Ann. Funct. Anal. 8 (2017), no. 1, 27–37
http://dx.doi.org/10.1215/20088752-3720566
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

CONE ISOMORPHISMS AND EXPRESSIONS OF SOME COMPLETELY POSITIVE MAPS

XIUHONG SUN[1,2] and YUAN LI[1*]

Communicated by S.-H. Kye

Abstract. Let $B(\mathcal{H})$, $K(\mathcal{H})$ and $T(\mathcal{H})$ be the set of all bounded linear operators, compact operators, and trace-class operators on the Hilbert space $\mathcal{H}$. The cone of all completely positive maps from $K(\mathcal{H})$ into $T(\mathcal{K})$ and all normal completely positive maps from $B(\mathcal{K})$ into $T(\mathcal{H})$ is denoted by $CP(K(\mathcal{H}), T(\mathcal{K}))$ and $NCP(B(\mathcal{K}), T(\mathcal{H}))$, respectively. In this note, the order structures of the positive cones $CP(K(\mathcal{H}), T(\mathcal{K}))$ and $NCP(B(\mathcal{K}), T(\mathcal{H}))$ are investigated. First, we show that $CP(K(\mathcal{H}), T(\mathcal{K}))$, $NCP(B(\mathcal{K}), T(\mathcal{H}))$, and $T(\mathcal{K} \otimes \mathcal{H})^+$ are cone-isomorphic. Then we give the operator sum representation for the map $\Phi \in CP(K(\mathcal{H}), T(\mathcal{K}))$.

1. Introduction and preliminaries

The study of positive maps and completely positive maps are essential and useful in both mathematics and quantum theory. Many interesting results of (completely) positive maps in operator algebras were obtained in [1], [2], [4], [11], [12], [14], [15]. Some equivalent conditions and properties such as the interpolation problem and fixed points of completely positive maps were obtained in [5], [7], [8], [10], [16]. Moreover, recent investigations in (completely) positive maps are being used in quantum entanglement theory (see [13]).

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, and let $\mathcal{B}(\mathcal{H})$ ($\mathcal{B}(\mathcal{H}, \mathcal{K})$) be the set of all bounded linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$). For an operator $A \in \mathcal{B}(\mathcal{H})$, the adjoint of A is denoted by A^*. We write $A \geq 0$ if A is a positive operator,

Received Feb. 19, 2016; Accepted May 21, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47L05; Secondary 47L50.
Keywords. positive cones, completely positive maps, trace-class operators.

meaning $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Let $K(\mathcal{H},\mathcal{K})$ and $T(\mathcal{H},\mathcal{K})$ be the set of all compact operators and all trace-class operators, respectively, from $\mathcal{H}$ to $\mathcal{K}$. It is well known that $K(\mathcal{H})$ is a nonunital C*-algebra and that $T(\mathcal{H})$ is only a nonunital Banach*-algebra. For $x \in \mathcal{H}$ and $y \in \mathcal{K}$, $x \otimes y$ denotes the one rank linear operator $x \otimes y(z) := \langle z, y\rangle x$ $(z \in \mathcal{K})$.

If W is a real vector space, then a cone in W is a nonempty subset $C \subseteq W$ with the following two properties:

(a) $\lambda\nu \in C$ whenever $\lambda \in \mathbb{R}^+ := [0,\infty)$ and $\nu \in C$;
(b) $\mu + \nu \in C$ whenever $\mu, \nu \in C$, and in particular, C is called a *proper cone* if $C \cap (-C) = \{0\}$.

Let V be a complex vector space. An involution on V is a conjugate linear map $* : V \to V$ given by $v \mapsto v^*$ such that $v^{**} = v$ and $(v + \lambda\mu)^* = v^* + \overline{\lambda}\mu^*$ for all $\lambda \in \mathbb{C}$ and $v, \mu \in V$. The complex vector space V together with the involution map is called a $*$-*vector space*. If V is a $*$-vector space, then we let $V_h = \{v \in V : v = v^*\}$ be the real vector space of self-adjoint elements of V. Note that V_h is a real vector space. An ordered $*$-vector space (V, V^+) is a pair consisting of a $*$-vector space V and a proper cone $V^+ \subseteq V_h$. In any ordered $*$-vector space, we may define a partial ordering $\geq$ on V_h by defining $v \geq w$ (or, equivalently, $w \leq v$) if and only if $v - w \in V^+$. Note that $v \in V^+$ if and only if $v \geq 0$. In this case, V^+ is called the cone of positive elements of V. So, we also denote by $B(\mathcal{H})^+$, $K(\mathcal{H})^+$, and $T(\mathcal{H})^+$ the cone of positive elements of $B(\mathcal{H})$, $K(\mathcal{H})$, and $T(\mathcal{H})$, respectively.

Let (V, V^+) and (W, W^+) be two ordered $*$-vector spaces, respectively. A linear map $\phi : V \to W$ is called *positive* if $\phi(V^+) \subseteq W^+$. Moreover, ϕ is an order isomorphism if ϕ is bijective, and both ϕ and ϕ^{-1} are positive. Also, $M_{n,m}(V)$ denote the set of all $n \times m$ matrices with entries in V. The natural addition and scalar multiplication turn $M_{n,m}(V)$ into a complex vector space. We also write $M_{n,m} = M_{n,m}(\mathbb{C})$ and use the identifications $M_{n,m}(V) = M_{n,m} \otimes V = V \otimes M_{n,m}$. As usual, let $\{E_{ij}\}$ denote the canonical matrix unit system; so we denote $\sum_{i,j=1}^{n} E_{ij} \otimes a_{ij} := (a_{ij})_{n\times n} \in M_n(V)$, where $a_{ij} \in V$, for $1 \leq i, j \leq n$. Furthermore, we define a $*$-operation on $M_{n,m}(V)$ by letting $(a_{ij})^*_{n\times m} = (a^*_{ji})_{m\times n}$ and XA be the element of $M_{l,n}(V)$ whose (i,j) entry $(XA)_{i,j}$ equals $\sum_{k=1}^{m} x_{ik}a_{kj}$, for $X = (x_{ij})_{l\times m} \in M_{l,m}$ and $A = (a_{ij})_{m\times n} \in M_{m,n}(V)$. The definition of multiplication by scalar matrices on the right is done in a similar way.

Let $n \geq 1$ be an integer, and denote by $M_n(\mathcal{B}(\mathcal{H}))$ the von Neumann algebra of $n \times n$ matrices whose entries are in $\mathcal{B}(\mathcal{H})$. Let $\Phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a linear map. Then Φ induces a map $\Phi_n : M_n(\mathcal{B}(\mathcal{H})) \longrightarrow M_n(\mathcal{B}(\mathcal{H}))$ by the formula

$$\Phi_n\big((a_{i,j})\big) = \big(\Phi(a_{i,j})\big) \quad \text{for } (a_{i,j}) \in M_n\big(\mathcal{B}(\mathcal{H})\big).$$

If every Φ_n is a positive map, then Φ is called *completely positive*. It is well known that $K(\mathcal{H})^* \simeq T(\mathcal{H})$ and $T(\mathcal{H})^* \simeq \mathcal{B}(\mathcal{H})$, where V^* denotes the set of all bounded linear functions on the vector space V. Also, $\Phi : B(\mathcal{K}) \to T(\mathcal{H})$ is said to be *normal* if Φ is continuous with respect to the W^* topology, where the W^* topology on $T(\mathcal{H})$ is induced by the identity $K(\mathcal{H})^* \simeq T(\mathcal{H})$. Normal completely positive maps on $\mathcal{B}(\mathcal{H})$ were characterized by Kraus in [6, Theorem 3.3] as follows.

Lemma 1.1 (Kraus theorem). *Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be contractive ($\|\Phi\| \le 1$). Then Φ is a normal completely positive map if and only if there exists a sequence $\{A_i\}_{i=1}^{\infty}$ of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ such that, for all $X \in \mathcal{B}(\mathcal{H})$,*

$$\Phi(X) = \sum_{i=1}^{\infty} A_i X A_i^* \quad \textit{with} \ \sum_{i=1}^{\infty} A_i A_i^* \le I.$$

The sequence $\{A_i\}_{i=1}^{\infty}$ is not necessarily unique, and $\sum_{i=1}^{\infty} A_i A_i^* \le I$ in the strong operator topology. The family $\{A_i\}_{i=1}^{\infty}$ is called a *family of Kraus operators* for Φ. In finite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, two equivalent conditions for completely positive maps are also obtained in [1, Theorems 1, 2].

Lemma 1.2 (Choi theorem). *Let $\mathcal{H}$ ($\dim \mathcal{H} = n$) and $\mathcal{K}$ be finite-dimensional Hilbert spaces. If $\Phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ is a linear map, then the following statements are equivalent:*

(a) *Φ is completely positive;*
(b) *$\sum_{i,j=1}^{n} e_{ij} \otimes \Phi(e_{ij}) \ge 0$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of $\mathcal{H}$ and $e_{ij} := e_i \otimes e_j$;*
(c) *there exists a finite sets of operators $A_i \in B(\mathcal{H}, \mathcal{K})$ such that $\Phi(X) = \sum_{i=1}^{s} A_i X A_i^*$ ($s < \infty$).*

Definition 1.3. Let V^+ and W^+ be two proper cones. A bijective map $\phi : V^+ \to W^+$ is called a *cone isomorphism* if the following two conditions are satisfied:

(1) $\phi(\lambda\nu) = \lambda\phi(\nu)$ for all $\lambda \in \mathbb{R}^+$ and $\nu \in V^+$,
(2) $\phi(\mu + \nu) = \phi(\mu) + \phi(\nu)$ whenever $\mu, \nu \in V^+$.

Clearly, if $\phi : V^+ \to W^+$ is cone-isomorphic, then $\phi(\mu) \ge \phi(\nu)$ if and only if $\mu \ge \nu$. Also, we consider V^+ and W^+ to be cone-isomorphic and denote $V^+ \simeq W^+$ if there exists a cone isomorphism between two proper cones V^+ and W^+. For convenience, we also denote

$$\begin{aligned} &CP\big(K(\mathcal{H}), T(\mathcal{K})\big) \\ &\quad := \big\{\Phi : \Phi \text{ is the completely positive linear map from } K(\mathcal{H}) \text{ into } T(\mathcal{K})\big\}, \\ &NCP\big(B(\mathcal{K}), T(\mathcal{H})\big) \\ &\quad := \big\{\Phi : \Phi \text{ is the normal completely positive linear map} \\ &\qquad \text{from } B(\mathcal{K}) \text{ into } T(\mathcal{H})\big\}. \end{aligned}$$

The purpose of the present article is to consider the order structures of the above-mentioned cones. We mainly investigate the relationship between two cones $CP(K(\mathcal{H}), T(\mathcal{K}))$ and $NCP(B(\mathcal{K}), T(\mathcal{H}))$. The relations $CP(K(\mathcal{H}), T(\mathcal{K})) \simeq NCP(B(\mathcal{K}), T(\mathcal{H})) \simeq T(\mathcal{K} \otimes \mathcal{H})^+$ are shown. Also, we get that $\Phi \in CP(K(\mathcal{H}), T(\mathcal{K}))$ if and only if there exists $V_i \in B(\mathcal{H}, \mathcal{K})$ such that $\Phi(X) = \sum_{i=1}^{s} V_i X V_i^*$ and $\sum_{i=1}^{s} V_i V_i^* \in T(\mathcal{K})$, where $s \le \infty$.

Remark 1.4. It is clear that $B(\mathcal{H})^+$, $K(\mathcal{H})^+$, and $T(\mathcal{H})^+$ are proper cones. However, they are not cone-isomorphic, when $\mathcal{H}$ is an infinite-dimensional separable Hilbert space. Assume that $B(\mathcal{H})^+ \simeq K(\mathcal{H})^+$. Then there exists a bijective map $\phi : B(\mathcal{H})^+ \to K(\mathcal{H})^+$, which implies that $\phi(I) \in K(\mathcal{H})^+$. For any finite-rank orthogonal projection P, we have $\phi^{-1}(P) \in B(\mathcal{H})^+$. As $\phi^{-1}(P) \le \|\phi^{-1}(P)\| I$, then $P \le \|\phi^{-1}(P)\|\phi(I)$, so by the *range inclusion theorem* (see [3, Theorem 1]), we get $R(\phi(I)^{1/2}) \supseteq R(P)$ for any finite-rank orthogonal projection P, where $R(T)$ denotes the range of an operator T. Thus $R(\phi(I)^{1/2}) = \mathcal{H}$, which is a contradiction to the fact that $\phi(I)^{1/2} \in K(\mathcal{H})^+$. Similarly, we also can show that $B(\mathcal{H})^+$ and $T(\mathcal{H})^+$ are not cone-isomorphic.

If $K(\mathcal{H})^+ \simeq T(\mathcal{H})^+$, then we get a bijective map $\psi : K(\mathcal{H})^+ \to T(\mathcal{H})^+$ such that (a) and (b) of Definition 1.3 hold. Clearly, ψ can be extended to a positive linear map from $K(\mathcal{H})$ into $T(\mathcal{H})$, so by the following Lemma 2.1, we conclude that there exists $M > 0$ such that

$$\big\|\psi(A)\big\|_1 \le M\|A\| \quad \text{for all } A \in K(\mathcal{H})^+.$$

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$, and let P_n be the n-rank orthogonal projection spanned by the vectors $\{e_i\}_{i=1}^{n}$, for $n = 1, 2, \dots$. Thus for any positive integer N, we have

$$\operatorname{tr}\Big[\sum_{n=1}^{N} \psi(e_n \otimes e_n)\Big] = \operatorname{tr}\Big[\psi\Big(\sum_{n=1}^{N} e_n \otimes e_n\Big)\Big] = \Big\|\psi\Big(\sum_{n=1}^{N} e_n \otimes e_n\Big)\Big\|_1 \le M,$$

which implies that $\operatorname{tr}[\sum_{n=1}^{\infty} \psi(e_n \otimes e_n)] \le M$, so $\sum_{n=1}^{\infty} \psi(e_n \otimes e_n) \in T(\mathcal{H})^+$. Then for any positive integer N,

$$K(\mathcal{H})^+ \ni \psi^{-1}\Big(\sum_{n=1}^{\infty} \psi(e_n \otimes e_n)\Big) \ge \psi^{-1}\Big(\sum_{n=1}^{N} \psi(e_n \otimes e_n)\Big) = \sum_{n=1}^{N} (e_n \otimes e_n) = P_N,$$

which induces that for $m = 1, 2, \dots$,

$$\Big\langle \psi^{-1}\Big(\sum_{n=1}^{\infty} \psi(e_n \otimes e_n)\Big) e_m, e_m \Big\rangle \ge \langle P_N e_m, e_m\rangle \ge 1 \quad \text{for } N \ge m.$$

This is a contradiction to the fact that

$$\lim_{m\to\infty} \Big\langle \psi^{-1}\Big(\sum_{n=1}^{\infty} \psi(e_n \otimes e_n)\Big) e_m, e_m \Big\rangle \le \lim_{m\to\infty} \Big\|\psi^{-1}\Big(\sum_{n=1}^{\infty} \psi(e_n \otimes e_n)\Big) e_m\Big\| = 0,$$

since $\psi^{-1}(\sum_{n=1}^{\infty} \psi(e_n \otimes e_n)) \in K(\mathcal{H})^+$.

The proof also shows that the ordered $*$-vector spaces $(\mathcal{B}(\mathcal{H}), B(\mathcal{H})^+)$, $(K(\mathcal{H}), K(\mathcal{H})^+)$, and $(T(\mathcal{H}), T(\mathcal{H})^+)$ are not order-isomorphic to each other. Indeed, if $\phi : \mathcal{B}(\mathcal{H})(K(\mathcal{H}), T(\mathcal{H})) \to K(\mathcal{H})(T(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ is a order isomorphism, then

$$\phi|_{\mathcal{B}(\mathcal{H})^+}(\phi|_{K(\mathcal{H})^+}, \phi|_{T(\mathcal{H})^+}) : \mathcal{B}(\mathcal{H})^+\big(K(\mathcal{H})^+, T(\mathcal{H})^+\big) \to K(\mathcal{H})^+\big(T(\mathcal{H})^+, \mathcal{B}(\mathcal{H})^+\big)$$

is a cone isomorphism.

Lemma 1.5. *Let $A \in B(\mathcal{H})^+$ and $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Then $\sum_{i,j=1}^{\infty} \operatorname{tr}(Ae_{ij})e_{ij} \in B(\mathcal{H})^+$.*

It is clear that $A = \sum_{i,j=1}^{\infty} \operatorname{tr}(Ae_{ji})e_{ij} \geq 0$ and $\sum_{i,j=1}^{n} \operatorname{tr}(Ae_{ji})e_{ij} \geq 0$, so by the positive property of the transpose matrix for a positive matrix, we get

$$\sum_{i,j=1}^{n} \operatorname{tr}(Ae_{ij})e_{ij} \geq 0 \qquad \text{and} \qquad \Big\|\sum_{i,j=1}^{n} \operatorname{tr}(Ae_{ij})e_{ij}\Big\| = \Big\|\sum_{i,j=1}^{n} \operatorname{tr}(Ae_{ji})e_{ij}\Big\| \leq \|A\|$$

for all n. So, the Lemmas 2.2 and 2.3 imply that $\sum_{i,j=1}^{\infty} \operatorname{tr}(Ae_{ij})e_{ij} \in B(\mathcal{H})^+$.

2. Main results

To get the main results, we need the following lemmas.

Lemma 2.1. *Let $\Phi : K(\mathcal{H}) \to T(\mathcal{K})$ be a positive linear map. Then $\sup\{\|\Phi(A)\|_1 : A \in K(\mathcal{H})$ with $\|A\| \leq 1\} < \infty$.*

Proof. The proof is similar to the proof of [9, Lemma 2.3]. □

Lemma 2.2 ([9, Lemma 3.3])**.** *Let $P_n, A \in B(\mathcal{H})$, $n = 1, 2, \ldots$. If P_n are orthogonal projections such that P_n converges to the unit operator I in the weak operator topology, then $A \geq 0$ if and only if $P_nAP_n \geq 0$ for all $n = 1, 2, \ldots$.*

Lemma 2.3 ([9, Lemma 2.6])**.** *Let*

$$\widetilde{A} := \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} & \cdots \\ A_{21} & A_{22} & \cdots & A_{2m} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}$$

and $M > 0$, where all $A_{ij} \in B(\mathcal{H})$. If $\|\sum_{i,j=1}^{n} E_{ij} \otimes A_{ij}\| \leq M$ for all n and $A_{ij} = A_{ji}^*$ for all i, j, then $\widetilde{A} \in B(\bigoplus_{n=1}^{\infty} \mathcal{H}_n)$ with $\|\widetilde{A}\| \leq M$, where all $\mathcal{H}_n = \mathcal{H}$.

The following is our main result. The matrix-ordered spaces (M_n, M_n^+) and $(M_n^d, (M_n^d)^+)$ are completely order-isomorphic (see [13, Theorem 6.2]), where M_n^d is the dual space of M_n. The motivation of Theorem 2.4 is to consider the similar result for the infinite-dimensional case.

Theorem 2.4. *We have*

$$CP\big(K(\mathcal{H}), T(\mathcal{K})\big) \simeq T(\mathcal{K} \otimes \mathcal{H})^+.$$

Proof. Let $\Phi \in CP(K(\mathcal{H}), T(\mathcal{K}))$ and $\{f_i\}_{i=1}^{\infty}$ be the orthonormal basis of $\mathcal{K}$. Denote

$$\Phi_{ij}(X) = \big\langle \Phi(X)f_j, f_i \big\rangle \quad \text{for } X \in K(\mathcal{H}), i, j = 1, 2, \ldots. \tag{2.1}$$

Then by Lemma 2.1, Φ_{ij} is a bounded linear functional on $K(\mathcal{H})$, so there exist operators $T_{ij} \in T(\mathcal{H})$ such that

$$\Phi_{ij}(X) = \operatorname{tr}(T_{ij}X) \quad \text{for } i, j = 1, 2, \ldots.$$

Define

$$T=\sum_{i,j=1}^{\infty} f_{ij}\otimes T_{ji}=\begin{pmatrix} T_{11} & T_{21} & \cdots & T_{m1} & \cdots \\ T_{12} & T_{22} & \cdots & T_{m2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ T_{1m} & T_{2m} & \cdots & T_{mm} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}, \tag{2.2}$$

where $f_{ij} := f_i \otimes f_j$. We will show that $T \in T(\mathcal{K}\otimes\mathcal{H})^+$.

If Φ is positive, then Φ_{ii} are positive, so T_{ii} are positive trace-class operators for $i = 1, 2, \dots$. Let P_n be the n-rank orthogonal projection spanned by the orthonormal vectors $\{f_i\}_{i=1}^n$, for $n = 1, 2, \dots$. Then $P_n\Phi P_n \in CP(K(\mathcal{H}), B(P_n\mathcal{K}))$ and for $X \in K(\mathcal{H})$,

$$P_n\Phi(X)P_n=\sum_{i,j=1}^{n} f_{ij}\otimes \Phi_{ij}(X)=\begin{pmatrix} \Phi_{11}(X) & \Phi_{12}(X) & \cdots & \Phi_{1n}(X) \\ \Phi_{21}(X) & \Phi_{22}(X) & \cdots & \Phi_{2n}(X) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n1}(X) & \Phi_{n2}(X) & \cdots & \Phi_{nn}(X) \end{pmatrix},$$

which implies that

$$\sum_{i,j=1}^{n} f_{ij}\otimes T_{ji}=\begin{pmatrix} T_{11} & T_{21} & \cdots & T_{n1} \\ T_{12} & T_{22} & \cdots & T_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix} \in B(P_n\mathcal{K}\otimes\mathcal{H})^+ \tag{2.3}$$

for $n = 1, 2, \dots$. Indeed, the positivity (2.3) follows from the fact that

$$\sum_{i,j=1}^{n} \operatorname{tr}(T_{ij}S_{ij}) = \operatorname{tr}\Big[\Big(\sum_{i,j=1}^{n} E_{ij}\otimes T_{ji}\Big)\Big(\sum_{i,j=1}^{n} E_{ij}\otimes S_{ij}\Big)\Big] \geq 0,$$

for all $\sum_{i,j=1}^n E_{ij}\otimes S_{ij} \in M_n(K(\mathcal{H}))^+$. Since $P_n\Phi P_n \in CP(K(\mathcal{H}), B(P_n\mathcal{K}))$, then $\sum_{i,j=1}^n E_{ij}\otimes P_n\Phi P_n(S_{ij}) \in M_n(B(P_n\mathcal{K}))^+$, that is

$$\mathcal{M}=\begin{pmatrix} \operatorname{tr}(T_{11}S_{11}) & \cdots & \operatorname{tr}(T_{1n}S_{11}) & \cdots & \operatorname{tr}(T_{11}S_{1n}) & \cdots & \operatorname{tr}(T_{1n}S_{1n}) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \operatorname{tr}(T_{n1}S_{11}) & \cdots & \operatorname{tr}(T_{nn}S_{11}) & \cdots & \operatorname{tr}(T_{n1}S_{1n}) & \cdots & \operatorname{tr}(T_{nn}S_{1n}) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \operatorname{tr}(T_{11}S_{n1}) & \cdots & \operatorname{tr}(T_{1n}S_{n1}) & \cdots & \operatorname{tr}(T_{11}S_{nn}) & \cdots & \operatorname{tr}(T_{1n}S_{nn}) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \operatorname{tr}(T_{n1}S_{n1}) & \cdots & \operatorname{tr}(T_{nn}S_{n1}) & \cdots & \operatorname{tr}(T_{n1}S_{nn}) & \cdots & \operatorname{tr}(T_{nn}S_{nn}) \end{pmatrix} \geq 0.$$

Also, for $\alpha_i \in \mathbb{C}^n$, we suppose that $\alpha_i = (\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{ni})$ where $i = 1, 2, \dots, n$. Thus

$$\sum_{l,m=1}^{n} \operatorname{tr}\Big[T_{lm}\Big(\sum_{i,j=1}^{n}\alpha_{li}S_{ij}\overline{\alpha_{mj}}\Big)\Big] = \sum_{i,j=1}^{n}\sum_{l,m=1}^{n}\alpha_{li}\operatorname{tr}(T_{lm}S_{ij})\overline{\alpha_{mj}}$$
$$= (\alpha_1, \alpha_2, \ldots, \alpha_n)\mathcal{M}\begin{pmatrix}\overline{\alpha_1}^t \\ \overline{\alpha_2}^t \\ \vdots \\ \overline{\alpha_n}^t\end{pmatrix} \geq 0. \tag{2.4}$$

Particularly, for $i = 1, 2, \ldots, n$, set

$$\alpha_{li} = \delta_{li} \quad \text{(Kronecker delta)},$$

and then $\sum_{i,j=1}^{n}\alpha_{li}S_{ij}\overline{\alpha_{mj}} = S_{lm}$, so $\sum_{i,j=1}^{n}\operatorname{tr}(T_{ij}S_{ij}) \geq 0$ follows from inequality (2.4). Also, the positivity of (2.3) implies $T_{ij} = T_{ji}^*$, for $i, j = 1, 2, \ldots, n$.

Let $S \in K(\mathcal{H})^+$; it is clear that $\Phi(S) \geq 0$, which yields

$$\sum_{i=1}^{\infty}\operatorname{tr}(T_{ii}S) = \operatorname{tr}\big(\Phi(S)\big) = \big\|\Phi(S)\big\|_1 \leq M\|S\|,$$

from Lemma 2.1, so $\sum_{i=1}^{\infty}\operatorname{tr}(T_{ii}) \leq M$. Thus the positivity of (2.3) implies that

$$\Big\|\sum_{i,j=1}^{n} f_{ij}\otimes T_{ji}\Big\| \leq \Big\|\sum_{i,j=1}^{n} f_{ij}\otimes T_{ji}\Big\|_1 = \sum_{i=1}^{n}\operatorname{tr}(T_{ii}) \leq M,$$

for $n = 1, 2, \ldots$, so Lemma 2.3 induces that $T \in B(\mathcal{K}\otimes\mathcal{H})$ is well defined. It is easy to verify that $P_n \otimes I$ converge to $I \otimes I$ in the weak operator topology and $(P_n\otimes I)T(P_n\otimes I) = \sum_{i,j=1}^{n} f_{ij}\otimes T_{ji}$. Then by Lemma 2.2 and the positivity of (2.3), we conclude that T is positive, which implies that $T \in T(\mathcal{K}\otimes\mathcal{H})^+$, as $\sum_{i=1}^{\infty}\operatorname{tr}(T_{ii}) \leq M$. Defining $\Gamma : CP(K(\mathcal{H}), T(\mathcal{K})) \to T(\mathcal{K}\otimes\mathcal{H})^+$ as $\Gamma(\Phi) = T$, where T has form (2.2), we obtain that Γ is well defined and injective.

In the following, we show that Γ is surjective. If $\widetilde{T} \in T(\mathcal{K}\otimes\mathcal{H})^+$, then suppose that $\widetilde{T} = \sum_{i,j=1}^{\infty} f_{ij}\otimes\widetilde{T}_{ij}$, so $(P_{f_i}\otimes I)\widetilde{T}(P_{f_j}\otimes I)$ are trace class operators, which implies that $\widetilde{T}_{ij} \in T(\mathcal{H})$ for $i, j = 1, 2, \ldots$, where P_{f_i} denotes the orthogonal projection on the subspace spanned by f_i. Also, define the linear functional on $K(\mathcal{H})$ by

$$\Psi_{ij}(X) = \operatorname{tr}(\widetilde{T}_{ji}X) \quad \text{for all } X \in K(\mathcal{H}).$$

Denoting $\Psi(X) = \sum_{i,j=1}^{\infty}\Psi_{ij}(X)f_{ij}$, we need to show that $\Psi(X) \in T(\mathcal{K})$ for all $X \in K(\mathcal{H})$. Let $S \in K(\mathcal{H})^+$ be arbitrary. Then by [9, Remark 2.8],

$$T(\mathcal{K}\otimes\mathcal{H})^+ \ni (I\otimes S^{1/2})\widetilde{T}(I\otimes S^{1/2}) = \sum_{i,j=1}^{\infty} f_{ij}\otimes(S^{1/2}\widetilde{T}_{ij}S^{1/2})$$

implies that $\sum_{i,j=1}^{\infty}\operatorname{tr}(\widetilde{T}_{ij}S)f_{ij} \in T(\mathcal{K})^+$. Lemma 1.5 yields $\sum_{i,j=1}^{\infty}\operatorname{tr}(\widetilde{T}_{ji}S)f_{ij} \in T(\mathcal{K})^+$; that is, $\Psi(S) = \sum_{i,j=1}^{\infty}\Psi_{ij}(S)f_{ij} \in T(\mathcal{K})^+$. For a general operator $X \in K(\mathcal{H})$, we write $X = X_1 - X_2 + \sqrt{-1}(X_3 - X_4)$, where $X_i \in K(\mathcal{H})^+$, and thus

$$\Psi(X) = \Psi(X_1) - \Psi(X_2) + \sqrt{-1}\big[\Psi(X_3) - \Psi(X_4)\big] \in T(\mathcal{K}).$$

Again suppose that $\sum_{i,j=1}^m E_{ij}\otimes S_{ij}\in M_m(K(\mathcal{H}))^+$. Here, we need to show that $\sum_{i,j=1}^m E_{ij}\otimes \Psi(S_{ij})\in M_m(T(\mathcal{K}))^+$. Clearly, $\sum_{i,j=1}^m E_{ij}\otimes \Psi(S_{ij})\in M_m(T(\mathcal{K}))$. We claim that

$$\sum_{i,j=1}^m E_{ij}\otimes P_n\Psi(S_{ij})P_n\in M_m\big(B(P_n\mathcal{K})\big)^+ \quad \text{for } n=1,2,\dots. \tag{2.5}$$

Indeed, the proof of this claim is similar to equation (2.4). Let $\alpha_i\in P_n\mathcal{K}$, and $\alpha_i=(\alpha_{1i},\alpha_{2i},\dots,\alpha_{ni})$ in the orthonormal basis $\{f_i\}_{i=1}^n$, for $i=1,2,\dots,m$. Then

$$\begin{aligned}
&\sum_{l,s=1}^n \operatorname{tr}\Big[\widetilde{T}_{sl}\Big(\sum_{i,j=1}^m \alpha_{li}S_{ij}\overline{\alpha_{sj}}\Big)\Big]\\
&\quad=\sum_{i,j=1}^m\sum_{l,s=1}^n \alpha_{li}\operatorname{tr}(\widetilde{T}_{sl}S_{ij})\overline{\alpha_{sj}}\\
&\quad=(\alpha_1,\alpha_2,\dots,\alpha_m)\Big(\sum_{i,j=1}^m E_{ij}\otimes P_n\Psi(S_{ij})P_n\Big)\begin{pmatrix}\overline{\alpha_1}^t\\ \overline{\alpha_2}^t\\ \vdots\\ \overline{\alpha_m}^t\end{pmatrix}.
\end{aligned}$$

It is easy to verify that $\sum_{l,s=1}^n E_{ls}\otimes(\sum_{i,j=1}^m \alpha_{li}S_{ij}\overline{\alpha_{sj}})\in M_n(K(\mathcal{H}))^+$ and that $\sum_{l,s=1}^n E_{ls}\otimes\widetilde{T}_{ls}\in M_n(T(\mathcal{H}))^+$, as $T(\mathcal{K}\otimes\mathcal{H})^+\ni\widetilde{T}=\sum_{i,j=1}^\infty f_{ij}\otimes\widetilde{T}_{ij}$. Thus

$$\sum_{l,s=1}^n \operatorname{tr}\Big[\widetilde{T}_{sl}\Big(\sum_{i,j=1}^m \alpha_{li}S_{ij}\overline{\alpha_{sj}}\Big)\Big]\ge 0,$$

so equation (2.5) holds, which says that

$$\begin{pmatrix}
P_n\Psi(S_{11})P_n & P_n\Psi(S_{12})P_n & \cdots & P_n\Psi(S_{1m})P_n\\
P_n\Psi(S_{21})P_n & P_n\Psi(S_{22})P_n & \cdots & P_n\Psi(S_{2m})P_n\\
\vdots & \vdots & \cdots & \vdots\\
P_n\Psi(S_{m1})P_n & P_n\Psi(S_{m2})P_n & \cdots & P_n\Psi(S_{mm})P_n
\end{pmatrix}\ge 0 \quad \text{for } n=1,2,\dots.$$

Using Lemma 2.2 again, we have $(\Psi(S_{ij}))_{m\times m}\ge 0$. So Ψ is completely positive, which induces that Γ is surjective. Then by equations (2.1) and (2.2), clearly Γ is a cone isomorphism. □

Remark 2.5. The cone isomorphisms between $CP(K(\mathcal{H}),T(\mathcal{K}))$ and $T(\mathcal{K}\otimes\mathcal{H})^+$ are not canonical. That is, the cone isomorphisms are dependent of the choice of the orthonormal basis.

The following is another main result of this note.

Proposition 2.6. *We have $NCP(B(\mathcal{K}),T(\mathcal{H}))\simeq T(\mathcal{K}\otimes\mathcal{H})^+$.*

Proof. Let $\Phi\in CP(K(\mathcal{H}),T(\mathcal{K}))$, and define the map $\Phi^\dagger$ from $B(\mathcal{K})$ into $T(\mathcal{H})$ as

$$\operatorname{tr}\big[\Phi^\dagger(X)A\big]=\operatorname{tr}\big[X\Phi(A)\big] \quad \text{for all } A\in K(\mathcal{H}), X\in B(\mathcal{K}). \tag{2.6}$$

Then it is easy to see that $\Phi^\dagger$ is well defined and positive. We claim that $\Phi^\dagger$ is completely positive. Actually, suppose that $(A_{ij})_{m\times m} \in M_m(B(\mathcal{K}))^+$ and also that $(K_{ij})_{m\times m} \in M_m(K(\mathcal{H}))^+$. It is easy to see that

$$\mathrm{tr}\big[\big(\Phi^\dagger(A_{ij})\big)_{m\times m}(K_{ij})_{m\times m}\big] = \mathrm{tr}\Big(\sum_{i,j=1}^{m} \Phi^\dagger(A_{ij})K_{ji}\Big) = \mathrm{tr}\Big(\sum_{i,j=1}^{m} A_{ij}\Phi(K_{ji})\Big) \geq 0,$$

as $\Phi \in CP(K(\mathcal{H}), T(\mathcal{K}))$ implies $(\Phi(K_{ij}))_{m\times m} \in M_m(T(\mathcal{K}))^+$. Thus $(\Phi^\dagger(A_{ij}))_{m\times m} \in M_m(T(\mathcal{H}))^+$. Let $A_\tau \longrightarrow_{W^*} A$ be a convergent net of $B(\mathcal{K})$. Then

$$\mathrm{tr}\big[\Phi^\dagger(A_\tau)S\big] = \mathrm{tr}\big[A_\tau\Phi(S)\big] \longrightarrow \mathrm{tr}\big[A\Phi(S)\big] = \mathrm{tr}\big[\Phi^\dagger(A)S\big],$$

for any $S \in K(\mathcal{H})$, so $\Phi^\dagger(A_\tau) \longrightarrow_{W^*} \Phi^\dagger(A)$, which says the map $\Gamma(\Phi) = \Phi^\dagger$ from $CP(K(\mathcal{H}), T(\mathcal{K}))$ into $NCP(B(\mathcal{K}), T(\mathcal{H}))$ is well defined and injective.

To show that Γ is surjective, suppose that $\Psi \in NCP(B(\mathcal{K}), T(\mathcal{H}))$. For all $Y \in K(\mathcal{H})$, define the linear functional Θ_Y on $B(\mathcal{K})$ by

$$\Theta_Y(X) = \mathrm{tr}\big[\Psi(X)Y\big] \quad \text{for all } X \in B(\mathcal{K}).$$

If $X_\tau \longrightarrow_{W^*} X$ is a convergent net of $B(\mathcal{K})$, then $\Psi(X_\tau) \longrightarrow_{W^*} \Psi(X)$, so

$$\Theta_Y(X_\tau) = \mathrm{tr}\big[\Psi(X_\tau)Y\big] \longrightarrow \mathrm{tr}\big[\Psi(X_\tau)Y\big] = \Theta_Y(X).$$

Thus Θ_Y is the normal linear functional on $B(\mathcal{K})$, which yields that there exists the unique element $\widetilde{\Psi}(Y) \in T(\mathcal{K})$ such that

$$\Theta_Y(X) = \mathrm{tr}\big[X\widetilde{\Psi}(Y)\big] \quad \text{for all } X \in B(\mathcal{K}), \tag{2.7}$$

so the map $\widetilde{\Psi}$ is well defined from $K(\mathcal{H})$ into $T(\mathcal{K})$. Also, it is easy to verify that $\widetilde{\Psi}$ is completely positive as $\Psi \in CP(B(\mathcal{K}), T(\mathcal{H}))$. By equations (2.6) and (2.7), we have $\widetilde{\Psi}^\dagger = \Psi$, which implies that $\Gamma(\Phi) = \Phi^\dagger$ is a cone isomorphism form $CP(K(\mathcal{H}), T(\mathcal{K}))$ onto $NCP(B(\mathcal{K}), T(\mathcal{H}))$. Then the desired result follows from Theorem 2.4. □

In the following, we also get another characterization of normal completely positive maps from $B(\mathcal{H})$ into $T(\mathcal{K})$, which is relevant to the Kraus theorem.

Proposition 2.7. *We have that $\Phi \in NCP(B(\mathcal{H}), T(\mathcal{K}))$ if and only if there exists $V_i \in B(\mathcal{H}, \mathcal{K})$ such that $\Phi(X) = \sum_{i=1}^{s} V_iXV_i^*$ and $\sum_{i=1}^{s} V_iV_i^* \in T(\mathcal{K})$, where $s \leq \infty$.*

Proof. Necessity case: This is obvious by the Kraus theorem and the fact that Φ is a bounded map and the weak* topology on the $T(\mathcal{K})$ is stronger than the weak* topology on the $(T(\mathcal{K}) \subseteq)B(\mathcal{K})$, as $T(\mathcal{K}) \subseteq K(\mathcal{K})$.

Sufficiency case: For any $S \in B(\mathcal{H})^+$, we have

$$\Phi(S) = \sum_{i=1}^{s} V_iSV_i^* \leq \sum_{i=1}^{s} V_i\|S\|V_i^* = \|S\|\sum_{i=1}^{s} V_iV_i^*,$$

which yields $\mathrm{tr}(\Phi(S)) < \infty$, so by the operator decomposition property

$$\Phi(Y) \in T(\mathcal{K}) \quad \text{for all } Y \in B(\mathcal{H}).$$

If Φ has this form, it is easy to see that Φ is completely positive. Let $A_\tau \longrightarrow_{W*} A$ be a convergent net. Then $\Phi(A_\tau) = \sum_{i=1}^s V_i A_\tau V_i^*$ implies that

$$\operatorname{tr}\Big[\Phi(A_\tau)X\Big] = \operatorname{tr}\Big[\sum_{i=1}^s V_i A_\tau V_i^* X\Big] = \operatorname{tr}\Big[\Big(\sum_{i=1}^s V_i^* X V_i\Big) A_\tau\Big] \quad \text{for } X \in K(\mathcal{K}),$$

so $\Phi(A_\tau) \longrightarrow_{W*} \Phi(A)$, as $\sum_{i=1}^s V_i^* X V_i \in T(\mathcal{H})$. □

In the following, we give the concrete expressions of the maps in $CP(K(\mathcal{H}), T(\mathcal{K}))$.

Proposition 2.8. *We have that $\Phi \in CP(K(\mathcal{H}), T(\mathcal{K}))$ if and only if there exist $V_i \in B(\mathcal{H}, \mathcal{K})$ such that $\Phi(X) = \sum_{i=1}^s V_i X V_i^*$ and $\sum_{i=1}^s V_i V_i^* \in T(\mathcal{K})$, where $s \leq \infty$.*

Proof. According to Proposition 2.6, we conclude that $\Phi \in CP(K(\mathcal{H}), T(\mathcal{K}))$ if and only if $\Phi^\dagger \in NCP(B(\mathcal{K}), T(\mathcal{H}))$. Thus Proposition 2.7 implies that there exist $S_i \in T(\mathcal{K}, \mathcal{H})$ such that $\Phi^\dagger(X) = \sum_{i=1}^s S_i X S_i^*$ and $\sum_{i=1}^s S_i S_i^* \in T(\mathcal{H})$, where $s \leq \infty$, so by equation (2.6),

$$\Phi(Y) = \sum_{i=1}^s S_i^* Y S_i \quad \text{for } Y \in K(\mathcal{H}).$$

Letting $V_i = S_i^*$ and noting that $\sum_{i=1}^s S_i^* S_i \in T(\mathcal{K})$ if and only if $\sum_{i=1}^s S_i S_i^* \in T(\mathcal{H})$, we get the desired result. □

Acknowledgments. The authors thank one of the reviewers for efficient and careful comments and suggestions which greatly improved the readability of this article.

This work was supported by National Natural Science Foundation of China (NSFC) grants no. 11571211 and 11671242, and by the Fundamental Research Funds for the Central Universities grants GK201603001 and GK201601004.

References

1. M.-D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra Appl. **10** (1975), 285–290. Zbl 0327.15018. MR0376726. 27, 29
2. J. B. Conway, *A Course in Operator Theory*, Grad. Stud. Math. **21**, Amer. Math. Soc., Providence, 2000. Zbl 0936.47001. MR1721402. 27
3. R. G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415. Zbl 0146.12503. MR0203464. 30
4. E. G. Effros and Z.-J. Ruan, *Operator Spaces*, London Math. Soc. Monogr. Ser. **23**, Oxford University Press, New York, 2000. Zbl 0969.46002. MR1793753. 27
5. M. H. Hsua, D. W. Kuo, and M. C. Tsai, *Completely positive interpolations of compact, trace-class and Schatten-p class operators*, J. Funct. Anal. **267** (2014), no. 4, 1205–1240. Zbl 1302.15027. MR3217062. DOI 10.1016/j.jfa.2014.05.013. 27
6. K. Kraus, *General state changes in quantum theory*, Ann. Physics **64** (1971), 311–335. Zbl 1229.81137. MR0292434. 29
7. C.-K. Li and Y.-T. Poon, *Interpolation by completely positive maps*, Linear Multilinear Algebra **59** (2011), no. 10, 1159–1170. Zbl 1236.15004. MR2837768. DOI 10.1080/03081087.2011.585987. 27

8. Y. Li, *Fixed points of dual quantum operations*, J. Math. Anal. Appl. **382** (2011), no. 1, 172–179. Zbl 1230.81006. MR2805505. DOI 10.1016/j.jmaa.2011.04.047. 27
9. Y. Li and H.-K. Du, *Interpolations of entanglement breaking channels and equivalent conditions for completely positive maps*, J. Funct. Anal. **268** (2015), no. 11, 3566–3599. Zbl 1321.81011. MR3336734. DOI 10.1016/j.jfa.2015.01.008. 31, 33
10. B. Magajna, *Fixed points of normal completely positive maps on* B($\mathcal{H}$), J. Math. Anal. Appl. **389** (2012), no. 2, 1291–1302. Zbl 1266.47059. MR2879297. DOI 10.1016/j.jmaa.2012.01.007. 27
11. V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Stud. Adv. Math. **78**, Cambridge University Press, Cambridge, 2002. Zbl 1029.47003. MR1976867. 27
12. V. Paulsen and F. Shultz, *Complete positivity of the map from a basis to its dual basis*, J. Math. Phys. **54** (2013), no. 7, art. ID 072201. Zbl 1296.47034. MR3114209. DOI 10.1063/1.4812329. 27
13. V. Paulsen, I. Todorov, and M. Tomforde, *Operator system structures on ordered spaces*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 1, 25–49. Zbl 1213.46050. MR2747723. DOI 10.1112/plms/pdq011. 27, 31
14. E. Størmer, *Extension of positive maps into* $B(\mathcal{H})$, J. Funct. Anal. **66** (1986), no. 2, 235–254. Zbl 0637.46061. MR0832990. DOI 10.1016/0022-1236(86)90072-8. 27
15. E. Størmer, *Positive Linear Maps of Operator Algebras*, Springer Monogr. Math., Springer, Heidelberg, 2013. Zbl 1269.46003. MR3012443. DOI 10.1007/978-3-642-34369-8. 27
16. X.-H. Sun and Y. Li, *The range of generalized quantum operations*, Linear Algebra Appl. **452** (2014), 120–129. Zbl 1291.81032. MR3201093. DOI 10.1016/j.laa.2014.03.041. 27

[1]School of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710062, People's Republic of China.
E-mail address: xiuhongsun@aliyun.com

[2]School of Science, Xi'an University of Science and Technology, Xi'an 710054, People's Republic of China.
E-mail address: liyuan0401@aliyun.com