Ann. Funct. Anal. 8 (2017), no. 2, 168–176
http://dx.doi.org/10.1215/20088752-3784269
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

PERSPECTIVES AND COMPLETELY POSITIVE MAPS

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Communicated by J.-C. Bourin

Abstract. We study the filtering of the perspective of a regular operator map of several variables through a completely positive linear map. By this method we are able to extend known operator inequalities of two variables to several variables, with applications in the theory of operator means of several variables. We also extend Lieb and Ruskai's convexity theorem from two to $n+1$ operator variables for any natural number n.

1. Introduction

We study the filtering of a regular operator map through a completely positive linear map Φ. A main result is the inequality

$$F\big(\Phi(A_1),\ldots,\Phi(A_k)\big) \le \Phi\big(F(A_1,\ldots,A_k)\big),$$

where $A_1,\ldots,A_k$ are positive definite operators on a Hilbert space of finite dimension, and F is a positively homogeneous convex regular operator map of k variables. If G_k denotes any of the various geometric means of k variables studied in the literature, we obtain as a special case the inequality

$$\Phi\big(G_k(A_1,\ldots,A_k)\big) \le G_k\big(\Phi(A_1),\ldots,\Phi(A_k)\big).$$

This inequality extends a result in the literature for $k=2$, for geometric means of k variables that may be obtained inductively by the power mean of two variables, and for means that are limits of such means, including the Karcher mean (see [3]).

Received Aug. 8, 2016; Accepted Aug. 9, 2016.
2010 *Mathematics Subject Classification.* 47A63.
Keywords. partial traces of operator means, Lieb and Ruskai's convexity theorem.

We extend Lieb and Ruskai's convexity theorem from two to $n+1$ operator variables. For $n=2$, we obtain in particular that the map

$$L(A,B,C)=C^*B^{-1/2}(B^{1/2}A^{-1}B^{1/2})^{1/2}B^{-1/2}C$$

is convex in arbitrary C and positive definite A and B. In addition,

$$L\bigl(\Phi(A),\Phi(B),\Phi(C)\bigr)\le\Phi\bigl(L(A,B,C)\bigr)$$

for a completely positive linear map Φ between operators acting on finite-dimensional Hilbert spaces. In particular, this includes quantum channels and partial traces. For commuting A and B, the generalized Lieb–Ruskai map reduces to

$$L(A,B,C)=C^*A^{-1/2}B^{-1/2}C.$$

In particular, $L(A,A,C)=C^*A^{-1}C$.

2. Preliminaries

Let $\mathcal{D}\subseteq B(\mathcal{H})\times\cdots\times B(\mathcal{H})$ be a convex domain, where $B(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$.

In [8, Definition 2.1] we defined the notion of a regular map $F\colon\mathcal{D}\to B(\mathcal{H})$, generalizing Davis's idea of a spectral function for functions of one variable, the notion of a regular matrix map of two variables (by the author [7]), and the notion of a regular operator map of two variables (by the author and Effros in [6, Definition 2.1]). Loosely speaking, a regular map is unitarily invariant and reduces block matrices in a simple and natural way. It retains regularity when compressed to a subspace.

Although we often restrict the study to finite-dimensional spaces, it is convenient to consider only such regular maps that may be defined also on an infinite-dimensional Hilbert space $\mathcal{H}$. Since $\mathcal{H}$ in this case is isomorphic to $\mathcal{H}\oplus\mathcal{H}$, this allows us to use block matrix techniques without imposing dimensionality conditions. Furthermore, it implies that a regular map is well defined regardless of the underlying Hilbert space. We may thus port a regular map unambiguously from one Hilbert space to another. In this article, we only consider domains of the form

$$\mathcal{D}^k(\mathcal{H})=\bigl\{(A_1,\dots,A_k)\mid A_1,\dots,A_k\ge 0\bigr\}$$

of k-tuples of positive semidefinite operators, or domains

$$\mathcal{D}^k_+(\mathcal{H})=\bigl\{(A_1,\dots,A_k)\mid A_1,\dots,A_k> 0\bigr\}$$

of k-tuples of positive definite and invertible operators acting on a Hilbert space $\mathcal{H}$. The latter is the natural type of domain for perspectives.

2.1. Jensen's inequality for regular operator maps.

The following result was proved for $\mathcal{H}=\mathcal{K}$ in [8, Theorem 2.2(i)]. It is just an exercise to generalize the statement and obtain the following.

Lemma 2.1. *Let $F\colon \mathcal{D}^k(\mathcal{H}) \to B(\mathcal{H})_{\rm sa}$ be a convex regular map, and take a contraction $C\colon \mathcal{H} \to \mathcal{K}$ of $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. If $F(0,\dots,0) \le 0$, then the inequality*

$$F(C^*A_1C,\dots,C^*A_kC) \le C^*F(A_1,\dots,A_k)C$$

holds for k-tuples $(A_1,\dots,A_k)$ in $\mathcal{D}^k(\mathcal{K})$.

The next result reduces to [8, Theorem 2.2(ii)] for $\mathcal{H} = \mathcal{K}$ and $n = 2$. Since the generalization is quite straightforward, we leave the proof to the reader.

Theorem 2.2 (Jensen's inequality for regular operator maps)**.** *Let $F\colon \mathcal{D}^k(\mathcal{H}) \to B(\mathcal{H})_{\rm sa}$ be a convex regular map, and let $C_1,\dots,C_n\colon \mathcal{H} \to \mathcal{K}$ be mappings of $\mathcal{H}$ into (possibly another) Hilbert space $\mathcal{K}$ such that*

$$C_1^*C_1 + \cdots + C_n^*C_n = 1_{\mathcal{H}}.$$

Then the inequality

$$F\Big(\sum_{i=1}^n C_i^*A_{i1}C_i,\dots,\sum_{i=1}^n C_i^*A_{ik}C_i\Big) \le \sum_{i=1}^n C_i^*F(A_{i1},\dots,A_{ik})C_i$$

holds for k-tuples $(A_{i1},\dots,A_{ik})$ in $\mathcal{D}^k(\mathcal{K})$ for $i = 1,\dots,n$.

Corollary 2.3. *Let $\Phi\colon B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive unital linear map between operators on Hilbert spaces of finite dimension, and let F be a convex regular map. Then*

$$F\big(\Phi(A_1),\dots,\Phi(A_k)\big) \le \Phi\big(F(A_1,\dots,A_k)\big)$$

for $(A_1,\dots,A_k) \in \mathcal{D}_k(\mathcal{H})$.

Proof. By Choi's decomposition theorem, there exist operators $C_1,\dots,C_n$ in $B(\mathcal{K},\mathcal{H})$ with $C_1^*C_1 + \cdots + C_n^*C_n = 1_{\mathcal{K}}$ such that

$$\Phi(A) = \sum_{i=1}^n C_i^*AC_i \quad \text{for } A \in B(\mathcal{H}).$$

The statement now follows by Theorem 2.2 by choosing

$$(A_{i1},\dots,A_{ik}) = (A_1,\dots,A_k)$$

for $i = 1,\dots,n$. □

Davis [4, Main Corollary] proved that $f(\Phi(A)) \le \Phi(f(A))$ for an operator-convex function f with $f(0) = 0$ and a completely positive linear map Φ with $\Phi(1) \le 1$. Jensen's operator inequality is the slightly more general statement

$$f\Big(\sum_{i=1}^n C_i^*A_iC_i\Big) \le \sum_{i=1}^n C_i^*f(A_i)C_i$$

for tuples $(A_1,\dots,A_n)$ and operators $C_1,\dots,C_n$ with $C_1^*C_1 + \cdots + C_n^*C_n = 1$ (see [9, Theorem 2.1(iii)] and [10]). Jensen's inequality for regular operator maps may similarly be considered a generalization of Corollary 2.3.

3. Perspectives

We introduced the perspective (see [8, Definition 3.1]) of a regular operator map of k variables as a generalization of the operator perspective of a function of one variable defined by Effros [5]. A key result is that the perspective $\mathcal{P}_F$ of a convex regular operator map $F\colon \mathcal{D}_+^k(\mathcal{H}) \to B(\mathcal{H})$ of k variables is a convex positively homogenous regular operator map of $k+1$ variables (see [8, Theorem 3.2]).

Theorem 3.1. *Let $\Phi\colon B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive linear map between operators on Hilbert spaces of finite dimension, and let $F\colon \mathcal{D}_+^k(\mathcal{H}) \to B(\mathcal{H})$ be a convex regular map. Then*

$$\mathcal{P}_F\bigl(\Phi(A_1),\dots,\Phi(A_{k+1})\bigr) \le \Phi\bigl(\mathcal{P}_F(A_1,\dots,A_{k+1})\bigr)$$

for operators $(A_1,\dots,A_{k+1})$ in $\mathcal{D}_+^{k+1}(\mathcal{H})$, where $\mathcal{P}_F$ is the perspective of F.

Proof. We first assume that Φ is faithful, that is, that $\Phi(1_{\mathcal{H}})$ is an invertible operator on $\mathcal{K}$. This may be obtained by properly compressing $\mathcal{K}$. We then extend an idea of Ando [1, p. 211] for functions of one variable to regular operator maps. To a fixed positive definite $B \in B(\mathcal{H})$ we set

$$\Psi(X) = \Phi(B)^{-1/2}\Phi(B^{1/2}XB^{1/2})\Phi(B)^{-1/2},$$

noting that $\Psi\colon B(\mathcal{H}) \to B(\mathcal{K})$ is a unital linear map. By the definition of complete positivity, we realize that Ψ is also completely positive. Since F is convex, we may thus apply Corollary 2.3 and obtain

$$\begin{aligned} &F\bigl(\Psi(B^{-1/2}A_1B^{-1/2}),\dots,\Psi(B^{-1/2}A_kB^{-1/2})\bigr)\\ &\quad\le \Psi\bigl(F(B^{-1/2}A_1B^{-1/2},\dots,B^{-1/2}A_kB^{-1/2})\bigr).\end{aligned}$$

Inserting Ψ, we obtain the inequality

$$\begin{aligned} &F\bigl(\Phi(B)^{-1/2}\Phi(A_1)\Phi(B)^{-1/2},\dots,\Phi(B)^{-1/2}\Phi(A_k)\Phi(B)^{-1/2}\bigr)\\ &\quad\le \Phi(B)^{-1/2}\Phi\bigl(B^{1/2}F(B^{-1/2}A_1B^{-1/2},\dots,B^{-1/2}A_kB^{-1/2})B^{1/2}\bigr)\Phi(B)^{-1/2}.\end{aligned}$$

By multiplying from the left and from the right with $\Phi(B)^{1/2}$, we obtain

$$\begin{aligned} &\mathcal{P}_F\bigl(\Phi(A_1),\dots,\Phi(A_k),\Phi(B)\bigr)\\ &\quad= \Phi(B)^{1/2}F\bigl(\Phi(B)^{-1/2}\Phi(A_1)\Phi(B)^{-1/2},\dots,\Phi(B)^{-1/2}\Phi(A_k)\Phi(B)^{-1/2}\bigr)\Phi(B)^{1/2}\\ &\quad\le \Phi\bigl(B^{1/2}F(B^{-1/2}A_1B^{-1/2},\dots,B^{-1/2}A_kB^{-1/2})B^{1/2}\bigr)\\ &\quad= \Phi\bigl(\mathcal{P}_F(A_1,\dots,A_k,B)\bigr),\end{aligned}$$

which is the assertion. If Φ is not faithful, then we obtain equality on the null space of $\Phi(1_{\mathcal{H}})$ by the calculation convention $\mathcal{P}_F(0,\dots,0)=0$. □

Note that we do not require Φ to be unital or trace-preserving in the above theorem.

Theorem 3.2. *Let $\Phi\colon B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive linear map between operators on Hilbert spaces of finite dimension, and let $F\colon \mathcal{D}_+^{k+1}(\mathcal{H}) \to B(\mathcal{H})$ be a convex and positively homogeneous regular map. Then*

$$F\bigl(\Phi(A_1),\dots,\Phi(A_{k+1})\bigr) \le \Phi\bigl(F(A_1,\dots,A_{k+1})\bigr)$$

for positive definite $A_1,\dots,A_{k+1} \in B(\mathcal{H})$.

Proof. We proved elsewhere (see [8, Proposition 3.3]) that a convex and positively homogeneous regular map F of $k+1$ variables is the perspective of its restriction

$$G(A_1,\dots,A_k) = F(A_1,\dots,A_k,1)$$

to k variables. Since $G\colon \mathcal{D}_+^k(\mathcal{H}) \to B(\mathcal{H})$ is convex and regular, the assertion follows from Theorem 3.1. □

Note that there is equality on the null space of $\Phi(1_{\mathcal{H}})$ due to homogeneity.

Remark 3.3. A geometric mean G of several variables is an example of a concave positively homogeneous regular map. The inequality in Theorem 3.2 thus yields

$$G\bigl(\Phi(A_1),\dots,\Phi(A_k)\bigr) \ge \Phi\bigl(G(A_1,\dots,A_k)\bigr).$$

This result was proved [3, Theorem 4.1] for all geometric means that may be obtained inductively by an application of the power mean of two variables. By a limiting argument, this was then extended to the Karcher mean. However, there exist geometric means that cannot be obtained in this way, for example, the means introduced in [8, Section 4.2].

4. Lieb and Ruskai's convexity theorem

Lieb and Ruskai [12, Theorem 1] proved convexity of the map

$$L(A,K) = K^*A^{-1}K$$

in pairs (A,K) of bounded linear operators on a Hilbert space, where A is positive definite and invertible. Subsequently, Ando gave a very elegant proof of this result in [1, Theorem 1]. If K is positive definite, then we may write

$$KA^{-1}K = K^{1/2}(K^{-1/2}AK^{-1/2})^{-1}K^{1/2}$$

as the perspective of the function $t \to t^{-1}$. Since this function is operator-convex, we obtain convexity of the perspective $L(A,K)$ if K is restricted to positive definite operators. This, however, is enough to obtain the general result. Indeed, the set of (K,A) where $\|K\| < 1$ and $A \ge 1$ is convex, and the embedding

$$K \to \begin{pmatrix} A & K^* \\ K & A \end{pmatrix} > 0 \tag{4.1}$$

is affine into positive definite operators. It thus follows that

$$\begin{aligned}(K,A) \to \begin{pmatrix} A & K^* \\ K & A \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}^{-1}\begin{pmatrix} A & K^* \\ K & A \end{pmatrix} \\ = \begin{pmatrix} A + K^*A^{-1}K & 2K^* \\ 2K & A + KA^{-1}K^* \end{pmatrix}\end{aligned}$$

is convex in the specified set. In particular, $(K,A) \to K^*A^{-1}K$ is convex.

M. B. Ruskai kindly informed the author that she and Lieb obtained their much cited convexity result unaware that it was proved much earlier in another context by Kiefer [11].

Proposition 4.1. *Let $\Phi\colon B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive linear map between operators on Hilbert spaces of finite dimension. The inequality*

$$\Phi(K)^*\Phi(A)^{-1}\Phi(K) \le \Phi(K^*A^{-1}K)$$

is valid for positive definite A and arbitrary K.

Proof. If we restrict K to positive definite operators, then the inequality is already contained in Theorem 3.1. The same block matrix construction as in (4.1) applied to the completely positive linear map $\Phi \otimes 1_2$ then leads to the inequality

$$\Phi(A) + \Phi(K^*)\Phi(A)^{-1}\Phi(K) \le \Phi(A + K^*A^{-1}K)$$

for $A \ge 1$ and $\|K\| < 1$, and the statement follows. □

Note that the above inequality was obtained in [1, Corollary 3.1] if K is positive definite (see also [12, Theorems 2 and 3]).

There is another way to consider Lieb and Ruskai's convexity theorem which points to generalizations of the result to more than two operators. The geometric mean G_1 of one positive definite operator is trivially given by $G_1(A) = A$. It is a concave regular map and its inverse

$$A \to G_1(A)^{-1} = A^{-1}$$

is thus a convex regular map. The perspective

$$\mathcal{P}_{G_1^{-1}}(A, B) = B^{1/2}G_1(B^{-1/2}AB^{-1/2})^{-1}B^{1/2} = BA^{-1}B = L(A, B)$$

is therefore a convex regular map by [8, Theorem 3.2], and it is increasing when filtered through a completely positive linear map by Theorem 3.1. A similar construction may be carried out for any number of operator variables.

Theorem 4.2. *Let G_n be a positively homogeneous concave regular operator map which is self-dual, congruence-invariant, and extends the function*

$$(t_1, \ldots, t_n) \to t_1^{1/n} \cdots t_n^{1/n} \quad t_1, \ldots, t_n > 0$$

to operators (see the discussions in [2] and [8]). The operator map

$$L(A_1, \ldots, A_n, C) = CG_n(A_1, \ldots, A_n)^{-1}C$$

is then convex in positive definite and invertible operators.

Proof. The (geometric) mean G_n is a positive concave and regular map. The inverse

$$G_n(A_1, \ldots, A_n)^{-1} = G_n(A_1^{-1}, \ldots, A_n^{-1})$$

is therefore convex and regular. The perspective

$$\begin{aligned}&\mathcal{P}_{G_n^{-1}}(A_1,\dots,A_n,C)\\&\quad=C^{1/2}G_n(C^{-1/2}A_1C^{-1/2},\dots,C^{-1/2}A_nC^{-1/2})^{-1}C^{1/2}\\&\quad=C^{1/2}G_n(C^{1/2}A_1^{-1}C^{1/2},\dots,C^{1/2}A_n^{-1}C^{1/2})C^{1/2}\\&\quad=CG_n(A_1^{-1},\dots,A_n^{-1})C=CG_n(A_1,\dots,A_n)^{-1}C\\&\quad=L(A_1,\dots,A_n,C),\end{aligned}$$

where we used self-duality and congruence-invariance of the geometric mean. It now follows, by [8, Theorem 3.2], that L is a convex regular map. $\square$

Remark 4.3. It is interesting to note that Theorem 4.2 alternatively may be obtained by adapting the arguments of Ando in [1, Theorem 1], and that this way of reasoning even imparts convexity of the map

$$L(A,B,C)=C^*G_2(A,B)^{-1}C,$$

where C now is arbitrary and A,B are positive definite and invertible. The argument uses the well-known fact that a block matrix of the form

$$\begin{pmatrix}A & C\\ C^* & B\end{pmatrix},$$

where A is positive definite and invertible, is positive semidefinite if and only if $B\ge C^*A^{-1}C$. Indeed, by taking $\lambda\in[0,1]$ and setting

$$\begin{aligned}&C=\lambda C_1+(1-\lambda)C_2,\\&T=\lambda C_1^*G_2(A_1,B_1)^{-1}C_1+(1-\lambda)C_2^*G_2(A_2,B_2)^{-1}C_2,\end{aligned}$$

we obtain the equality

$$\begin{aligned}X&=\begin{pmatrix}\lambda G_2(A_1,B_1)+(1-\lambda)G_2(A_2,B_2) & C\\ C^* & T\end{pmatrix}\\&=\lambda\begin{pmatrix}G_2(A_1,B_1) & C_1\\ C_1^* & C_1^*G_2(A_1,B_1)^{-1}C_1\end{pmatrix}\\&\quad+(1-\lambda)\begin{pmatrix}G_2(A_2,B_2) & C_2\\ C_2^* & C_2^*G_2(A_2,B_2)^{-1}C_2\end{pmatrix}.\end{aligned}$$

Since the two last block matrices by construction are positive semidefinite, we obtain that the block matrix X is positive semidefinite. Therefore,

$$T\ge C^*\big(\lambda G_2(A_1,B_1)+(1-\lambda)G_2(A_2,B_2)\big)^{-1}C.$$

We thus obtain

$$\begin{aligned}&\lambda L(A_1,B_1,C_1)+(1-\lambda)L(A_2,B_2,C_2)\\&\quad=\lambda C_1^*G_2(A_1,B_1)^{-1}C_1+(1-\lambda)C_2^*G_2(A_2,B_2)^{-1}C_2=T\\&\quad\ge C^*\big(\lambda G_2(A_1,B_1)+(1-\lambda)G_2(A_2,B_2)\big)^{-1}C\\&\quad\ge C^*G_2\big(\lambda A_1+(1-\lambda)A_2,\lambda B_1+(1-\lambda)B_2\big)^{-1}C\\&\quad=L(\lambda A_1+(1-\lambda)A_2,\lambda B_1+(1-\lambda B_2),\lambda C_1+(1-\lambda C_2),\end{aligned}$$

where in the last inequality we used concavity of the geometric mean and operator convexity of the inverse function.

It seems mysterious that in the last proof we only used concavity of G_2, while in Theorem 4.2 we used self-duality and congruence-invariance in addition. However, if we want $L(A,B,C)$ to be positively homogeneous, then G_2 must have the same property; and if we also want G_2 to be an extension of the geometric mean of positive numbers, then the geometric mean of two operators is the only solution satisfying all these requirements (see [8, Proposition 3.3]). This way of reasoning extends to any number of variables, and we obtain the following.

Corollary 4.4. *Let G_n be any geometric mean of n positive semidefinite and invertible operators. The operator function*

$$L(A_1,\dots,A_n,C) = C^*G_n(A_1,\dots,A_n)^{-1}C \tag{4.2}$$

is convex in arbitrary C and positive definite and invertible $A_1,\dots,A_n$ acting on a Hilbert space.

Since L is positively homogeneous, we furthermore obtain the following.

Corollary 4.5. *Let $\Phi\colon B(\mathcal{H})\to B(\mathcal{K})$ be a completely positive linear map between operators on Hilbert spaces of finite dimension. The inequality*

$$L\big(\Phi(C),\Phi(A_1),\dots,\Phi(A_n)\big) \le \Phi\big(L(C,A_1,\dots,A_n)\big)$$

is valid for positive definite $A_1,\dots,A_n$ and C.

It is known that the geometric mean of two variables is the unique extension of the function $(t,s)\to t^{1/2}s^{1/2}$ to a positively homogeneous, regular, and concave operator map (see [6]). Therefore,

$$L(A,B,C) = CB^{-1/2}(B^{1/2}A^{-1}B^{1/2})^{1/2}B^{-1/2}C$$

is the only sensible extension of Lieb and Ruskai's map to three positive definite and invertible operators with symmetry condition $L(A,B,C)=L(B,A,C)$. Without the symmetry condition, there are other solutions. The weighted geometric mean,

$$G_2(\alpha;A,B) = B^{1/2}(B^{-1/2}AB^{-1/2})^{\alpha}B^{1/2} \quad 0\le\alpha\le 1,$$

is the perspective of the operator concave function $t\to t^{\alpha}$ and is therefore concave and congruent-invariant (see [6], [7]). It is also manifestly self-dual. We can therefore apply a proof similar to the one used in Theorem 4.2 and obtain that the map

$$L(\alpha;A,B,C) = CB^{-1/2}(B^{1/2}A^{-1}B^{1/2})^{\alpha}B^{-1/2}C$$

is convex in positive semidefinite and invertible operators. Furthermore, it is positively homogeneous and therefore increasing when filtered through a completely positive linear map between operators on finite-dimensional Hilbert spaces. It reduces to

$$L(\alpha;A,B,C) = CA^{-\alpha}B^{-(1-\alpha)}C$$

for commuting A and B.

It is known that for $n \geq 3$ there exist many different extensions of the real function $(t_1, \ldots, t_n) \to t_1^{1/n} \cdots t_n^{1/n}$ to an operator mapping G_n satisfying the conditions in Theorem 4.2 (see [8]). Note that if $A_1, \ldots, A_n$ commute, then

$$L(A_1, \ldots, A_n, C) = C^* A_1^{-1/n} \cdots A_n^{-1/n} C$$

and, in particular, $L(A, \ldots, A, C) = C^* A^{-1} C$.

Acknowledgments. This work was supported by Grant-in-Aid for Scientific Research of Japan grant 26400104 and by National Science Foundation of China grant 11301025.

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