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A NOTE ON THE ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS ON BMOA

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Abstract. We give some new estimates for the essential norm of weighted composition operators on the space *BMOA*. As a corollary, we obtain a new characterization for the compactness of weighted composition operators on the space *BMOA*.

1. Introduction

Let X be a Banach space. The essential norm $\|T\|_{e,X}$ of a bounded linear operator $T : X \to X$ is its distance to the set of compact operators K on X, that is,

$$\|T\|_{e,X} = \inf\big\{\|T - K\|_X : K \text{ is compact}\big\},$$

where $\|\cdot\|_X$ is the operator norm.

Let $\mathbb{D}$ denote the unit disk in the complex plane and let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. Throughout the present article, $S(\mathbb{D})$ denotes the set of analytic self-maps of $\mathbb{D}$. For a function $u \in H(\mathbb{D})$ and a map $\varphi \in S(\mathbb{D})$, we define the weighted composition operator uC_φ, induced by u and φ, as

$$(uC_\varphi f)(z) = u(z) \cdot f\big(\varphi(z)\big), \quad f \in H(\mathbb{D}).$$

It is clear that the weighted composition operator uC_φ is the generalization of the composition operator and the multiplication operator.

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For $0 < p < \infty$, the Hardy space H^p is the space that consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_p = \sup_{0<r<1} \Big(\frac{1}{2\pi} \int_0^{2\pi} \big| f(re^{i\theta}) \big|^p \, d\theta \Big)^{1/p} < \infty.$$

An $f \in H(\mathbb{D})$ is said to *belong* to the *BMOA* space if (see [5])

$$\|f\|_* = \sup_{a\in\mathbb{D}} \big\| f \circ \sigma_a - f(a) \big\|_2 < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the automorphism of $\mathbb{D}$ exchanging 0 for a. It is easy to check that $BMOA$ is a Banach space under the norm $\|f\| = |f(0)| + \|f\|_*$. If $f \in H(\mathbb{D})$ such that $\lim_{|a|\to 1} \|f \circ \sigma_a - f(a)\|_2 = 0$, then we say that f belongs to $VMOA$.

By Littlewood's subordination theorem, we see that C_φ is bounded on $BMOA$ for any $\varphi \in S(\mathbb{D})$. The compactness of the operator $C_\varphi : BMOA \to BMOA$ was studied in [1], [4], [9], [10], and [11]. Combining the results in [1] and [9], Wulan in [10] showed that $C_\varphi : BMOA \to BMOA$ is compact if and only if

$$\lim_{n\to\infty} \|\varphi^n\|_* = 0 \qquad \text{and} \qquad \lim_{|a|\to 1} \|\sigma_a \circ \varphi\|_* = 0.$$

In [11], Wulan, Zheng, and Zhu further showed that $C_\varphi : BMOA \to BMOA$ is compact if and only if $\lim_{n\to\infty} \|\varphi^n\|_* = 0$, and they asked whether $\lim_{|a|\to 1} \|\sigma_a \circ \varphi\|_* = 0$ alone would characterize the compactness of $C_\varphi : BMOA \to BMOA$. This is affirmatively answered in [8]. Moreover, they also showed that $C_\varphi : BMOA \to BMOA$ is compact if and only if

$$\lim_{|\varphi(a)|\to 1} \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_2 = 0.$$

For $t \in (0,1)$, define

$$E(\varphi, a, t) = \big\{ \xi \in \partial\mathbb{D} : \big| (\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a)(\xi) \big| > t \big\}.$$

Set

$$\begin{aligned} \alpha(a) &= \big|u(a)\big| \cdot \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_2, \\ \beta(a) &= \log \frac{2}{1-|\varphi(a)|^2} \big\| u \circ \sigma_a - u(a) \big\|_2. \end{aligned}$$

In [6], Laitila provided several characterizations for the boundedness and compactness of the operator $uC_\varphi : BMOA \to BMOA$. For example, he showed that $uC_\varphi : BMOA \to BMOA$ is compact if and only if

$$\lim_{|\varphi(a)|\to 1} \alpha(a) = 0, \qquad \lim_{|\varphi(a)|\to 1} \beta(a) = 0,$$

and

$$\lim_{r\to 1} \limsup_{t\to 1} \sup_{|\varphi(a)|\le r} \Big(\int_{E(\varphi,a,t)} \big| u\big(\sigma_a(e^{i\theta})\big) \big|^4 \frac{d\theta}{2\pi} \Big)^{1/4} = 0. \tag{1.1}$$

In [2], Colonna used the idea of [11, p. 3826] and showed that $uC_\varphi : BMOA \to BMOA$ is compact if and only if

$$\lim_{n\to\infty} \|u\varphi^n\|_* = 0 \qquad \text{and} \qquad \lim_{|\varphi(a)|\to 1} \beta(a) = 0, \tag{1.2}$$

as well as

$$\lim_{n\to\infty} \|u\varphi^n\|_* = 0 \qquad \text{and} \qquad \lim_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_* = 0, \tag{1.3}$$

where

$$g_a(z) = \Big(\log \frac{2}{1-\overline{\varphi(a)}z}\Big)^2 \Big(\log \frac{2}{1-|\varphi(a)|^2}\Big)^{-1}.$$

Motivated by (1.2), Laitila and Lindström in [7] gave some estimates for norm and essential norm of the weighted composition operator $uC_\varphi : BMOA \to BMOA$. Among other results, they showed that, under the assumption of the boundedness of uC_φ on $BMOA$,

$$\|uC_\varphi\|_{e,BMOA} \approx \limsup_{n\to\infty} \|u\varphi^n\|_* + \limsup_{|\varphi(a)|\to 1} \beta(a). \tag{1.4}$$

Hence a natural question, motivated by (1.1) and (1.3), is whether we can give two estimates for the essential norm of the weighted composition uC_φ on $BMOA$ by using $\|u\varphi^n\|_*$ and $\|uC_\varphi g_a\|_*$, as well as $\alpha(a)$, $\|uC_\varphi g_a\|_*$ and

$$\sup_{|\varphi(a)|\le r} \Big(\int_{E(\varphi,a,t)} \big|u\big(\sigma_a(e^{i\theta})\big)\big|^4 \frac{d\theta}{2\pi}\Big)^{1/4}?$$

In the following, we give three different characterizations for the essential norm of the operator $uC_\varphi : BMOA \to BMOA$. This gives an affirmative answer to the above question by making a minor modification of the method used in [7]. Our main result is stated as follows.

Theorem 1. *Let $u \in H(\mathbb{D})$ and let $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : BMOA \to BMOA$ is bounded. Then*

$$\begin{aligned} \|uC_\varphi\|_{e,BMOA} &\approx \limsup_{n\to\infty} \|u\varphi^n\|_* + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_* \\ &\approx \limsup_{|\varphi(a)|\to 1} \alpha(a) + \limsup_{|\varphi(a)|\to 1} \beta(a) + \gamma \\ &\approx \limsup_{|\varphi(a)|\to 1} \alpha(a) + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_* + \gamma, \end{aligned}$$

where

$$\gamma := \limsup_{r\to 1} \limsup_{t\to 1} \sup_{|\varphi(a)|\le r} \Big(\int_{E(\varphi,a,t)} \big|u\big(\sigma_a(e^{i\theta})\big)\big|^4 \frac{d\theta}{2\pi}\Big)^{1/4}.$$

Throughout the rest of this article, the notation $a \lesssim b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \lesssim b$ and $b \lesssim a$ hold, then one says that $a \approx b$.

2. Proof of main result

In this section, we give a proof for our main result. For that purpose, we need the following lemmas.

Lemma 1 ([7, Lemma 4]). *Let $\varphi \in S(\mathbb{D})$ and let $u \in H(\mathbb{D})$. The following statements hold.*

(i) *For $a \in \mathbb{D}$, let $f_a(z) = \sigma_{\varphi(a)} - \varphi(a)$. Then*

$$\alpha(a) \lesssim \frac{\beta(a)}{\log \frac{2}{1-|\varphi(a)|^2}} + \|uC_\varphi f_a\|_*.$$

(ii) *For $a \in \mathbb{D}$, let $g_a = \frac{h_a^2}{h_a(\varphi(a))}$, where $h_a(z) = \log \frac{2}{1-\overline{\varphi(a)}z}$. Then*

$$\beta(a) \lesssim \alpha(a) + \big\|\big(g_a \circ \varphi \circ \sigma_a - g_a\big(\varphi(a)\big)\big) \cdot \big(u \circ \sigma_a - u(a)\big)\big\|_2 + \|uC_\varphi g_a\|_*.$$

(iii) *For all $f \in BMOA$ and $a \in \mathbb{D}$,*

$$\big\|(uC_\varphi f) \circ \sigma_a - (uC_\varphi f)(a)\big\|_2 \lesssim \big\|\big(u \circ \sigma_a - u(a)\big) \cdot \big(f \circ \varphi \circ \sigma_a - f\big(\varphi(a)\big)\big)\big\|_2 + \big(\alpha(a) + \beta(a)\big)\|f\|_*.$$

(iv) *For all $f \in BMOA$ and $a \in \mathbb{D}$,*

$$\big\|\big(u \circ \sigma_a - u(a)\big) \cdot \big(f \circ \varphi \circ \sigma_a - f\big(\varphi(a)\big)\big)\big\|_2 \lesssim \|f\|_* \min\Big\{\sup_{a\in\mathbb{D}} \beta(a), \frac{\|uC_\varphi\|_{BMOA}}{\sqrt{\log \frac{2}{1-|\varphi(a)|^2}}}\Big\}.$$

Lemma 2 ([7, Lemma 9]). *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then*

$$\limsup_{r\to 1} \limsup_{t\to 1} \sup_{|\varphi(a)|\le r} \Big(\int_{E(\varphi,a,t)} \big|u\big(\sigma_a(e^{i\theta})\big)\big|^4 \frac{d\theta}{2\pi}\Big)^{1/4} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_*.$$

Lemma 3 ([7, Lemma 5]). *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : BMOA \to BMOA$ is bounded. Then*

$$\limsup_{|\varphi(a)|\to 1} \|uC_\varphi f_a\|_* \le 2 \limsup_{n\to\infty} \|u\varphi^n\|_*.$$

Lemma 4. *A sequence $\{f_n\}$ in VMOA converges weakly to 0 in BMOA if and only if $\sup_n \|f_n\|_* < \infty$ and $f_n \to 0$ pointwise in $\mathbb{D}$.*

Proof. From Theorem 9.28 in [12], we see that the dual space of *VMOA* is H^1. Proposition 1.2 in [3] dictates that f_n converges weakly to 0 in *VMOA* if and only if $\sup_n \|f_n\|_* < \infty$ and $f_n \to 0$ pointwise in $\mathbb{D}$. Now consider the sequence $\{f_n\}$ as belonging to *BMOA*. It is easy to see that weak convergence in *BMOA* is equivalent to weak convergence in *VMOA*. In one direction, restrict an arbitrary functional on *BMOA* to a functional on *VMOA*; in the other direction, use the Hahn–Banach theorem to extend an arbitrary functional on *VMOA* to a functional on *BMOA*. □

Now we are in a position to prove the main result in this article.

Proof of Theorem 1. Let $a_n \in \mathbb{D}$ such that $|\varphi(a_n)| \to 1$ as $n \to \infty$. It was shown in [6] that $g_{a_n} \in VMOA$ and $\sup_n \|g_{a_n}\|_* < \infty$ and $g_{a_n} \to 0$ pointwise in $\mathbb{D}$. By Lemma 4, we see that $g_{a_n} \to 0$ weakly in $BMOA$ as $n \to \infty$. Hence,

$$\|uC_\varphi\|_{e,BMOA} \gtrsim \limsup_{n\to\infty} \|uC_\varphi g_{a_n}\|_* = \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_*. \tag{2.1}$$

Set $f_n(z) = z^n$. It is well known that $f_n \in VMOA$, $\sup_n \|f_n\|_* < \infty$, and $f_n \to 0$ pointwise in $\mathbb{D}$. Also, by Lemma 4 we see that $f_n \to 0$ weakly in $BMOA$ as $n \to \infty$. Then

$$\|uC_\varphi\|_{e,BMOA} \gtrsim \limsup_{n\to\infty} \|uC_\varphi f_n\|_* = \limsup_{n\to\infty} \|u\varphi^n\|_*. \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$\|uC_\varphi\|_{e,BMOA} \gtrsim \limsup_{n\to\infty} \|u\varphi^n\|_* + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_*. \tag{2.3}$$

From item (i) of Lemma 1, we see that

$$\alpha(a) \lesssim \frac{\beta(a)}{\log \frac{2}{1-|\varphi(a)|^2}} + \|uC_\varphi f_a\|_*,$$

which together with Lemma 3 implies that

$$\limsup_{|\varphi(a)|\to 1} \alpha(a) \lesssim \limsup_{|\varphi(a)|\to 1} \|uC_\varphi f_a\|_* \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_*. \tag{2.4}$$

Here we use the fact that $\sup_{a\in\mathbb{D}} \beta(a) < \infty$, by the assumption that $uC_\varphi : BMOA \to BMOA$ is bounded (see Theorem 1 of [7]).

From items (ii) and (iv) of Lemma 1, we see that

$$\begin{aligned}\beta(a) &\lesssim \alpha(a) + \big\|\big(g_a \circ \varphi \circ \sigma_a - g_a\big(\varphi(a)\big)\big) \cdot \big(u \circ \sigma_a - u(a)\big)\big\|_2 + \|uC_\varphi g_a\|_* \\ &\lesssim \alpha(a) + \|g_a\|_* \frac{\|uC_\varphi\|_{BMOA}}{\sqrt{\log \frac{2}{1-|\varphi(a)|^2}}} + \|uC_\varphi g_a\|_*,\end{aligned}$$

which implies that

$$\limsup_{|\varphi(a)|\to 1} \beta(a) \lesssim \limsup_{|\varphi(a)|\to 1} \alpha(a) + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_*. \tag{2.5}$$

By Lemma 2, (2.3), (2.4), and (2.5), we have

$$\begin{aligned}\|uC_\varphi\|_{e,BMOA} &\gtrsim \limsup_{|\varphi(a)|\to 1} \alpha(a) + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_* + \gamma \\ &\gtrsim \limsup_{|\varphi(a)|\to 1} \alpha(a) + \limsup_{|\varphi(a)|\to 1} \beta(a) + \gamma.\end{aligned}$$

Next we give the upper estimates for $\|uC_\varphi\|_{e,BMOA}$. From the proof of Lemma 10 of [7], we have

$$\|uC_\varphi\|_{e,BMOA} \lesssim \sup_{|\varphi(a)|>r}\Big(\alpha(a)+\beta(a)+\frac{\|uC_\varphi\|_{BMOA}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\Big) + \sup_{|\varphi(a)|\le r}\Big(\int_{E(\varphi,a,t)}\big|u\big(\sigma_a(e^{i\theta})\big)\big|^4\frac{d\theta}{2\pi}\Big)^{1/4}, \tag{2.6}$$

which implies that

$$\|uC_\varphi\|_{e,BMOA} \lesssim \limsup_{|\varphi(a)|\to 1}\alpha(a)+\limsup_{|\varphi(a)|\to 1}\beta(a)+\gamma.$$

By (1.4), (2.4), and (2.5), we get

$$\begin{aligned}\|uC_\varphi\|_{e,BMOA} &\lesssim \limsup_{|\varphi(a)|\to 1}\beta(a)+\limsup_{n\to\infty}\|u\varphi^n\|_*\\ &\lesssim \limsup_{|\varphi(a)|\to 1}\alpha(a)+\limsup_{|\varphi(a)|\to 1}\|uC_\varphi g_a\|_*+\limsup_{n\to\infty}\|u\varphi^n\|_*\\ &\lesssim \limsup_{|\varphi(a)|\to 1}\|uC_\varphi g_a\|_*+\limsup_{n\to\infty}\|u\varphi^n\|_*.\end{aligned}$$

Also, by items (ii) and (iv) of Proposition 2.12 and by (2.6), we have

$$\begin{aligned}&\|uC_\varphi\|_{e,BMOA}\\ &\quad\lesssim \sup_{|\varphi(a)|>r}\Big(\alpha(a)+\beta(a)+\frac{\|uC_\varphi\|_{BMOA}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\Big)\\ &\qquad+\sup_{|\varphi(a)|\le r}\Big(\int_{E(\varphi,a,t)}\big|u\big(\sigma_a(e^{i\theta})\big)\big|^4\frac{d\theta}{2\pi}\Big)^{1/4}\\ &\quad\lesssim \sup_{|\varphi(a)|>r}\Big(\alpha(a)+\|uC_\varphi g_a\|_*+\big\|\big(u\circ\sigma_a-u(a)\big)\cdot\big(g_a\circ\varphi\circ\sigma_a-g_a\big(\varphi(a)\big)\big)\big\|_2\\ &\qquad+\frac{\|uC_\varphi\|_{BMOA}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\Big)+\sup_{|\varphi(a)|\le r}\Big(\int_{E(\varphi,a,t)}\big|u\big(\sigma_a(e^{i\theta})\big)\big|^4\frac{d\theta}{2\pi}\Big)^{1/4}\\ &\quad\lesssim \sup_{|\varphi(a)|>r}\Big(\alpha(a)+\|uC_\varphi g_a\|_*+\|g_a\|_*\frac{\|uC_\varphi\|_{BMOA}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}+\frac{\|uC_\varphi\|_{BMOA}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\Big)\\ &\qquad+\sup_{|\varphi(a)|\le r}\Big(\int_{E(\varphi,a,t)}\big|u\big(\sigma_a(e^{i\theta})\big)\big|^4\frac{d\theta}{2\pi}\Big)^{1/4},\end{aligned}$$

which implies that

$$\|uC_\varphi\|_{e,BMOA} \lesssim \limsup_{|\varphi(a)|\to 1}\alpha(a)+\limsup_{|\varphi(a)|\to 1}\|uC_\varphi g_a\|_*+\gamma.$$

The proof is complete. □

From Theorem 1, we immediately get the following new characterization of the compactness of the operator $uC_\varphi : BMOA \to BMOA$.

Corollary 1. *Let $u \in H(\mathbb{D})$ and let $\varphi \in S(\mathbb{D})$ such that uC_φ is bounded on BMOA. Then the operator $uC_\varphi : BMOA \to BMOA$ is compact if and only if*

$$\limsup_{|\varphi(a)|\to 1} \alpha(a) = 0, \qquad \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_* = 0$$

and

$$\limsup_{r\to 1} \limsup_{t\to 1} \sup_{|\varphi(a)|\le r} \Big(\int_{E(\varphi,a,t)} \big|u\big(\sigma_a(e^{i\theta})\big)\big|^4 \frac{d\theta}{2\pi} \Big)^{1/4} = 0.$$

Remark 1. From Theorem 1, we see that

$$\begin{aligned} \|C_\varphi\|_{e,BMOA} &\approx \limsup_{|\varphi(a)|\to 1} \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_2 \\ &\quad + \limsup_{r\to 1} \limsup_{t\to 1} \sup_{|\varphi(a)|\le r} \big(m\big(E(\varphi,a,t)\big)\big)^{1/4}. \end{aligned} \tag{2.7}$$

In [4], it was shown that

$$\|C_\varphi\|_{e,BMOA} \approx \limsup_{n\to\infty} \|\varphi^n\|$$

and that

$$\begin{aligned} \|C_\varphi\|_{e,BMOA} &\approx \limsup_{|\varphi(a)|\to 1} \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_2 \\ &\quad + \limsup_{r\to 1} \limsup_{|z|\to 1} \sup_{|\varphi(a)|\le r} \sqrt{\frac{N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, z)}{-\log|z|}}. \end{aligned}$$

Here

$$N(\varphi, w) = \sum_{\varphi(z)=w} \log \frac{1}{|w|}, \quad w \in \mathbb{D} \setminus \varphi(0).$$

The example of (2.7) can be seen as a new estimate for the essential norm of the operator $C_\varphi : BMOA \to BMOA$.

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