Ann. Funct. Anal. 7 (2016), no. 4, 573–577
http://dx.doi.org/10.1215/20088752-3661494
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

A NOTE ON WEAK*-CONVERGENCE IN $h^1(\mathbb{R}^d)$

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Communicated by V. Valov

Abstract. We give a very simple proof of a result by Dafni that states that the weak*-convergence is true in the local Hardy space $h^1(\mathbb{R}^d)$.

1. Introduction

A famous and classical result of Fefferman [3, Theorem 1] states that the John–Nirenberg space $\mathrm{BMO}(\mathbb{R}^d)$ is the dual of the Hardy space $H^1(\mathbb{R}^d)$. It is also well known that $H^1(\mathbb{R}^d)$ is one of the few examples of separable, nonreflexive Banach space which is a dual space. In fact, let $C_c(\mathbb{R}^d)$ be the space of all continuous functions with compact support, and denote by $\mathrm{VMO}(\mathbb{R}^d)$ the closure of $C_c(\mathbb{R}^d)$ in $\mathrm{BMO}(\mathbb{R}^d)$. Coifman and Weiss showed in [1] that $H^1(\mathbb{R}^d)$ is the dual space of $\mathrm{VMO}(\mathbb{R}^d)$, which gives to $H^1(\mathbb{R}^d)$ a richer structure than $L^1(\mathbb{R}^d)$. For example, the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are not bounded on $L^1(\mathbb{R}^d)$ but instead are bounded on $H^1(\mathbb{R}^d)$. In addition, the weak*-convergence is true in $H^1(\mathbb{R}^d)$ (see [5]), which is useful in the application of Hardy spaces to compensated compactness and in studying the endpoint estimates for commutators of singular integral operators (see [6], [7]). Recently, Dafni showed in [2] that the local Hardy space $h^1(\mathbb{R}^d)$ of Goldberg [4] is in fact the dual space of $\mathrm{vmo}(\mathbb{R}^d)$, the closure of $C_c(\mathbb{R}^d)$ in $\mathrm{bmo}(\mathbb{R}^d)$. Moreover, the weak*-convergence is true in $h^1(\mathbb{R}^d)$. More precisely, in [2], Dafni proved the following.

Theorem 1.1 ([2, Theorem 9]). *The space* $h^1(\mathbb{R}^d)$ *is the dual space of* $\mathrm{vmo}(\mathbb{R}^d)$.

Received Jan. 21, 2016; Accepted Apr. 8, 2016.
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2010 *Mathematics Subject Classification.* Primary 42B30; Secondary 46E15.
Keywords. H^1, BMO, VMO, Banach–Alaoglu.

Theorem 1.2 ([2, Theorem 11]). *Suppose that $\{f_n\}_{n=1}^\infty$ is a bounded sequence in $h^1(\mathbb{R}^d)$ and that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \mathbb{R}^d$. Then $f \in h^1(\mathbb{R}^d)$ and $\{f_n\}_{n=1}^\infty$ weak*-converges to f; that is, for every $\phi \in \mathrm{vmo}(\mathbb{R}^d)$, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\phi(x)\,dx = \int_{\mathbb{R}^d} f(x)\phi(x)\,dx.$$

The aim of the present paper is to give very simple proofs of the two above theorems. To this end, we first recall some definitions of the function spaces. As usual, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class of test functions on $\mathbb{R}^d$. The subset $\mathcal{A}$ of $\mathcal{S}(\mathbb{R}^d)$ is then defined by

$$\mathcal{A} = \big\{\phi \in \mathcal{S}(\mathbb{R}^d) : \big|\phi(x)\big| + |\nabla\phi(x)| \le (1+|x|^2)^{-(d+1)}\big\},$$

where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_d)$ denotes the gradient. We define

$$\mathfrak{M}f(x) := \sup_{\phi\in\mathcal{A}} \sup_{|y-x|<t} \big|f * \phi_t(y)\big| \qquad \text{and} \qquad \mathfrak{m}f(x) := \sup_{\phi\in\mathcal{A}} \sup_{|y-x|<t<1} \big|f * \phi_t(y)\big|,$$

where $\phi_t(\cdot) = t^{-d}\phi(t^{-1}\cdot)$. The space $H^1(\mathbb{R}^d)$ is the space of all integrable functions f such that $\mathfrak{M}f \in L^1(\mathbb{R}^d)$ equipped with the norm $\|f\|_{H^1} = \|\mathfrak{M}f\|_{L^1}$. The space $h^1(\mathbb{R}^d)$ denotes the space of all integrable functions f such that $\mathfrak{m}f \in L^1(\mathbb{R}^d)$ equipped with the norm $\|f\|_{h^1} = \|\mathfrak{m}f\|_{L^1}$.

We remark that the local real Hardy space $h^1(\mathbb{R}^d)$, first introduced by Goldberg in [4], is larger than $H^1(\mathbb{R}^d)$ and allows more flexibility, since global cancellation conditions are not necessary. For example, the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is contained in $h^1(\mathbb{R}^d)$ but not in $H^1(\mathbb{R}^d)$, and multiplication by cutoff functions preserves $h^1(\mathbb{R}^d)$ but not $H^1(\mathbb{R}^d)$. Thus it makes $h^1(\mathbb{R}^d)$ more suitable for working in domains and on manifolds.

It is well known (see [3]) that the dual space of $H^1(\mathbb{R}^d)$ is $\mathrm{BMO}(\mathbb{R}^d)$, the space of all locally integrable functions f with

$$\|f\|_{\mathrm{BMO}} := \sup_B \frac{1}{|B|}\int_B \Big|f(x) - \frac{1}{|B|}\int_B f(y)\,dy\Big|\,dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^d$. It was also shown in [4] that the dual space of $h^1(\mathbb{R}^d)$ can be identified with the space $\mathrm{bmo}(\mathbb{R}^d)$, consisting of locally integrable functions f with

$$\|f\|_{\mathrm{bmo}} := \sup_{|B|\le 1} \frac{1}{|B|}\int_B \Big|f(x) - \frac{1}{|B|}\int_B f(y)\,dy\Big|\,dx + \sup_{|B|\ge 1}\frac{1}{|B|}\int_B \big|f(x)\big|\,dx < \infty,$$

where the suprema are taken over all balls $B \subset \mathbb{R}^d$.

It is clear that, for any $f \in H^1(\mathbb{R}^d)$ and $g \in \mathrm{bmo}(\mathbb{R}^d)$,

$$\|f\|_{h^1} \le \|f\|_{H^1} \qquad \text{and} \qquad \|g\|_{\mathrm{BMO}} \le \|g\|_{\mathrm{bmo}}.$$

Recall that the space $\mathrm{VMO}(\mathbb{R}^d)$ (resp., $\mathrm{vmo}(\mathbb{R}^d)$) is the closure of $C_c(\mathbb{R}^d)$ in $(\mathrm{BMO}(\mathbb{R}^d), \|\cdot\|_{\mathrm{BMO}})$ (resp., $(\mathrm{bmo}(\mathbb{R}^d), \|\cdot\|_{\mathrm{bmo}})$). The following theorem is due to Coifman and Weiss.

Theorem 1.3 ([1, Theorem 4.1]). *The space $H^1(\mathbb{R}^d)$ is the dual space of $\mathrm{VMO}(\mathbb{R}^d)$.*

Throughout this article, C denotes a positive geometric constant which is independent of the main parameters, but it may change from line to line.

2. Proof of Theorems 1.1 and 1.2

In this section, we fix $\varphi \in C_c(\mathbb{R}^d)$ with supp $\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$. Let $\psi := \varphi * \varphi$. The following lemma is due to Goldberg.

Lemma 2.1 ([4, Lemmas 4, 5]). *There exists a positive constant $C = C(d,\varphi)$ such that*

(i) *for any $f \in L^1(\mathbb{R}^d)$,*

$$\|\varphi * f\|_{h^1} \le C\|f\|_{L^1};$$

(ii) *for any $g \in h^1(\mathbb{R}^d)$,*

$$\|g - \psi * g\|_{H^1} \le C\|g\|_{h^1}.$$

As a consequence of Lemma 2.1(ii), for any $\phi \in C_c(\mathbb{R}^d)$,

$$\|\phi - \overline{\psi} * \phi\|_{\mathrm{bmo}} \le C\|\phi\|_{\mathrm{BMO}}, \tag{2.1}$$

so here and hereafter, $\overline{\psi}(x) := \psi(-x)$ for all $x \in \mathbb{R}^d$.

Proof of Theorem 1.1. Since $\mathrm{vmo}(\mathbb{R}^d)$ is a subspace of $\mathrm{bmo}(\mathbb{R}^d)$, which is the dual space of $h^1(\mathbb{R}^d)$, every function f in $h^1(\mathbb{R}^d)$ determines a bounded linear functional on $\mathrm{vmo}(\mathbb{R}^d)$ of norm bounded by $\|f\|_{h^1}$.

Conversely, given a bounded linear functional L on $\mathrm{vmo}(\mathbb{R}^d)$, we have

$$\big|L(\phi)\big| \le \|L\|\|\phi\|_{\mathrm{vmo}} \le \|L\|\|\phi\|_{L^\infty}$$

for all $\phi \in C_c(\mathbb{R}^d)$. This implies (see [8]) that there exists a finite signed Radon measure μ on $\mathbb{R}^d$ such that, for any $\phi \in C_c(\mathbb{R}^d)$,

$$L(\phi) = \int_{\mathbb{R}^d} \phi(x)\,d\mu(x).$$

Moreover, the total variation of μ, $|\mu|(\mathbb{R}^d)$ is bounded by $\|L\|$. Therefore,

$$\|\psi * \mu\|_{h^1} = \big\|\varphi * (\varphi * \mu)\big\|_{h^1} \le C\|\varphi * \mu\big\|_{L^1} \le C|\mu|(\mathbb{R}^d) \le C\big\|L\| \tag{2.2}$$

by Lemma 2.1. On the other hand, by (2.1) we have

$$\begin{aligned}\big|(L - \psi * \mu)(\phi)\big| &= \big|L(\phi - \overline{\psi} * \phi)\big| \le \|L\|\|\phi - \overline{\psi} * \phi\|_{\mathrm{vmo}} \\ &\le C\|L\|\|\phi\|_{\mathrm{BMO}}\end{aligned}$$

for all $\phi \in C_c(\mathbb{R}^d)$. Consequently, by Theorem 1.3, there exists a function h belonging to $H^1(\mathbb{R}^d)$ such that $\|h\|_{H^1} \le C\|L\|$ and

$$(L - \psi * \mu)(\phi) = \int_{\mathbb{R}^d} h(x)\phi(x)\,dx$$

for all $\phi \in C_c(\mathbb{R}^d)$. This, together with (2.2), allows us to conclude that

$$L(\phi) = \int_{\mathbb{R}^d} f(x)\phi(x)\,dx$$

for all $\phi \in C_c(\mathbb{R}^d)$, where $f := h + \psi * \mu \in h^1(\mathbb{R}^d)$ satisfying $\|f\|_{h^1} \le \|h\|_{H^1} + \|\psi * \mu\|_{h^1} \le C\|L\|$. The proof of Theorem 1.1 is thus completed. □

In order to prove Theorem 1.2, we also need the following lemma due to Jones and Journé.

Lemma 2.2 ([5, p. 137])**.** *Suppose that $\{f_n\}_{n=1}^\infty$ is a bounded sequence in $H^1(\mathbb{R}^d)$ and that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in H^1(\mathbb{R}^d)$ and $\{f_n\}_{n=1}^\infty$ weak*-converges to f; that is, for every $\phi \in \mathrm{VMO}(\mathbb{R}^d)$, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\phi(x)\,dx = \int_{\mathbb{R}^d} f(x)\phi(x)\,dx.$$

Proof of Theorem 1.2. Let $\{f_{n_k}\}_{k=1}^\infty$ be an arbitrary subsequence of $\{f_n\}_{n=1}^\infty$. As $\{f_{n_k}\}_{k=1}^\infty$ is a bounded sequence in $h^1(\mathbb{R}^d)$, by Theorem 1.1 and the Banach–Alaoglu theorem there exists a subsequence $\{f_{n_{k_j}}\}_{j=1}^\infty$ of $\{f_{n_k}\}_{k=1}^\infty$ such that $\{f_{n_{k_j}}\}_{j=1}^\infty$ weak*-converges to g for some $g \in h^1(\mathbb{R}^d)$. Therefore, for any $x \in \mathbb{R}^d$,

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} f_{n_{k_j}}(y)\psi(x-y)\,dy = \int_{\mathbb{R}^d} g(y)\psi(x-y)\,dy.$$

This implies that $\lim_{j\to\infty}[f_{n_{k_j}}(x) - (f_{n_{k_j}} * \psi)(x)] = f(x) - (g * \psi)(x)$ for almost every $x \in \mathbb{R}^d$. Hence, by Lemma 2.1(ii) and Lemma 2.2, we have

$$\|f - g*\psi\|_{H^1} \le \sup_{j\ge1} \|f_{n_{k_j}} - f_{n_{k_j}} * \psi\|_{H^1} \le C \sup_{j\ge1} \|f_{n_{k_j}}\|_{h^1} < \infty.$$

Moreover,

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} \big[f_{n_{k_j}}(x) - (f_{n_{k_j}} * \psi)(x)\big]\phi(x)\,dx = \int_{\mathbb{R}^d} \big[f(x) - (g*\psi)(x)\big]\phi(x)\,dx$$

for all $\phi \in C_c(\mathbb{R}^d)$. As a consequence, we obtain that

$$\begin{aligned} \|f\|_{h^1} &\le \|f - g*\psi\|_{h^1} + \|g*\psi\|_{h^1} \le \|f - g*\psi\|_{H^1} + C\|g\|_{h^1} \\ &\le C \sup_{j\ge1} \|f_{n_{k_j}}\|_{h^1} < \infty, \end{aligned}$$

and moreover,

$$\begin{aligned} &\lim_{j\to\infty} \int_{\mathbb{R}^d} f_{n_{k_j}}(x)\phi(x)\,dx \\ &\quad = \lim_{j\to\infty} \int_{\mathbb{R}^d} \big[f_{n_{k_j}}(x) - (f_{n_{k_j}} * \psi)(x)\big]\phi(x)\,dx + \lim_{j\to\infty} \int_{\mathbb{R}^d} f_{n_{k_j}}(x)(\overline{\psi} * \phi)(x)\,dx \\ &\quad = \int_{\mathbb{R}^d} \big[f(x) - (g*\psi)(x)\big]\phi(x)\,dx + \int_{\mathbb{R}^d} g(x)(\overline{\psi} * \phi)(x)\,dx \\ &\quad = \int_{\mathbb{R}^d} f(x)\phi(x)\,dx \end{aligned}$$

since $\{f_{n_{k_j}}\}_{j=1}^\infty$ weak*-converges to g in $h^1(\mathbb{R}^d)$. This allows us to complete the proof of Theorem 1.2 since $\{f_{n_k}\}_{k=1}^\infty$ is an arbitrary subsequence of $\{f_n\}_{n=1}^\infty$. □

Acknowledgments. This work was completed when Ky was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM), and he would like to thank that institution for its support and hospitality. The authors would also like to thank the referees for their careful reading and helpful suggestions. Hung's work was partially supported by Vietnam National Foundation for Science and Technology Development grant No. 101.02-2014.51.

References

1. R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645. Zbl 0358.30023. MR0447954. 573, 574
2. G. Dafni, *Local VMO and weak convergence in h^1*, Canad. Math. Bull. **45** (2002), no. 1, 46–59. Zbl 1004.42020. MR1884133. DOI 10.4153/CMB-2002-005-2. 573, 574
3. C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), no. 4, 587–588. Zbl 0229.46051. MR0280994. 573, 574
4. D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42. Zbl 0409.46060. MR0523600. 573, 574, 575
5. P. W. Jones and J.-L. Journé, *On weak convergence in $H^1(\boldsymbol{R}^d)$*, Proc. Amer. Math. Soc. **120** (1994), no. 1, 137–138. Zbl 0814.42011. MR1159172. DOI 10.2307/2160177. 573, 576
6. L. D. Ky, *Bilinear decompositions and commutators of singular integral operators*, Trans. Amer. Math. Soc. **365** (2013), no. 6, 2931–2958. Zbl 1272.42010. MR3034454. DOI 10.1090/S0002-9947-2012-05727-8. 573
7. L. D. Ky, *Endpoint estimates for commutators of singular integrals related to Schrödinger operators*, Rev. Mat. Iberoam. **31** (2015), no. 4, 1333–1373. Zbl 06539486. MR3438392. DOI 10.4171/RMI/871. 573
8. H. L. Royden, *Real Analysis*, 3rd ed., Macmillan, New York, 1988. Zbl 0704.26006. MR1013117. 575

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