Ann. Funct. Anal. 9 (2018), no. 1, 111–122
https://doi.org/10.1215/20088752-2017-0028
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON SOLVING PROXIMAL SPLIT FEASIBILITY PROBLEMS AND APPLICATIONS

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Communicated by B. Ricceri

Abstract. We study the problem of proximal split feasibility of two objective convex functions in Hilbert spaces. We prove that, under suitable conditions, certain strong convergence theorems of the Halpern-type algorithm present solutions to the proximal split feasibility problem. Finally, we provide some related applications as well as numerical experiments.

1. Introduction and preliminaries

Let H_1 and H_2 be real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator.

Now we consider the *proximal split feasibility problem*. Find a minimizer x^* of f such that Ax^* minimizes g; that is, find $x^* \in \arg\min f$ such that

$$Ax^* \in \arg\min g, \tag{1.1}$$

where $\arg\min f = \{x \in H_1 : f(x) \leq f(y), \forall y \in H_1\}$, and where $\arg\min g = \{x \in H_2 : g(x) \leq g(y), \forall y \in H_2\}$. In what follows, $\Omega = \arg\min f \cap A^{-1}(\arg\min g)$ will denote the solution set of the problem (1.1).

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [4]. It concerns modeling inverse problems which

Copyright 2018 by the Tusi Mathematical Research Group.
Received Nov. 14, 2016; Accepted Feb. 28, 2017.
First published online Aug. 14, 2017.
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2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H05, 47J25.
Keywords. split feasibility problem, strong convergence, Halpern-type algorithm, proximity operator.

arise from phase retrievals and in medical image reconstruction, especially intensity-modulated therapy (see [3]). Due to its applications, this problem has been discussed in many works published in recent years (see, for example, [2], [5], [10], [12], [13]).

Let C be a nonempty, closed, and convex subset of a real Hilbert space H with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|. \tag{1.2}$$

Then P_C is called the *metric projection* of H onto C. For any $x \in H$, we know that

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \tag{1.3}$$

for all $y \in C$.

If f and g are the indicator functions of two nonempty, closed, and convex sets $C \subset H_1$ and $Q \subset H_2$, respectively, then

$$f(x) = \delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$g(x) = \delta_Q(x) = \begin{cases} 0 & \text{if } x \in Q, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the problem (1.1) becomes the following convex minimization problem. Find $x^* \in C$ such that

$$Ax^* \in Q. \tag{1.4}$$

This problem is called *the split feasibility problem*. A classical way to solve the problem (1.4) is to use the CQ algorithm introduced by Byrne [1, p. 442], which is defined in the following manner: $x_1 \in H_1$, and

$$x_{n+1} = P_C\big(x_n - \mu_n A^*(I - P_Q)Ax_n\big) \tag{1.5}$$

for each $n \geq 1$, where the stepsize $\mu_n \in (0, \frac{2}{\|A\|^2})$ and P_C, P_Q are the metric projections on C and Q, respectively.

It is noted that, in general, the operator norm $\|A\|$ or the largest eigenvalue of A^*A may not be calculated easily. To overcome this difficulty, López et al. [9, Algorithm 3.1] suggested the following algorithm: let $x_1 \in H_1$, and assume that $\{x_n\} \subset C$ has been constructed and that $\nabla h(x_n) \neq 0$. Then compute x_{n+1} via the rule

$$x_{n+1} = P_C\big(x_n - \mu_n A^*(I - P_Q)Ax_n\big) \tag{1.6}$$

for each $n \geq 1$, where $\mu_n = \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}$ with $0 < \rho_n < 4$ and $h(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$. It was proved that, if $\inf_n \rho_n(4 - \rho_n) > 0$, then the sequence $\{x_n\}$ defined by (1.6) converges weakly to a solution of (1.4).

Recall that the *subdifferential* of $F : H \to \mathbb{R} \cup \{+\infty\}$ at x is defined by

$$\partial F(x) = \big\{y \in H : F(x) + \langle y, z - x\rangle \leq F(z), \forall z \in H\big\}. \tag{1.7}$$

The *proximity operator* of F is defined by

$$\operatorname{prox}_{\lambda F}(x) = \arg\min_{y\in H}\Big\{F(y) + \frac{1}{2\lambda}\|x-y\|^2\Big\} \tag{1.8}$$

for any $\lambda > 0$. It is seen that

$$0 \in \partial F(x^*) \Longleftrightarrow x^* = \operatorname{prox}_{\lambda F}(x^*). \tag{1.9}$$

Hence the minimizers of any functions are the fixed point of its proximity operator. Moreover, the proximity operator of F is firmly nonexpansive, namely,

$$\big\langle \operatorname{prox}_{\lambda F}(x) - \operatorname{prox}_{\lambda F}(y), x-y\big\rangle \geq \big\|\operatorname{prox}_{\lambda F}(x) - \operatorname{prox}_{\lambda F}(y)\big\|^2 \tag{1.10}$$

for all $x, y \in H$, which is equivalent to

$$\begin{aligned}&\big\|\operatorname{prox}_{\lambda F}(x) - \operatorname{prox}_{\lambda F}(y)\big\|^2\\ &\quad\leq \|x-y\|^2 - \big\|(I-\operatorname{prox}_{\lambda F})(x) - (I-\operatorname{prox}_{\lambda F})(y)\big\|^2\end{aligned} \tag{1.11}$$

for all $x, y \in H$. Also, the complement $I - \operatorname{prox}_{\lambda F}$ is firmly nonexpansive. This suggests that we should employ the technique in fixed point theory for solving the convex minimization feasibility problem (see [6]).

Recently, Moudafi and Thakur [11, p. 2102] proposed the following split proximal algorithm: $x_1 \in H_1$ and

$$x_{n+1} = \operatorname{prox}_{\lambda\mu_n f}\big(x_n - \mu_n A^*(I-\operatorname{prox}_{\lambda g})Ax_n\big), \tag{1.12}$$

where the stepsize

$$\mu_n = \rho_n \frac{h(x_n) + l(x_n)}{\theta^2(x_n)} \tag{1.13}$$

with

$$0 < \rho_n < 4, \qquad h(x_n) = \frac{1}{2}\big\|(I-\operatorname{prox}_{\lambda g})Ax_n\big\|^2, \tag{1.14}$$

$$l(x_n) = \frac{1}{2}\big\|(I-\operatorname{prox}_{\mu_n\lambda f})x_n\big\|^2 \tag{1.15}$$

and

$$\theta(x_n) = \sqrt{\big\|\nabla h(x_n)\big\|^2 + \big\|\nabla l(x_n)\big\|^2}. \tag{1.16}$$

They proved that, if $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ for some $\epsilon > 0$ small enough, then the sequence $\{x_n\}$ generated by (1.12) converges weakly to a solution of (1.1). We observe, however, that the stepsize sequence $\{\mu_n\}$, which appeared in (1.13), seems to be implicit because of the terms $l(x_n)$ and $\theta(x_n)$.

In order to solve the proximal split feasibility problem, we introduce a Halpern-type algorithm and prove its strong convergence under the condition on the stepsize suggested by López et al. [9, Theorem 3.5]. Then we provide some numerical experiments to support our main result. In order to complete the proof, we need the following lemma proved by He and Yang [8].

Lemma 1.1 ([8, Lemma 7]). *Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\delta_n, \tag{1.17}$$

and

$$s_{n+1} \leq s_n - \eta_n + t_n \tag{1.18}$$

for each $n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0,1)$, where $\{\eta_n\}$ is a sequence of nonnegative real numbers, and where $\{\delta_n\}$ and $\{t_n\}$ are real sequences such that

(a) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(b) $\lim_{n\to\infty} t_n = 0$, *and*
(c) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

2. Main results

Let H_1 and H_2 be real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and convex functions, and let $A : H_1 \to H_2$ be a bounded linear operator. We introduce the following results.

Algorithm I.

Step 1. Choose an initial point $x_0 \in H_1$.

Step 2. Assume that $\{x_n\}$ has been constructed. Set

$$h(x_n) = \frac{1}{2}\big\|(I - \operatorname{prox}_{\lambda g})Ax_n\big\|^2 \tag{2.1}$$

with $\|\nabla h(x_n)\| \neq 0$ for each $n \geq 1$.

We compute x_{n+1} in the following manner:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2} A^*(I - \operatorname{prox}_{\lambda g})Ax_n\Big), \tag{2.2}$$

for each $n \geq 1$, where $u \in H_1$ is fixed, $\lambda > 0$, $\{\alpha_n\} \subset (0,1)$, $\{\rho_n\} \subset (0,4)$.

Theorem 2.1. *Suppose that $\Omega \neq \emptyset$, and assume that $\{\alpha_n\}$ and $\{\rho_n\}$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and*
(C3) $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$.

Then the sequence $\{x_n\}$ converges strongly to $z = P_\Omega u$.

Proof. Let $z = P_\Omega u$. Then $z = \operatorname{prox}_{\lambda f} z$, and $Az = \operatorname{prox}_{\lambda g} Az$. Note that

$$\nabla h(x_n) = A^*(I - \operatorname{prox}_{\lambda g})Ax_n. \tag{2.3}$$

Thus, since $I - \mathrm{prox}_{\lambda g}$ is firmly nonexpansive, by using (1.10), we have

$$\begin{aligned}\big\langle \nabla h(x_n), x_n - z\big\rangle &= \big\langle A^*(I - \mathrm{prox}_{\lambda g})Ax_n, x_n - z\big\rangle \\ &= \big\langle (I - \mathrm{prox}_{\lambda g})Ax_n, Ax_n - Az\big\rangle \\ &= \big\langle (I - \mathrm{prox}_{\lambda g})Ax_n - (I - \mathrm{prox}_{\lambda g})Az, Ax_n - Az\big\rangle \\ &\geq \big\|(I - \mathrm{prox}_{\lambda g})Ax_n\big\|^2 = 2h(x_n). \end{aligned} \tag{2.4}$$

Then by using (2.4), we obtain

$$\begin{aligned}&\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\|^2 \\ &\quad = \|x_n - z\|^2 + \rho_n^2 \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} - 2\rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\big\langle \nabla h(x_n), x_n - z\big\rangle \\ &\quad \leq \|x_n - z\|^2 + \rho_n^2 \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} - 4\rho_n \frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \\ &\quad = \|x_n - z\|^2 - \rho_n(4 - \rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2}. \end{aligned} \tag{2.5}$$

Since $\{\rho_n\} \subset (0, 4)$, it then follows that

$$\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\| \leq \|x_n - z\|. \tag{2.6}$$

Next we show that $\{x_n\}$ is bounded. Consider

$$\begin{aligned}&\|x_{n+1} - z\| \\ &\quad = \Big\|\alpha_n(u - z) \\ &\qquad + (1 - \alpha_n)\Big(\mathrm{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}A^*(I - \mathrm{prox}_{\lambda g})Ax_n\Big) - z\Big)\Big\| \\ &\quad \leq \alpha_n\|u - z\| + (1 - \alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\| \\ &\quad \leq \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \end{aligned} \tag{2.7}$$

It follows, by induction, that

$$\|x_n - z\| \leq \max\big\{\|u - z\|, \|x_0 - z\|\big\}; \tag{2.8}$$

hence $\{x_n\}$ is bounded. Using (1.11) and (2.5), we see that

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\quad \leq (1 - \alpha_n)\Big\|\mathrm{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big) - z\Big\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\quad \leq (1 - \alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) - z\Big\|^2 \\ &\qquad - (1 - \alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \end{aligned}$$

$$
\begin{aligned}
&- \operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big)\Big\|^2 \\
&+ 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\
\leq\ & (1-\alpha_n)\|x_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \\
&- (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \\
&- \operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big)\Big\|^2 \\
&+ 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \qquad (2.9)
\end{aligned}
$$

Next we set

$$
s_n = \|x_n - z\|^2, \qquad \gamma_n = \alpha_n, \qquad (2.10)
$$

$$
\delta_n = 2\langle u - z, x_{n+1} - z\rangle, \qquad t_n = 2\alpha_n\langle u - z, x_{n+1} - z\rangle \qquad (2.11)
$$

and

$$
\begin{aligned}
\eta_n = (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} &+ (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \\
&- \operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big)\Big\|^2. \qquad (2.12)
\end{aligned}
$$

From (2.9), it follows that

$$
\|x_{n+1} - z\|^2 \leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \qquad (2.13)
$$

and that

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad\leq \|x_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{h^2(x_n)}{\|\nabla h(x_n)\|^2} \\
&\qquad - (1-\alpha_n)\Big\|x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n) \\
&\qquad - \operatorname{prox}_{\lambda f}\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2}\nabla h(x_n)\Big)\Big\|^2 \\
&\qquad + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \qquad (2.14)
\end{aligned}
$$

It is easy to check that $\lim_{n\to\infty} t_n = 0$ and that $\sum_{n=0}^{\infty}\gamma_n = \infty$ by using (C1) and (C2), respectively. In order to apply Lemma 1.1, we need to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\delta_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$.

Suppose that $\lim_{k\to\infty}\eta_{n_k} = 0$ for any subsequence $\{n_k\}$ of $\{n\}$. By (C1) and (C3), it follows that

$$
\lim_{k\to\infty}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} = 0 \qquad (2.15)
$$

and that

$$\lim_{k\to\infty}\Big\| x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k}) - \operatorname{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| = 0. \tag{2.16}$$

We note that $\{\nabla h(x_{n_k})\}$ is bounded. Indeed, by the Lipschitzian continuity of ∇h and by the boundedness of $\{x_{n_k}\}$, we obtain

$$\begin{aligned}\big\|\nabla h(x_{n_k})\big\| &\le \big\|\nabla h(x_{n_k}) - \nabla h(z)\big\| + \big\|\nabla h(z)\big\| \\ &\le \|A\|^2\|x_{n_k} - z\| + \big\|\nabla h(z)\big\|.\end{aligned} \tag{2.17}$$

Hence, by (2.15), we obtain

$$\lim_{k\to\infty} h(x_{n_k}) = 0 \tag{2.18}$$

for any subsequence $\{n_k\}$ of $\{n\}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \rightharpoonup x^*$, and

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle = \lim_{i\to\infty}\langle u - z, x_{n_i} - z\rangle. \tag{2.19}$$

By the lower semicontinuity of h, we have

$$0 \le h(x^*) \le \liminf_{i\to\infty} h(x_{n_i}) = \lim_{i\to\infty} h(x_{n_i}) = 0. \tag{2.20}$$

Hence we have

$$h(x^*) = \frac{1}{2}\big\|(I - \operatorname{prox}_{\lambda g})Ax^*\big\| = 0. \tag{2.21}$$

Thus Ax^* is a fixed point of the proximity operator g; that is, $0 \in \partial g(Ax^*)$, or Ax^* is a minimizer of g.

Next we show that x^* is also a minimizer of f. Observe that

$$\begin{aligned}&\|x_{n_k} - \operatorname{prox}_{\lambda f} x_{n_k}\| \\ &\quad\le \Big\|x_{n_k} - \Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| \\ &\qquad + \Big\|x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k}) \\ &\qquad - \operatorname{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| \\ &\qquad + \Big\|\operatorname{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) - \operatorname{prox}_{\lambda f} x_{n_k}\Big\| \\ &\quad\le 2\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} + \Big\|x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k}) \\ &\qquad - \operatorname{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\|.\end{aligned} \tag{2.22}$$

This implies, by (2.15) and (2.16), that

$$\lim_{k\to\infty}\|x_{n_k} - \operatorname{prox}_{\lambda f} x_{n_k}\| = 0 \tag{2.23}$$

for any subsequence $\{n_k\}$ of $\{n\}$. Note that $\mathrm{prox}_{\lambda f}$ is nonexpansive and that $x_{n_i} \rightharpoonup x^*$. Thus, by the demiclosedness principle (see [7]), we conclude that x^* is a fixed point of the proximity operator of f. This shows that x^* is also a minimizer of f. Hence $x^* \in \Omega$. On the other hand, we observe that

$$\begin{aligned}
&\|x_{n_k+1} - x_{n_k}\| \\
&\quad \le \alpha_{n_k}\|u - x_{n_k}\| \\
&\qquad + (1-\alpha_{n_k})\Big\|\mathrm{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) - x_{n_k}\Big\| \\
&\quad \le \alpha_{n_k}\|u - x_{n_k}\| + (1-\alpha_{n_k})\Big\|\mathrm{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) \\
&\qquad - \Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| \\
&\qquad + (1-\alpha_{n_k})\Big\|\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) - x_{n_k}\Big\| \\
&\quad = \alpha_{n_k}\|u - x_{n_k}\| + (1-\alpha_{n_k})\Big\|\mathrm{prox}_{\lambda f}\Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big) \\
&\qquad - \Big(x_{n_k} - \rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|^2}\nabla h(x_{n_k})\Big)\Big\| \\
&\qquad + (1-\alpha_{n_k})\rho_{n_k}\frac{h(x_{n_k})}{\|\nabla h(x_{n_k})\|} \\
&\to 0
\end{aligned} \tag{2.24}$$

as $k \to \infty$. Thus, by (1.3), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle u - z, x_{n_k+1} - z\rangle &= \limsup_{k\to\infty}\langle u - z, x_{n_k} - z\rangle \\
&\le \limsup_{n\to\infty}\langle u - z, x_n - z\rangle \\
&= \lim_{i\to\infty}\langle u - z, x_{n_i} - z\rangle \\
&= \langle u - z, x^* - z\rangle \\
&\le 0.
\end{aligned} \tag{2.25}$$

This implies that

$$\limsup_{k\to\infty}\delta_{n_k} \le 0 \tag{2.26}$$

for any subsequence $\{n_k\}$. Therefore, by using Lemma 1.1, we conclude that $s_n = \|x_n - z\|^2 \to 0$. Hence $x_n \to z = P_\Omega u$. This completes the proof. □

When $f = \delta_C$ and $g = \delta_Q$ are indicator functions of nonempty, closed, and convex sets C and Q of H_1 and H_2, respectively, we obtain the following results.

Algorithm II.

Step 1. Choose an initial point $x_0 \in H_1$.

Step 2. Assume that $\{x_n\} \subseteq C$ has been constructed. Set $h(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ with $\|\nabla h(x_n)\| \neq 0$. Compute x_{n+1} in the following manner:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)P_C\Big(x_n - \rho_n \frac{h(x_n)}{\|\nabla h(x_n)\|^2} A^*(I - P_Q)Ax_n\Big) \tag{2.27}$$

for each $n \geq 1$, where $u \in C$ is fixed, where $\{\alpha_n\} \subset (0,1)$ and where $\{\rho_n\} \subset (0,4)$.

Corollary 2.2. *Suppose that $\Theta = C \cap A^{-1}(Q) \neq \emptyset$, and assume that $\{\alpha_n\}$ and $\{\rho_n\}$ satisfy the conditions (C1)–(C3). Then the sequence $\{x_n\}$ converges strongly to $z = P_\Theta u$.*

Remark 2.3. In the case of $\|\nabla h(x_n)\| = 0$, we see that Algorithm I reduces to the following: $x_0 \in H_1$, and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\operatorname{prox}_{\lambda f} x_n \tag{2.28}$$

for each $n \geq 1$, where $u \in H_1$ is fixed, where $\{\alpha_n\} \subset (0,1)$, and where $\lambda > 0$. If the sequence $\{\alpha_n\}$ satisfies (C1) and (C2), then the sequence $\{x_n\}$ converges strongly to $z = P_{\arg\min f} u$. Since ∇h is continuous, it follows that $\nabla h(x_n) \to \nabla h(z)$. Thus we obtain $\nabla h(z) = 0$ because $\|\nabla h(x_n)\| = 0$. This shows that Az is a minimizer of g. Hence $\{x_n\}$ converges strongly to a solution of (1.1).

Remark 2.4. We highlight our work with the following conclusions.

(1) In this paper, we have established strong convergence theorems for solving the proximal split feasibility problem of two convex functions. These theorems mainly improve and generalize the results obtained by Byrne [1], López et al. [9], and Moudafi and Thakur [11].

(2) We obtain strong convergence theorems by using a simpler and more explicit method than that of Moudafi and Thakur [11] whose approach may require an implicit computation.

3. Numerical examples

In this section, we give numerical examples to support our main theorem.

Example 3.1. Let $f : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$, and let $g : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(x) = \|x\|^2 + (2,4,-5)x + 10 \tag{3.1}$$

and

$$g(x) = \|x\|^2 - (8,10,-8)x - 5, \tag{3.2}$$

respectively. Let $A = \left(\begin{smallmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 2 & 1 & 0 \end{smallmatrix}\right)$. Solve the following proximal split feasibility problem:

Find $x^* \in \mathbb{R}^3$ such that x^* minimizes f and Ax^* minimizes g.

We can check that $x^* = (-1,-2,2.5)$ is a minimizer of f such that $Ax^* = (4,5,-4)$ minimizes g. We next show the convergence behavior of the sequence in Algorithm I by using our conditions. Let $u = (1,1,1)$, and let $x_0 = (-2,4,-3)$.

TABLE 1. Numerical results for Algorithm I

n	x_n	Ax_n	$f(x_n)$	$g(A(x_n))$
1	$(-2.00000, 4.00000, -3.00000)$	$(-8.00000, 2.00000, 0.00000)$	66.000000	107.000000
5	$(-1.00362, -1.95651, 2.47069)$	$(3.93775, 5.01684, -3.96376)$	-1.247236	-61.994527
10	$(-0.99977, -1.99961, 2.49974)$	$(3.99971, 4.99992, -3.99915)$	-1.250000	-61.999999
15	$(-0.99987, -1.99973, 2.49987)$	$(3.99988, 5.00016, -3.99947)$	-1.250000	-62.000000
20	$(-0.99989, -1.99981, 2.49988)$	$(3.99987, 4.99997, -3.99960)$	-1.250000	-62.000000
25	$(-0.99992, -1.99984, 2.49992)$	$(3.99992, 5.00009, -3.99969)$	-1.250000	-62.000000
30	$(-0.99993, -1.99988, 2.49992)$	$(3.99991, 4.99998, -3.99973)$	-1.250000	-62.000000
35	$(-0.99994, -1.99989, 2.49995)$	$(3.99995, 5.00006, -3.99978)$	-1.250000	-62.000000
40	$(-0.99995, -1.99991, 2.49994)$	$(3.99994, 4.99999, -3.99980)$	-1.250000	-62.000000
45	$(-0.99996, -1.99991, 2.49996)$	$(3.99996, 5.00005, -3.99983)$	-1.250000	-62.000000
50	$(-0.99996, -1.99993, 2.49995)$	$(3.99995, 4.99999, -3.99984)$	-1.250000	-62.000000

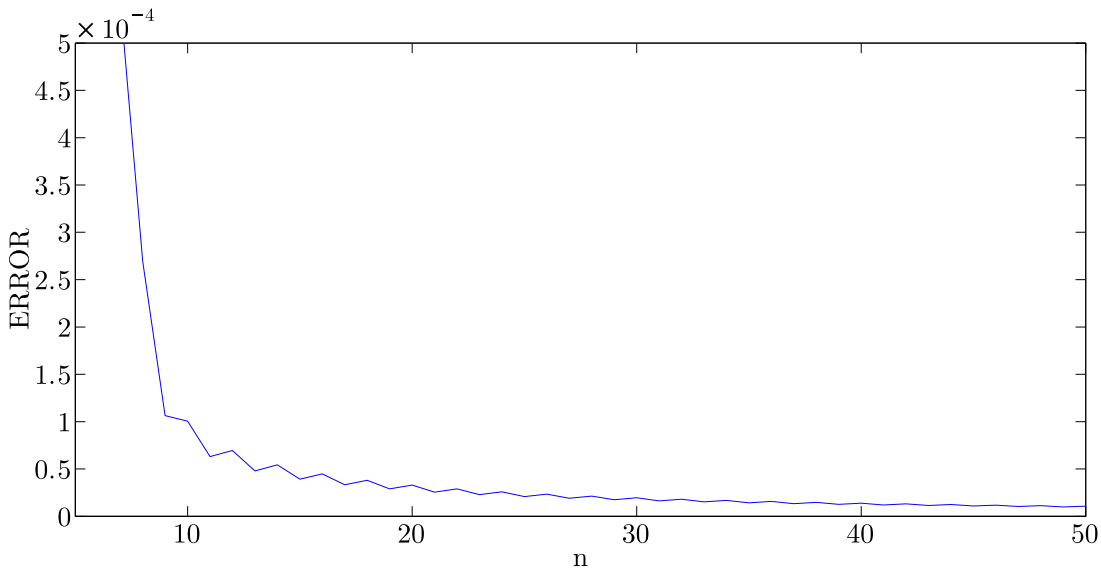

FIGURE 1. Error plotting of Table 1.

Choose $\lambda = 1$, $\alpha_n = \frac{10^{-3}}{n+1}$, and $\rho_n = 2$ for all $n \in \mathbb{N}$. Computing Algorithm I iteratively, we obtain the following numerical results.

From Table 1, the minimum values of f and g are -1.25 and -62, respectively. The errors of $\|x_{n+1} - x_n\|_2$ are plotted in Figure 1.

Example 3.2. Solve the following unconstrained linear equation system: find x^* in $\mathbb{R}^5$ such that $Ax^* = b$, where

$$A = \begin{pmatrix} 2 & 3 & -1 & 1 & 6 \\ -2 & -4 & 1 & -2 & 5 \\ -1 & -2 & -2 & -5 & 2 \\ 5 & 1 & -3 & 3 & -3 \\ 4 & 2 & 4 & 2 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} -20 \\ 21 \\ 6 \\ -15 \\ 18 \end{pmatrix}.$$

TABLE 2. Numerical results for Algorithm II

n	x_n^T	$\|x_{n+1} - x_n\|_2$
1	$(-3.00000, 1.00000, 4.00000, -2.00000, 0.00000)$	1.906981E+01
50	$(0.70266, -1.46469, 2.52074, -0.85842, 2.42700)$	1.678374E−02
100	$(0.82396, -1.68393, 2.71636, -0.91545, 2.25244)$	5.873192E−03
150	$(0.89597, -1.81311, 2.83231, -0.94998, 2.14917)$	2.054463E−03
200	$(0.93856, -1.88954, 2.90090, -0.97041, 2.08809)$	7.183362E−04
250	$(0.96375, -1.93475, 2.94148, -0.98250, 2.05196)$	2.510188E−04
300	$(0.97865, -1.96149, 2.96548, -0.98965, 2.03060)$	8.764489E−05
350	$(0.98746, -1.97732, 2.97968, -0.99389, 2.01797)$	3.056299E−05
400	$(0.99268, -1.98669, 2.98809, -0.99640, 2.01049)$	1.063548E−05
450	$(0.99577, -1.99226, 2.99308, -0.99789, 2.00606)$	3.687309E−06
500	$(0.99761, -1.99558, 2.99606, -0.99878, 2.00341)$	1.269242E−06
550	$(0.99874, -1.99762, 2.99789, -0.99934, 2.00180)$	4.299900E−07

Let $u = (1, 1, 1, 1, 1)^T$, and let $x_0 = (-3, 1, 4, -2, 0)^T$. Choose $\lambda = 1$, $\alpha_n = \frac{10^{-5}}{\sqrt{n+1}}$ and $\rho_n = 2$ for all $n \in \mathbb{N}$. Computing Algorithm II iteratively, we obtain the following numerical results.

From Table 2, the solution of the linear equation system is $(1, -2, 3, -1, 2)^T$.

Acknowledgments. Cholamjiak's research was supported by the Thailand Research Fund and the Commission on Higher Education under Grant MRG5980248. Furthermore, Cho's work was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF), which is funded by the Ministry of Science, ICT, and Future Planning (2014R1A2A2A01002100).

REFERENCES

1. C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems **18** (2002), no. 2, 441–453. Zbl 0996.65048. MR1910248. DOI 10.1088/0266-5611/18/2/310. 112, 119
2. C. Byrne, Y. Censor, A. Gibali, and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), no. 4, 759–775. Zbl 1262.47073. MR3015119. 112
3. Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol. **51** (2006) 2353–2365. 112
4. Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms **8** (1994), no. 2–4, 221–239. Zbl 0828.65065. MR1309222. DOI 10.1007/BF02142692. 111
5. P. L. Combettes and V. R. Wajs, *Signal recovery by proximal forward-backward splitting*, Multiscale Model. Simul. **4** (2005), no. 4, 1168–1200. Zbl 1179.94031. MR2203849. DOI 10.1137/050626090. 112
6. A. Gibali and A. Moudafi, *From implicit convex feasibility to convex minimization*, Trans. Math. Program Appl. **5** (2017), 25–45. 113
7. K. Goebel and W. A. Kirk, *Topics on Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990. Zbl 0708.47031. MR1074005. DOI 10.1017/CBO9780511526152. 118

8. S. He and C. Yang, *Solving the variational inequality problem defined on intersection of finite level sets*, Abstr. Appl. Anal. **2013** (2013), art. ID 942315. Zbl 1273.47099. MR3055865. 113, 114
9. G. López, V. Martín-Márquez, F. Wang, and H. K. Xu, *Solving the split feasibility problem without prior knowledge of matrix norms*, Inverse Problems **28** (2012), no. 8, art. ID 085004. Zbl 1262.90193. MR2948743. DOI 10.1088/0266-5611/28/8/085004. 112, 113, 119
10. A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl. **150** (2011), no. 2, 275–283. Zbl 1231.90358. MR2818920. DOI 10.1007/s10957-011-9814-6. 112
11. A. Moudafi and B. S. Thakur, *Solving proximal split feasibility problems without prior knowledge of operator norms*, Optim. Lett. **8** (2014), no. 7, 2099–2110. Zbl 1317.49019. MR3263242. DOI 10.1007/s11590-013-0708-4. 113, 119
12. H. K. Xu, *Iterative methods for the split feasibility problem in infinite dimensional Hilbert spaces*, Inverse Problems **26** (2010), art. ID 105018. Zbl 1213.65085. MR2719779. DOI 10.1088/0266-5611/26/10/105018. 112
13. J. Zhao and Q. Yang, *Several solution methods for the split feasibility problem*, Inverse Problems **21** (2005), no. 5, 1791–1799. Zbl 1080.65035. MR2173423. DOI 10.1088/0266-5611/21/5/017. 112

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