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SEMIFINITE TRACIAL SUBALGEBRAS

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Abstract. Let $\mathcal{M}$ be a semifinite von Neumann algebra, and let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. We show that $\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$ if and only if it has the unique normal state extension property and is a τ-maximal tracial subalgebra, which is also equivalent to $\mathcal{A}$ having the unique normal state extension property and satisfying L_2-density.

1. Introduction

The noncommutative Hardy space theory has undergone considerable development since the seminal paper by Arveson [1] in 1967 which introduced the notion of finite, maximal, subdiagonal algebras $\mathcal{A}$ of $\mathcal{M}$ as noncommutative analogues of weak* Dirichlet algebras. Many classical results of Hardy space have been successfully transferred to the noncommutative setting (cf., e.g., [3], [4]). In [4], among other things, Blecher and Labuschagne transferred a large part of the circle of theorems characterizing weak* Dirichlet algebras to Arveson's noncommutative setting of subalgebras of finite von Neumann algebras. In [3], the first author and Ospanov proved that if a tracial subalgebra $\mathcal{A}$ has L_E-factorization, then $\mathcal{A}$ is a subdiagonal algebra, where E is a symmetric quasi Banach space on $[0, 1]$.

We continue this line of investigation. The aim of this paper is to prove some characterizations of subdiagonal algebras of semifinite von Neumann algebras. We will define the semifinite version of tracial subalgebras of semifinite von Neumann algebras.

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This article is organized as follows. Section 2 contains some preliminary definitions. In Section 3, we prove that if a tracial subalgebra $\mathcal{A}$ has the unique normal state extension property and τ-maximal or satisfies the L_2-density, then $\mathcal{A}$ is a subdiagonal algebra.

2. Preliminaries

We use standard notation and notions from noncommutative L_p-spaces theory (see, e.g., [5], [8]).Throughout this article, we denote by $\mathcal{M}$ a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a normal faithful semifinite trace τ. A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, then x is said to be *τ-measurable* if for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{H}) \sqsubseteq D(x)$ and $\tau(e^\perp) < \varepsilon$ (where for any projection e we let $e^\perp = 1 - e$). The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product. For a positive self-adjoint operator $x = \int_0^\infty \lambda \,\mathrm{d}e_\lambda$ (the spectral decomposition) affiliated with $\mathcal{M}$, we set

$$\tau(x) = \sup_n \tau\Big(\int_0^n \lambda \,\mathrm{d}e_\lambda\Big) = \int_0^\infty \lambda \, d\tau(e_\lambda).$$

For $0 < p < \infty$, $L_p(\mathcal{M})$ is defined as the set of all τ-measurable operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau\big(|x|^p\big)^{\frac{1}{p}} < \infty.$$

In addition, we put $L_\infty(\mathcal{M}) = \mathcal{M}$, and we denote by $\|\cdot\|_\infty$ $(= \|\cdot\|)$ the usual operator norm. It is well known that $L_p(\mathcal{M})$ is a Banach space under $\|\cdot\|_p$ $(1 \le p \le \infty)$ satisfying all the expected properties such as duality.

In the following, $[K]_p$ denotes the closed linear span of K in $L_p(\mathcal{M})$ (relative to the w*-topology for $p = \infty$), and $J(K)$ is the family of the adjoints of the elements of K.

Henceforth we will assume that $\mathcal{D}$ is a von Neumann subalgebra of $\mathcal{M}$ such that the restriction of τ to $\mathcal{D}$ is still semifinite. Let $\mathcal{E}$ be the (unique) normal positive faithful conditional expectation of $\mathcal{M}$ with respect to $\mathcal{D}$ such that $\tau \circ \mathcal{E} = \tau$.

Definition 2.1. A w*-closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ is called a *subdiagonal algebra* of $\mathcal{M}$ with respect to $\mathcal{E}$ (or $\mathcal{D}$) if

(i) $\mathcal{A} + J(\mathcal{A})$ is w*-dense in $\mathcal{M}$,
(ii) $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$, $\forall x, y \in \mathcal{A}$,
(iii) $\mathcal{A} \cap J(\mathcal{A}) = \mathcal{D}$.

$\mathcal{D}$ is then called the *diagonal* of $\mathcal{A}$.

It is proved by Ji [6] that a semifinite subdiagonal algebra $\mathcal{A}$ is automatically maximal; that is, $\mathcal{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is a subdiagonal algebra with respect to $\mathcal{E}$.

Since $\mathcal{D}$ is semifinite, we can choose an increasing family of $\{e_i\}_{i\in I}$ of τ-finite projections in $\mathcal{D}$ such that $e_i \to 1$ strongly, where 1 is the identity of $\mathcal{M}$ (see Theorem 2.5.6 in [9]). Throughout, the $\{e_i\}_{i\in I}$ will be used to indicate this net.

Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{M}$ such that the restriction of τ to $\mathcal{B}$ is still semifinite, and let $\mathcal{N}$ be a subset of $\mathcal{M}$ containing $\mathcal{B}$. We call subset $\mathcal{N}$ *$\mathcal{B}$-invariant* if $\mathcal{B}\mathcal{N}\mathcal{B} \subseteq \mathcal{N}$. We call $\Phi : \mathcal{N} \to \mathcal{B}$ the *conditional expectation* if $\Phi(asb) = a\Phi(s)b$ for all $a, b \in \mathcal{B}, s \in \mathcal{N}$. We say that $\Phi : \mathcal{N} \to \mathcal{B}$ is normal if, for any net $\{x_\alpha\}_{\alpha\in\Lambda} \subset \mathcal{N}$ with $\sup_{\alpha\in\Lambda} x_\alpha \in \mathcal{N}$, the following equality holds: $\Phi(\sup_{\alpha\in\Lambda} x_\alpha) = \sup_{\alpha\in\Lambda} \Phi(x_\alpha)$.

Lemma 2.2. *Let $\mathcal{N}$ be a weak*-closed $\mathcal{B}$-invariant subset of $\mathcal{M}$, and let $\Phi : \mathcal{N} \to \mathcal{B}$ be a normal "conditional expectation" which is preserved by τ. Then $\Phi(a) = a$ for all $a \in \mathcal{B}$, and $\Phi \circ \Phi = \Phi$.*

Proof. Let e be a τ-finite projection in $\mathcal{B}$, and let

$$\mathcal{M}_e = e\mathcal{M}e, \qquad \mathcal{N}_e = e\mathcal{N}e, \qquad \mathcal{B}_e = e\mathcal{B}e,$$

and Φ_e be the restriction of Φ to $\mathcal{N}_e$. Then $\mathcal{N}_e$ is a weak*-closed $\mathcal{B}_e$-invariant subset of $\mathcal{M}_e$, and Φ_e is a normal "conditional expectation." Hence, we have that

$$\begin{aligned}
\tau\big(\big|\Phi_e(e) - e\big|^2\big) &= \tau\big(\big(\Phi_e(e) - e\big)^*\big(\Phi_e(e) - e\big)\big) = \tau\big(\big(\Phi_e(e)^* - e\big)\big(\Phi_e(e) - e\big)\big) \\
&= \tau\big(\Phi_e(e)^*\Phi_e(e)\big) - \tau\big(\Phi_e(e)^*e\big) - \tau\big(e\Phi_e(e)\big) + \tau(e) \\
&= \tau\big(\Phi_e\big(\Phi_e(e)^*e\big)\big) - \tau\big(\Phi_e(e)^*e\big) - \tau\big(\Phi_e(e)\big) + \tau(e) \\
&= \tau\big(\Phi_e(e)^*e\big) - \tau\big(\Phi_e(e)^*e\big) - \tau(e) + \tau(e) = 0.
\end{aligned}$$

From faithfulness of τ, it follows that $\Phi_e(e) = e$. Since $\mathcal{B}$ is semifinite, we can choose an increasing family of $\{e_\alpha\}_{\alpha\in\Lambda}$ of τ-finite projections in $\mathcal{B}$ such that $e_\alpha \to 1$ strongly. Therefore,

$$\Phi(1) = \Phi\big(\sup_{\alpha\in\Lambda} e_\alpha\big) = \sup_{\alpha\in\Lambda} \Phi(e_\alpha) = \sup_{\alpha\in\Lambda} \Phi_{e_\alpha}(e_\alpha) = \sup_{\alpha\in\Lambda} e_\alpha = 1.$$

From this follows that

$$\Phi(a) = \Phi(a1) = a\Phi(1) = a \quad \text{for all } a \in \mathcal{B}$$

and

$$\Phi\big(\Phi(x)\big) = \Phi\big(\Phi(x)1\big) = \Phi(x)\Phi(1) = \Phi(x) \quad \text{for all } x \in \mathcal{N}. \qquad \square$$

Lemma 2.3. *There is at most one normal "conditional expectation" from any weak*-closed $\mathcal{B}$-invariant subset $\mathcal{N}$ of $\mathcal{M}$ containing $\mathcal{B}$ onto $\mathcal{B}$ which is preserved by τ.*

Proof. Suppose that Φ, Ψ are normal conditional expectations of $\mathcal{N}$ onto $\mathcal{B}$ which is preserved by τ. Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an increasing family of τ-finite projections in $\mathcal{B}$ such that $e_\alpha \to 1$ strongly. Then, using the conditional expectation property, we have for $x \in \mathcal{N}$ and $\alpha \in \Lambda$ that

$$\begin{aligned}
&\tau\big(\big|\Phi(e_\alpha x e_\alpha) - \Psi(e_\alpha x e_\alpha)\big|^2\big) \\
&\quad = \tau\big(\big(\Phi(e_\alpha x e_\alpha) - \Psi(e_\alpha x e_\alpha)\big)^*\big(\Phi(e_\alpha x e_\alpha) - \Psi(e_\alpha x e_\alpha)\big)\big)
\end{aligned}$$

$$\begin{aligned}&= \tau\big(\Phi(e_\alpha x e_\alpha)^*\Phi(e_\alpha x e_\alpha)\big) - \tau\big(\Psi(e_\alpha x e_\alpha)^*\Phi(e_\alpha x e_\alpha)\big)\\&\quad - \tau\big(\Phi(e_\alpha x e_\alpha)^*\Psi(e_\alpha x e_\alpha)\big) + \tau\big(\Psi(e_\alpha x e_\alpha)^*\Psi(e_\alpha x e_\alpha)\big)\\&= \tau\big(\Phi(e_\alpha x e_\alpha)^* e_\alpha x e_\alpha\big) - \tau\big(\Psi(e_\alpha x e_\alpha)^* e_\alpha x e_\alpha\big)\\&\quad - \tau\big(\Phi(e_\alpha x e_\alpha)^* e_\alpha x e_\alpha\big) + \tau\big(\Psi(e_\alpha x e_\alpha)^* e_\alpha x e_\alpha\big) = 0.\end{aligned}$$

Hence $\Phi(e_\alpha x e_\alpha) = \Psi(e_\alpha x e_\alpha)$, and so $e_\alpha\Phi(x)e_\beta = e_\alpha e_\beta\Phi(x)e_\beta = e_\alpha e_\beta\Psi(x)e_\beta = e_\alpha\Psi(x)e_\beta$ for any $\alpha \le \beta$. Therefore, for any $\xi \in \mathcal{H}$, we have that

$$e_\alpha\Phi(x)\xi = \lim_{\beta\in\Lambda} e_\alpha\Phi(x)e_\beta\xi = \lim_{\beta\in\Lambda} e_\alpha\Psi(x)e_\beta\xi = e_\alpha\Psi(x)\xi.$$

It follows that $e_\alpha\Phi(x) = e_\alpha\Psi(x)$, and so $\Phi = \Psi$. □

Definition 2.4. A weak*-closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ is called a *tracial subalgebra* of $\mathcal{M}$ with respect to Φ (or $\Delta = \mathcal{A} \cap J(\mathcal{A})$) if

(i) $\Delta(\mathcal{A})$ is semifinite,
(ii) $\Phi : \mathcal{A} \to \Delta(\mathcal{A})$ is a normal homomorphism,
(iii) $\tau(x) = \tau(\Phi(x))$, $\forall x \in \mathcal{A}$.

We claim that if $\mathcal{A}$ is a tracial subalgebra of a von Neumann algebra $\mathcal{M}$, then the map Φ in Definition 2.4 is a unique normal homomorphism. Indeed, the conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\Delta(\mathcal{A})$ restricts to a normal "conditional expectation" from $\mathcal{A}$ onto $\Delta(\mathcal{A})$. Clearly, Φ is a normal "conditional expectation" from $\mathcal{A}$ onto $\Delta(\mathcal{A})$. The claim then follows by Lemma 2.3. Hence we may write Φ as $\mathcal{E}$ and write $\Delta(\mathcal{A})$ as $\mathcal{D}$. Therefore, A tracial subalgebra $\mathcal{A}$ of a von Neumann algebra $\mathcal{M}$ is a subdiagonal algebra of $\mathcal{M}$ if and only if $\mathcal{A} + J(\mathcal{A})$ is w*-dense in $\mathcal{M}$.

It is well known that $\mathcal{E}$ extends to a contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{D})$ for every $1 \le p \le \infty$. The extension will still be denoted by $\mathcal{E}$.

Let $\mathcal{A}_0 = \mathcal{A} \cap \ker(\mathcal{E})$. We call $\mathcal{A}$ *τ-maximal* if

$$\mathcal{A} = \big\{x \in \mathcal{M} : \tau(xy) = 0, \forall y \in \mathcal{A}_0\big\}.$$

We say that a tracial subalgebra $\mathcal{A}$ of $\mathcal{M}$ satisfies the L_2- density if $\mathcal{A}\cap L_2(\mathcal{M}) + J(\mathcal{A}) \cap L_2(\mathcal{M})$ is dense in $L_2(\mathcal{M})$ in the usual Hilbert space norm on that space.

Given a projection e in $\mathcal{D}$, let

$$\mathcal{M}_e = e\mathcal{M}e, \qquad \mathcal{A}_e = e\mathcal{A}e, \qquad \mathcal{D}_e = e\mathcal{D}e,$$

and let $\mathcal{E}_e$ be the restriction of $\mathcal{E}$ to $\mathcal{M}_e$. Then we have the following results.

Lemma 2.5. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$ with respect to $\mathcal{D}$, and let e be a projection in $\mathcal{D}$. We have that*

(i) *$\mathcal{A}_e$ is a tracial subalgebra of $\mathcal{M}_e$ with respect to $\mathcal{E}_e$ (or $\mathcal{D}_e$);*
(ii) *$(\mathcal{A}_e)_0 = e\mathcal{A}_0e$;*
(iii) *if $\mathcal{A}$ is τ-maximal, then $\mathcal{A}_e$ is τ-maximal;*
(iv) *if $\mathcal{A}$ satisfies L_2- density, then $\mathcal{A}_e$ satisfies L_2-density.*

Proof. Using the methods as in the proofs (i) and (ii) of Lemma 3.1 in [2], we obtain (i) and (ii).

(iii) It is clear that $\mathcal{A}_e \subseteq \{x \in \mathcal{M}_e : \tau(xa) = 0, \forall a \in (\mathcal{A}_e)_0\}$. Conversely, let $x \in \mathcal{M}_e$ and $\tau(xa) = 0$ for all $a \in (\mathcal{A}_e)_0\}$. Then

$$\tau(xy) = \tau(exey) = \tau(xeye) = 0, \quad y \in \mathcal{A}_0.$$

Hence $x \in \mathcal{A}$ since $\mathcal{A}$ is τ-maximal, and so $x \in \mathcal{A}_e$.

(iv) From (i) and (ii) it follows that $[(\mathcal{A}_e)_0]_2 = e[\mathcal{A}_0 \cap L_2(\mathcal{M})]_2 e$, $[(J(\mathcal{A}_e))_0]_2 = e[J(\mathcal{A}_0) \cap L_2(\mathcal{M})]_2 e$ and $[\mathcal{D}_e]_2 = e[\mathcal{D} \cap L_2(\mathcal{M})]_2 e$. On the other hand, $L_2(\mathcal{M}_e) = eL_2(\mathcal{M})e$. Hence $\mathcal{A}_e + J(\mathcal{A}_e)$ is dense in $L_2(\mathcal{M}_e)$ in the usual Hilbert space norm on that space. $\square$

3. Characterizations of subdiagonal algebra

Proposition 3.1. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$.*
(ii) *For any $i \in I$, $\mathcal{A}_{e_i}$ is a subdiagonal algebra of $\mathcal{M}_{e_i}$.*

Proof. (i) $\Rightarrow$ (ii) follows from (i) of Lemma 3.1 in [2].

(ii) $\Rightarrow$ (i) Since $e_i \to 1$ strongly, we get $\lim_i \|xe_i - x\|_1 = 0$ and $\lim_i \|e_i x - x\|_1 = 0$ for any $x \in L_1(\mathcal{M})$ (cf. Lemma 2.3 in [7]). Hence, for any $y \in \mathcal{M}$, we have that

$$\begin{aligned}\lim_i \big|\tau\big((y - e_i y e_i)x\big)\big| &\leq \lim_i \big|\tau\big((y - ye_i)x\big)\big| + \lim_i \big|\tau\big((ye_i - e_i y e_i)x\big)\big| \\ &\leq \|y\|_\infty \big(\lim_i \|x - e_i x\|_1 + \lim_i \big\|e_i(x - xe_i)\big\|_1\big) = 0.\end{aligned}$$

Thus $\bigcup_{i\in I} \mathcal{M}_{e_i}$ is weak*-dense in $\mathcal{M}$. On the other hand, $\mathcal{A}_{e_i} + J(\mathcal{A}_{e_i})$ is weak*-dense in $\mathcal{M}_{e_i}$ ($\forall i \in I$), and so $\bigcup_{i\in I}(\mathcal{A}_{e_i} + J(\mathcal{A}_{e_i}))$ is weak*-dense in $\mathcal{M}$. Therefore, $\mathcal{A} + J(\mathcal{A})$ is weak*-dense in $\mathcal{M}$; that is, $\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$. $\square$

Definition 3.2. Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$ with respect to $\mathcal{D}$. We say that $\mathcal{A}$ has the unique normal state extension property if it satisfies the following:

$$\text{If } x \in L_1(\mathcal{M})_+ \text{ and } \tau(xa) = 0 \text{ for all } a \in \mathcal{A}_0, \text{then } x \in L_1(\mathcal{D}).$$

Remark 3.3. In [4], for a tracial subalgebra $\mathcal{A}$ of a finite von Neumann algebra $\mathcal{M}$, the unique normal state extension property is defined by the following condition:

$$\text{If } x \in L_1(\mathcal{M})_+ \text{ and } \tau(xa) = \tau(a) \text{ for all } a \in \mathcal{A}, \text{then } x = 1.$$

By Lemma 4.1 in [4], this definition is equivalent to our definition of the unique normal state extension property.

Lemma 3.4. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$ with respect to $\mathcal{D}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ has the unique normal state extension property.*
(ii) *For any $i \in I$, $\mathcal{A}_{e_i}$ has the unique normal state extension property.*

Proof. (i) $\Rightarrow$ (ii) Let $i \in I$. Suppose that $x \in L_1(\mathcal{M}_{e_i})_+$ and $\tau(xa) = 0$ for all $a \in (\mathcal{A}_{e_i})_0$. By (ii) of Lemma 2.5, we have $\tau(xe_i a e_i) = 0$ for all $a \in \mathcal{A}_0$. Hence $x \in L_1(\mathcal{D})$, and so $x \in L_1(\mathcal{D}_{e_i})$.

(ii) $\Rightarrow$ (i) If $x \in L_1(\mathcal{M})_+$ and $\tau(xa) = 0$ for all $a \in \mathcal{A}_0$, then $\tau(xe_i a e_i) = 0$ for all $a \in \mathcal{A}_0$ and $i \in I$. It follows that $\tau(e_i x e_i a) = 0$ for all $a \in (\mathcal{A}_{e_i})_0$ and

$i \in I$. Hence $e_i x e_i \in L_1(\mathcal{D}_{e_i})$ for all $i \in I$. Since $e_i x e_i \to x$ in norm in $L_1(\mathcal{M})$, we conclude that $x \in L_1(\mathcal{D})$. □

Theorem 3.5. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$ with respect to $\mathcal{D}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$.*
(ii) *$\mathcal{A}$ is a τ-maximal tracial subalgebra of $\mathcal{M}$ satisfying the unique normal state extension property.*
(iii) *$\mathcal{A}$ satisfies the L_2-density and the unique normal state extension property.*

Proof. (i) ⇒ (ii), (iii) are trivial.

(ii) ⇒ (i) Let $i \in I$. By Lemma 2.5 and 3.4, we know that $\mathcal{A}_{e_i}$ is a τ-maximal tracial subalgebra of $\mathcal{M}_{e_i}$ satisfying the unique normal state extension property. Using Theorem 1.1 in [4], we obtain that $\mathcal{A}_{e_i}$ is a subdiagonal algebra of $\mathcal{M}_{e_i}$, and so, by Proposition 3.1, it follows that $\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$.

(iii) ⇒ (i) Similar to the above, we use Theorem 1.1 in [4], Lemmas 2.5 and 3.4, and Proposition 3.1 to obtain the desired result. □

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