Ann. Funct. Anal. 7 (2016), no. 2, 371–380
http://dx.doi.org/10.1215/20088752-3544893
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

RIEMANN SURFACES AND AF-ALGEBRAS

IGOR NIKOLAEV

In memory of Ola Bratteli

Communicated by M. de Jeu

Abstract. For a generic set in the Teichmüller space, we construct a covariant functor with the range in a category of the AF-algebras; the functor maps isomorphic Riemann surfaces to the stably isomorphic AF-algebras. In the special case of genus one, one gets a functor between the category of complex tori and the Effros–Shen algebras.

1. Introduction

The aim of our paper is to construct a functor from the set of generic Riemann surfaces to a category of operator algebras known as the *AF-algebras*; for the sake of clarity, consider the simplest example. We shall write $\{\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau \mid \Im(\tau) > 0\}$ to denote a lattice in the complex plane $\mathbb{C}$. Let $\mathbb{C}/\Lambda_\tau$ be a complex torus corresponding to Λ_τ, that is, the Riemann surface of genus $g = 1$. We shall write $\{\mathbb{A}_\theta \mid \theta \in \mathbb{R}\}$ to denote an AF-algebra defined by the inductive limit of positive isomorphisms:

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}} \dots, \tag{1.1}$$

where the regular continued fraction $[a_0, a_1, a_2, \dots]$ converges to θ; we refer the reader to Bratteli [3] for a definition of the AF-algebras and Effros and Shen [6] for the properties of algebra $\mathbb{A}_\theta$ (the *Effros–Shen algebra*). Recall that $\mathbb{C}/\Lambda_\tau$ and

Received Aug. 13, 2015; Accepted Nov. 3, 2015.
2010 *Mathematics Subject Classification.* Primary 46L85; Secondary 14H55.
Keywords. Riemann surfaces, AF-algebras.

$\mathbb{C}/\Lambda_{\tau'}$ are isomorphic complex tori if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ for some integers a, b, c, and d, such that $ad - bc = \pm 1$; here, an isomorphism means a conformal map between the Riemann surfaces $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda_{\tau'}$. It is a deep and amazing fact that the same is true of the Effros–Shen algebras. Namely, recall that the C^*-algebras $\mathbb{A}$ and $\mathbb{A}'$ are called *stably isomorphic* (Morita equivalent) if $\mathbb{A} \otimes \mathbb{K} \cong \mathbb{A}' \otimes \mathbb{K}$, where $\mathbb{K}$ is the C^*-algebra of compact operators on a Hilbert space H. It is known that the Effros–Shen algebras $\mathbb{A}_\theta$ and $\mathbb{A}_{\theta'}$ are stably isomorphic if and only if $\theta' = \frac{a\theta+b}{c\theta+d}$ for some integers a, b, c, and d, such that $ad - bc = \pm 1$ (see, e.g., [6, pp. 199–201]). (A relation between complex tori and continued fractions was already known to Klein [10].) One may wonder if there exists a functor from the category of complex tori (resp., Riemann surfaces) to the category of Effros–Shen algebras (resp., AF-algebras) such that isomorphisms between the Riemann surfaces generate stable isomorphisms between the corresponding AF-algebras.

In the present paper we construct a covariant functor F from a generic set of the Riemann surfaces of genus $g \geq 1$ to a category of the so-called *toric AF-algebras* (to be specified below); the functor maps isomorphic Riemann surfaces to the stably isomorphic toric AF-algebras (Theorem 1.1). To formulate our results, denote by $T(g)$ the Teichmüller space of genus $g \geq 1$, and let $S \in T(g)$ be a Riemann surface. Let $q \in H^0(S, \Omega^{\otimes 2})$ be a holomorphic quadratic differential on the Riemann surface S such that all zeros of q are simple (see [13]). By $\widetilde{S}$ we denote a double cover of S ramified over the zeros of q. Note that there is an involution on the homology groups $H_*(\widetilde{S})$ induced by the covering map $\widetilde{S} \to S$. Let $H_1^{\mathrm{odd}}(\widetilde{S})$ be the odd part of the first (integral) homology of $\widetilde{S}$ with respect to this involution relative to the zeros of q. By the formulas for the relative homology, one gets $H_1^{\mathrm{odd}}(\widetilde{S}) \cong \mathbb{Z}^n$, where $n = 6g - 6$ if $g \geq 2$ and $n = 2$ if $g = 1$. It is known that

$$\mathrm{Hom}\big(H_1^{\mathrm{odd}}(\widetilde{S}); \mathbb{R}\big) - \{0\} \cong T(g), \tag{1.2}$$

where 0 is the zero homomorphism (see [9]). Fix a basis in homology group $H_1^{\mathrm{odd}}(\widetilde{S})$ and, in view of (1.2), denote by $(\lambda_1, \ldots, \lambda_n)$ its image in $\mathbb{R}$ such that $\lambda_1 \neq 0$; let $\theta = (\theta_1, \ldots, \theta_{n-1})$ be a vector with the coordinates $\theta_i = \lambda_{i-1}/\lambda_1$. We shall consider the following Jacobi–Perron continued fraction:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \tag{1.3}$$

where $b_i = (b_1^{(i)}, \ldots, b_n^{(i)})^T$ is a vector of the nonnegative integers, I is the unit matrix, and $\mathbb{I} = (0, \ldots, 0, 1)^T$; we refer the reader to Bernstein [2] for the theory of such fractions. Finally, consider an AF-algebra defined by the following inductive limit of positive isomorphisms:

$$\mathbb{Z}^n \xrightarrow{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix}} \mathbb{Z}^n \xrightarrow{\begin{pmatrix} 0 & 1 \\ I & b_2 \end{pmatrix}} \mathbb{Z}^n \xrightarrow{\begin{pmatrix} 0 & 1 \\ I & b_3 \end{pmatrix}} \cdots \tag{1.4}$$

(see, e.g., [5]). We shall denote such an algebra by $\mathbb{A}_\theta$ and refer to $\mathbb{A}_\theta$ as a *toric AF-algebra*. Notice that if $g = 1$, then the Jacobi–Perron continued fraction coincides with a regular continued fraction; thus, for $g = 1$, the toric AF-algebra is isomorphic to an Effros–Shen algebra $\mathbb{A}_\theta$, and hence our notation.

Let $F : T(g) \to \{\text{toric } AF\text{-algebras}\}$ be a map acting by the formula $(\lambda_1, \dots, \lambda_n) \mapsto \mathbb{A}_\theta$, where $\theta = (\theta_1, \dots, \theta_n)$. Let V be the maximal subset of $T(g)$ such that every Riemann surface $S \in V$ corresponds to a convergent Jacobi–Perron fraction, and let $W = F(V)$. Our main result is as follows.

Theorem 1.1. *The set V is a generic subset of $T(g)$, and map F has the following properties: (i) $V \cong W \times (0, \infty)$ is a trivial fiber bundle, whose projection map $\pi : V \to W$ coincides with F; and (ii) F is a covariant functor which maps isomorphic Riemann surfaces $S, S' \in V$ to the stably isomorphic toric AF-algebras $\mathbb{A}_\theta, \mathbb{A}_{\theta'} \in W$.*

The article is organized as follows. Preliminary facts are reviewed in Section 2. Theorem 1.1 is proved in Section 3.

2. Preliminaries

2.1. **Measured foliations and $T(g)$.** A measured foliation, $\mathbb{F}$, on a surface X is a partition of X into the singular points $x_1, \dots, x_n$ of order $k_1, \dots, k_n$ and regular leaves (1-dimensional submanifolds). On each open cover U_i of $X - \{x_1, \dots, x_n\}$ there exists a nonvanishing real-valued closed 1-form ϕ_i such that (i) $\phi_i = \pm\phi_j$ on $U_i \cap U_j$, and (ii) at each x_i there exists a local chart $(u, v) : V \to \mathbb{R}^2$ such that, for $z = u + iv$, it holds that $\phi_i = \mathrm{Im}(z^{\frac{k_i}{2}} dz)$ on $V \cap U_i$ for some branch of $z^{\frac{k_i}{2}}$. The pair (U_i, ϕ_i) is called an *atlas for measured foliation* $\mathbb{F}$. Finally, a measure μ is assigned to each segment $(t_0, t) \in U_i$, which is transverse to the leaves of $\mathbb{F}$ via the integral $\mu(t_0, t) = \int_{t_0}^{t} \phi_i$. The measure is invariant along the leaves of $\mathbb{F}$; hence the name. We refer the reader to Thurston [14] and Fathi, Laudenbach, and Poénaru [8] for a systematic account of measured foliations.

Let S be a Riemann surface, and let $q \in H^0(S, \Omega^{\otimes 2})$ be a holomorphic quadratic differential on S. The lines $\mathrm{Re}\, q = 0$ and $\mathrm{Im}\, q = 0$ define a pair of measured foliations on R, which are transversal to each other outside the set of singular points. The set of singular points is common to both foliations and coincides with the zeros of q. The above measured foliations are said to represent the vertical and horizontal trajectory structure of q, respectively. Let $T(g)$ be the Teichmüller space of the topological surface X of genus $g \geq 1$, that is, the space of the complex structures on X. Consider the vector bundle $p : Q \to T(g)$ over $T(g)$, whose fiber above a point $S \in T(g)$ is the vector space $H^0(S, \Omega^{\otimes 2})$. Given nonzero $q \in Q$ above S, we can consider a horizontal measured foliation $\mathbb{F}_q \in \Phi_X$ of q, where Φ_X denotes the space of equivalence classes of measured foliations on X. If $\{0\}$ is the zero section of Q, the above construction defines a map $Q - \{0\} \longrightarrow \Phi_X$. For any $\mathbb{F} \in \Phi_X$, let $E_{\mathbb{F}} \subset Q - \{0\}$ be the fiber above $\mathbb{F}$. In other words, $E_{\mathbb{F}}$ is a subspace of the holomorphic quadratic forms whose horizontal trajectory structure coincides with the measured foliation $\mathbb{F}$. Note that if $\mathbb{F}$ is a measured foliation with the simple zeros (a generic case), then $E_{\mathbb{F}} \cong \mathbb{R}^n - 0$, while $T(g) \cong \mathbb{R}^n$, where $n = 6g - 6$ if $g \geq 2$ and $n = 2$ if $g = 1$.

Lemma 2.1 (Hubbard and Masur [9, Main Theorem])**.** *The restriction of p to $E_{\mathbb{F}}$ defines a homeomorphism (an embedding) $h_{\mathbb{F}} : E_{\mathbb{F}} \to T(g)$.*

The Hubbard–Masur result implies that the measured foliations parameterize the space $T(g) - \{pt\}$, where $pt = h_{\mathbb{F}}(0)$; indeed, denote by $\mathbb{F}'$ a vertical trajectory structure of q. Since $\mathbb{F}$ and $\mathbb{F}'$ define q, and $\mathbb{F} = \mathrm{Const}$ for all $q \in E_{\mathbb{F}}$, one gets a homeomorphism between $T(g) - \{pt\}$ and Φ_X, where $\Phi_X \cong \mathbb{R}^n - 0$ is the space of equivalence classes of the measured foliations $\mathbb{F}'$ on X. Note that the above parameterization depends on a foliation $\mathbb{F}$. However, there exists a unique canonical homeomorphism $h = h_{\mathbb{F}}$ as follows. Let $\mathrm{Sp}(S)$ be the length spectrum of the Riemann surface S, and let $\mathrm{Sp}(\mathbb{F}')$ be the set of positive reals $\inf \mu(\gamma_i)$, where γ_i runs over all simple closed curves, which are transverse to the foliation $\mathbb{F}'$. A canonical homeomorphism $h = h_{\mathbb{F}} : \Phi_X \to T(g) - \{pt\}$ is defined by the formula $\mathrm{Sp}(\mathbb{F}') = \mathrm{Sp}(h_{\mathbb{F}}(\mathbb{F}'))$ for $\forall \mathbb{F}' \in \Phi_X$. Thus, the following corollary is true.

Corollary 2.2. *There exists a unique homeomorphism $h : \Phi_X \to T(g) - \{pt\}$.*

Recall that Φ_X is the space of equivalence classes of measured foliations on the topological surface X. Following Douady and Hubbard [4], we consider a coordinate system on Φ_X suitable for the proof of Theorem 1.1. For clarity, let us make a generic assumption that $q \in H^0(S, \Omega^{\otimes 2})$ is a nontrivial holomorphic quadratic differential with only simple zeros. We wish to construct a Riemann surface of $\sqrt{q}$, which is a double cover of S with ramification over the zeros of q. Such a surface, denoted by $\widetilde{S}$, is unique and has an advantage of carrying a holomorphic differential ω such that $\omega^2 = q$. We further denote by $\pi : \widetilde{S} \to S$ the covering projection. The vector space $H^0(\widetilde{S}, \Omega)$ splits into the direct sum $H^0_{\mathrm{even}}(\widetilde{S}, \Omega) \oplus H^0_{\mathrm{odd}}(\widetilde{S}, \Omega)$ in view of the involution π^{-1} of $\widetilde{S}$, and the vector space $H^0(S, \Omega^{\otimes 2}) \cong H^0_{\mathrm{odd}}(\widetilde{S}, \Omega)$. Let $H_1^{\mathrm{odd}}(\widetilde{S})$ be the odd part of the homology of $\widetilde{S}$ relative to the zeros of q. Consider the pairing $H_1^{\mathrm{odd}}(\widetilde{S}) \times H^0(S, \Omega^{\otimes 2}) \to \mathbb{C}$ defined by the integration $(\gamma, q) \mapsto \int_\gamma \omega$. We shall take the associated map $\psi_q : H^0(S, \Omega^{\otimes 2}) \to \mathrm{Hom}(H_1^{\mathrm{odd}}(\widetilde{S}); \mathbb{C})$, and let $h_q = \mathrm{Re}\, \psi_q$.

Lemma 2.3 (Douady and Hubbard [4])**.** *The map*

$$h_q : H^0(S, \Omega^{\otimes 2}) \longrightarrow \mathrm{Hom}\big(H_1^{\mathrm{odd}}(\widetilde{S}); \mathbb{R}\big) \tag{2.1}$$

is an $\mathbb{R}$-isomorphism.

Since each $\mathbb{F} \in \Phi_X$ is the vertical foliation $\mathrm{Re}\, q = 0$ for a $q \in H^0(S, \Omega^{\otimes 2})$, the Douady–Hubbard lemma implies that $\Phi_X \cong \mathrm{Hom}(H_1^{\mathrm{odd}}(\widetilde{S}); \mathbb{R})$. By formulas for the relative homology, one finds that $H_1^{\mathrm{odd}}(\widetilde{S}) \cong \mathbb{Z}^n$, where $n = 6g - 6$ if $g \geq 2$ and $n = 2$ if $g = 1$. Finally, each $h \in \mathrm{Hom}(\mathbb{Z}^n; \mathbb{R})$ is given by the reals $\lambda_1 = h(e_1), \ldots, \lambda_n = h(e_n)$, where $(e_1, \ldots, e_n)$ is a basis in $\mathbb{Z}^n$. The numbers $(\lambda_1, \ldots, \lambda_n)$ are the coordinates in the space Φ_X and, in view of Corollary 2.2, in the Teichmüller space $T(g)$.

2.2. **The Jacobi–Perron continued fraction.** Let $a_1, a_2 \in \mathbb{N}$ such that $a_2 \leq a_1$. Recall that the greatest common divisor of a_1, a_2, $\mathrm{GCD}(a_1, a_2)$ can be determined from the Euclidean algorithm

$$\begin{cases} a_1 = a_2 b_1 + r_3, \\ a_2 = r_3 b_2 + r_4, \\ r_3 = r_4 b_3 + r_5, \\ \vdots \\ r_{k-3} = r_{k-2} b_{k-1} + r_{k-1}, \\ r_{k-2} = r_{k-1} b_k, \end{cases}$$

where $b_i \in \mathbb{N}$ and $\mathrm{GCD}(a_1, a_2) = r_{k-1}$. The Euclidean algorithm can be written as the regular continued fraction

$$\theta = \frac{a_1}{a_2} = b_1 + \cfrac{1}{b_2 + \cfrac{1}{+\cdots+\frac{1}{b_k}}} = [b_1, \dots, b_k]. \tag{2.2}$$

If a_1, a_2 are noncommensurable, in the sense that $\theta \in \mathbb{R} - \mathbb{Q}$, then the Euclidean algorithm never stops and $\theta = [b_1, b_2, \dots]$. Note that the regular continued fraction can be written in matrix form:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_k \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.3}$$

The Jacobi–Perron algorithm and the connected (multidimensional) continued fraction generalize the Euclidean algorithm to the case $\mathrm{GCD}(a_1, \dots, a_n)$ when $n \geq 2$. Namely, let $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{R} - \mathbb{Q}$, and $\theta_{i-1} = \frac{\lambda_i}{\lambda_1}$, where $1 \leq i \leq n$. The continued fraction

$$\begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & b_1^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1}^{(1)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & b_1^{(k)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1}^{(k)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

where $b_i^{(j)} \in \mathbb{N} \cup \{0\}$, is called the *Jacobi–Perron algorithm (JPA)*. Unlike the regular continued fraction algorithm, the JPA may diverge for certain vectors $\lambda \in \mathbb{R}^n$. However, for points of a generic subset of $\mathbb{R}^n$, the JPA converges. The convergence of the JPA algorithm can be characterized in terms of the measured foliations. Let $\mathbb{F} \in \Phi_X$ be a measured foliation on the surface X of genus $g \geq 1$. Recall that $\mathbb{F}$ is called *uniquely ergodic* if every invariant measure of $\mathbb{F}$ is a multiple of the Lebesgue measure. It is known that there exists a generic subset $V \subset \Phi_X$ such that each $\mathbb{F} \in V$ is uniquely ergodic (see [12], [15]). We let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the vector with coordinates $\lambda_i = \mu(\gamma_i)$, where $\gamma_i \in H_1^{\mathrm{odd}}(\widetilde{S})$; by an abuse of notation, we shall say that $\lambda \in V$. In view of a bijection between measured foliations and the interval exchange transformations (see [12]), the following characterization of convergence of the JPA is true.

Lemma 2.4 (Bauer [1, Theorem 4])**.** *The JPA converges if and only if* $\lambda \in V \subset \mathbb{R}^n$.

3. Proof of Theorem 1.1

Let us outline the proof. We shall consider the following sets of objects:

(i) generic Riemann surfaces V;
(ii) pseudolattices $\mathbb{P}L$ (see [11]);
(iii) projective pseudolattices $\mathbb{P}PL$;
(iv) toric AF-algebras W.

The proof takes the following steps:

(a) Show that $V \cong \mathbb{P}L$ are equivalent categories such that isomorphic Riemann surfaces $S, S' \in V$ map to isomorphic pseudolattices $PL, PL' \in \mathbb{P}L$.
(b) A noninjective functor $F : \mathbb{P}L \to \mathbb{P}PL$ is constructed. The F maps isomorphic pseudolattices to isomorphic projective pseudolattices and $\mathrm{Ker}\, F \cong (0, \infty)$.
(c) Show that a subcategory $U \subseteq \mathbb{P}PL$ and W are the equivalent categories.

In other words, we have the following diagram:

$$V \xrightarrow{\alpha} \mathbb{P}L \xrightarrow{F} U \xrightarrow{\beta} W, \tag{3.1}$$

where α is an injective map, β is a bijection, and $\mathrm{Ker}\, F \cong (0, \infty)$.

(i) Category V. A *Riemann surface* is a triple (X, S, j), where X is a topological surface of genus $g \geq 1$, $j : X \to S$ is a complex (conformal) parameterization of X, and S is a Riemann surface. A *morphism* of Riemann surfaces $(X, S, j) \to (X, S', j')$ is a biholomorphic map modulo the ones that are isotopic to the identity map with respect to a fixed topological marking of X. A category of generic Riemann surfaces V consists of $Ob(\mathbb{S})$, which are Riemann surfaces $S \in V \subset T(g)$ and morphisms $H(S, S')$ between $S, S' \in Ob(V)$ that coincide with the morphisms specified above. For any $S, S', S'' \in Ob(\mathbb{S})$ and any morphisms $\varphi' : S \to S'$, $\varphi'' : S' \to S''$, a morphism $\phi : S \to S''$ is the composite of φ' and φ'', which we write as $\phi = \varphi''\varphi'$. The identity morphism, 1_S, is a morphism $H(S, S)$.

(ii) Category $\mathbb{P}L$. A *pseudolattice* (of rank n) is a triple $(\Lambda, \mathbb{R}, j)$, where $\Lambda \cong \mathbb{Z}^n$ and $j : \Lambda \to \mathbb{R}$ is a homomorphism. A morphism of pseudolattices $(\Lambda, \mathbb{R}, j) \to (\Lambda, \mathbb{R}, j')$ is a commutative diagram:

$$\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{j} & \mathbb{R} \\
\Big\downarrow \varphi & & \Big\downarrow \psi \\
\mathbb{Z}^n & \xrightarrow{j'} & \mathbb{R}
\end{array}$$

where φ is a group homomorphism and ψ is an inclusion map; that is, $j'(\Lambda') \subseteq j(\Lambda)$. Any isomorphism class of a pseudolattice contains a representative given by $j : \mathbb{Z}^n \to \mathbb{R}$ such that

$$j(1, 0, \dots, 0) = \lambda_1, \qquad j(0, 1, \dots, 0) = \lambda_2, \qquad \dots, \qquad j(0, 0, \dots, 1) = \lambda_n,$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are positive reals. The pseudolattices of rank n make up a category, which we denote by $\mathbb{P}L_n$.

The following lemma says that the $\mathbb{Z}$-module $\mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ is an invariant of the isomorphism class of the Riemann surface S; in other words, the action of the mapping class group $\mathrm{Mod}(X)$ on such a module corresponds to a transformation of the basis of the module.

Lemma 3.1. *Let $g \geq 2$ (resp., $g = 1$) and $n = 6g-6$ (resp., $n = 2$). There exists an injective covariant functor $\alpha : V \to \mathbb{P}L_n$ which maps isomorphic Riemann surfaces $S, S' \in V$ to the isomorphic pseudolattices $PL, PL' \in \mathbb{P}L_n$.*

Proof. Let $\alpha : T(g) - \{pt\} \to \mathrm{Hom}(H_1^{\mathrm{odd}}(\widetilde{S}); \mathbb{R}) - 0$ be a Hubbard–Masur map. Since α is a homeomorphism between the respective spaces, we conclude that α is an injective map. The first claim of the lemma is proved.

Let us show that α sends morphisms of $\mathbb{S}$ to morphisms of $\mathbb{P}L$. Let $\varphi \in \mathrm{Mod}(X)$ be a diffeomorphism of X. Suppose that all the zeros of measured foliations are generic (simple), and let $p : \widetilde{X} \to X$ be the double cover of X. (Note that the case of torus does not require a double cover, and thus one can assert $p = \mathrm{Id}$ in the argument below.) Denote by $\widetilde{\varphi}$ a diffeomorphism of $\widetilde{X}$, which makes the following diagram commutative:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "p"] \\ X \arrow[r, "\varphi"] & X \end{tikzcd}$$

One can consider the effect of $\varphi, \widetilde{\varphi}$, and p on the respective (relative) integral homology groups:

$$\begin{tikzcd} H_1^{\mathrm{odd}}(\widetilde{X}) \oplus H_1^{\mathrm{even}}(\widetilde{X}) \arrow[r, "\widetilde{\varphi}_*"] \arrow[d, "p_*"] & H_1^{\mathrm{odd}}(\widetilde{X}) \oplus H_1^{\mathrm{even}}(\widetilde{X}) \arrow[d, "p_*"] \\ H_1(X, \mathrm{Sing}\,\mathbb{F}) \arrow[r, "\varphi_*"] & H_1(X, \mathrm{Sing}\,\mathbb{F}) \end{tikzcd}$$

where $\mathrm{Ker}\, p_* \cong H_1^{\mathrm{even}}(\widetilde{X})$. Since $p_* : H_1^{\mathrm{odd}}(\widetilde{X}) \to H_1(X, \mathrm{Sing}\,\mathbb{F})$ is an isomorphism, we conclude that $\widetilde{\varphi}_* \in \mathrm{GL}_n(\mathbb{Z})$, where $n = \dim H_1^{\mathrm{odd}}(\widetilde{X})$. It is easy to see that $\widetilde{\varphi}_*$ acts on a pseudolattice by a transformation of its basis, and therefore $\widetilde{\varphi}_* \in \mathrm{Mor}(\mathbb{P}L)$.

Let us show that α is a functor; indeed, let $S, S' \in V$ be isomorphic Riemann surfaces such that $S' = \varphi(S)$ for a $\varphi \in \mathrm{Mod}(X)$. Let a_{ij} be the elements of matrix

$\widetilde{\varphi}_* \in \mathrm{GL}_n(\mathbb{Z})$. Recall that

$$\lambda_i = \int_{\gamma_i} \phi \tag{3.2}$$

for a closed 1-form $\phi = \mathrm{Re}\,\omega$ and $\gamma_i \in H_1^{\mathrm{odd}}(\widetilde{X})$. Then

$$\gamma_j = \sum_{i=1}^n a_{ij}\gamma_i, \quad j = 1, \dots, n, \tag{3.3}$$

are the elements of a new basis in $H_1^{\mathrm{odd}}(\widetilde{X})$. By the integration rules,

$$\lambda'_j = \int_{\gamma_j} \phi = \int_{\sum a_{ij}\gamma_i} \phi = \sum_{i=1}^n a_{ij}\lambda_i. \tag{3.4}$$

Finally, let $j(\Lambda) = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ and $j'(\Lambda) = \mathbb{Z}\lambda'_1 + \cdots + \mathbb{Z}\lambda'_n$. Since $\lambda'_j = \sum_{i=1}^n a_{ij}\lambda_i$ and $(a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$, we conclude that

$$j(\Lambda) = j'(\Lambda). \tag{3.5}$$

In other words, the $\mathbb{Z}$-module $\mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ is an invariant of $\mathrm{Mod}(X)$. In particular, the pseudolattices $(\Lambda, \mathbb{R}, j)$ and $(\Lambda, \mathbb{R}, j')$ are isomorphic. Hence, $\alpha : V \to \mathbb{P}L$ maps isomorphic Riemann surfaces to the isomorphic pseudolattices; that is, α is a functor.

Finally, let us show that α is a covariant functor; indeed, let $\varphi_1, \varphi_2 \in \mathrm{Mor}(\mathbb{S})$. Then $\alpha(\varphi_1\varphi_2) = (\widetilde{\varphi_1\varphi_2})_* = (\widetilde{\varphi}_1)_*(\widetilde{\varphi}_2)_* = \alpha(\varphi_1)\alpha(\varphi_2)$. Lemma 3.1 follows. □

(iii) Category $\mathbb{P}PL$. A *projective pseudolattice* (of rank n) is a triple $(\Lambda, \mathbb{R}, j)$, where $\Lambda \cong \mathbb{Z}^n$ and $j : \Lambda \to \mathbb{R}$ is a homomorphism. A morphism of projective pseudolattices $(\Lambda, \mathbb{C}, j) \to (\Lambda, \mathbb{R}, j')$ is a commutative diagram:

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "j"] \arrow[d, "\varphi"] & \mathbb{R} \arrow[d, "\psi"] \\ \mathbb{Z}^n \arrow[r, "j'"] & \mathbb{R} \end{tikzcd}$$

where φ is a group homomorphism and ψ is an $\mathbb{R}$-linear map. (Notice that, unlike the case of pseudolattices, ψ is a scaling map as opposed to an inclusion map. This allows the two pseudolattices to be projectively equivalent while being distinct in the category $\mathbb{P}L_n$.) It is not hard to see that any isomorphism class of a projective pseudolattice contains a representative given by $j : \mathbb{Z}^n \to \mathbb{R}$ such that

$$j(1, 0, \dots, 0) = 1, \qquad j(0, 1, \dots, 0) = \theta_1, \qquad \dots, \qquad j(0, 0, \dots, 1) = \theta_{n-1},$$

where the θ_i's are positive reals. The projective pseudolattices of rank n make up a category which we denote by $\mathbb{P}PL_n$.

(iv) Category W. Let $\theta = (\theta_1, \dots, \theta_{n-1})$. Then toric AF-algebras $\mathbb{A}_\theta$ make a category; morphisms in the category are stable isomorphisms between toric AF-algebras. We shall denote such a category by W_n.

Lemma 3.2. *Let $U_n \subseteq \mathbb{P}PL_n$ be a subcategory consisting of the projective pseudolattices $PPL = PPL(1, \theta_1, \dots, \theta_{n-1})$ for which the Jacobi–Perron fraction of the vector $(1, \theta_1, \dots, \theta_{n-1})$ converges to the vector. Define a map $\beta : U_n \to W_n$ by the formula $PPL(1, \theta_1, \dots, \theta_{n-1}) \mapsto \mathbb{A}_\theta$. Then β is a bijective functor that maps isomorphic projective pseudolattices to the stably isomorphic toric AF-algebras.*

Proof. It is evident that β is injective and surjective. Let us show that β is a functor; indeed, every totally ordered abelian group of rank n has form $\mathbb{Z} + \theta_1\mathbb{Z} + \cdots + \mathbb{Z}\theta_{n-1}$ (see, e.g., [5, Corollary 4.7]). The latter is a projective pseudolattice PPL from the category U_n. On the other hand, each PPL defines a stable isomorphism class of the AF-algebra $\mathbb{A}_{\theta_1,\dots,\theta_{n-1}} \in W_n$ (see [7]). Therefore, β maps isomorphic projective pseudolattices (from the set U_n) to the stably isomorphic toric AF-algebras, and vice versa. Lemma 3.2 follows. $\square$

Let $PL(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{P}L_n$ and $PPL(1, \theta_1, \dots, \theta_{n-1}) \in \mathbb{P}PL_n$. To finish the proof of Theorem 1.1, it remains to show the following.

Lemma 3.3. *Let $F : \mathbb{P}L_n \to \mathbb{P}PL_n$ be a map given by formula*

$$PL(\lambda_1, \lambda_2, \dots, \lambda_n) \mapsto PPL\Big(1, \frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_n}{\lambda_1}\Big).$$

Then $\operatorname{Ker} F = (0, \infty)$ *and F is a functor which maps isomorphic pseudolattices to isomorphic projective pseudolattices.*

Proof. Indeed, F can be thought of as a map from $\mathbb{R}^n$ to $\mathbb{R}P^{n-1}$. Hence, $\operatorname{Ker} F = \{\lambda_1 : \lambda_1 > 0\} \cong (0, \infty)$. The second part of the lemma is evident. $\square$

Assuming $n = 6g - 6$ (resp., $n = 2$) for $g \geq 2$ (resp., $g = 1$), one gets items (i) and (ii) of the second part of Theorem 1.1 from Lemmas 3.1–3.3; the first part of Theorem 1.1 (i.e., that V is generic) follows from Lemma 2.4. $\square$

References

1. M. Bauer, *A characterization of uniquely ergodic interval exchange maps in terms of the Jacobi–Perron algorithm*, Bol. Soc. Bras. Mat. **27** (1996), no. 2, 109–128. Zbl 0877.11044. MR1418928. DOI 10.1007/BF01259355. 375
2. L. Bernstein, *The Jacobi-Perron Algorithm, Its Theory and Applications*, Lecture Notes in Math. **207**, Springer, Berlin, 1971. MR0285478. 372
3. O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234. Zbl 0264.46057. MR0312282. 371
4. A. Douady and J. Hubbard, *On the density of Strebel differentials*, Invent. Math. **30** (1975), no. 2, 175–179. Zbl 0371.30017. MR0396936. 374
5. E. G. Effros, *Dimensions and C^*-Algebras*, CBMS Reg. Conf. Ser. Math. **46**, Amer. Math. Soc., Providence, 1981. Zbl 0475.46050. MR0623762. 372, 379
6. E. G. Effros and C. L. Shen, *Approximately finite C^*-algebras and continued fractions*, Indiana Univ. Math. J. **29** (1980), no. 2, 191–204. Zbl 0457.46046. MR0563206. DOI 10.1512/iumj.1980.29.29013. 371, 372

7. G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44. Zbl 0323.46063. MR0397420. 379
8. A. Fathi, F. Laudenbach, and V. Poénaru, *Travaux de Thurston sur les surfaces*, Astérisque **66**–**67**, Soc. Math. France, Montrouge, 1979. MR0568308. 373
9. J. Hubbard and H. Masur, *Quadratic differentials and foliations*, Acta Math. **142** (1979), nos. 3–4, 221–274. Zbl 0415.30038. MR0523212. DOI 10.1007/BF02395062. 372, 374
10. F. Klein, *Ausgewälte Kapitel der Zahlentheorie*, Math. Ann. **48** (1896/97). 372
11. Y. I. Manin, "Real multiplication and noncommutative geometry (ein Alterstraum)" in *Legacy of Niels Hendrik Abel*, Springer, Berlin, 2004, 685–727. MR2077591. 376
12. H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) **115** (1982), no. 1, 169–200. Zbl 0497.28012. MR0644018. DOI 10.2307/1971341. 375
13. K. Strebel, *Quadratic Differentials*, Ergeb. Math. Grenzgeb. (3) **5**, Springer, Berlin, 1984. MR0743423. DOI 10.1007/978-3-662-02414-0. 372
14. W. P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431. Zbl 0674.57008. MR0956596. DOI 10.1090/S0273-0979-1988-15685-6. 373
15. W. A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2) **115** (1982), no. 1, 201–242. Zbl 0486.28014. MR0644019. DOI 10.2307/1971391. 375

The Fields Institute for Research in Mathematical Sciences, Toronto, ON, Canada.

E-mail address: igor.v.nikolaev@gmail.com