\eta_j)\le\sum_{j=1}^{+\infty}\varphi(\xi_j^*\xi_j)=\varphi\big(\langle x,x\rangle\big).\tag{3.3}$$

This proves that seminorms (3.1) are well defined, and also that $\tau\subset\tau_2$.

To prove $\tau_1\subset\tau$, pick $y\in l^2(\mathcal{A})$, $y=(\eta_1,\eta_2,\dots)$. The sequence ζ_j given by $\zeta_j=\eta_j/\varphi(\eta_j^*\eta_j)^{1/2}$ if $\varphi(\eta_j^*\eta_j)\ne0$, and $\zeta_j=0$ otherwise, obviously fulfills (3.2).

Hence

$$
\begin{aligned}
\big|\varphi\big(\langle y,x\rangle\big)\big| &= \Big|\varphi\Big(\sum_{j=1}^{+\infty}\eta_j^*\xi_j\Big)\Big| = \Big|\sum_{j=1}^{+\infty}\varphi(\eta_j^*\eta_j)^{1/2}\varphi(\zeta_j^*\xi_j)\Big| \\
&\le \Big(\sum_{j=1}^{+\infty}\varphi(\eta_j^*\eta_j)\Big)^{1/2}\Big(\sum_{j=1}^{+\infty}\big|\varphi(\zeta_j^*\xi_j)\big|^2\Big)^{1/2} = \varphi\big(\langle y,y\rangle\big)^{1/2}p_{\varphi,z}(x),
\end{aligned}
$$

finishing the proof. □

Remark 3.2. The dual module of the module $\mathcal{M}$ is defined as the module of all $\mathcal{A}$-linear and $\mathcal{A}$-valued bounded functionals. It is denoted by $\mathcal{M}'$. The module $\mathcal{M}$ always can be embedded in $\mathcal{M}'$ via $\mathcal{M} \ni y \mapsto \Lambda_y \in \mathcal{M}'$, $\Lambda_y(x) = \langle y,x\rangle$. If this embedding is onto, the module $\mathcal{M}$ is called *self-dual*. It is worth mentioning that the problem of self-duality, even if the underlying algebra $\mathcal{A}$ is commutative, is still actual (see, e.g., some recent results [3, Theorem 3.3(2)], [11, Theorems 4.5 and 4.6]). It is also well known that $l^2(\mathcal{A})$ is not self-dual, except when the algebra $\mathcal{A}$ is finite-dimensional. Namely, $l^2(\mathcal{A})'$ can be described as the module of all sequences $x = (\xi_1,\xi_2,\dots)$ such that the sequence of sums $\sum_{j=1}^n\xi_j^*\xi_j$ is norm bounded (see [8, Proposition 2.5.5]). A careful reading of the proof of Proposition 3.1 reveals that nothing is changed if we replace $l^2(\mathcal{A})$ by $l^2(\mathcal{A})'$. Indeed, the entire proof does not depend on the norm convergence of the series $\sum_{j=1}^{+\infty}\xi_j^*\xi_j$.

Proposition 3.3. *The unit ball in $l^2(\mathcal{A})$ is not complete with respect to τ unless $\mathcal{A}$ is finite-dimensional. Its completion is the unit ball in the dual module $l^2(\mathcal{A})'$.*

Proof. First, we prove that the unit ball in $l^2(\mathcal{A})$ is dense in the unit ball in $l^2(\mathcal{A})'$. Let $x \in l^2(\mathcal{A})'$, $x = (\xi_1,\xi_2,\dots)$. Since the sequence of sums $\sum_{j=1}^n\xi_j^*\xi_j$ is bounded, it is convergent in strong (or weak, ultraweak, etc.) topology. By normality of φ we have

$$
\varphi\Big(\sum_{j=1}^{+\infty}\xi_j^*\xi_j\Big) = \sum_{j=1}^{+\infty}\varphi(\xi_j^*\xi_j),
$$

implying that $\varphi(\sum_{j=n}^{+\infty}\xi_j^*\xi_j) \to 0$, as $n \to +\infty$. Thus, by the inequality (3.3) $(\xi_1,\xi_2,\dots,\xi_n,0,0,\dots) \to x$ in each seminorm of the form (3.1).

Next, we prove that $l^2(\mathcal{A})'$ is complete. Let $x^\alpha = (\xi_1^\alpha,\xi_2^\alpha,\dots)$ be a Cauchy net in the unit ball. Choosing an arbitrary normal state, with $\eta_k = 1$, $\eta_j = 0$ for $j \ne k$, we obtain that ξ_k^α is a Cauchy net in the weak-$*$ topology in the unit ball in $\mathcal{A}$. Hence, it is convergent, say, $\xi_k^\alpha \to \xi_k$ in the weak-$*$ topology.

Since multiplying is ultraweakly continuous, for any $n \in \mathbf{N}$ and for all η_j which satisfy (3.2), we have

$$
\sum_{j=1}^{k}\big|\varphi(\eta_j^*\xi_j^\alpha)\big|^2 \to \sum_{j=1}^{k}\big|\varphi(\eta_j^*\xi_j)\big|^2.
$$

Choosing $\eta_j = \xi_j/\varphi(\xi_j^*\xi_j)^{1/2}$ if $\varphi(\xi_j^*\xi_j) \neq 0$ and $\eta_j = 0$ otherwise, we get

$$\sum_{j=1}^{k} \varphi(\xi_j^*\xi_j) = \sum_{j=1}^{k} \big|\varphi(\eta_j^*\xi_j)\big|^2 = \lim_{\alpha} \sum_{j=1}^{k} \big|\varphi(\eta_j^*\xi_j^\alpha)\big|^2 \leq \|x\| \leq 1.$$

Taking the limit as $k \to +\infty$, we conclude that $x = (\xi_1, \xi_2, \dots) \in l^2(\mathcal{A})'$. To see that x is the limit of the Cauchy net x^α, it is enough to take the limit over β in

$$\sum_{j=1}^{k} \big|\varphi(\eta_j^*\xi_j^\alpha) - \varphi(\eta_j^*\xi_j^\beta)\big|^2 \leq \sum_{j=1}^{+\infty} \big|\varphi(\eta_j^*\xi_j^\alpha) - \varphi(\eta_j^*\xi_j^\beta)\big|^2 < \varepsilon,$$

and finally the limit as $k \to +\infty$. □

Next, we want to study the restriction of τ to the module $\mathcal{A}^n$ seen as a submodule of $l^2(\mathcal{A})$ consisting of those x for which $\xi_j = 0$ for all $j > n$.

Proposition 3.4.

(a) *On $\mathcal{A}^n$, the weak PF and our topology coincide; that is, we have $\tau_1|_{\mathcal{A}^n} = \tau|_{\mathcal{A}^n}$.*

(b) *The embedding $i : \mathcal{A}^n \to l^2(\mathcal{A})$, $i(\xi_1, \dots, \xi_n) = (\xi_1, \dots, \xi_n, 0, \dots)$, is continuous with respect to $(\tau|_{\mathcal{A}^n}, \tau)$.*

Proof. (a) We already have $\tau_1 \subseteq \tau$. Let us prove the converse. An arbitrary seminorm of the form (3.1) restricted to $\mathcal{A}^n$ has the form

$$p_{\varphi,y}(x) = \sqrt{\sum_{j=1}^{n} \big|\varphi(\eta_j^*\xi_j)\big|^2}.$$

Consider the vectors $y_j = (0, \dots, 0, \eta_j, 0, \dots, 0)$, where η_j is the jth entry. Then

$$p_{\varphi,y}(x) = \sqrt{\sum_{j=1}^{n} \big|\varphi\big(\langle y_j, x\rangle\big)\big|^2} \leq \sum_{j=1}^{n} \big|\varphi\big(\langle y_j, x\rangle\big)\big|,$$

from which we conclude that $p_{\varphi,y}$ is continuous with respect to τ_1.

(b) One can easily check that

$$i^{-1}\Big(\big\{x \mid p_{\varphi,\eta_1,\dots,\eta_n,\dots}(x) < \varepsilon\big\}\Big) = \big\{(\xi_1, \dots, \xi_n) \mid p_{\varphi,\eta_1,\dots,\eta_n}(\xi_1, \dots, \xi_n) < \varepsilon\big\}.$$ □

Proposition 3.5. *The unit ball in $\mathcal{A}^n$ is compact with respect to $\tau|_{\mathcal{A}^n}$. Since $\mathcal{A}^n$ is self-dual, the unit ball is also complete and hence totally bounded.*

Proof. In the case $n = 1$, both topologies τ and τ_1 are generated by seminorms $\xi \mapsto |\varphi(\eta^*\xi)|$, $\eta \in \mathcal{A}$, φ normal state. It is easy to verify that these topologies are exactly the weak-$*$ topology on $\mathcal{A}$. Therefore, in this special case the conclusion follows by the Banach–Alaoglu theorem.

To obtain the result in the general case, consider the product topology on $\mathcal{A}^n = \mathcal{A} \times \cdots \times \mathcal{A}$. Basic neighborhoods of zero have the form $\{(\xi_1, \xi_2, \dots, \xi_n) \mid$

$\forall j = 1, 2, \dots, n |\varphi_j(\eta_j^* \xi_j)| < \varepsilon_j\}$. Due to the inequalities

$$\max_{1\le j\le n}\bigl|\varphi(\eta_j^*\xi_j)\bigr| \le \sqrt{\sum_{j=1}^n \bigl|\varphi(\eta_j^*\xi_j)\bigr|^2} \le \sqrt{n}\max_{1\le j\le n}\bigl|\varphi(\eta_j^*\xi_j)\bigr|,$$

the topology τ is weaker than the product topology. Since the product of unit balls is compact in stronger product topology, and since τ is Hausdorff, we conclude that τ coincides with the product topology on the product of unit balls. Therefore, it remains to show that the unit ball in $\mathcal{A}^n$ is closed in the product of n unit balls in $\mathcal{A}$, that is, that its complement is open.

Let $z = (\zeta_1, \dots, \zeta_n) \in \mathcal{A}^n$, $\|z\| > 1$, be arbitrary. Let $\varepsilon > 0$ be a number less than $(\|z\|^2 - \|z\|)/2\sqrt{n}$, and let φ be the normal state that attains its norm at $\langle z, z\rangle = \zeta_1^*\zeta_1 + \cdots + \zeta_n^*\zeta_n$ up to $\varepsilon\sqrt{n}$; that is, $\varphi(\langle z, z\rangle) > \|z\|^2 - \varepsilon\sqrt{n}$. Consider the seminorm

$$p_{\varphi,z}(x) = \sqrt{\bigl|\varphi(\zeta_1^*\xi_1)\bigr|^2 + \cdots + \bigl|\varphi(\zeta_n^*\xi_n)\bigr|^2}, \quad x = (\xi_1, \xi_2, \dots, \xi_n).$$

We claim that the open set

$$G = \bigl\{x \bigm| p_{\varphi,z}(x - z) < \varepsilon\bigr\}$$

does not intersect the unit ball B. Indeed, let $x \in G$. Then by the classic Cauchy–Schwarz inequality, we have

$$\begin{aligned}\varepsilon^2 > p_{\varphi,z}(x-z)^2 &= \bigl|\varphi(\zeta_1^*\xi_1) - \varphi(\zeta_1^*\zeta_1)\bigr|^2 + \cdots + \bigl|\varphi(\zeta_n^*\xi_n) - \varphi(\zeta_n^*\zeta_n)\bigr|^2\\ &\ge \frac{1}{n}\bigl|\varphi(\zeta_1^*\xi_1) + \cdots + \varphi(\zeta_n^*\xi_n) - \varphi(\zeta_1^*\zeta_1) - \cdots - \varphi(\zeta_n^*\zeta_n)\bigr|^2\end{aligned}$$

or

$$\begin{aligned}\varepsilon\sqrt{n} &> \bigl|\varphi(\zeta_1^*\xi_1 + \cdots + \zeta_n^*\xi_n) - \varphi\bigl(\langle z, z\rangle\bigr)\bigr|\\ &\ge \|z\|^2 - \varepsilon\sqrt{n} - \bigl|\varphi(\zeta_1^*\xi_1 + \cdots + \zeta_n^*\xi_n)\bigr|;\end{aligned}$$

that is,

$$\|z\|^2 - 2\varepsilon\sqrt{n} < \bigl|\varphi(\zeta_1^*\xi_1 + \cdots + \zeta_n^*\xi_n)\bigr| = \bigl|\varphi\bigl(\langle z, x\rangle\bigr)\bigr|. \tag{3.4}$$

However, $\varphi(\langle z, x\rangle)$ is a semi-inner product and it satisfies the Cauchy–Schwarz inequality

$$\bigl|\varphi\bigl(\langle z, x\rangle\bigr)\bigr|^2 \le \varphi\bigl(\langle z, z\rangle\bigr)\varphi\bigl(\langle x, x\rangle\bigr) \le \|z\|^2\|x\|^2. \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$\|z\|\|x\| > \|z\|^2 - 2\varepsilon\sqrt{n},$$

and taking into account how ε is chosen, we have

$$\|x\| > \frac{1}{\|z\|}\bigl(\|z\|^2 - 2\varepsilon\sqrt{n}\bigr) > 1.$$

Therefore, $x \notin B$, implying that B is a closed set. The proof is complete. $\square$

Proposition 3.6. *The unit ball in $l^2(\mathcal{A})$ is not totally bounded in τ.*

Proof. Let $e_j = (0, \dots, 0, 1, 0, \dots)$, where 1 (the unit of the algebra $\mathcal{A}$) stands at the jth entry. Let φ be an arbitrary normal state, and consider the seminorm $p = p_{\varphi,1,1,\dots}$ given by $p(x)^2 = \sum_{j=1}^{+\infty} |\varphi(\xi_j)|^2$. We claim that the sequence e_j is totally discrete in p. Indeed, $p(e_i - e_j)^2 = |\varphi(1)|^2 + |\varphi(-1)|^2 = 2$; that is, $p(e_i - e_j) = \sqrt{2}$. Hence, the set $\{e_j \mid j \geq 1\}$ is not totally bounded in p and also in τ. The same is valid for a larger set—the unit ball. □

4. "Compact" operators

Let $y, z \in l^2(\mathcal{A})$. The operator $l^2(\mathcal{A}) \to l^2(\mathcal{A})$, $x \mapsto z\langle y, x\rangle$ is adjointable (its adjoint is $x \mapsto y\langle z, x\rangle$) and bounded. The closed linear hull of such operators is called the *algebra of "compact" operators*. We say that the operator $T \in B^a(l^2(\mathcal{A}))$ is *compact* if its image of any (norm) bounded set is a totally bounded set in the topology τ described in Section 3. For the operator $T \in B^a(l^2(\mathcal{A}))$, it is enough to map the unit ball into a totally bounded set to be a compact operator.

Remark 4.1. Totally bounded and relatively compact sets differ in the general case (whenever the unit ball is not complete). Also, throughout the literature, there is a certain ambiguity between the terms *completely continuous*, *compact*, and *precompact* when applied to operators. Although it seems that the terms *completely continuous* and *precompact* are more accurate, we found that *compact* is more convenient for our purposes.

Before we prove that any "compact" operator is compact, we need a few lemmas

Lemma 4.2. *For $S, T \subseteq l^2(\mathcal{A})$ and a seminorm p, denote*

$$d_p(S, T) = \sup_{x \in S} \inf_{y \in T} p(x - y)$$

(and note that d_p is not symmetric). Let $S \subseteq l^2(\mathcal{A})$. If for all seminorms p of the form (3.1) and all $\varepsilon > 0$ there is a totally bounded set $S_{p,\varepsilon}$ such that

$$d_p(S, S_{p,\varepsilon}) < \varepsilon, \tag{4.1}$$

then S is also totally bounded.

Proof. Denote

$$B_p(x; \varepsilon) = \{y \in l^2(\mathcal{A}) \mid p(x - y) < \varepsilon\}.$$

The condition (4.1) gives

$$S \subseteq \bigcup_{x \in S_{p,\varepsilon/2}} B_p(x; \varepsilon/2), \tag{4.2}$$

for all $\varepsilon > 0$. Let $\varepsilon > 0$ be arbitrary. The set $S_{p,\varepsilon/2}$ is totally bounded in p and hence there is a finite set $\{c_1, \dots, c_m\}$ such that the union of balls $B_p(c_j; \varepsilon/2)$ covers $S_{p,\varepsilon/2}$. By (4.2), the union of balls $B_p(c_j; \varepsilon)$ covers S. □

Lemma 4.3. *Let $T_\alpha : l^2(\mathcal{A}) \to l^2(\mathcal{A})$ be a net of compact operators such that $T_\alpha x \to Tx$ in τ uniformly with respect to $\|x\| < 1$. Then T is also compact.*

Proof. For any $\varepsilon > 0$ and any seminorm p of the form (3.1), there is α such that $\sup_{\|x\|<1} p(Tx - T_\alpha x) < \varepsilon$. Therefore,

$$d_p\big(T\big(B_{\|\cdot\|}(0;1)\big), T_\alpha\big(B_{\|\cdot\|}(0;1)\big)\big) \leq \varepsilon$$

and the conclusion follows from Lemma 4.2. □

Corollary 4.4. *Let $S \subseteq l^2(\mathcal{A})$ be a set such that, for all $\varepsilon > 0$, there is a totally bounded (in τ) set S_ε such that*

$$d(S, S_\varepsilon) = \sup_{x \in S} \inf_{y \in S_\varepsilon} \|x - y\| < \varepsilon.$$

Then S is also totally bounded in τ. Also, let $T_n : l^2(\mathcal{A}) \to l^2(\mathcal{A})$ be a sequence of compact operators that converges to T in the operator norm. Then T is also compact.

Proof. Both conclusions follow from the fact that τ is coarser than the norm topology. □

Lemma 4.5. *Let T_1 and T_2 be compact operators, and let $u_1, u_2 \in \mathcal{A}$. Then $T_1 u_1 + T_2 u_2$ is also compact.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since T_1 and T_2 are compact there is a finite $\varepsilon/2\|u_1\|$ net for $T_1(B_{\|\cdot\|}(0;1))$, say, $c_1, c_2, \dots, c_n$, and a finite $\varepsilon/2\|u_2\|$ net for $T_2(B_{\|\cdot\|}(0;1))$, say, $d_1, \dots, d_m$. Then the set $\{c_i u_1 + d_j u_2 \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is a finite ε net for $(T_1 u_1 + T_2 u_2)(B_{\|\cdot\|}(0;1))$. Indeed, if $x \in B_{\|\cdot\|}(0;1)$, then there is i and j such that $\|T_1 x - c_i\| < \varepsilon/2\|u_1\|$ and $\|T_2 x - d_j\| < \varepsilon/2\|u_2\|$. Hence

$$\big\|(T_1 x u_1 + T_2 x u_2) - (c_i u_1 + d_j u_2)\big\| \leq \|T_1 x - c_i\|\|u_1\| + \|T_2 x - d_j\|\|u_2\| < \varepsilon. \ \square$$

Theorem 4.6. *Let $T : l^2(\mathcal{A}) \to l^2(\mathcal{A})$ be a "compact" operator. Then T is compact.*

Proof. In view of Lemmas 4.3 and 4.5, it is enough to prove that operators of the form $x \mapsto \Theta_{y,z}(x) = z\langle y, x\rangle$ are compact. In the *special case*, where $z = e_j\zeta$, $\zeta \in \mathcal{A}$, it immediately follows from Proposition 3.5. Indeed, then $\Theta_{y,e_j\zeta}(B_{\|\cdot\|}(0;1))$ is contained in the ball of radius $\|\Theta_{y,e_j\zeta}\|$ in $\mathcal{A}^1$ which is totally bounded. In the *general case*, let $z = (\zeta_1, \zeta_2, \dots)$. Then $z = \sum_{j=1}^{+\infty} e_j \zeta_j$, where the series converges in the norm. Since $\|\Theta_{y,z} - \Theta_{y,z'}\| \leq \|y\|\|z - z'\|$, we have

$$\Theta_{y,z} = \lim_{n \to +\infty} \sum_{j=1}^{n} \Theta_{y,e_j\zeta_j}$$

and the conclusion follows from the special case and Lemmas 4.3 and 4.5. □

Remark 4.7. In the case where $\mathcal{A}$ is only a C^*-algebra and not a W^*-algebra, we cannot use Proposition 3.5 since it relies on the property that $\mathcal{A}$ has a predual. Moreover, the definition of the topology τ becomes inappropriate since it uses the notion of normal state. Of course, we can substitute normal states by general states. However, states (not necessarily normal) belong to the dual, not to the predual, and hence they do not generate weak-$*$ topology on the single algebra $\mathcal{A}$. Thus, we have no compactness results. Nevertheless, we hope that something can be done using the enveloping W^*-algebra A^{**}, but we leave this for future work.

The converse is true in the special case where $\mathcal{A} = B(H)$ is the full algebra of all bounded linear operators on a Hilbert space H. Before we prove such a result, we need a technical lemma.

Lemma 4.8. *Let $\mathcal{A} = B(H)$, and let $a_j \in \mathcal{A}$, $j \geq 1$ be positive elements with $\|a_j\| > \delta$. There is a normal state φ and unitary elements $u_j, v_j \in \mathcal{A}$ such that $|\varphi(v_j^* a_j u_j)| > \delta$.*

Remark 4.9. Actually, we can choose φ to be a vector state, and we can also choose $u_j = v_j$.

Proof. Let $\psi \in H$ be a unit vector, and let φ be the corresponding vector state, that is, $\varphi(a) = \langle a\psi, \psi \rangle$. For all a_j let h_j be a unit vector such that $\langle a_j h_j, h_j \rangle > \delta$. As is easy to see, there is a unitary u_j such that $u_j \psi = h_j$. Thus, we have $\varphi(u_j^* a_j u_j) = \langle u_j^* a_j u_j \psi, \psi \rangle = \langle a_j h_j, h_j \rangle > \delta$. □

Theorem 4.10. *Let $\mathcal{A} = B(H)$, and let $T : l^2(\mathcal{A}) \to l^2(\mathcal{A})$ be a compact operator. Then T is "compact."*

Proof. Let P_k denote the projection to the first k coordinates, that is, $P_k(\xi_1, \xi_2, \dots) = (\xi_1, \dots, \xi_k, 0, 0, \dots)$. It is well known that all P_k's are "compact."

Suppose that T is not "compact." Then

$$\delta = \inf_{k \geq 1} \left\| (I - P_k)T \right\| > 0.$$

Indeed, otherwise either for some k we have $(I - P_k)T = 0$ and hence $T = P_k T$ is "compact," or there is a sequence of positive integers k_n such that $\|T - P_{k_n} T\| \to 0$, from which it follows that T is "compact."

To simplify the calculations, assume that $\|T\| = 1$. Then immediately, $\delta \leq 1$.

Define the sequence of projections $Q_n \in \{P_1, P_2, \dots\}$ and the sequences of vectors x_n, y_n, and $z_n \in l^2(\mathcal{A})$ in the following way. Let $Q_0 = 0$. If Q_{n-1} is already defined, there is $x_n \in l^2(\mathcal{A})$ such that $\|x_n\| = 1$ and $\|(I - Q_{n-1})Tx_n\| > \delta/2$. Denote $y_n = Tx_n$. Then, by $\|I - Q_{n-1}\| = 1$,

$$\|y_n\| \geq \left\| (I - Q_{n-1}) y_n \right\| > \frac{\delta}{2}.$$

Since $\lim_{k \to +\infty} \|(I - P_k)(I - Q_{n-1}) y_n\| = 0$, there is a positive integer k_n such that $\|(I - P_{k_n})(I - Q_{n-1}) y_n\| < \delta^2/8 \leq \delta/8$. Define $Q_n = P_{k_n}$ and

$$z_n = Q_n (I - Q_{n-1}) y_n. \tag{4.3}$$

The sequences y_n and z_n have the following properties. First, by definition, the inequalities

$$\left\| (I - Q_n)(I - Q_{n-1}) y_n \right\| < \frac{\delta^2}{8} \leq \frac{\delta}{8}, \tag{4.4}$$

$$\|z_n\| \leq \|y_n\| \leq \|T\| \|x_n\| = 1, \tag{4.5}$$

$$\|z_n\| \geq \left\| (I - Q_{n-1}) y_n \right\| - \left\| (I - Q_n)(I - Q_{n-1}) y_n \right\| > \frac{\delta}{2} - \frac{\delta}{8} = \frac{3\delta}{8} \tag{4.6}$$

hold. Second,

$$\langle z_n, y_n \rangle = \langle z_n, z_n \rangle. \tag{4.7}$$

Indeed, since $z_n = Q_n(I - Q_{n-1})y_n$, we have

$$\begin{aligned}\langle z_n, y_n\rangle &= \big\langle Q_n(I - Q_{n-1})y_n, y_n\big\rangle \\ &= \big\langle Q_n(I - Q_{n-1})y_n, (I - Q_{n-1})Q_n y_n\big\rangle = \langle z_n, z_n\rangle.\end{aligned}$$

Third, for $m > n$, we have

$$\big\|\langle z_m, y_n\rangle\big\| < \frac{\delta^2}{8}. \tag{4.8}$$

Indeed, for such m and n, we have $Q_{n-1} \leq Q_n \leq Q_{m-1}$; that is, $I - Q_{m-1} \leq I - Q_n \leq I - Q_{n-1}$, implying that $I - Q_{m-1} = (I - Q_{m-1})(I - Q_n)(I - Q_{n-1})$, and thus that

$$\begin{aligned}\langle z_m, y_n\rangle &= \big\langle (I - Q_{m-1})z_m, y_n\big\rangle \\ &= \big\langle z_m, (I - Q_{m-1})(I - Q_n)(I - Q_{n-1})y_n\big\rangle \\ &= \big\langle z_m, (I - Q_n)(I - Q_{n-1})y_n\big\rangle.\end{aligned}$$

Therefore, by (4.4) and (4.5),

$$\big\|\langle z_m, y_n\rangle\big\| \leq \|z_m\|\big\|(I - Q_n)(I - Q_{n-1})y_n\big\| \leq \frac{\delta^2}{8}.$$

Let us construct a seminorm p, continuous in τ, and a totally discrete sequence from $T(\overline{B_{\|\cdot\|}(0;1)})$. Since by (4.6) $\|z_n\|^2 = \|\langle z_n, z_n\rangle\| > (3\delta/8)^2$, we can choose φ and $\upsilon_j, \nu_j \in \mathcal{A}$ according to Lemma 4.8 such that

$$\varphi\big(\upsilon_n^*\langle z_n, z_n\rangle\nu_n\big) > \frac{9\delta^2}{64}. \tag{4.9}$$

Consider the seminorm p given by

$$p(x) = \sqrt{\sum_{j=1}^{+\infty}\big|\varphi\big(\langle z_j\upsilon_j, x\rangle\big)\big|^2}.$$

By (4.3) there is a sequence $\zeta_j \in \mathcal{A}$ such that

$$z_k = (0, \dots, 0, \zeta_{k_{n-1}+1}, \dots, \zeta_{k_n}, 0, \dots).$$

Define $\omega_j = \zeta_j\upsilon_n/\varphi(\upsilon_n^*\zeta_j^*\zeta_j\upsilon_n)^{1/2}$, for $j = k_{n-1}+1, \dots, k_n$. Obviously $\varphi(\omega_j^*\omega_j) = 1$. Also, for $x = (\xi_1, \xi_2, \dots)$ we have

$$\begin{aligned}\big|\varphi\big(\langle z_n\upsilon_n, x\rangle\big)\big|^2 &= \Big|\sum_{j=k_{n-1}+1}^{k_n}\varphi(\upsilon_n^*\zeta_j^*\zeta_j\upsilon_n)^{1/2}\varphi(\omega_j^*\xi_j)\Big|^2 \\ &\leq \sum_{j=k_{n-1}+1}^{k_n}\varphi(\upsilon_n^*\zeta_j^*\zeta_j\upsilon_n)\sum_{j=k_{n-1}+1}^{k_n}\big|\varphi(\omega_j^*\xi_j)\big|^2 \\ &= \varphi\big(\upsilon_n^*\langle z_n, z_n\rangle\upsilon_n\big)\sum_{j=k_{n-1}+1}^{k_n}\big|\varphi(\omega_j^*\xi_j)\big|^2.\end{aligned}$$

Including (4.5), we obtain $\varphi(\upsilon_n^*\langle z_n, z_n\rangle\upsilon_n) \le \|\upsilon_n^*\langle z_n, z_n\rangle\upsilon_n\| = \|z_n\|^2 \le 1$, and hence

$$p(x)^2 = \sum_{n=1}^{+\infty}\bigl|\varphi\bigl(\langle z_n\upsilon_n, x\rangle\bigr)\bigr|^2 \le \sum_{j=1}^{+\infty}\bigl|\varphi(\omega_j^*\xi_j)\bigr|^2 = p_{\varphi,\omega_1,\dots,\omega_n,\dots}(x)^2.$$

Thus, we conclude that p is well defined and also that it is continuous with respect to τ. Also, $\|x_n\nu_n\| = \|x_n\|$; that is, $y_n\nu_n = Tx_n\nu_n \in T(\overline{B(0;1)})$. Finally, we prove that $y_n\nu_n$ is a totally discrete sequence. Indeed, for $m > n$, we have

$$\begin{aligned} p(y_m\nu_m - y_n\nu_n) &\ge \bigl|\varphi\bigl(\langle z_m\upsilon_m, y_m\nu_m - y_n\nu_n\rangle\bigr)\bigr| \\ &\ge \bigl|\varphi\bigl(\upsilon_m^*\langle z_m, y_m\rangle\nu_m\bigr)\bigr| - \bigl|\varphi\bigl(\upsilon_m^*\langle z_m, y_n\rangle\nu_n\bigr)\bigr|. \end{aligned}$$

However, by (4.7) and (4.9),

$$\bigl|\varphi\bigl(\upsilon_m^*\langle z_m, z_m\rangle\nu_m\bigr)\bigr| > \frac{9\delta^2}{64}$$

and, by (4.8),

$$\bigl|\varphi\bigl(\upsilon_m^*\langle z_m, y_n\rangle\nu_n\bigr)\bigr| \le \bigl\|\langle z_m, y_n\rangle\bigr\| < \frac{\delta^2}{8}.$$

Therefore,

$$p(y_m\nu_m - y_n\nu_n) > \frac{9\delta^2}{64} - \frac{\delta^2}{8} = \frac{\delta^2}{64}. \qquad \square$$

5. An example and a comment

The proof of Theorem 4.10 depends on Lemma 4.8. Hence it is valid for all unital W^*-algebras that satisfy the mentioned lemma. We do not know how to describe such algebras, but it should be noted that Lemma 4.8 does not hold for infinite-dimensional commutative W^*-algebras.

Example 5.1. In any infinite-dimensional commutative W^*-algebra $\mathcal{A}$, there is a sequence p_j of nontrivial mutually orthogonal projections. Since $\sum_{j=1}^{n} p_j$ is an increasing sequence, $p = \sum_{j=1}^{+\infty} p_j \in \mathcal{A}$. Therefore, for an arbitrary normal state φ, the series $\sum_{j=1}^{+\infty}\varphi(p_j)$ is convergent. The algebra is commutative, and for all unitary υ_j, ν_j, we have

$$\bigl|\varphi(\upsilon_j p_j \nu_j)\bigr| = \bigl|\varphi(p_j\upsilon_j\nu_j)\bigr| \le \varphi(p_j)^{1/2}\varphi(\nu_j^*\upsilon_j^*\upsilon_j\nu_j)^{1/2} \to 0.$$

Thus, Lemma 4.8 is not valid for commutative W^*-algebras. Moreover, we can use this sequence of projections to construct an operator which is compact, but is not “compact.” Indeed, let $T : l^2(\mathcal{A}) \to l^2(\mathcal{A})$ be the operator defined by

$$Tx = T(\xi_1, \xi_2, \dots) = (p_1\xi_1, p_2\xi_2, \dots).$$

Then, T is not “compact.” Indeed, if it is “compact,” for all $\varepsilon > 0$ there is an operator S of the form $S = \sum_{j=1}^{n}\lambda_j\Theta_{y_j,z_j}$ such that $\|T - S\| < \varepsilon/3$. Since $P_k z_j - z_j \to 0$, as $k \to +\infty$ for all $1 \le j \le n$ implies $\|P_kS - S\| \to 0$, there is k large enough such that $\|P_kS - S\| < \varepsilon/3$ and then $\|T - P_kT\| \le \|T -$

$S\| + \|S - P_kS\| + \|P_k(S-T)\| < \varepsilon$. However, as it is easy to see, $\|T - P_kT\| \geq \|Te_{k+1} - P_kTe_{k+1}\| = \|p_k\| = 1$.

On the other hand, for an arbitrary seminorm of the form (3.1), we have $p((T - P_kT)x) \to 0$ uniformly with respect to $x \in B_{\|\cdot\|}(0;1)$. Indeed, $\mathcal{A}$ is commutative and therefore $\xi_j^*\xi_j\eta_j^*\eta_j \leq \|\xi_j\|^2\eta_j^*\eta_j$; furthermore, $\varphi(\xi_j^*\xi_j\eta_j^*\eta_j) \leq \|\xi_j\|^2 \sup_j \varphi(\eta_j^*\eta_j) \leq 1$, by $\|x\| < 1$ and (3.2). Thus, we have

$$p\big((T - P_kT)x\big)^2 = \sum_{j=k+1}^{+\infty} \big|\varphi(\eta_j^* p_j \xi_j)\big|^2 \leq \sum_{j>k} \varphi(p_j)\varphi(\xi_j^*\xi_j\eta_j^*\eta_j) \leq \sum_{j>k} \varphi(p_j) \to 0.$$

Hence, T is compact by Lemma 4.3.

Remark 5.2. The topology τ defined in this article highly depends on coordinates, and therefore it is inappropriate for Hilbert modules other than $l^2(\mathcal{A})$. One might try to define a topology by seminorms

$$p_{\varphi,z_j}(x) = \sqrt{\sum_{j=1}^{+\infty} \big|\varphi\big(\langle z_j, x\rangle\big)\big|^2}, \tag{5.1}$$

where φ is a normal state and z_j is an orthogonal sequence, that satisfies $\sup_{j\geq 1} \varphi(\langle z_j, z_j\rangle) = 1$. These seminorms are generalizations of those given by (3.1). Indeed, seminorms (5.1) become seminorms (3.1) by choosing $z_j = e_j\eta_j$.

However, such new topology is in the case of $l^2(\mathcal{A})$ larger than τ, even if we suppose that z_j is moreover orthonormal. Namely, if $\mathcal{A} = B(H)$, H infinite-dimensional, there is a Cuntz ∞-tuple, that is, a sequence of isometries v_j satisfying $v_j^*v_j = 1$ and $\sum_{j=1}^{+\infty} v_jv_j^* = 1$. Then, it is easy to see that $x_j = (v_j, 0, 0, \dots)$ is orthonormal. But in the seminorm p_{φ,x_j} of the form (5.1), the sequence x_j itself is totally discrete.

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Faculty of Mathematics, University of Belgrade, Studentski trg 16-18, 11000 Belgrade, Serbia.

E-mail address: keckic@matf.bg.ac.rs; zlatkol@matf.bg.ac.rs