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THE FPP FOR LEFT REVERSIBLE SEMIGROUPS IN ℓ_1 ENDOWED WITH DIFFERENT LOCALLY CONVEX TOPOLOGIES

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Abstract. In this article we will consider locally convex topologies τ on ℓ_1 which are coarser than the weak topology on the unit ball and such that the unit vector basic sequence (e_n) is τ-convergent. We characterize these topologies depending on the τ-fixed point property for left reversible semigroups on $(\ell_1, \|\cdot\|_1)$. We will apply our results to the case of different weak* topologies on ℓ_1.

1. Introduction and preliminaries

Let S be a semigroup. It is said that S is a *semitopological semigroup* if S is equipped with a Hausdorff topology such that, for each $a \in S$, the two mappings from S into S defined by $s \to as$ and $s \to sa$ are continuous. A semitopological semigroup S is said to be *left reversible* if any two nonempty closed right ideals of S have a nonempty intersection. In other words, a semitopological semigroup is left reversible if

$$\overline{aS} \cap \overline{bS} \neq \emptyset$$

for every $a, b \in S$. Clearly, every abelian semitopological semigroup and every semitopological group are left reversible. Also, left amenable and, in particular, amenable semitopological semigroups are left reversible.

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Let C be a subset of a Banach space $(X, \|\cdot\|)$, and let S be a semitopological semigroup. A nonexpansive action of S on the set C is a map $\phi : S \times C \to C$, denoted by $\phi(s, u) = s(u)$ (or su), which satisfies the following:

(i) $ts(u) = t(su)$ for all $t, s \in S$ and $u \in C$;
(ii) for all $u_0 \in C$, the function $s \in S \to \phi(s, u_0) \in C$ is continuous;
(iii) for every $s \in S$, the mapping $u \in C \to s(u) \in C$ is nonexpansive.

A subset C is said to *verify the fixed point property* for left reversible semigroups if, for every left reversible semitopological semigroup S and for every nonexpansive action $\phi : S \times C \to C$, the set $\mathrm{Fix}(S) := \{u \in C : t(u) = u, \forall t \in S\}$ is nonempty.

Definition 1.1. Let X be a Banach space, and let τ be a topology on X. It is said that X has the τ-fixed point property (τ-FPP) for left reversible semigroups if every closed, convex, and bounded subset C which is τ-compact and τ-sequentially compact has the fixed point property for left reversible semigroups.

Notice that if X is a separable Banach space and τ is a topology weaker than the norm topology, then every τ-sequentially compact subset is τ-compact. For metrizable topologies and for the weak topologies, both compactness concepts coincide.

Given a nonexpansive mapping T, if we replace the left reversible semigroup by the discrete and abelian semigroup $\{T, T^2, T^3, \dots\}$ acting from C to C, Definition 1.1 becomes the usual definition of the τ-FPP for nonexpansive mappings.

In 1965, W. Kirk proved that every Banach space with weak normal structure satisfies the w-FPP for unique nonexpansive mappings. In a similar way it can be proved that weak* normal structure implies the weak*-FPP in dual Banach spaces.

In the 1970s, Kirk's result was generalized by T.-C. Lim [8] and by R. Holmes and A. Lau [3] in the setting of nonexpansive actions for left reversible semigroups; that is, a weak normal structure implies the w-FPP for left reversible semigroups.

In the case of dual Banach spaces, although particular examples of dual Banach spaces are known to verify the w^*-FPP for left reversible semigroups (see, e.g., [2], [5], [7], [9], [10]), it is still an open problem whether every dual Banach space with a weak* normal structure satisfies the weak*-fixed point property for left reversible semigroups (see [5, Open Problem 6.3]).

In 1980, T.-C. Lim [7] proved that the sequence space ℓ_1 satisfies the weak*-FPP for left reversible semigroups, where the weak* topology is the $\sigma(\ell_1, c_0)$-topology; that is, we are considering c_0 as the predual of ℓ_1. However, there exist uncountably many nonisometric (and nonisomororphic) Banach spaces whose duals can be isometrically identified with ℓ_1. Another well-known predual of ℓ_1 is c, the Banach space of all real convergent sequences. It is also known that ℓ_1 fails to have the $\sigma(\ell_1, c)$-FPP for nonexpansive mappings (see [7]), so ℓ_1 automatically fails the $\sigma(\ell_1, c)$-FPP for left reversible semigroups. Notice that when we consider the $\sigma(\ell_1, c_0)$-topology, the basic sequence (e_n) is w^*-convergent to the null vector, while if we consider the $\sigma(\ell_1, c)$-topology, the sequence (e_n) is w^*-convergent to the vector $(1, 0, 0, \dots)$.

In this article, we will consider locally convex topologies τ on ℓ_1 that are coarser than the weak topology on the unit ball and such that the basic sequence (e_n) is convergent; that is, there exists some $e \in \ell_1$ such that $\tau - \lim_n e_n = e$. For these topologies we will study the τ-FPP for left reversible semigroups. In fact, we will characterize the topologies for which the τ-FPP for left reversible semigroups holds according to the vector e. The results are an extension of those obtained in [4] (see Theorem 8) for the case of a unique nonexpansive mapping.

Furthermore, in the case where weak* topologies are considered (i.e., $\tau = \sigma(\ell_1, X)$ when $X^* = \ell_1$), we check that X is isometric to $W_f = \{x \in c : \langle f, x\rangle = 0\}$ for some $f \in \ell_1$ with $\|f\|_1 = 1$. In this case, we characterize the $\sigma(\ell_1, X)$-FPP for left reversible semigroups according to the coordinates of the vector f.

2. Main results

We need to introduce the following notation.

Let X be a Banach space, and let $\{B_s\}_{s\in A}$ be a decreasing net of bounded subsets of X. For $x \in X$ and $s \in A$ we consider

$$r_s(x) := \sup\big\{\|x - y\| : y \in B_s\big\},$$
$$r(x) := \inf\big\{r_s(x) : s \in A\big\} = \lim_s r_s(x).$$

We define the asymptotic radius and the asymptotic center of a set C with respect to the family $\{B_s\}_{s\in A}$ as

$$r_0 := \inf\big\{r(x) : x \in C\big\},$$
$$\mathcal{AC}\big(\{B_s\}_{s\in A}, C\big) := \big\{x \in C : r(x) = r_0\big\}.$$

The following technical lemma was proved in [2]. We include the proof for the sake of completeness.

Lemma 2.1. *Let C be a convex bounded subset of X. Let $\{B_s\}_{s\in A}$ be a decreasing net of subsets of C such that $\mathcal{AC}(\{B_s\}_{s\in A}, C) = C$. If C is (norm) separable, then there exists $\{x_n\} \subset C$ such that*

$$\lim_n \|x_n - x\| = r_0$$

for every $x \in C$, where r_0 denotes the asymptotic radius of C with respect to the net $\{B_s\}_s$.

Proof. Let $\{y_n\}$ be a dense sequence in C, and define $\overline{y_n} = \sum_{i=1}^n \frac{y_i}{n}$.

Since $y_1 \in C = \mathcal{AC}(\{B_s\}_s, C)$, we can find $s_1 \in S$ such that

$$r_0 \le r_{s_1}(y_1) \le r_0 + 1.$$

Then select any $x_1 \in B_{s_1}$ such that $\|x_1 - y_1\| \ge r_0 - 1$.

Assume that we have constructed $x_1, x_2, \dots, x_{n-1}$ such that, for all $1 \le j \le k \le n - 1$, we have

$$r_0 - \frac{2}{k} + \frac{1}{k^2} \le \|x_k - y_j\| \le r_0 + \frac{1}{k^2}.$$

Take $s_n \in S$ such that $r_{s_n}(y_i) \le r_0 + \frac{1}{n^2}$, $i = 1, \dots, n$ and $r_{s_n}(\overline{y_n}) \le r_0 + \frac{1}{n^2}$. Select $x_n \in B_{s_n}$ such that $r_0 - \frac{1}{n^2} \le \|x_n - \overline{y_n}\|$.

Fix $k \le n$. We then have the following inequalities:

$$\begin{aligned} r_0 - \frac{1}{n^2} \le \|x_n - \overline{y_n}\| &\le \sum_{i=1}^{n} \frac{\|x_n - y_i\|}{n} \\ &= \frac{\|x_n - y_k\|}{n} + \sum_{i=1, i\neq k}^{n} \frac{\|x_n - y_i\|}{n} \\ &\le \frac{\|x_n - y_k\|}{n} + \sum_{i=1, i\neq k}^{n} \frac{r_0 + \frac{1}{n^2}}{n} \\ &= \frac{\|x_n - y_k\|}{n} + \Big(\frac{n-1}{n}\Big)\Big(r_0 + \frac{1}{n^2}\Big). \end{aligned}$$

From this we obtain that $\frac{r_0}{n} - \frac{2}{n^2} + \frac{1}{n^3} \le \frac{\|x_n - y_k\|}{n}$, and it follows that

$$r_0 - \frac{2}{n} + \frac{1}{n^2} \le \|x_n - y_k\| \le r_{s_n}(y_k) \le r_0 + \frac{1}{n^2}.$$

Thus, for a fixed k, it easily follows that $\lim_{n\to\infty} \|x_n - y_k\| = r_0$. Since $\{y_n\}$ is dense in C, we deduce $\lim_{n\to\infty} \|x_n - x\| = r_0$ for all $x \in C$. □

On the other hand, it is clear that

$$\mathcal{AC}\big(\{B_s\}_{s\in A}, C\big) = \bigcap_{n\in\mathbb{N}} \Big\{x \in C : r(x) \le r_0 + \frac{1}{n}\Big\}.$$

Therefore, if the set C is τ-sequentially compact and the function $r(\cdot)$ is τ-sequentially lower semicontinuous (τ-slsc), then the asymptotic center $\mathcal{AC}(\{B_s\}_{s\in A}, C)$ is a nonempty, τ-sequentially compact set. If C is convex, then so is $\mathcal{AC}(\{B_s\}_{s\in A}, C)$. Recall that $r(\cdot)$ is said to be τ-*slsc* if $r(x) \le \lim_n r(x_n)$ whenever $\tau - \lim_n x_n = x$.

Let $\{x_n\}$ be a bounded sequence. We define the type function associated to the sequence $\{x_n\}_n$ by

$$\Gamma(x) = \limsup_n \|x - x_n\|, \quad x \in X.$$

In the case where $\tau - \lim_n x_n = 0$, we say that Γ is a τ-*null-type function*.

In [2, Lemma 3.4] it is proved that the function $r(\cdot)$ is always τ-slsc if and only if the type functions $\Gamma(\cdot)$ are τ-slsc. If we assume that the topology τ is translation invariant and that the set C is τ-sequentially compact, then we can assure that the set $\mathcal{AC}(\{B_s\}_{s\in A}, C)$ is nonempty whenever the τ-null type functions are τ-slsc.

Every left reversible semitopological semigroup S becomes a directed set when the following partial order is defined:

$$a, b \in S, \quad a \ge b \Longleftrightarrow aS \subset \overline{bS}.$$

Let C be subset of X, let S be a left reversible semitopological semigroup, and consider a nonexpansive action of S acting on C.

For a fixed element $u \in C$, define $W_s := \overline{sS(u)}$, where the closure is taken for the norm topology. The sets $\{W_s : s \in S\}$ form a nondecreasing family of subsets of C. In this case we define $r(x) = \lim_s r(x, W_s)$. It is not difficult to check that

$$r_{ts}(tx) \le r_s(x)$$

for all $t, s \in S$ and $x \in C$; therefore, $r(tx) = \inf_s r_s(tx) \le \inf_s r_s(x) = r(x)$ for every $t \in S$, and this implies that the set $\mathcal{AC}(\{W_s\}_{s\in A}, C)$ is S-invariant, whenever it is nonempty.

We now prove the following.

Theorem 2.2. *Let τ be a convex topology in the real Banach space ℓ_1 that is coarser than the weak topology on the unit ball. Assume that the sequence (e_n) converges to some $e \in \ell_1$ with respect to τ. Then ℓ_1 has the τ-FPP for left reversible semigroups if and only if one of the following conditions holds:*

(i) $\|e\|_1 < 1$,
(ii) $\|e\|_1 = 1$ *and the set* $N^+ = \{n \in \mathbb{N} : e(n) \ge 0\}$ *is finite.*

Proof. Let us prove that either condition (i) or (ii) implies the τ-FPP for left reversible semigroups.

Let S be a left reversible semigroup generating a nonexpansive action over a closed convex bounded subset $C \subset \ell_1$, which is τ-compact.

Let $\mathcal{F}$ be the family of nonempty, convex, τ-closed, and S-invariant subsets of C. Ordering this family by inclusion and using Zorn's lemma, we obtain a set which is minimal with respect to being nonempty, convex, τ-closed, and S-invariant. We can then assume that C is the minimal set.

Fix some $u \in C$, and define the subsets $\{W_s\}_{s\in S}$ as before. Consider the asymptotic center $\mathcal{AC}(\{W_s\}_{s\in S}, C)$. As long as it is nonempty, this set is a convex τ-compact, τ-sequentially compact, and S-invariant subset of C. In order to prove that it is nonempty, we will check that the function $r(\cdot)$ is τ-slsc. According to the above observations, it suffices to check that the τ-null type functions are τ-slsc. We will prove this in two steps.

Step 1. The norm is τ-slsc

We argue in the same way as in the proof of [4, Theorem 8]:

For $z = (z(k))_k \in \ell_1$ define $\gamma(z) := \sum_{k=1}^{\infty} z(k)$.

Consider a bounded sequence $(x_n) \subset \ell_1$ which converges to some $x \in \ell_1$ with respect to τ. Without loss of generality we can assume that (x_n) converges to some y for the weak* topology $\sigma(\ell_1, c_0)$. In the remainder of this proof we will consider the w^*-topology $\sigma(\ell_1, c_0)$ and we say that a sequence is *w^*-convergent* if it is convergent for the $\sigma(\ell_1, c_0)$-topology. Notice that, for w^*-null sequences $(z_n) \subset \ell_1$, it holds that

$$\limsup_n \|z_n + z\|_1 = \limsup_n \|z_n\|_1 + \|z\|_1$$

for all $z \in \ell_1$. We will use this property several times in the proof.

Consider the sequence $\hat{y}_n = x_n - y$, which is w^*-null convergent and converges to $\hat{y} = x - y$ with respect to τ. Up to a subsequence, we can assume that $\gamma = \lim_n s(\hat{y}_n)$ exists. From [4], we deduce that $\hat{y} = \gamma e$. In this case,

$$\|\hat{y}\|_1 = \|\gamma e\|_1 = |\gamma|\|e\|_1 \le |\gamma| = \lim_n \big|\gamma(\hat{y}_n)\big| \le \liminf_n \|\hat{y}_n\|_1.$$

Therefore,

$$\begin{aligned}\|x\|_1 &\leq \|x-y\|_1 + \|y\| \leq \liminf_n \|x_n - y\|_1 + \|y\|_1 \\ &= \liminf_n \|x_n - y + y\|_1 = \liminf_n \|x_n\|_1.\end{aligned}$$

Step 2. The τ-null functions are τ-slsc

Let (x_n) be a τ-null bounded sequence in ℓ_1. Let (y_m) be a sequence in ℓ_1 such that $\tau - \lim_m y_m = y$. Taking a subsequence, if necessary, we can assume that (x_n) converges weak* to some $x \in \ell_1$. Then

$$\begin{aligned}\Gamma(y) = \limsup_n \|x_n - y\|_1 &= \limsup_n \|x_n - x + x - y\|_1 \\ &= \limsup_n \|x_n - x\|_1 + \|x-y\|_1 \\ &\leq \limsup_n \|x_n - x\|_1 + \liminf_m \|x - y_m\| \\ &= \liminf_m \limsup_n \|x_n - y_m\|_1 = \liminf_m \Gamma(y_m).\end{aligned}$$

Therefore, since C is a minimal set and $\mathcal{AC}(\{W_s\}_{s\in S}, C)$ is nonempty, it turns out that $C = \mathcal{AC}(\{W_s\}_{s\in S}, C)$. Let r be the asymptotic radius of C with respect to $\{W_s\}_{s\in S}$. If $r = 0$, then the action has a fixed point, since in this case $\mathcal{AC}(\{W_s\}_{s\in S}, C)$ is a singleton. Indeed, using the triangular inequality, if $x, y \in \mathcal{AC}(\{W_s\}_{s\in S}, C)$, $\|x-y\|_1 \leq r(x) + r(y) = 0$, and $x = y$.

Assume otherwise that $r > 0$. Applying Lemma 2.1, there exists a sequence $(x_n) \subset C$ such that

$$\lim_n \|x_n - x\|_1 = r$$

for all $x \in C$. Since C is τ-sequentially compact, we can assume that (x_n) tends to some $x \in C$ with respect to τ, it is weak*-convergent to some $y \in \ell_1$, and there exists $\gamma = \lim_n \gamma(x_n - y)$. In this case, $x - y = \gamma e$. On the one hand,

$$\begin{aligned}r = \lim_n \|x_n - x\|_1 &= \lim_n \left\|x_n - y + (y - x)\right\|_1 \\ &= \limsup_n \|x_n - y\|_1 + \|y - x\|_1 \\ &\leq \limsup_n \|x_n - y\|_1 + \limsup_m \|y - x_m\|_1 \\ &= \limsup_m \limsup_n \|x_n - x_m\|_1 = r.\end{aligned}$$

From the above we deduce that $\limsup_n \|x_n - y\|_1 = r/2$ and that $\|x - y\|_1 = r/2$. On the other hand,

$$\begin{aligned}\frac{r}{2} &= \|x-y\|_1 = \|\gamma e\|_1 = |\gamma| \|e\|_1 \\ &= \left|\lim_n \gamma(x_n - y)\right| \|e\|_1 \leq \limsup_n \|x_n - y\|_1 \|e\|_1 \\ &\leq \frac{r}{2}\|e\|_1.\end{aligned}$$

The above implies that $\|e\|_1 \geq 1$, and then (i) does not hold. According to the hypotheses $\|e\|_1 = 1$, $|\gamma| = r/2$ and the set N^+ is finite.

Consider the case $\gamma = r/2$. Let $m = \max N^+$. We define $y_n := x_n - y$ and consider the sequence (y_n), which is weak* null, and $\lim \gamma(y_n) = r/2$. We can choose $n_0 \in \mathbb{N}$ such that $\gamma(y_{n_0}) > r/4$ and $\|P_m(y_{n_0})\|_1 < r/8$. For such n_0,

$$\sum_{k=m+1}^{\infty} y_{n_0}(k) = \sum_{k=1}^{\infty} y_{n_0}(k) - \sum_{k=1}^{m} y_{n_0}(k) \geq \frac{r}{4} - \frac{r}{8} = \frac{r}{8} > 0,$$

which implies that $\{k > m : y_{n_0}(k) > 0\}$ is nonempty, and therefore the set $B = \{k \in \mathbb{N} : e(k)y_{n_0}(k) < 0\}$ is nonempty. Since for every $a, b \in \mathbb{R}$ with $ab < 0$ we have the equality $|a + b| = |a| + |b| - 2\min\{|a|, |b|\}$, we obtain

$$\|\gamma e + y_n\|_1 = \|\gamma e\|_1 + \|y_n\|_1 - 2c,$$

where $c = \sum_{k \in B} \min\{|\gamma e(k)|, |y_{n_0}(k)|\} > 0$. Then we have

$$\begin{aligned}
\frac{r}{2} &= \lim_n \left\| \frac{x + x_{n_0}}{2} - x_n \right\|_1 = \lim_n \left\| \frac{x - y + x_{n_0} - y}{2} - (x_n - y) \right\|_1 \\
&= \left\| \frac{x - y + x_{n_0} - y}{2} \right\|_1 + \lim_n \|x_n - y\|_1 \\
&= \frac{1}{2}\|\gamma e + y_{n_0}\|_1 + \lim_n \|y_n\|_1 \\
&= \frac{1}{2}\big(\|\gamma e\|_1 + \|y_{n_0}\|_1 - 2c\big) + \lim_n \|y_n\|_1 \\
&= \frac{1}{2}\big(\|\gamma e\|_1 + \|y_{n_0}\|_1 + 2\lim_n \|y_n\|_1\big) - c \\
&= \frac{1}{2}\Big(\lim_n \big\|y_n - (x - y)\big\|_1 + \lim_n \|y_n - y_{n_0}\|_1\Big) - c \\
&= \frac{1}{2}\Big(\lim_n \|x_n - x\|_1 + \lim_n \|x_n - x_{n_0}\|_1\Big) - c \\
&= \frac{r}{2} - c < \frac{r}{2},
\end{aligned}$$

which is a contradiction, and then the action has a fixed point as we wanted to prove. If $\gamma = -r/2$ the proof is similar.

On the other hand, according to [4, Theorem 7], if $\|e\|_1 > 1$, then ℓ_1 fails the τ-FPP for nonexpansive mappings and therefore for left reversible semigroups as well. Hence the proof is finished. $\square$

In particular, if we consider the $\sigma(\ell_1, c_0)$ topology, the sequence (e_n) converges to the zero vector, condition (i) is satisfied and ℓ_1 verifies the $\sigma(\ell_1, c_0)$-FPP for left reversible semigroups (this was first proved by T.-C. Lim [7]). In the case of the $\sigma(\ell_1, c)$-topology, the sequence (e_n) converges to the vector $e = (1, 0, 0, \ldots)$, and neither (i) nor (ii) are satisfied.

Let $f = (f_n) \in \ell_1$ with $\|f\|_1 = 1$. We denote by

$$W_f = \operatorname{Ker}(f) = \big\{x = (x_n) \in c : \langle f, x\rangle = 0\big\},$$

where the duality is given by

$$\langle f, x\rangle = f_1 \lim_n x_n + \sum_{n=1}^{\infty} f_{i+1} x_i.$$

Notice that, if $f = e_1$, then $W_f = c_0$. In fact, from [1, Corollary 4.4], we have a complete isometric description of the preduals of ℓ_1 under the additional assumption that the standard basis is $\sigma(\ell_1, X)$-convergent. In fact, the following can be proved.

Lemma 2.3 ([1, Corollary 4.4]). *Let X be a Banach space such that $X^* = \ell_1$. If the standard basis (e_n) of ℓ_1 is a $\sigma(\ell_1, X)$-convergent sequence, then there exists some $f \in \ell_1$ with $\|f\|_1 = 1$ such that X is isometric to W_f.*

Moreover, according to the results in [1], the hyperplanes W_f for which W_f^* is isometric to ℓ_1, can be classified into three different classes:

(1) W_f is isometric to c_0 and then ℓ_1 has the $\sigma(\ell_1, W_f)$-FPP for left reversible semigroups.
(2) W_f is isometric to c and then ℓ_1 fails to have the $\sigma(\ell_1, W_f)$-FPP for left reversible semigroups.
(3) W_f is isometric neither to c nor c_0. This is equivalent to the fact that $\frac{1}{2} \leq |f(1)| < 1$ and $|f(n)| < \frac{1}{2}$ for every $n \geq 2$. In this case, the sequence (e_n) converges to the vector

$$e = \Big(-\frac{f(2)}{f(1)}, -\frac{f(3)}{f(1)}, -\frac{f(4)}{f(1)}, \dots\Big)$$

with respect to the $\sigma(\ell_1, W_f)$-topology. Finally, using Theorem 2.2 we can deduce the following.

Corollary 2.4. *Let W_f be an hyperplane contained in c for which W_f^* is isometric to ℓ_1, where $f = (f(n))_n$ with $\|f\|_1 = 1$. The space ℓ_1 has the $\sigma(\ell_1, W_f)$-FPP for left reversible semigroups if and only if one of the following conditions holds:*

(i) $|f(1)| > \frac{1}{2}$,
(ii) $|f(1)| = \frac{1}{2}$ *and the set* $\{n \in \mathbb{N} : \operatorname{sign}(f(n)) \neq \operatorname{sign}(f(1))\}$ *is finite.*

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