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LIFTING PROBLEMS FOR NORMED SPACES

NIELS GRØNBÆK

Dedicated to Professor Anthony To-Ming Lau

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Abstract. A classical theorem of G. Köthe states that the Banach spaces X with the property that all bounded linear maps $X \to Y$ into an arbitrary Banach space Y can be lifted with respect to bounded linear surjections onto Y are up to topological linear isomorphism precisely the spaces $\ell^1(A)$. We extend this result to the category of normed linear spaces and bounded linear maps. This answers a question raised by A. Ya. Helemskiĭ.

1. Introduction

Extension and lifting problems are at the core of the theory of topological vector spaces and continuous linear maps. We start by considering these problems in the setting of Banach spaces and bounded linear maps. Let F be a Banach space with a closed subspace E. An *extension problem* is a diagram

$$\begin{tikzcd} E \arrow[r, "\iota"] \arrow[d, "\psi"'] & F \\ X & \end{tikzcd} \tag{1.1}$$

and a *solution* is a bounded linear map $\tilde{\psi}\colon F \to X$ such that $\psi = \iota\tilde{\psi}$. A rather straightforward application of Hahn–Banach's theorem yields the result that if $X = \ell^\infty(A)$, then all extension problems can be solved. In fact, we have solutions

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$\tilde{\psi}\colon F \to \ell^\infty(A)$ which additionally satisfy $\|\tilde{\psi}\| = \|\psi\|$. The natural question—Which are the Banach spaces, X, such that all extension problems (1.1) can be solved with $\|\tilde{\psi}\| = \|\psi\|$?—was addressed early in the history of Banach spaces by R. S. Phillips in [9] and answered in full by L. Nachbin in [7].

We consider the dual problem: For a Banach space Y with a closed subspace H, a diagram

$$\begin{tikzcd} & X \arrow[d, "\phi"] \\ Y \arrow[r, "q"] & Y/H \end{tikzcd}$$

is a *lifting problem* and a *solution* is a bounded linear map $\tilde{\phi}\colon X \to Y$ such that $q\tilde{\phi} = \phi$. Following Köthe [6], for $\lambda > 1$ we call a Banach space X an $\mathfrak{H}_\lambda$-space if all lifting problems can be solved with $\|\tilde{\phi}\| < \lambda\|\phi\|$. Although it was not explicitly stated, A. Grothedieck showed in [2] that if X is an $\mathfrak{H}_\lambda$-space for all $\lambda > 1$, then X is isometrically isomorphic to $\ell^1(A)$ for a set A of the appropriate cardinality. In the article [6], Köthe extended Grothendieck's result to include the following: If X is a $\mathfrak{H}_\lambda$-space for some $\lambda > 1$, then X is topologically isomorphic to $\ell^1(A)$. (However, Grothendieck's result does not follow directly from Köthe's.)

Extension and lifting problems have equal importance in the category of operator spaces and completely bounded maps, but are of considerable higher complexity as is witnessed by the fundamental Arveson–Wittstock theorem [10, Section 2.3], a Hahn–Banach-type theorem in which $\mathbb{C}$ is replaced by the operator space of all bounded operators on a separable Hilbert space. In a number of papers (see [3], [4], [5]) Helemskiĭ developed a category framework for discussing extension and lifting problems in the classical setting of normed spaces as well as in the setting of operator spaces, in particular the Arveson–Wittstock theorem. Helemskiĭ points out that in this framework certain lifting results hold only if noncomplete normed spaces are taken into account. (For a detailed discussion, see [4].) In the present article we will solve the noncomplete equivalent of the problem that Köthe solved for Banach spaces. (This problem was posited by Helemskiĭ in a private communication, and our sketch of a solution has been cited as "preprint (unpublished)" in [3].)

1.1. **Definitions and preliminary results.** We will work in the category of normed spaces and continuous linear maps.

Definition 1.1. The category with objects normed spaces and morphisms continuous linear maps is **Norm**. Equivalence is denoted by $\cong$ and direct sum by $\oplus$ (i.e., for objects E, F, their direct sum $E \oplus F$ as a vector space is the Cartesian product $E \times F$, and the norm is given as

$$\big\|(e, f)\big\| = \|e\| + \|f\|, \quad e \in E, f \in F.$$

We phrase our main problem in category terms.

Definition 1.2. A *lifting problem* for an object $E \in \mathbf{Norm}$ is a diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle \phi} \\ Y & \xrightarrow{\psi} & Z \end{array}$$

with the map $\psi \colon Y \to Z$ open. A *solution* is a morphism $\tilde{\phi} \colon E \to Y$ so that $\psi\tilde{\phi} = \phi$.

The object $E \in \mathbf{Norm}$ is *topologically projective* if all lifting problems for E can be solved (see [3]).

Remark 1.3. One may consider lifting problems without the requirement that the map $\psi \colon Y \to Z$ is open. However, simple examples show that then the only projective objects would be the finite-dimensional spaces.

For a set A, we denote its cardinality by $\#A$. The cardinality of the set of natural numbers is $\aleph_0$.

We need the following concepts from the theory of normed spaces (over $\mathbb{C}$).

Definition 1.4. Let E be a normed space. For a family of vectors $(x_i)_{i\in I}$ in E, we set $]x_i[_{i\in I} = \mathrm{lin}\{x_i \mid i \in I\}$. The *dimension* of E, $\dim E$, is the minimal cardinality $\#I$ such that $]x_i[_{i\in I}$ is dense in E.

Remark 1.5. This is a generalization of the finite-dimensional concept. For $\dim E \geq \aleph_0$, the dimension is the minimal cardinality of a dense subset.

A crucial part of our argument hinges on the following concept from [1].

Definition 1.6. Let $0 < \sigma < 1$. A *σ-net* in E is a family $(x_i)_{i\in I}$ such that

$$\begin{aligned} \|x_i\| &= 1, \quad i \in I, \\ \|x_i - x_j\| &\geq \sigma, \quad i \neq j. \end{aligned}$$

Zorn's lemma provides the existence of maximal σ-nets for each σ. The following is stated in Lemma 6.1 of [1] without proof.

Lemma 1.7. *Suppose that* $\dim E \geq \aleph_0$. *If* $(x_i)_{i\in I}$ *is a maximal σ-net for some* $0 < \sigma < 1$*, then* $\#I = \dim E$.

Proof. Since a dense subset of E must intersect each of the disjoint balls $\{y \mid \|y - x_i\| < \frac{\sigma}{2}\}, i \in I$, we have $\dim E \geq \#I$ by Remark 1.5. For the reverse inequality, let H be the closure of $]x_i[_{i\in I}$ and suppose that $H \neq E$. Choose $\varepsilon > 0$ so that $\frac{1}{1+\varepsilon} \geq \sigma$ and choose $y \in E$ so that $\|y + H\| = 1$ in the quotient E/H. Further, choose $h \in H$ so that $\|y + h\| \leq 1 + \varepsilon$ in E, and set $x = \frac{1}{\|y+h\|}(y + h)$. Then $\|x\| = 1$, and for all $i \in I$, we have

$$\|x - x_i\| = \frac{1}{\|y+h\|}\big\|y + h - \|y+h\|x_i\big\| \geq \frac{1}{\|y+h\|}\|y + H\| \geq \frac{1}{1+\varepsilon} \geq \sigma,$$

contradicting that $(x_i)_{i\in I}$ is maximal. It follows that $]x_i[_{i\in I}$ is dense in E, so that $\dim E \leq \#I$. $\square$

We now define the objects in **Norm**, which are absolutely central to our result.

Definition 1.8. Let A be a set, and let $\mathbb{C}^{(A)}$ denote the vector space of finitely supported functions $A \to \mathbb{C}$. The functions $e_a, a \in A$ are the indicator functions of singletons $\{a\}$. We equip $\mathbb{C}^{(A)}$ with the classical norms to obtain $\ell_0^p(A) = (\mathbb{C}^{(A)}, \|\cdot\|_p), 1 \le p \le \infty$. The canonical dualities are

$$\langle x, x^* \rangle = \sum_{a \in A} x(a)x^*(a), \quad x \in \ell_0^p(A), x^* \in \ell_0^q(A), \frac{1}{p} + \frac{1}{q} = 1.$$

As these spaces up to isometry depend only on $\#A$, we will also use the notation $\ell_0^p(d)$ for a cardinal d.

For $f \in \mathbb{C}^{(A)}$ we set $\operatorname{supp} f = \{a \in A \mid f(a) \neq 0\}$, and for a subset $M \subseteq \mathbb{C}^{(A)}$ we set $\operatorname{supp} M = \bigcup_{f \in M} \operatorname{supp} f$.

We need the following observation.

Lemma 1.9. *Let Y be a subspace of $\ell_0^1(d)$ with $\dim Y \ge \aleph_0$. Then $\# \operatorname{supp} Y = \dim Y$.*

Proof. This is straightforward if $Y = \ell_0^1(d)$. Since, in general, Y is isometric to a subspace of $\ell_0^1(\operatorname{supp} Y)$, we have $\dim Y \le \# \operatorname{supp} Y$. For the reverse, if $E \subseteq Y$ spans a dense subspace of Y, then $\operatorname{supp} E = \operatorname{supp} Y$. Since each $x \in E$ is finitely supported $\# \operatorname{supp} E \le \#E$, so $\# \operatorname{supp} Y \le \dim Y$. □

We note the universal property of $\ell_0^1(A)$-spaces.

Proposition 1.10. *Let $\iota\colon A \to \ell_0^1(A)$ be the canonical mapping $a \mapsto e_a$. If $\phi\colon A \to X$ is a bounded mapping into a normed space, then there is a unique bounded linear map $\tilde{\phi}\colon \ell_0^1(A) \to X$ of the same bound such that $\phi = \tilde{\phi}\iota$.*

For this reason, we will call the spaces $\ell_0^1(A)$ the *free objects* of **Norm**. We now have the well-known and easy, though important, corollary that free objects are projective and generate the category **Norm**, and that projective objects precisely are retracts of free objects.

Corollary 1.11. *The category* **Norm** *has enough projectives.*

(1) *The spaces $\ell_0^1(A)$ are topologically projective.*
(2) *For each object $Y \in$ **Norm** there is a cardinal d and an open map $\ell_0^1(d) \to Y$.*

Furthermore

(3) *An object $E \in$ **Norm** is topologically projective if and only if there is a cardinal d and an object $X \in$ **Norm** so that*

$$\ell_0^1(d) \cong E \oplus X.$$

Proof. To prove the first statement, consider a lifting problem

$$\begin{array}{ccc} & & \ell_0^1(A) \\ & & \downarrow{\scriptstyle \phi} \\ Y & \xrightarrow{\psi} & Z \end{array}$$

Since $\psi\colon Y \to Z$ is open, there is a bounded set $\{y_a \mid a \in A\} \subseteq Y$ such that $\psi(y_a) = \phi(e_a)$ for each $a \in A$. The linear map $\tilde{\phi}\colon \ell_0^1(A) \to Y$ given by $\tilde{\phi}(e_a) = y_a$ is bounded and solves the lifting problem.

To prove (2), let $S(Y)$ be the unit sphere of Y. By Proposition 1.10 there is a bounded linear map $\kappa_Y\colon \ell_0^1(S(Y)) \to Y$ such that $\kappa_Y(e_y) = y$ for each $y \in S(Y)$. This map is clearly open; in fact, it takes the unit ball of $\ell_0^1(S(Y))$ to the unit ball of Y.

It is obvious that a retract of a topologically projective object is itself topologically projective. Hence to conclude (3), let $p\colon E \to \ell_0^1(S(E))$ be a solution to the lifting problem

$$\begin{tikzcd} & E \arrow[d, "\mathrm{id}"] \\ \ell_0^1\big(S(E)\big) \arrow[r, "\kappa_E"] & E \end{tikzcd}$$

Then $p\kappa_E$ is a bounded projection of $\ell_0^1(S(E))$ onto a subspace isomorphic to E. $\square$

2. Main result

We are now ready to prove our main result.

Theorem 2.1. *A normed space E is topologically projective if and only if it is isomorphic to $\ell_0^1(\dim(E))$.*

Clearly, if a normed space E is isomorphic to $\ell_0^1(d)$ for a cardinal d, then $d = \dim E$, and we have already noted in Corollary 1.11 that the spaces $\ell_0^1(d)$ are topologically projective for all cardinals d. We will prove the converse implication by adapting Köthe's proof of the corresponding Banach version. The first step is the proposition below proved by transfinite induction. As we are dealing with ℓ_0^1-spaces rather than ℓ^1-spaces, the induction step is more involved to compensate for the fact that we are restricted to elements with finite support. Likewise as in Köthe's work, the second step is through Pełczyński's decomposition method [8], which applies equally well in the noncomplete setting.

Proposition 2.2. *Let Y be a subspace of $\ell_0^1(d)$ for some cardinal d. There is then a subspace of Y, which is complemented in $\ell_0^1(d)$ and isomorphic to $\ell_0^1(\dim Y)$.*

Proof. This is trivial if $\dim Y < \aleph_0$, so we assume that $\dim Y \geq \aleph_0$. Let $\#A = d$, and consider $Y \subseteq \ell_0^1(A)$. Without loss of generality we may assume that $A = \operatorname{supp} Y$.

Let ξ be the smallest ordinal with $\#\xi = \dim Y$, and let $\varepsilon > 0$. We will use transfinite induction to define $y_\alpha \in Y, z_\alpha \in \ell_0^1(A), z_\alpha^* \in \ell_0^\infty(A)$ for all $\alpha < \xi$ such that with the notation

$$B_\alpha = \operatorname{supp}]y_\beta[_{\beta\leq\alpha}, \qquad B'_\alpha = \operatorname{supp}]y_\beta[_{\beta<\alpha},$$
$$F_\alpha = \{x \in \ell_0^1(A) \mid \operatorname{supp} x \subseteq B_\alpha\}, \qquad F'_\alpha = \{x \in \ell_0^1(A) \mid \operatorname{supp} x \subseteq B'_\alpha\},$$
$$G_\alpha = \bigcap_{\beta\leq\alpha} \ker z_\beta^* \cap F_\alpha, \qquad G'_\alpha = \bigcap_{\beta<\alpha} \ker z_\beta^* \cap F'_\alpha,$$

the statements below, (i)–(vii) are true:

(i) $\|y_\alpha\| = 1$,
(ii) $\operatorname{supp} z_\alpha \subseteq \operatorname{supp} y_\alpha$ and $z_\alpha(b) = y_\alpha(b)$ for $b \in \operatorname{supp} z_\alpha$,
(iii) $\operatorname{supp}]z_\beta[_{\beta\le\alpha} = B_\alpha$,
(iv) $\operatorname{supp} z_\alpha \cap B'_\alpha = \emptyset$,
(v) $\|y_\alpha - z_\alpha\| < \varepsilon$,
(vi) $\operatorname{supp} z^*_\alpha = \operatorname{supp} z_\alpha, \|z^*_\alpha\| = \frac{1}{\|z_\alpha\|}, \langle z_\alpha, z^*_\alpha\rangle = 1$,
(vii) $]y_\beta[_{\beta\le\alpha} + G_\alpha = F_\alpha$.

For ease of reference, we denote the conjunction of the seven statements (i)–(vii) by $\mathcal{S}(\alpha)$.

Suppose this can be accomplished (i.e., we have found elements $y_\alpha, z_\alpha, z^*_\alpha$ so that $\mathcal{S}(\alpha)$ is true for all $\alpha < \xi$). By (iii) above, $z_\alpha \in F'_\xi$ for all $\alpha < \xi$. We note that for $\beta < \alpha$,

$$F_\beta \subseteq F_\alpha,$$

since $B_\beta \subseteq B_\alpha$, and that

$$G_\beta \subseteq G_\alpha, \tag{2.1}$$

since by (iii), (iv), and (vi) $\operatorname{supp} z^*_\gamma \cap B_\beta = \emptyset$ for all $\beta < \gamma \le \alpha$, from which we get

$$G_\beta \subseteq \bigcap_{\gamma\le\alpha} \ker z^*_\gamma \cap F_\beta \subseteq \bigcap_{\gamma\le\alpha} \ker z^*_\gamma \cap F_\alpha = G_\alpha.$$

It then follows that

$$G'_\xi = \bigcup_{\alpha<\xi} G_\alpha. \tag{2.2}$$

Since $\#B'_\xi = \dim Y$, we have $F'_\xi \cong \ell^1_0(d)$ isometrically. On F'_ξ we define a linear map by

$$P\colon x \mapsto \sum_{\alpha<\xi} \langle x, z^*_\alpha\rangle z_\alpha.$$

Note that the series has at most finitely many nonzero terms. By (iv) and (vi), we have $\langle z_\beta, z^*_\alpha\rangle = \delta_{\alpha\beta}$ (Kronecker's delta symbol). By (v) and (i), (iv), and (ii), we have

$$(1-\varepsilon)\sum_{\alpha<\xi} |\lambda_\alpha| \le \sum_{\alpha<\xi} |\lambda_\alpha| \|z_\alpha\| = \Big\|\sum_{\alpha<\xi} \lambda_\alpha z_\alpha\Big\| \le \sum_{\alpha<\xi} |\lambda_\alpha|. \tag{2.3}$$

Here and throughout, we adopt the convention that $\sum_{j\in J}\cdots$ denote sums with at most finitely many nonzero terms even though $\#J \ge \aleph_0$. Hence $]z_\alpha[_{\alpha<\xi}$ is isomorphic to $\ell^1_0(d)$ and P is a projection of $\|P\| = 1$ onto $]z_\alpha[_{\alpha<\xi}$. We get

$$F'_\xi =]z_\alpha[_{\alpha<\xi} \oplus G'_\xi.$$

By (vii) and (2.2), we have

$$F'_\xi = \bigcup_{\alpha<\xi} F_\alpha = \bigcup_{\alpha<\xi} \big(]y_\beta[_{\beta\le\alpha} + G_\alpha\big) =]y_\beta[_{\beta<\xi} + G'_\xi. \tag{2.4}$$

Consider the continuous linear map $T\colon F'_\xi \to F'_\xi$ given by

$$z_\alpha + g \mapsto y_\alpha + g, \quad g \in G'_\xi.$$

By (2.4), T is surjective. Furthermore by (v) and (2.3), we have

$$\begin{aligned}\Big\|(\mathbf{1}-T)\Big(\sum_{\alpha<\xi}\lambda_\alpha z_\alpha + g\Big)\Big\| &= \Big\|\sum_{\alpha<\xi}\lambda_\alpha(z_\alpha - y_\alpha)\Big\| \\ &\le \sum_{\alpha<\xi}|\lambda_\alpha|\varepsilon \le \frac{\varepsilon}{1-\varepsilon}\Big\|\sum_{\alpha<\xi}\lambda_\alpha z_\alpha\Big\| \\ &\le \frac{\varepsilon}{1-\varepsilon}\Big\|\sum_{\alpha<\xi}\lambda_\alpha z_\alpha + g\Big\|.\end{aligned}$$

If $\varepsilon < \frac{1}{2}$, then T is bounded below. Altogether T is an isomorphism, $]y_\alpha[_{\alpha<\xi} \cong\,]z_\alpha[_{\alpha<\xi} \cong \ell^1_0(d)$ and $F'_\xi \cong\,]y_\alpha[_{\alpha<\xi} \oplus G'_\xi$. Since F'_ξ is complemented in $\ell^1_0(d)$ so is $]y_\alpha[_{\alpha<\xi}$.

So to finish the proof we must show that the transfinite induction indeed can be accomplished. Suppose that for some $\alpha < \xi$ we have found $y_\beta, z_\beta, z^*_\beta$ so that $\mathcal{S}(\beta)$ is true for all $\beta < \alpha$. First, we consider the case $\#\alpha < \aleph_0$. Let $P_\alpha\colon \ell^1_0(A) \to F'_\alpha$ be the projection $x \mapsto \chi_{B'_\alpha}x$, where $\chi_{B'_\alpha}$ is an indicator function. Since F'_α is finite-dimensional and $\dim Y \ge \aleph_0$, we may choose $y_\alpha \in Y \cap \ker P_\alpha$ with $\|y_\alpha\| = 1$. Put $z_\alpha = y_\alpha$ and choose z^*_α in accordance with (vi). Then clearly (i)–(vii) are satisfied. If $\dim Y = \aleph_0$, we are done by ordinary induction.

Otherwise consider the case $\#\alpha \ge \aleph_0$. We decompose each $x \in \ell^1_0(A)$ as $x = x' + x''$ with $\operatorname{supp} x' \subseteq B'_\alpha, \operatorname{supp} x'' \subseteq A\setminus B'_\alpha$, that is, with P_α as above $x' = P_\alpha(x)$. Let $0 < \sigma < 1$ and let $(x_i)_{i\in I}$ be a maximal σ-net for Y. Further, let $\delta > 0$. Since $\dim F'_\alpha < \dim Y$, there are i_1, i_2 so that $\|x'_{i_1} - x'_{i_2}\| < \delta$. (Otherwise there would be $\#I = \dim Y$ disjoint balls of radii $< \delta/2$.) Put $y_\alpha = \|x_{i_1} - x_{i_2}\|^{-1}(x_{i_1} - x_{i_2})$ and $z_\alpha = \|x_{i_1} - x_{i_2}\|^{-1}(x''_{i_1} - x''_{i_2})$. Then clearly (i)–(iv) are satisfied. Since

$$\|y_\alpha - z_\alpha\| = \|x_{i_1} - x_{i_2}\|^{-1}\|x'_{i_1} - x'_{i_2}\| < \frac{\delta}{\sigma},$$

we can also satisfy (v) by choosing σ, δ appropriately, and clearly we may choose z^*_α to satisfy (vi). To show that (vii) is satisfied, let $b \in B_\alpha$. If $b \in B_\beta$ for some $\beta < \alpha$, then the corresponding canonical ℓ^1_0-basis vector $e_b \in\,]y_\gamma[_{\gamma\le\beta} + G_\beta \subseteq\,]y_\gamma[_{\gamma\le\alpha} + G_\alpha$. If $b \in \operatorname{supp} z_\alpha$, write $e_b = z^*_\alpha(b)z_\alpha + (e_b - z^*_\alpha(b)z_\alpha)$. Since $\operatorname{supp} z^*_\beta \cap \operatorname{supp} z_\alpha = \emptyset$ for $\beta < \alpha$ we have $e_b - z^*_\alpha(b)z_\alpha \in G_\alpha$. Since $\operatorname{supp}(z_\alpha - y_\alpha) \subseteq B'_\alpha$ there is by the induction hypothesis (vii) some $\beta < \alpha$ so that $z_\alpha - y_\alpha \in\,]y_\gamma[_{\gamma\le\beta} + G_\beta$. Hence $z_\alpha \in\,]y_\beta[_{\beta\le\alpha} + G_\alpha$. Since $B_\alpha = B'_\alpha \cup \operatorname{supp} z_\alpha$, we have shown that $F_\alpha (=\,]e_b[_{b\in B_\alpha}) =\,]y_\beta[_{\beta\le\alpha} + G_\alpha$, and thereby completed the induction. □

Theorem 2.1 is now proved using the Pełczyński decomposition method [8] as in [6].

Proposition 2.3. *Let Y be a complemented subspace of $\ell^1_0(d)$ for some cardinal d. Then $Y \cong \ell^1_0(\dim Y)$.*

Proof. If $\dim Y < \aleph_0$, this is elementary. By Proposition 1.9, we may assume that $d = \dim Y$. By Proposition 2.2, we may write $Y \cong \ell^1_0(d) \oplus X_1$ for a suitable closed subspace $X_1 \subseteq Y$. From this we get $\ell^1_0(d) \oplus Y \cong \ell^1_0(d) \oplus \ell^1_0(d) \oplus X_1 \cong \ell^1_0(d) \oplus X_1 \cong Y$. By assumption, there is a subspace X of $\ell^1_0(d)$ so that $\ell^1_0(d) \cong Y \oplus X$.

We now use the fact that $\ell^1_0(d) \cong (\ell^1_0(d) \oplus \cdots \oplus \ell^1_0(d) \oplus \cdots)^1_0$, where $(\cdots \oplus X_n \oplus \cdots)^1_0$ denotes the algebraic direct sum of $X_n, n \in \mathbb{N}$ with norm $\|\sum_{n\in\mathbb{N}} x_n\| = \sum_{n\in\mathbb{N}} \|x_n\|$. From this we get the following:

$$\begin{aligned}
\ell^1_0(d) &\cong \ell^1_0(d) \oplus \big(\ell^1_0(d) \oplus \cdots \oplus \ell^1_0(d) \oplus \cdots\big)^1_0 \\
&\cong \ell^1_0(d) \oplus \big((Y \oplus X) \oplus \cdots \oplus (Y \oplus X) \oplus \cdots\big)^1_0 \\
&\cong \big(\ell^1_0(d) \oplus Y\big) \oplus \big((X \oplus Y) \oplus \cdots \oplus (X \oplus Y) \oplus \cdots\big)^1_0 \\
&\cong Y \oplus \big((X \oplus Y) \oplus \cdots \oplus (X \oplus Y) \oplus \cdots\big)^1_0 \\
&\cong Y \oplus \big((Y \oplus X) \oplus \cdots \oplus (Y \oplus X) \oplus \cdots\big)^1_0 \\
&\cong Y \oplus \ell^1_0(d) \cong Y. \qquad \square
\end{aligned}$$

3. Concluding discussion

The definition of $\mathfrak{H}_\lambda$-spaces obviously extends to **Norm**. It is straightforward, using Proposition 1.10, that the spaces $\ell^1_0(d)$ are $\mathfrak{H}_\lambda$-spaces for every $\lambda > 1$. It follows that if E is topologically projective then E is a $\mathfrak{H}_\lambda$-space for all $\lambda > \operatorname{dist}(E, \ell^1_0(d))$, the Banach–Mazur distance between E and ℓ^1_0. The aforementioned result by A. Grothendieck raises the following.

Conjecture 3.1. *If a normed space E is a $\mathfrak{H}_\lambda$-space in* **Norm** *for all $\lambda > 1$, then E is isometrically isomorphic to $\ell^1_0(\dim E)$.*

This conjecture is also raised in [4], where spaces which are $\mathfrak{H}_\lambda$-spaces for each $\lambda > 1$ are called *extremely projective normed spaces*.

Grothendieck's approach uses completeness of the involved spaces fundamentally by proving that E is 1-complemented in $(\ell^1)^{**}$, and then showing that E is discretely supported in an appropriate sense. Of course if a (noncomplete) normed space is a $\mathfrak{H}_\lambda$-space for all $\lambda > 1$, then the same is true for its completion, so E is norm dense in $\ell^1(\dim E)$ and isomorphic to $\ell^1_0(\dim E)$. To prove that E is isometric to $\ell^1_0(d)$ it seems that one would need to control the support of functions in E in its embedding in $\ell^1(d)^{**}$, that is, in $M(\beta d)$, the Radon measures on the Stone–Čeck compactification. At the moment this seems intractable.

As pointed out in Remark 1.3 the unrestricted lifting problem is of minor interest. A standard approach to get a fruitful notion of projectivitity is to consider only lifting problems which are retractions in some functorial picture, typically by applying some forgetful functor. In [4] the author discusses this in terms of so-called *rigged categories* and projectivity with regard to a rig. As a partial converse to Conjecture 3.1, one may ask the following: Which rigs of **Norm** ensure that projectives with respect to the rig are isometric to $\ell^1_0(d)$-spaces? One answer is given in Theorem 3.5 of [4], which states that metrically projective normed spaces are isometric to $\ell^1_0(d)$-spaces.

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Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark.

E-mail address: gronbaek@math.ku.dk