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EXTENSION OF THE BEST POLYNOMIAL APPROXIMATION OPERATOR IN VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. Let Ω be a bounded measurable set in $\mathbb{R}^n$. The best polynomial approximation operator was recently extended by Cuenya from L^p to L^{p-1}.

In this paper, we extend the operator of the best polynomial approximation from $L^{p(\cdot)}(\Omega)$ to the space $L^{p(\cdot)-1}(\Omega)$.

1. Introduction

Let Ω be a bounded measurable set in $\mathbb{R}^n$. Given a measurable function $p : \Omega \longrightarrow (0,+\infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that, for some positive $\lambda > 0$,

$$\int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}\,dx < +\infty.$$

If $1 \leq p(x) < \infty$, then this set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}\,dx \leq 1\Big\}.$$

Assume that

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

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Note that if $0 < p_- \le p_+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is the space (not necessary normed) of all measurable functions f such that $\int_\Omega |f(x)|^{p(x)}\,dx < \infty$.

The Lebesgue spaces with variable exponent and the corresponding variable Sobolev spaces are of interest for their applications to modeling problems in physics and in studying variational integrals and partial differential equations with nonstandard growth condition. Most of the problems in the development of the theory of $L^{p(\cdot)}$-spaces arise from the fact that these spaces are not translation-invariant (if $p \ne \text{const}$). The use of convolution is also limited; indeed, Young's inequality $\|f * g\|_{p(\cdot)} \le c\|f\|_{p(\cdot)} \cdot \|g\|_1$ holds if and only if $p(\cdot)$ is constant (see [2], [5]).

Variable exponent Lebesgue spaces on the real line have been developed by Sharapudinov [11]. These investigations originated in a paper by Tsenov [13]. The question raised by Tsenov and solved by Sharapudinov is the minimization of

$$\int_0^1 \bigl|f(x) - g(x)\bigr|^{p(x)}\,dx,$$

where f is a fixed function and g varies over a finite-dimensional subspace of $L^{p(\cdot)}([0,1])$.

Let Π^m be the space of all polynomials (algebraic or trigonometric), defined on $\mathbb{R}^n$ and of degree at most m. A polynomial $Q \in \Pi^m$ is called the *best approximation* of $f \in L^{p(\cdot)}(\Omega)$ $(1 < p_- \le p_+ < \infty)$ if and only if

$$\int_\Omega \bigl|f(x) - Q(x)\bigr|^{p(x)}\,dx = \inf_{S \in \Pi^m} \int_\Omega \bigl|f(x) - S(x)\bigr|^{p(x)}\,dx. \tag{1.1}$$

For $f \in L^{p(\cdot)}(\Omega)$ we set $E(f)$ as the set of all polynomials Q that satisfy (1.1).

It is well known (see [8], [12] in the case of $p(\cdot) = \text{const}$, $1 < p < \infty$, and [11] in the case of $L^{p(\cdot)}[0;1]$, $1 < p_- \le p_+ < \infty$) that $Q \in \Pi^m$ is the best approximation of $f \in L^{p(\cdot)}(\Omega)$ $(1 < p_- \le p_+ < \infty)$ if and only if

$$\int_\Omega p(x)\bigl|f(x) - Q(x)\bigr|^{p(x)-1}\operatorname{sign}\bigl(f(x) - Q(x)\bigr)S(x)\,dx = 0, \tag{1.2}$$

for every $S \in \Pi^m$. Such a polynomial Q always exists and it is unique. Let us define the operator $T(f) := Q$. We observe that if $1 < p_- \le p_+ < \infty$, then the left member of (1.2) is defined even if $f \in L^{p(\cdot)-1}(\Omega)$.

Cuenya in [3] proved that, for each $f \in L^{p-1}(\Omega), p > 1$, there exists a polynomial $Q \in \Pi^m$ satisfying (1.2). This polynomial will be called an *extended polynomial approximant*. Cuenya proved that the extended polynomial approximant is unique. Denote by $\overline{T}(f)$ the solution of (1.2) for $f \in L^{p-1}(\Omega)$ where $1 < p < \infty$. The operator $\overline{T} : L^{p-1} \to \Pi^m$ is continuous and, as a consequence, $\overline{T} : L^{p-1} \to \Pi^m$ is the unique extension of T preserving the property of continuity.

In the case of Π^0 (class of constant functions) the operator $\overline{T}$ for $1 \le p < \infty$ was studied in [10]. In that paper the authors extended the best constant approximation operator to $L^{p-1}(\Omega)$ if $p > 1$, and to the space of measurable functions that are finite almost everywhere if $p = 1$. Later, in [6] and [7] the authors considered the operator $\overline{T}$ defined in Orlicz spaces, and in [9] the operator $\overline{T}$ was studied in Orlicz–Lorentz spaces.

In the present paper we extend the operator of the best polynomial approximation from $L^{p(\cdot)}(\Omega)$ to the space $L^{p(\cdot)-1}(\Omega)$, $1 < p_- \leq p_+ < \infty$.

Throughout, we use C to stand for an absolute positive constant, which may have different values in different occurrences.

2. Existence of the best polynomial approximation operator in $L^{p(\cdot)}(\Omega)$

We begin with the existence of the best polynomial approximation operator of functions in $L^{p(\cdot)}(\Omega)$. Note that, for this space in the case of $\Omega = [0;1]$, the existence of the best polynomial approximation was shown by Sharapudinov [11]. An analogous result holds for $L^{p(\cdot)}(\Omega)$. The next two theorems follow standard techniques. However, for the sake of completeness, detailed proofs of them are included.

Theorem 2.1. *Let $f \in L^{p(\cdot)}(\Omega)$, $1 \leq p_- \leq p_+ < \infty$. Then there exists $Q \in \Pi^m$ such that*

$$\int_\Omega \bigl|f(x) - Q(x)\bigr|^{p(x)}\,dx = \inf_{S \in \Pi^m} \int_\Omega \bigl|f(x) - S(x)\bigr|^{p(x)}\,dx.$$

Proof. Indeed, let $I = \inf_{S\in\Pi^m} \int_\Omega |f(x) - S(x)|^{p(x)}\,dx$. Then there exists a sequence of polynomials $\{S_n | n \in \mathbb{N}\} \subset \Pi^n$ such that

$$\int_\Omega \bigl|f(x) - S_n(x)\bigr|^{p(x)}\,dx \to I, \quad n \to +\infty.$$

Since $|t|^{p(x)}$ is convex with respect to t for all fixed x, we have

$$\begin{aligned}\int_\Omega \bigl|S_n(x)/2\bigr|^{p(x)}\,dx &\leq \int_\Omega \bigl(\bigl|S_n(x)/2 - f(x)/2\bigr| + \bigl|f(x)/2\bigr|\bigr)^{p(x)}\,dx \\ &\leq \frac{1}{2}\int_\Omega \bigl|S_n(x) - f(x)\bigr|^{p(x)}\,dx + \frac{1}{2}\int_\Omega \bigl|f(x)\bigr|^{p(x)}\,dx.\end{aligned}$$

By the last estimation we conclude that $\|S_n\|_{p(\cdot)} \leq C_1(f)$, where $C_1(f)$ is constant depending on f. Since Π^m is a finite-dimensional space, all norms defined on it are equivalent and, consequently, we have $\|S_n\|_\infty \leq C_2(f)$, where

$$\|S_n\|_\infty = \sup_{x\in\Omega} \bigl|S_n(x)\bigr|.$$

Therefore, we can choose a subsequence $\{S_{n_k} | k \in \mathbb{N}\}$ which converges uniformly to $Q \in \Pi^n$.

By Lebesgue's dominated convergence theorem we have

$$I = \lim_{k\to+\infty} \int_\Omega \bigl|f(x) - S_{n_k}(x)\bigr|^{p(x)}\,dx = \int_\Omega \bigl|f(x) - Q(x)\bigr|^{p(x)}\,dx. \qquad \square$$

The next theorem gives a necessary and sufficient condition for Q to be the best polynomial approximation.

Theorem 2.2. *Let $f \in L^{p(\cdot)}(\Omega)$, $1 < p_- \leq p_+ < +\infty$. Then $Q \in \Pi^m$ is in $E(f)$ if and only if for every $S \in \Pi^m$ we have*

$$\int_\Omega p(x)\big|f(x) - Q(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q(x)\big)S(x)\,dx = 0. \tag{2.1}$$

Proof. At first we prove necessity. For $Q \in E(f)$ and $S \in \Pi^m$ we denote

$$F_S(t) := \int_\Omega \big|f(x) - Q(x) + tS(x)\big|^{p(x)}\,dx.$$

Let us prove that we can differentiate this function at the point 0. By using the well-known inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ and the mean value theorem, for all fixed x and for $|t| \leq 1$, we get

$$\begin{aligned}
&\Big|\frac{|f(x) - Q(x) + tS(x)|^{p(x)} - |f(x) - Q(x)|^{p(x)}}{t}\Big| \\
&\quad = \frac{|t| \cdot p(x) \cdot |f(x) - Q(x) + \xi S(x)|^{p(x)-1}|\operatorname{sign}(f(x) - Q(x) + \xi S(x))S(x)|}{|t|} \\
&\quad \leq C_0 \cdot \big|S(x)\big|\Big(\big|f(x) - Q(x)\big|^{p(x)-1} + \big|S(x)\big|^{p(x)-1}\Big).
\end{aligned}$$

Since $|S(x)|(|f(x) - Q(x)|^{p(x)-1} + |S(x)|^{p(x)-1})$ is an integrable function, we are allowed to differentiate inside the integral. Therefore,

$$F_S'(0) = \int_\Omega p(x)\big|f(x) - Q(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q(x)\big)S(x)\,dx.$$

Assuming that $F_S(0) = \min_{t\in\mathbb{R}} F_S(t)$, we have $F_S'(0) = 0$; this proves the necessity of condition (2.1).

Let us now prove sufficiency. Note that $F_S(t)$ on $\mathbb{R}$ is a convex function with respect to t. Indeed, for $a, b \geq 0$ such that $a + b = 1$, using convexity of the $|t|^{p(x)}$ for all fixed x and monotonicity of the integral, we have

$$\begin{aligned}
&F_S(at_1 + bt_2) \\
&\quad = \int_\Omega \big|f(x) - Q(x) + (at_1 + bt_2)S(x)\big|^{p(x)}\,dx \\
&\quad = \int_\Omega \big|(a+b)\big(f(x) - Q(x)\big) + at_1S(x) + bt_2S(x)\big|^{p(x)}\,dx \\
&\quad = \int_\Omega \big|a\big(f(x) - Q(x) + t_1S(x)\big) + b\big(f(x) - Q(x) + t_2S(x)\big)\big|^{p(x)}\,dx \\
&\quad \leq \int_\Omega \Big(a\big|f(x) - Q(x) + t_1S(x)\big|^{p(x)}\,dx + b\big|f(x) - Q(x) + t_2S(x)\big|^{p(x)}\Big)\,dx \\
&\quad = a\int_\Omega \big|f(x) - Q(x) + t_1S(x)\big|^{p(x)}\,dx + b\int_\Omega \big|f(x) - Q(x) + t_2S(x)\big|^{p(x)}\,dx \\
&\quad = aF_S(t_1) + bF_S(t_2).
\end{aligned}$$

Consequently, by condition (2.1) we conclude that $F'_S(0) = 0$. By combining the two facts that F_S is convex and $F'_S(0) = 0$, we conclude that

$$F_S(0) = \min_{t\in[0,\infty)} F_S(t).$$

This means that Q is the best polynomial approximation of f. The sufficiency of condition (2.1) is proved. □

The following theorem connects a modular of a function and a modular of the best polynomial approximation. In the case of $p(\cdot) = \text{const}$, the theorem was proved in [4]. Note that this estimation does not depend on the behavior of the exponent $p(\cdot)$.

Theorem 2.3. *Let $f \in L^{p(\cdot)}(\Omega)$, $1 < p_- \le p_+ < \infty$, and $Q \in E(f)$. Then*

$$\int_\Omega \big|Q(x)\big|^{p(x)-1}\big|S(x)\big|\,dx \le C\int_\Omega \big|f(x)\big|^{p(x)-1}\big|S(x)\big|\,dx, \tag{2.2}$$

for all $S \in \Pi^m$ such that S (or $-S$) and Q have the same sign for all $t \in \Omega$ where $Q(t)S(t) \ne 0$.

Proof. Suppose that $S \in \Pi^m$ and $S(x)Q(x) > 0$ when $S(x)Q(x) \ne 0$.

Let $A = \{x \in \Omega | f(x) > Q(x)\}$, $B = \Omega\backslash A$, and $H(x) = |f(x) - Q(x)|^{p(x)-1}S(x)$. Using (2.1), we obtain

$$\int_A p(x)H(x)\,dx = \int_B p(x)H(x)\,dx.$$

Let $A_1 = A \cap \{x \in \Omega|Q(x) \ge 0\}$, $A_2 = A\backslash A_1$, $B_1 = B \cap \{x \in \Omega|Q(x) \ge 0\}$, and $B_2 = B\backslash B_1$.

By the above equality we have

$$\begin{gathered}
\int_{A_1\cup A_2} p(x)H(x)\,dx = \int_{B_1\cup B_2} p(x)H(x)\,dx,\\
\int_{A_1} p(x)H(x)\,dx + \int_{A_2} p(x)H(x)\,dx = \int_{B_1} p(x)H(x) + \int_{B_2} p(x)H(x)\,dx, \qquad (2.3)\\
\int_{A_1} p(x)H(x)\,dx - \int_{B_2} p(x)H(x)\,dx = \int_{B_1} p(x)H(x)\,dx - \int_{A_2} p(x)H(x)\,dx.
\end{gathered}$$

Consider

$$\int_\Omega p(x)\big|Q(x)\big|^{p(x)-1}\big|S(x)\big|\,dx = \int_\Omega p(x)\big|Q(x) - f(x) + f(x)\big|^{p(x)-1}\big|S(x)\big|\,dx. \tag{2.4}$$

By using the well-known inequalities $(a+b)^{p-1} \le 2^{p-2}(a^{p-1} + b^{p-1})$, when $p - 1 \ge 1$, and $(a+b)^{p-1} \le a^{p-1} + b^{p-1}$, when $0 < p - 1 < 1$, and taking into account that $1 < p_- \le p_+ < +\infty$, we conclude, for all fixed x, that

$$\begin{aligned}
&\int_\Omega p(x)\big|Q(x) - f(x) + f(x)\big|^{p(x)-1}\big|S(x)\big|\,dx\\
&\quad\le C\Big(\int_\Omega \big|Q(x) - f(x)\big|^{p(x)-1}\big|S(x)\big|\,dx + \int_\Omega \big|f(x)\big|^{p(x)-1}\big|S(x)\big|\,dx\Big)
\end{aligned}$$

$$= C\Big(\sum_{i=1}^{2}\int_{A_i}\big|H(x)\big|\,dx + \sum_{i=1}^{2}\int_{B_i}\big|H(x)\big|\,dx + \int_{\Omega}\big|f(x)\big|^{p(x)-1}\big|S(x)\big|\,dx\Big). \quad (2.5)$$

Note that, for all $x \in A_1 \cup B_2$, we have $|Q - f| \leq |f|$. Then we obtain

$$\int_{A_1\cup B_2}\big|H(x)\big|\,dx \leq \int_{A_1\cup B_2}\big|f(x)\big|^{p(x)-1}\big|S(x)\big|\,dx. \quad (2.6)$$

Since $S(x)\cdot Q(x) \geq 0$, taking into account that for all $x \in A_2$, $Q(x) < 0$, and considering (2.4) and (2.6), we obtain

$$\begin{aligned}\int_{A_2}\big|H(x)\big|\,dx + \int_{B_1}\big|H(x)\big|\,dx &= \int_{A_2}\big(-H(x)\big)\,dx + \int_{B_1} H(x)\,dx \\ &= \int_{A_1} H(x)\,dx - \int_{B_2} H(x)\,dx \\ &= \int_{A_1}\big|H(x)\big|\,dx + \int_{B_2}\big|H(x)\big|\,dx \\ &\leq \int_{A_1\cup B_2}\big|f(x)\big|^{p(x)-1}\big|S(x)\big|\,dx.\end{aligned}$$

By (2.4), (2.5), (2.6), and the last estimation we get (2.2). We can obtain an estimation for all x, for which $Q(x)S(x) \leq 0$, in an analogous way. □

Corollary 2.4. *Let $f \in L^{p(\cdot)}(\Omega), 1 < p_- \leq p_+ < \infty$. If $Q \in E(f)$, then*

$$\int_{\Omega}\big|Q(x)\big|^{p(x)}\,dx \leq C\|Q\|_{\infty}\int_{\Omega}\big|f(x)\big|^{p(x)-1}\,dx.$$

Proof. If we take $Q = S$ in Theorem 2.3, then we obtain the desired result. □

3. Extension and uniqueness of the best polynomial approximation operator to $L^{p(\cdot)-1}(\Omega)$

Definition 3.1. Let $1 < p_- \leq p_+ < \infty$ and $f \in L^{p(\cdot)-1}(\Omega)$. We say that $Q \in \Pi^m$ is the *best polynomial approximant* of f if (1.2) holds.

In this section we will discuss the existence of the extended polynomial approximant in $L^{p(\cdot)-1}(\Omega)$ when $1 < p_- \leq p_+ < \infty$.

Theorem 3.2. *Let $f \in L^{p(\cdot)-1}(\Omega)$, $1 < p_- \leq p_+ < \infty$. Then there exists $Q \in \Pi^m$ such that for all $S \in \Pi^m$ the following holds:*

$$\int_{\Omega} p(x)\big|f(x) - Q(x)\big|^{p(x)-1}\operatorname{sign}\big(f(x) - Q(x)\big)S(x)\,dx = 0, \quad (3.1)$$

$$\int_{\Omega}\big|Q(x)\big|^{p(x)}\,dx \leq C\|Q\|_{\infty}\int_{\Omega}\big|f(x)\big|^{p(x)-1}\,dx. \quad (3.2)$$

While we prove Theorem 3.2, let us prove auxiliary lemmas.

Lemma 3.3. *Let* $\{f_n\}_{n=1}^{+\infty}$ *be a sequence of elements from* $L^{p(\cdot)}(\Omega)$, $1 < p_- \le p_+ < \infty$, *for which there exists a constant* $C_0 > 0$ *such that, for all* $n \in \mathbb{N}$,

$$\int_\Omega \bigl|f_n(x)\bigr|^{p(x)-1}\,dx \le C_0.$$

Then the set $\{\|Q\|_\infty : Q \in E(f_n), n \in \mathbb{N}\}$ *is bounded.*

Proof. By Corollary 2.4 we have

$$\int_\Omega \bigl|Q(x)\bigr|^{p(x)}\,dx \le C\|Q\|_\infty \int_\Omega \bigl|f_n(x)\bigr|^{p(x)-1}\,dx \le C \cdot C_0 \|Q\|_\infty. \tag{3.3}$$

On the other hand, since Π^m is a finite-dimensional space, norms $\|\cdot\|_\infty$ and $\|\cdot\|_{p(\cdot)}$ on Π^m are equivalent. Then from (3.3) we obtain

$$\int_\Omega \bigl|Q(x)\bigr|^{p(x)}\,dx \le C_1 \|Q\|_{p(\cdot)}.$$

Without restriction of generality suppose that $C_1\|Q\|_{p(\cdot)} \ge 1$. Thus, we have

$$1 \ge \int_\Omega \biggl|\frac{Q(x)}{(C_1\|Q\|_{p(\cdot)})^{1/p(x)}}\biggr|^{p(x)}\,dx \ge \int_\Omega \biggl|\frac{Q(x)}{(C_1\|Q\|_{p(\cdot)})^{1/p_-}}\biggr|^{p(x)}\,dx.$$

Consequently, by definition of the norm in the space $L^{p(\cdot)}(\Omega)$, we get

$$\|Q\|_{p(\cdot)} \le \bigl(C_1\|Q\|_{p(\cdot)}\bigr)^{1/p_-}.$$

This means that the set of $\|Q\|_{p(\cdot)}$ numbers is bounded. If we once more use the equivalency of $\|\cdot\|_\infty$ and $\|\cdot\|_{p(\cdot)}$ norms, then the proof is completed. □

Lemma 3.4. *Let* f_n, f *be functions in* $L^{p(\cdot)-1}(\Omega)$ *such that*

$$\int_\Omega \bigl|f_n(x) - f(x)\bigr|^{p(x)-1}\,dx \to 0, \tag{3.4}$$

and also let g_n, g *be measurable functions such that* $|g_n| \le C_0$ *for all* n *and* $g_n \to g$ *as* $n \to \infty$. *Then there exists a subsequence* n_k *such that*

$$\int_\Omega \bigl|f_{n_k}(x)\bigr|^{p(x)-1} g_{n_k}\,dx \to \int_\Omega \bigl|f(x)\bigr|^{p(x)-1} g(x)\,dx, \quad \text{as } k \to \infty.$$

Proof. For any measurable set $E \subset \Omega$ we have

$$\begin{aligned}\int_E \bigl|f_n(x)\bigr|^{p(x)-1}\,dx &= \int_E \bigl|f_n(x) - f(x) + f(x)\bigr|^{p(x)-1}\,dx \\ &\le C\Bigl(\int_E \bigl|f_n(x) - f(x)\bigr|^{p(x)-1}\,dx + \int_E \bigl|f(x)\bigr|^{p(x)-1}\,dx\Bigr).\end{aligned}$$

By (3.4) and the last estimation for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $E \subset \Omega$, $|E| < \delta$ and for any $n \in \mathbb{N}$, we have

$$\int_E \bigl|f_n(x)\bigr|^{p(x)-1}\,dx < \varepsilon.$$

Since $|f_n(x) - f(x)|^{p(x)-1}$ converges by $L^1(\Omega)$-norm to 0, then there exists a subsequence f_{n_k} which converges to f almost everywhere. Now, by Egorov's theorem, for $\delta > 0$ there exists $E \subset \Omega$, $|E| < \delta$, such that the sequence $\{|f_{n_k}(x)|^{p(x)-1} g_{n_k}(x)\}$ uniformly converges to $|f(x)|^{p(x)-1} g(x)$ on $\Omega \backslash E$.

We have

$$\begin{aligned}\int_\Omega \big|f_{n_k}(x)\big|^{p(x)-1} g_{n_k}(x)\,dx - \int_\Omega \big|f(x)\big|^{p(x)-1} g(x)\,dx \\ = \int_{\Omega\backslash E} \Big(\big|f_{n_k}(x)\big|^{p(x)-1} g_{n_k}(x) - \big|f(x)\big|^{p(x)-1} g(x)\Big)\,dx \\ + \int_E \Big(\big|f_{n_k}(x)\big|^{p(x)-1} g_{n_k}(x)\,dx - \big|f(x)\big|^{p(x)-1} g(x)\Big)\,dx.\end{aligned}$$

From the last equality, the proof of the lemma easily follows. ☐

Proof of Theorem 3.2. Supposing that $f \in L^{p(\cdot)-1}(\Omega)$, and considering the sequence

$$f_n = \min\big(\max(f, -n), n\big),$$

it is easy to see that $f_n \in L^{p(x)}(\Omega)$ for all $n \in \mathbb{N}$. Then by Theorems 2.1 and 2.2 there exists $Q_n \in \Pi^m$ such that we have

$$\int_\Omega p(x)\big|f_n(x) - Q_n(x)\big|^{p(x)-1} \operatorname{sign}\big(f_n(x) - Q_n(x)\big) S(x)\,dx = 0,$$

for all $n \in \mathbb{N}$ and $S \in \Pi^m$. Also, by Corollary 2.4 we have

$$\int_\Omega \big|Q_n(x)\big|^{p(x)}\,dx \le C\|Q_n\|_\infty \int_\Omega \big|f_n(x)\big|^{p(x)-1}\,dx.$$

Observe that

$$\lim_{n\to+\infty} \int_\Omega \big|f_n(x) - f(x)\big|^{p(x)-1}\,dx = 0.$$

Hence, it follows that there exists a positive number $C_0 > 0$ such that

$$\int_\Omega \big|f_n(x)\big|^{p(x)-1}\,dx \le C_0.$$

Then by Lemma 3.3 we obtain uniform boundedness of the sequence $\|Q_n\|_\infty$. Therefore, there exists a subsequence Q_{n_k} which uniformly converges on Ω to a polynomial $Q \in \Pi^m$. By using Lemma 3.4 and simple limiting arguments we obtain (3.1) and (3.2). ☐

Theorem 3.5. *For every $f \in L^{p(\cdot)-1}(\Omega), 1 < p_- \le p_+ < \infty$, there exists a unique extended best polynomial approximant.*

Proof. Let $f \in L^{p(\cdot)-1}(\Omega)$, let Q_1, Q_2 be extended polynomial approximants, and let $Q_1 \ne Q_2$. By (3.1), for all $S \in \Pi^m$, we have

$$\begin{aligned}\int_\Omega p(x)\big|f(x) - Q_1(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_1(x)\big) S(x)\,dx \\ = \int_\Omega p(x)\big|f(x) - Q_2(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_2(x)\big) S(x)\,dx = 0.\end{aligned}$$

Consider the polynomial $Q = Q_1 - Q_2$ and the sets

$$D = \{x \in \Omega | Q_1(x) > Q_2(x)\},$$
$$F = \{x \in \Omega | Q_1(x) < Q_2(x)\},$$
$$G = \{x \in \Omega | Q_1(x) = Q_2(x)\}.$$

On the set D we have $Q(x) > 0$ and $f(x) - Q_1(x) < f(x) - Q_2(x)$ and, thus, ($|z|^{p(x)-1} \operatorname{sign}(z)$ is monotone)

$$\begin{aligned}&\big|f(x) - Q_1(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_1(x)\big)Q(x)\\ &\quad < \big|f(x) - Q_2(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_2(x)\big)Q(x).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_D p(x)\big|f(x) - Q_1(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_1(x)\big)Q(x)\,dx\\ &\quad < \int_D p(x)\big|f(x) - Q_2(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_2(x)\big)Q(x)\,dx. \qquad (3.5)\end{aligned}$$

Analogously on the set F, we have $Q(x) < 0$ and $f(x) - Q_2(x) < f(x) - Q_1(x)$. Then

$$\begin{aligned}&\big|f(x) - Q_1(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_1(x)\big)Q(x)\\ &\quad < \big|f(x) - Q_2(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_2(x)\big)Q(x)\end{aligned}$$

and

$$\begin{aligned}&\int_F p(x)\big|f(x) - Q_1(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_1(x)\big)Q(x)\,dx\\ &\quad < \int_F p(x)\big|f(x) - Q_2(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_2(x)\big)Q(x)\,dx. \qquad (3.6)\end{aligned}$$

Note that $|G| = 0$; then by (3.5) and (3.6) we get

$$\begin{aligned}0 &= \int_\Omega p(x)\big|f(x) - Q_1(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_1(x)\big)Q(x)\\ &< \int_\Omega p(x)\big|f(x) - Q_2(x)\big|^{p(x)-1} \operatorname{sign}\big(f(x) - Q_2(x)\big)Q(x)\,dx = 0,\end{aligned}$$

which is a contradiction. □

Note that the extended polynomial approximant operator $\overline{T} : L^{p(\cdot)-1} \to \Pi^m$ is nonlinear. Next, we will show that this operator is continuous.

Theorem 3.6. *Let $h_n, h \in L^{p(\cdot)-1}(\Omega)$, $1 < p_- \le p_+ < \infty$, such that*

$$\int_\Omega \big|h_n(x) - h(x)\big|^{p(x)-1}\,dx \to 0, \quad n \to \infty;$$

then $\overline{T}(h_n) \to \overline{T}(h)$, $n \to \infty$.

Proof. Let us consider a $\{Q_n\}$-sequence of polynomials which are the extended approximants $Q_n = \overline{T}(h_n)$ for each h_n.

Analogously to the proof of Lemma 3.3, we can conclude that the sequence Q_n is uniformly bounded. Therefore, we can choose a subsequence Q_{n_k} which converges to a polynomial Q. Also, we can select a subsequence of h_{n_k}, which we will denote again by h_{n_k}, that converges to h almost everywhere. For $Q_{n_k} = \overline{T}(h_{n_k})$ and any $S \in \Pi^m$, by Theorem 3.2 we have

$$\int_\Omega p(x)\big|h_{n_k}(x) - Q_{n_k}(x)\big|^{p(x)-1} \operatorname{sign}\big(h_{n_k}(x) - Q_{n_k}(x)\big)S(x)\,dx = 0.$$

By using Lemma 3.4 we obtain

$$\int_\Omega p(x)\big|h(x) - Q(x)\big|^{p(x)-1} \operatorname{sign}\big(h(x) - Q(x)\big)S(x)\,dx = 0,$$

and taking into account Theorem 3.5, $Q = \overline{T}(h)$. According to the discussion, the limit (by norm of $C(\Omega)$) of any convergent subsequence of Q_n is Q. Therefore, we obtain the proof the theorem. □

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