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A NOTE ON RELATIVE COMPACTNESS IN $K(X,Y)$

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Abstract. For Banach spaces X and Y, let $K(X,Y)$ denote the space of all compact operators from X to Y endowed with the operator norm. We give sufficient conditions for subsets of $K(X,Y)$ to be relatively compact. We also give some necessary and sufficient conditions for the Dunford–Pettis relatively compact property of some spaces of operators.

1. Introduction and preliminaries

In this article, sufficient conditions are given for subsets of compact operators to be relatively compact. Also, the Dunford–Pettis relatively compact property and the Gelfand–Phillips property are studied in the context of spaces of operators.

Let X and Y denote Banach spaces, let X^* denote the continuous linear dual of X, and let B_X denote the unit ball of X. An operator $T : X \to Y$ will be a continuous and linear function. The set of all operators from X to Y will be denoted by $L(X,Y)$, and the subspace of compact operators will be denoted by $K(X,Y)$. The $w^* - w$ continuous (resp., compact) operators from X^* to Y will be denoted by $L_{w^*}(X^*,Y)$ (resp., $K_{w^*}(X^*,Y)$). The projective tensor product of X and Y will be denoted by $X \otimes_\pi Y$.

An operator $T : X \to Y$ is called *completely continuous (or Dunford–Pettis)* if T maps weakly convergent sequences to norm-convergent sequences. Let $CC(X,Y)$ denote the set of all completely continuous operators from X to Y.

A bounded subset A of X is called a *Dunford–Pettis* (DP) (resp., *limited*) subset of X if every weakly null (resp., w^*-null) sequence (x_n^*) in X^* tends to 0

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uniformly on A; that is,

$$\lim_n \Big(\sup\big\{\big|x_n^*(x)\big| : x \in A\big\}\Big) = 0.$$

A sequence (x_n) is *Dunford–Pettis* (DP) (resp., *limited*) if the set of its terms is a DP (resp., limited) set.

A bounded subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. Every DP (resp., limited) set is weakly precompact (e.g., see [14, p. 377]) (resp., [2]).

A Banach space X has the *Dunford–Pettis relatively compact property (DPrcP)* if every DP subset of X is relatively compact (see [7]). Schur spaces have the DPrcP.

A bounded subset A of X^* is called an *L-subset* of X^* if each weakly null sequence (x_n) in X tends to 0 uniformly on A; that is,

$$\lim_n \Big(\sup\big\{\big|x^*(x_n)\big| : x^* \in A\big\}\Big) = 0.$$

It is known that $\ell_1 \not\hookrightarrow X$ if and only if X^* has the DPrcP if and only if every L-subset of X^* is relatively compact (see [7], [6]).

A Banach space X has the *Gelfand–Phillips (GP) property* if every limited subset of X is relatively compact. The following spaces have the Gelfand–Phillips property: Schur spaces; spaces with w^*-sequential compact dual unit balls; separable spaces; reflexive spaces; spaces whose duals do not contain ℓ_1; subspaces of weakly compactly generated spaces; spaces whose duals have the Radon–Nikodym property; dual spaces X^* whith X not containining ℓ_1 (see [2], [6], [18, p. 31]).

A series $\sum x_n$ of elements of X is called *weakly unconditionally convergent* (wuc) if $\sum |x^*(x_n)| < \infty$ for each $x^* \in X^*$. An operator $T : X \to Y$ is called *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones. A Banach space X has property (V) if and only if every unconditionally converging operator with domain X is weakly compact (see [13]). Also, $C(K)$ spaces and reflexive spaces have property (V) (see [13]).

A Banach space X has the *Dunford–Pettis property (DPP)* if every weakly compact operator $T : X \to Y$ is completely continuous, for any Banach space Y (see [4]); $C(K)$ spaces and $L^1(\mu)$ spaces have the DPP (see [4]).

Numerous papers have investigated whether spaces of operators inherit the Dunford–Pettis relatively compact property or the Gelfand–Phillips property when the codomain and the dual of the domain possess the respective property (e.g., see the [5, Introduction and Section 2], [7], [8, Sections 2, 3], [10], and [17]).

In the present article, sufficient conditions for subsets of compact operators to be relatively compact and evaluation operators are used in spaces of operators to establish simple mapping results which extend and consolidate results in [10], [7], [8], [17], [16], and [5]. The Dunford–Pettis complete continuity of the evaluation operators is used in Theorem 3.3 and Theorem 3.7 to give sufficient conditions for $K(X,Y)$ and $K(X,Y^*)$ to have the DPrcP, thus extending results in [10] and [7]. The Dunford–Pettis complete continuity, the limited complete continuity, and the complete continuity of the evaluation operators are used in Theorem 3.12

to give sufficient conditions for $L(X,Y)$ and $L_{w^*}(X^*,Y)$ to have the DPrcP, the GP property, and the Schur property, thus extending results in [7], [8], [17], and [16]. The limited complete continuity of the evaluation operators is used in Theorem 3.16 to study the GP property of $K_{w^*}(X^*,Y)$, thus extending [5, Theorem 2.1].

2. Relative Compactness in spaces of compact operators

If H is a subset of $K(X,Y)$, $x \in X$ and $y^* \in Y^*$, then let $H(x) = \{T(x) : T \in H\}$, $H^*(y^*) = \{T^*(y^*) : T \in H\}$, $H(B_X) = \{T(x) : T \in H, x \in B_X\}$, and $H^*(B_{Y^*}) = \{T^*(y^*) : T \in H, y^* \in B_{Y^*}\}$. We begin by proving a theorem of Palmer [12] about relative compactness in $K(X,Y)$. The original proof used a variation of a general Arzela–Ascoli theorem. We give a different proof based on the following result, which gives a criterion for a set of compact operators to be weakly precompact. Our proof clarifies the connections between the weak precompactness and relative compactness of subsets of compact operators.

Theorem 2.1 ([9, Theorem 1])**.** *Let H be a bounded subset of $K(X,Y)$ such that*

(i) *$H(x)$ is weakly precompact for each $x \in X$, and*
(ii) *$H^*(y^*)$ is relatively weakly compact for each $y^* \in Y^*$.*

Then H is weakly precompact.

Theorem 2.2 ([12, Theorems 2.1, 2.2])**.** *A subset H of $K(X,Y)$ is relatively compact if and only if*

(a) *$H(B_X)$ is relatively compact in Y and*
(b) *$H^*(y^*)$ is relatively compact in X^*, for each $y^* \in Y^*$,*

or

(a)′ *$H^*(B_{Y^*})$ is relatively compact in X^* and*
(b)′ *$H(x)$ is relatively compact in Y, for each $x \in X$.*

Proof. Suppose that H is relatively compact in $K(X,Y)$. Let $y^* \in Y^*$ and $x \in X$. The maps $\psi_{y^*} : K(X,Y) \to X^*$, $\psi_{y^*}(T) = T^*(y^*)$ and $\phi_x : K(X,Y) \to Y$, $\psi_x(T) = T(x)$ are bounded linear operators. Hence $H^*(y^*)$ is relatively compact in X^* and $H(x)$ is relatively compact in Y.

Let $(T_n(x_n))$ be a sequence in $H(B_X)$. Since H is relatively compact, we can suppose (by passing to a subsequence) that $\|T_n - T\| \to 0$, where $T \in K(X,Y)$. For $n, m \in \mathbb{N}$ we have

$$\begin{aligned}
&\big\|T_n(x_n) - T_m(x_m)\big\| \\
&\quad \le \big\|(T_n - T_m)(x_n)\big\| + \big\|(T_m - T)(x_n - x_m)\big\| + \big\|T(x_n - x_m)\big\| \\
&\quad \le \|T_n - T_m\| + 2\|T_m - T\| + \big\|T(x_n) - T(x_m)\big\|.
\end{aligned}$$

Since $\|T_n - T\| \to 0$ and T is compact, some subsequence of $(T_n(x_n))$ is norm-convergent. Thus $H(B_X)$ is relatively compact. Since the map $T \to T^*$ from $K(X,Y)$ to $K(Y^*,X^*)$ is an isometry, H^* is relatively compact in $K(Y^*,X^*)$. Using an argument similar to the previous one, $H^*(B_{Y^*})$ is relatively compact.

Suppose that H satisfies (a) and (b). If H is not bounded, there is a sequence (T_n) in H so that $\|T_n\| > n$ for all n. Let (x_n) be a sequence in B_X such that

$\|T_n(x_n)\| > n$ for all n. Since $H(B_X)$ is relatively compact, without loss of generality we can assume that $(T_n(x_n))$ is convergent. Hence $(T_n(x_n))$ is bounded, a contradiction. Thus H is bounded.

Since $H(B_X)$ is relatively compact in Y, $H(x)$ is relatively compact in Y for each $x \in X$. Thus the assumptions of Theorem 2.1 are satisfied. By Theorem 2.1, H is weakly precompact. Let (T_n) be a sequence in H. Without loss of generality we can assume that (T_n) is weakly Cauchy.

Suppose that (T_n) has no norm-convergent subsequence. Choose $\epsilon > 0$ and let $\{n_k\}$ and $\{m_k\}$ be two sequences of positive integers such that $m_k < n_k < m_{k+1}$ and $\|T_{n_k} - T_{m_k}\| > \epsilon$ for all k. Let (x_k) be a sequence in B_X so that $\|T_{n_k}(x_k) - T_{m_k}(x_k)\| > \epsilon$ for all k.

For each $y^* \in Y^*$, the sequence $(T_n^*(y^*))$ has a convergent subsequence and is weakly Cauchy, and thus is convergent. Let $L \in L(Y^*, X^*)$ so that $T_n^*(y^*) \to L(y^*)$ for all $y^* \in Y^*$. For all $y^* \in Y^*$,

$$\left\langle T_{n_k}(x_k) - T_{m_k}(x_k), y^* \right\rangle \le \left\| T_{n_k}^*(y^*) - T_{m_k}^*(y^*) \right\| \to 0.$$

Thus $(T_{n_k}(x_k) - T_{m_k}(x_k))$ is weakly null in Y. Since $(T_{n_k}(x_k))$ and $(T_{m_k}(x_k))$ lie in $H(B_X)$ which is relatively compact, $\{T_{n_k}(x_k) - T_{m_k}(x_k) : k \in \mathbb{N}\}$ is relatively compact. Therefore $\|T_{n_k}(x_k) - T_{m_k}(x_k)\| \to 0$. This is a contradiction.

Now suppose that H satisfies (a)′ and (b)′. Note that H is bounded. Since $H^*(B_{Y^*})$ is relatively compact in X^*, $H^*(y^*)$ is relatively compact in X^* for each $y^* \in Y^*$. Let (T_n) be a sequence in H. By Theorem 2.1, we can assume that (T_n) is weakly Cauchy.

Suppose that (T_n) has no norm-convergent subsequence. Choose $\epsilon > 0$ and let $\{n_k\}$ and $\{m_k\}$ be two sequences of positive integers such that $m_k < n_k < m_{k+1}$ and $\|T_{n_k} - T_{m_k}\| > \epsilon$ for all k. Let (y_k^*) be a sequence in B_{Y^*} so that $\|T_{n_k}^*(y_k^*) - T_{m_k}^*(y_k^*)\| > \epsilon$ for all k. For each $x \in X$, the sequence $(T_n(x))$ has a convergent subsequence and is weakly Cauchy, and thus is convergent. Let $T \in L(X,Y)$ so that $T_n(x) \to T(x)$ for each $x \in X$. For all $x \in X$,

$$\left\langle T_{n_k}(x) - T_{m_k}(x), y^* \right\rangle \le \left\| T_{n_k}(x) - T_{m_k}(x) \right\| \to 0.$$

Hence $(T_{n_k}^*(y_k^*) - T_{m_k}^*(y_k^*))$ is w^*-null in X^*. Since $H^*(B_{Y^*})$ is relatively compact, $\{T_{n_k}^*(y_k^*) - T_{m_k}^*(y_k^*) : k \in \mathbb{N}\}$ is relatively compact. Therefore $\|T_{n_k}^*(y_k^*) - T_{m_k}^*(y_k^*)\| \to 0$. This contradiction concludes the proof. □

The following two theorems give criterions for relative compactness in the space $K_{w^*}(X^*, Y)$. We recall the following well-known isometries:

(1) $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$, $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$ $(T \to T^*)$,
(2) $W(X,Y) \simeq L_{w^*}(X^{**}, Y)$ and $K(X,Y) \simeq K_{w^*}(X^{**}, Y)$ $(T \to T^{**})$.

Theorem 2.3 ([15, Theorem 1.15])**.** *A subset H of $K_{w^*}(X^*, Y)$ is relatively compact if and only if*

(i) *$H(B_{X^*})$ is relatively compact in Y and*
(ii) *$H^*(y^*)$ is relatively compact in X for each $y^* \in Y^*$,*

or

(i)′ $H^*(B_{Y^*})$ *is relatively compact in* X *and*
(ii)′ $H(x^*)$ *is relatively compact in* Y *for each* $x^* \in X^*$.

Theorem 2.4. *Suppose that* X *is a Banach space. The Banach space* Y *has the Dunford–Pettis relatively compact property (DPrcP) (resp., the GP property) if and only if the following holds: if* H *is a subset of* $K_{w^*}(X^*, Y)$ *such that*

(i) $\lim_n(\sup\{\|T^*(y_n^*)\| : T \in H\}) = 0$ *for any weakly null (resp.,* w^**-null) sequence* (y_n^*) *in* Y^*, *and*
(ii) $H^*(y^*)$ *is relatively compact for any* $y^* \in Y^*$,

then H *is relatively compact in* $K_{w^*}(X^*, Y)$.

Proof. Suppose that H is a subset of $K_{w^*}(X^*, Y)$ satisfying (i) and (ii). (Note that H satisfies condition (ii) of Theorem 2.3.) Suppose that (y_n^*) is a weakly null (resp., w^*-null) sequence in Y^*. Since $\lim_n(\sup\{\|T^*(y_n^*)\| : T \in H\}) = 0$, we obtain

$$\begin{aligned}&\lim_n\big(\sup\{\big|\big\langle T(x^*), y_n^*\big\rangle\big| : T \in H, x^* \in B_{X^*}\}\big)\\ &\quad= \lim_n\big(\sup\{\big|\big\langle x^*, T^*(y_n^*)\big\rangle\big| : T \in H, x^* \in B_{X^*}\}\big) = 0.\end{aligned}$$

Hence $H(B_{X^*})$ is a DP (resp., limited) subset of Y, and thus relatively compact. By Theorem 2.3, H is relatively compact.

We show that if X and Y are such that a subset H of $K_{w^*}(X^*, Y)$ satisfying (i) and (ii) is relatively compact, then Y has the DPrcP (resp., the GP property). Let A be a DP (resp., limited) subset of Y, and let $x_0 \in X$, $\|x_0\| = 1$. For each $y \in A$, define $T_y : X^* \to Y$ by $T_y(x^*) = x^*(x_0)y$. Note that $T_y \in K_{w^*}(X^*, Y)$, and that $H = \{T_y : y \in A\}$ satisfies (ii). Further, if (y_n^*) is a weakly null (resp., w^*-null) sequence in Y^*, then $\sup_{y\in A} \|T_y^*(y_n^*)\| = \sup_{y\in A} \|y_n^*(y)x_0\| \to 0$, since A is a DP (resp., limited) subset of Y. Hence $H = \{T_y : y \in A\}$ is relatively compact in $K_{w^*}(X^*, Y)$. This implies that A is relatively compact. □

Theorem 2.5. *Suppose that* X *is a Banach space. The Banach space* Y *has the Dunford–Pettis relatively compact property (DPrcP) (resp., the GP property) if and only if the following holds:*

if H *is a subset of* $K(X, Y)$ *such that*

(i) $\lim_n(\sup\{\|T^*(y_n^*)\| : T \in H\}) = 0$ *for any weakly null (resp.,* w^**-null) sequence* (y_n^*) *in* Y^*, *and*
(ii) $H^*(y^*)$ *is relatively compact for any* $y^* \in Y^*$,

then H *is relatively compact in* $K(X, Y)$.

Proof. Suppose that H is a subset of $K(X, Y)$ satisfying (i) and (ii). Using the isometry $K(X, Y) \simeq K_{w^*}(X^{**}, Y)$ and Theorem 2.4, H is relatively compact.

If X and Y are such that a subset H of $K(X, Y)$ satisfying (i) and (ii) is relatively compact, then Y has the DPrcP (resp., the GP property), by the proof of Theorem 2.4. □

Theorem 2.6. *Suppose that* X *is a Banach space. The Banach space* Y *does not contain copies of* ℓ_1 *if and only if the following holds:*

if H is a subset of $K(X, Y^*)$ such that

(i) $\lim_n(\sup\{\|T^*(y_n)\| : T \in H\}) = 0$ *for any weakly null sequence (y_n) in Y, and*

(ii) $H^*(y^{**})$ *is relatively compact for any $y^{**} \in Y^{**}$,*

then H is relatively compact in $K(X, Y^*)$.

Proof. Suppose that H is a subset of $K(X, Y^*)$ satisfying (i) and (ii). We show that $H(B_X)$ is relatively compact in Y^*. Suppose that (y_n) is a weakly null sequence in Y. Since $\lim_n(\sup\{\|T^*(y_n)\| : T \in H\}) = 0$, it follows that

$$\begin{aligned}&\lim_n\big(\sup\big\{\big|\big\langle T(x), y_n\big\rangle\big| : T \in H, x \in B_X\big\}\big)\\&= \lim_n\big(\sup\big\{\big|\big\langle x, T^*(y_n)\big\rangle\big| : T \in H, x \in B_X\big\}\big) = 0.\end{aligned}$$

Then $H(B_X)$ is an L-subset of Y^*. Since Y does not contain copies of ℓ_1, $H(B_X)$ is relatively compact (see [6, Theorem 2]). By Theorem 2.2, H is relatively compact.

We show that if X and Y are such that a subset H of $K(X, Y^*)$ satisfying (i) and (ii) is relatively compact, then Y does not contain copies of ℓ_1. It is enough to show that every L-subset of Y^* is relatively compact [6]. Let A be an L-subset of Y^*, and let $x_0^* \in X^*$, $\|x_0^*\| = 1$. For each $y^* \in A$, define $S_{y^*} : X \to Y^*$ by $S_{y^*}(x) = x_0^*(x)y^*$. Note that S_{y^*} is compact, and that $H = \{S_{y^*} : y^* \in A\}$ satisfies (ii). Further, if (y_n) is a weakly null sequence in Y, then $\sup_{y^*\in A}\|S_{y^*}^*(y_n)\| = \sup_{y^*\in A}\|y^*(y_n)x_0^*\| \to 0$, since A is an L-set in Y^*. Hence $H = \{S_{y^*} : y^* \in A\}$ is relatively compact in $K(X, Y^*)$. This implies that A is relatively compact. □

Definition 2.7. A subset H of $K(X, Y)$ is called *sequentially weak-norm equicontinuous* (see [11]) if for each weakly null sequence (x_n) in X, the sequence $(T(x_n))$ converges to 0 uniformly for $T \in H$ (i.e., $\sup\{\|T(x_n)\| : T \in H\} \to 0$).

Corollary 2.8 ([11, Theorem 1])**.** *Suppose that X does not contain copies of ℓ_1 and Y is a Banach space. Let H be a subset of $K(X, Y)$ such that*

(i) *H is sequentially weak-norm equicontinuous and*

(ii) *$H(x)$ is relatively compact in Y, for each $x \in X$.*

Then H is relatively compact.

Proof. Let H be a subset of $K(X, Y)$ satisfying (i) and (ii). Suppose that (x_n) is a weakly null sequence in X. Since $\sup\{\|T(x_n)\| : T \in H\} \to 0$, we obtain

$$\begin{aligned}&\lim_n\big(\sup\big\{\big|\big\langle x_n, T^*(y^*)\big\rangle\big| : T \in H, y^* \in B_{Y^*}\big\}\big)\\&= \lim_n\big(\sup\big\{\big|\big\langle T(x_n), y^*\big\rangle\big| : T \in H, x \in B_X\big\}\big) = 0.\end{aligned}$$

Then $H^*(B_{Y^*})$ is an L-subset of X^*. Since X does not contain copies of ℓ_1, $H^*(B_{Y^*})$ is relatively compact (see [6]). By Theorem 2.2, H is relatively compact. □

3. Evaluation operators and the Dunford–Pettis relatively compact property

Suppose that X and Y are Banach spaces and M is a closed subspace of $L(X, Y)$. If $x \in X$ and $y^* \in Y^*$, then the evaluation operators $\phi_x : M \to Y$ and

$\psi_{y^*} : M \to X^*$ are defined by

$$\phi_x(T) = T(x), \qquad \psi_{y^*}(T) = T^*(y^*), \quad T \in M.$$

Definition 3.1. An operator $T : X \to Y$ is called *Dunford–Pettis completely continuous (DPcc)* if T carries weakly null DP sequences to norm null ones.

In the following we give some necessary and sufficient conditions for the Dunford–Pettis relatively compact property of some spaces of operators in terms of the Dunford–Pettis complete continuity of the evaluation operators.

We note that a space X has the DPrcP if and only if every weakly null DP sequence in X is norm null.

Lemma 3.2. *An operator $T : X \to Y$ is Dunford–Pettis completely continuous if and only if $T(A)$ is relatively compact in Y for each DP subset A of X.*

Proof. Suppose that $T : X \to Y$ is Dunford–Pettis completely continuous and let A be a DP subset of X. Let (x_n) be a sequence in A. Since DP sets are weakly precompact (see [14, p. 377]), (x_n) has a weakly Cauchy subsequence. Without loss of generality suppose that (x_n) is weakly Cauchy. Note that $A-A$ is a DP set. For any two subsequences (a_n) and (b_n) of (x_n), $(a_n - b_n)$ is weakly null and DP, and $\|T(a_n) - T(b_n)\| \to 0$. Then $(T(x_n))$ is norm Cauchy, hence norm-convergent in Y. Thus $T(A)$ is relatively compact.

Conversely, suppose that $T(A)$ is relatively compact for each DP subset A of X. Let (x_n) be a weakly null DP sequence in X. Since $(T(x_n))$ is relatively compact and weakly null, $\|T(x_n)\| \to 0$. Thus T is Dunford–Pettis completely continuous. $\square$

The following theorem extends [10, Theorem 3.8].

Theorem 3.3. *Suppose that Y has the DPrcP and that $L(Y^*, X^*) = CC(Y^*, X^*)$. If M is a closed subspace of $L(X,Y) = K(X,Y)$ such that the evaluation operator $\psi_{y^*} : M \to X^*$ is Dunford–Pettis completely continuous for each $y^* \in Y^*$, then M has the DPrcP.*

Proof. Let $T : X \to Y$ be an operator. Since $T^* : Y^* \to X^*$ is completely continuous, $T(B_X)$ is a DP subset of Y, and thus relatively compact. Hence T is compact. Thus $L(X,Y) = K(X,Y)$.

Let H be a DP subset of M. Let $y^* \in Y^*$. Since $\psi_{y^*} : M \to X^*$ is Dunford–Pettis completely continuous, $\psi_{y^*}(H) = H^*(y^*)$ is relatively compact. Hence H satisfies condition (ii) of Theorem 2.5. Suppose that condition (i) of Theorem 2.5 is not satisfied. Let $\epsilon > 0$, (y_n^*) be a weakly null sequence in Y^*, and (T_n) a sequence in H such that for each n,

$$\left\|T_n^*(y_n^*)\right\| > \epsilon.$$

Let (x_n) be a sequence in B_X such that $\langle T_n^*(y_n^*), x_n\rangle > \epsilon$.

Since $L(Y^*, X^*) = CC(Y^*, X^*)$, $(x_n \otimes y_n^*)$ is weakly null in $X \otimes_\pi Y^*$. Indeed, if $T \in (X \otimes_\pi Y^*)^* \simeq L(X, Y^{**})$ ([4, p. 230]), $T^*|_{Y^*} \in L(Y^*, X^*) = CC(Y^*, X^*)$ and

$$\langle x_n \otimes y_n^*, T\rangle \le \left\|T^*(y_n^*)\right\| \to 0.$$

Now $L(X,Y)$ embeds isometrically in $L(X,Y^{**})$ and (T_n) is a DP sequence in $L(X,Y^{**})$. Since a *DP* subset of a dual space is necessarily an L-subset of the dual space,

$$\langle T_n, x_n \otimes y_n^* \rangle = \langle T_n^*(y_n^*), x_n \rangle \to 0.$$

This is a contradiction. By Theorem 2.5, H is relatively compact. $\square$

Corollary 3.4. *Suppose that X^* and Y have the DPrcP and every operator $T : Y^* \to X^*$ is completely continuous. If M is a closed subspace of $L(X,Y) = K(X,Y)$, then M has the DPrcP.*

Proof. Since X^* has the DPrcP, $\psi_{y^*} : M \to X^*$ is Dunford–Pettis completely continuous for each $y^* \in Y^*$. Apply Theorem 3.3. $\square$

In [10, Theorem 3.8] it is shown that if X^* and Y have the DPrcP and if $L(Y^*, X^*) = K(Y^*, X^*)$, then $L(X,Y)$ has the DPrcP. Corollary 3.4 is more general than [10, Theorem 3.8], since the assumption $L(Y^*, X^*) = CC(Y^*, X^*)$ is more general than the assumption $L(Y^*, X^*) = K(Y^*, X^*)$. Let $X = c_0$ and $Y = E$ be the first Bourgain–Delbaen space [1]; that is, E is an infinite-dimensional separable $\mathcal{L}_\infty$-space with the Schur property. Since E^{**} is complemented in some $C(K)$ space (see [1, Proposition 1.23]), and $C(K)$ spaces have property (V) (see [13]), E^{**} has property (V) (see [13]). Since E^{**} is also nonreflexive, $c_0 \hookrightarrow E^{**}$ (see [13]), and thus ℓ_1 is complemented in E^* by a result of Bessaga–Pelczynski. Hence the projection $P : E^* \to \ell_1$ is a noncompact operator. Thus $L(Y^*, X^*) = CC(Y^*, X^*) \neq K(Y^*, X^*)$.

Corollary 3.5 ([7, Corollary 9])**.** *Suppose that X^* has Schur property and Y has the DPrcP. Then $L(X,Y) = K(X,Y)$ has the DPrcP.*

Proof. Since X^* has the Schur property, $L(Y^*, X^*) = CC(Y^*, X^*)$. Apply Corollary 3.4. $\square$

Lemma 3.6. *Suppose that $L(X, Y^*) = K(X, Y^*)$.*

(i) *If (x_n) is bounded in X and (y_n) is weakly null in Y, then $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$.*

(ii) *If (x_n) is weakly null in X and (y_n) is bounded in Y, then $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$.*

Proof. (i) Suppose that (x_n) is a sequence in B_X and (y_n) is weakly null in Y. If $T \in (X \otimes_\pi Y)^* \simeq L(X, Y^*)$ (see [4, p. 230]), then

$$\langle T, x_n \otimes y_n \rangle = \langle T^*(y_n), x_n \rangle \le \|T^*(y_n)\| \to 0,$$

since $T^*|_Y$ is completely continuous.

(ii) Suppose that (x_n) is weakly null in X and (y_n) is a sequence in B_Y. If $T \in (X \otimes_\pi Y)^* \simeq L(X, Y^*) = K(X, Y^*)$, then

$$\langle T, x_n \otimes y_n \rangle = \langle T(x_n), y_n \rangle \le \|T(x_n)\| \to 0,$$

since T is completely continuous. $\square$

We note that Lemma 3.6(i) holds if we only assume that $L(Y, X^*) = CC(Y, X^*)$ and Lemma 3.6(ii) holds if we only assume that $L(X, Y^*) = CC(X, Y^*)$.

Theorem 3.7. *Suppose that Y does not contain copies of ℓ_1, that $L(X, Y^*) = K(X, Y^*)$, and that M is a closed subspace of $L(X, Y^*)$ such that the evaluation operator $\psi_{y^{**}} : M \to X^*$ is Dunford–Pettis completely continuous for each $y^{**} \in Y^{**}$. Then M has the DPrcP and $X \otimes_\pi Y$ does not contain a copy of ℓ_1.*

Proof. Let (T_n) be a weakly null DP sequence in M so that $\|T_n\| = 1$ for each n. Let (x_n) be a sequence in B_X such that $\|T_n(x_n)\| > 1/2$ for each n. If $y^{**} \in Y^{**}$, then $\langle y^{**}, T_n(x_n)\rangle \le \|T_n^*(y^{**})\| \to 0$, since $\psi_{y^{**}} : M \to X^*$ is Dunford–Pettis completely continuous. Hence $(T_n(x_n))$ is weakly null in Y^*.

Let (y_n) be a weakly null sequence in Y. By Lemma 3.6 (i), $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$. Since (T_n) is a DP subset of $L(X, Y^*) \simeq (X \otimes_\pi Y)^*$,

$$\langle T_n, x_n \otimes y_n\rangle = \big\langle T_n(x_n), y_n\big\rangle \to 0.$$

Hence $(T_n(x_n))$ is an L-set in Y^*, and thus relatively compact [6]. Therefore $\|T_n(x_n)\| \to 0$, and we have a contradiction.

Since $L(X, Y^*) \simeq (X \otimes_\pi Y)^*$ has the DPrcP, $X \otimes_\pi Y$ does not contain copies of ℓ_1 (see [7]). □

Corollary 3.8 ([7, Theorem 3])**.** *Suppose that X and Y do not contain copies of ℓ_1 and that $L(X, Y^*) = K(X, Y^*)$. Then $X \otimes_\pi Y$ does not contain a copy of ℓ_1.*

Proof. Since X does not contain copies of ℓ_1, X^* has the DPrcP [7]. Hence $\psi_{y^{**}} : L(X, Y^*) \to X^*$ is Dunford–Pettis completely continuous for each $y^{**} \in Y^{**}$. Apply Theorem 3.7. □

Theorem 3.9. *Suppose that X does not contain copies of ℓ_1, $L(X, Y^{**}) = K(X, Y^{**})$, and suppose that M is a closed subspace of $L(X, Y)$ such that the evaluation operator $\phi_x : M \to Y$ is Dunford–Pettis completely continuous for each $x \in X$. Then M has the DPrcP.*

Proof. Let (T_n) be a weakly null DP sequence in M so that $\|T_n\| = 1$ for each n. Let (y_n^*) be a sequence in B_{Y^*} such that $\|T_n^*(y_n^*)\| > 1/2$ for each n. For each $x \in X$, $\phi_x : M \to Y$ is Dunford–Pettis completely continuous, hence $\|\phi_x(T_n)\| = \|T_n(x)\| \to 0$. Then $\langle T_n^*(y_n^*), x\rangle \le \|T_n(x)\| \to 0$, and thus $(T_n^*(y_n^*))$ is w^*-null in X^*.

Let (x_n) be a weakly null sequence in X. By Lemma 3.6 (ii), $(x_n \otimes y_n^*)$ is weakly null in $X \otimes_\pi Y^*$. Now $L(X, Y)$ embeds isometrically in $L(X, Y^{**})$ and (T_n) is a DP sequence in $L(X, Y^{**}) \simeq (X \otimes_\pi Y^*)^*$. Hence

$$\langle T_n, x_n \otimes y_n^*\rangle = \big\langle T_n^*(y_n^*), x_n\big\rangle \to 0.$$

Therefore $(T_n^*(y_n^*))$ is an L-set in X^*, and thus relatively compact [6]. Then $\|T_n^*(y_n^*)\| \to 0$, and we have a contradiction. □

Corollary 3.10.

(i) *Suppose that X does not contain copies of ℓ_1, that Y has the DPrcP, and that $L(X, Y^{**}) = K(X, Y^{**})$. Then any closed subspace of $L(X, Y)$ has the DPrcP.*

(ii) *Suppose that X does not contain copies of ℓ_1, that X has the DPP and property (V), that Y has the DPrcP, and that Y^{**} does not contain copies of c_0. Then any closed subspace of $L(X,Y)$ has the DPrcP.*

Proof. (i) Since Y has the DPrcP, $\phi_x : M \to Y$ is Dunford–Pettis completely continuous for each $x \in X$. Apply Theorem 3.9.

(ii) Let $T : X \to Y^{**}$ be an operator. Since $c_0 \not\hookrightarrow Y^{**}$, every wuc series in Y^{**} is unconditionally converging (see [4, p. 22]) and T is unconditionally converging. Since X has property (V), T is weakly compact (see [13]). Thus T is completely continuous, since X has the DPP. Then T is compact, since $\ell_1 \not\hookrightarrow X$ (see [14, p. 377]). Apply (i). $\square$

Definition 3.11. An operator $T : X \to Y$ is called *limited completely continuous* (lcc) if T maps weakly null limited sequences to norm null sequences [17].

Theorem 3.12.

(i) *Let X and Y be Banach spaces and let M be a closed subspace of $L(X,Y)$ such that the evaluation operator $\psi_{y^*} : M \to X^*$ is Dunford–Pettis completely continuous (resp., lcc) for each $y^* \in Y^*$. If M does not have the DPrcP (resp., the GP property), then there is a separable subspace Y_0 of Y and an operator $A : Y_0 \to c_0$ which is not completely continuous.*

(ii) *Let X and Y be Banach spaces and let M be a closed subspace of $L_{w^*}(X^*,Y)$ such that the evaluation operator $\psi_{y^*} : M \to X$ is Dunford–Pettis completely continuous (resp., lcc) for each $y^* \in Y^*$. If M does not have the DPrcP (resp., the GP property), then there is a separable subspace Y_0 of Y and an operator $A : Y_0 \to c_0$ which is not completely continuous.*

(iii) *Let X and Y be Banach spaces and let M be a closed subspace of $L_{w^*}(X^*,Y)$ such that the evaluation operator $\psi_{y^*} : M \to X$ is completely continuous for each $y^* \in Y^*$. If M does not have the Schur property, then there is a separable subspace Y_0 of Y and an operator $A : Y_0 \to c_0$ which is not completely continuous.*

Proof. (i) Suppose that M is a closed subspace of $L(X,Y)$ which does not have the DPrcP (resp., the GP property). Let (T_n) be a weakly null DP (resp., weakly null limited) sequence in M such that $\|T_n\| \not\to 0$. By passing to a subsequence, suppose that for some $\epsilon > 0$, $\|T_n\| > \epsilon$, for each n. Let (x_n) be a sequence in B_X so that $\|T_n(x_n)\| > \epsilon$ for each n.

Let $y^* \in Y^*$. Since $\psi_{y^*} : M \to X^*$ is Dunford–Pettis completely continuous (resp., lcc), $\|\psi_{y^*}(T_n)\| = \|T_n^*(y^*)\| \to 0$. Then $\langle y^*, T_n(x_n)\rangle = \langle T_n^*(y^*), x_n\rangle \leq \|T_n^*(y^*)\| \to 0$. Therefore $(y_n) := (T_n(x_n))$ is weakly null in Y. By the Bessaga–Pelczynski selection principle (see [3]), we may (and do) assume that (y_n) is a seminormalized weakly null basic sequence in Y. Let $Y_0 = [y_n]$ be the closed linear span of (y_n) and let (y_n^*) be the sequence of coefficient functionals associated with (y_n). Define $A : Y_0 \to c_0$ by $A(y) = (y_k^*(y))$, $y \in Y_0$. Note that $\|A(y_n)\| \geq 1$ for each n. Then A is a bounded linear operator defined on a separable space, and A is not completely continuous.

(ii) Suppose that M a closed subspace of $L_{w^*}(X^*, Y)$ which does not have the DPrcP (resp., the GP property). Let (T_n) be a weakly null DP (resp., weakly null limited) sequence in M such that $\|T_n\| \not\to 0$. By passing to a subsequence, suppose that for some $\epsilon > 0$, $\|T_n\| > \epsilon$, for each n. Let (x_n^*) be a sequence in B_{X^*} so that $\|T_n(x_n^*)\| > \epsilon$ for each n.

Let $y^* \in Y^*$. Since $\psi_{y^*} : M \to X$ is Dunford–Pettis completely continuous (resp., lcc), $\|\psi_{y^*}(T_n)\| = \|T_n^*(y^*)\| \to 0$. Then $\langle y^*, T_n(x_n^*)\rangle = \langle T_n^*(y^*), x_n^*\rangle \leq \|T_n^*(y^*)\| \to 0$. Therefore $(y_n) := (T_n(x_n^*))$ is weakly null in Y. Continue as in (i).

(iii) Suppose that M does not have the Schur property. Let (T_n) be a weakly null sequence in M such that $\|T_n\| \not\to 0$. By passing to a subsequence, suppose that for some $\epsilon > 0$, $\|T_n\| > \epsilon$, for each n. Let (x_n^*) be a sequence in B_{X^*} so that $\|T_n(x_n^*)\| > \epsilon$ for each n.

Let $y^* \in Y^*$. Since $\psi_{y^*} : M \to X$ is completely continuous, $\|\psi_{y^*}(T_n)\| = \|T_n^*(y^*)\| \to 0$. Then $\langle y^*, T_n(x_n^*)\rangle \leq \|T_n^*(y^*)\| \to 0$. Therefore $(y_n) := (T_n(x_n^*))$ is weakly null in Y. Continue as in (i). □

Corollary 3.13.

(i) *Suppose that X has the DPrcP (resp., the GP property) and that M is a closed subspace of $L_{w^*}(X^*, Y)$. If M does not have the DPrcP (resp., the GP property), then there is a separable subspace Y_0 of Y and an operator $A : Y_0 \to c_0$ which is not completely continuous.*

(ii) *Suppose that X has the DPrcP and that Y has the Schur property. If M is a closed subspace of $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$, then M has the DPrcP.*

(iii) *Suppose that M is a closed subspace of $L_{w^*}(X^*, Y)$ such that the evaluation operator $\phi_{x^*} : M \to Y$ is Dunford–Pettis completely continuous (resp., lcc) for each $x^* \in X^*$. If M does not have the DPrcP (resp., the GP property), then there is a separable subspace X_0 of X and an operator $A : X_0 \to c_0$ which is not completely continuous.*

(iv) *Suppose that X has the Schur property and that Y has the DPrcP. If M is a closed subspace of $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$, then M has the DPrcP (see [7, Theorem 7]).*

Proof. (i) Since X has the DPrcP (resp., the GP property), $\psi_{y^*} : M \to X$ is Dunford–Pettis completely continuous (resp., lcc). Apply Theorem 3.12.

(ii) Suppose that X has the DPrcP and that Y has the Schur property. Let $T \in L_{w^*}(X^*, Y)$. Since T is weakly compact and Y has the Schur property, T is compact. Suppose that M does not have the DPrcP. By Theorem 3.12, there is a noncompletely continuous operator defined on a closed linear subspace Y_0 of Y. This is a contradiction since Y has the Schur property.

(iii) Suppose that M is a closed subspace of $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$ which satisfies the assumptions. Apply Theorem 3.12.

(iv) By (ii) and the isometries (1) on p. 4, M has the DPrcP. □

Corollary 3.13 (iii) extends [17, Theorem 3.9], which shows that if X has the Schur property and if M is a closed subspace of $L_{w^*}(X^*, Y)$ such that the evaluation operator $\phi_{x^*} : M \to Y$ is limited completely continuous for each $x^* \in X^*$, then M has the GP property.

In [8, Theorem 2], it was shown that if X^* has the GP property and Y has the Schur property, then $L(X,Y)$ has the GP property. In [17, Theorem 3.8], it was shown that if Y has the Schur property and M is a closed subspace of $L(X,Y)$ such that the evaluation operators $\psi_{y^*} : M \to X^*$ are limited completely continuous, then M has the GP property. Corollary 3.14(i) extends [17, Theorem 3.8] and [8, Theorem 2].

Corollary 3.14.

(i) *Suppose that X^* has the DPrcP (resp., the GP property), and suppose that M is a closed subspace of $L(X,Y)$. If M does not have the DPrcP (resp., the GP property), then there is a separable subspace Y_0 of Y and an operator $A : Y_0 \to c_0$ which is not completely continuous.*

(ii) *Suppose that X^* has the DPrcP, and suppose that Y has the Schur property. Then $L(X,Y) = K(X,Y)$ has the DPrcP.*

Proof. (i) Since X^* has the DPrcP (resp., the GP property), $\psi_{y^*} : M \to X^*$ is Dunford–Pettis completely continuous (resp., lcc) for each $y^* \in Y^*$. Apply Theorem 3.12.

(ii) Let $T : X \to Y$ be an operator. Since T is completely continuous, $T^*(B_{Y^*})$ is an L-subset of X^*, and thus relatively compact (see [7]). Hence T^*, thus T, is compact. Thus $L(X,Y) = K(X,Y)$. Then $L(X,Y) = K(X,Y)$ has the DPrcP by (i). □

Using Theorem 3.12(iii), we obtain the following result.

Corollary 3.15. *We have the folllowing.*

(i) *$L_{w^*}(X^*,Y)$ has the Schur property if and only if X and Y have the Schur property (see [16]).*

(ii) *$L(X,Y)$ has the Schur property if and only if X^* and Y have the Schur property (see [16]).*

The following result extends [5, Theorem 2.1], which states that if X and Y have the GP property, then $K_{w^*}(X^*,Y)$ has the GP property.

Theorem 3.16.

(i) *If Y has the GP property and M is a closed subspace of $K_{w^*}(X^*,Y)$ such that the evaluation operator $\psi_{y^*} : M \to X$ is limited completely continuous for each $y^* \in Y^*$, then M has the GP property.*

(ii) *If Y has the GP property and M is a closed subspace of $K(X,Y)$ such that the evaluation operator $\psi_{y^*} : M \to X^*$ is limited completely continuous for each $y^* \in Y^*$, then M has the GP property.*

Proof. (i) Let H be a limited subset of M. Let $y^* \in Y^*$. Since $\psi_{y^*} : M \to X$ is limited completely continuous, $H^*(y^*) = \psi_{y^*}(H)$ is relatively compact ([17, Theorem 2.1]).

Let (y_n^*) be a w^*-null sequence in Y^*. By Theorem 2.4, it is enough to show that $(T^*(y_n^*))$ converges to zero uniformly for $T \in H$. Let $T \in H$. Since $T \in K_{w^*}(X^*,Y)$, T^* is w^*-norm sequentially continuous. Thus $\|\psi_{y_n^*}(T)\| = \|T^*(y_n^*)\| \to 0$, and $(\psi_{y_n^*})$ is a pointwise norm null sequence of operators. Since

H is limited, $(\psi_{y_n^*})$ converges uniformly on H (see [18, Proposition 1.1.2, p. 23]); that is,

$$\sup\{\|\psi_{y_n^*}(T)\| : T \in H\} = \sup\{\|T^*(y_n^*)\| : T \in H\} \to 0.$$

By Theorem 2.4, H is relatively compact. Thus M has the GP property.

(ii) Apply (i) and the isometry $K(X,Y) \simeq K_{w^*}(X^{**},Y)$. □

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