African Diaspora Journal of Mathematics
Proceedings of Geometry and Physics V
Volume 9, Number 2, pp. 74–81 (2010)

ISSN 1539-854X
www.african-j-math.org

Decomposability of a Poisson Tensor Could Be a Stable Phenomenon

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Abstract

In this paper, we develop one of the questions raised by the author in the mini-course he gave at the conference Geometry and Physics V held at the University Cheikh Anta Diop, Dakar in May 2007). Let Π be a Poisson tensor on a manifold M. We suppose that it is decomposable in a neighborhood U of a point m, i.e. we have $\Pi = X \wedge Y$ on U where X and Y are two vector fields. We will exhibit examples where every Poisson tensor near enough Π seems to be also decomposable in a neighborhood of a point which can be chosen arbitrarily near m; and this works even if M has a big dimension. This idea is a consequence of a cohomology calculation which can be interesting by itself.

AMS Subject Classification: 53D17; 14Fxx.

Keywords: Poisson tensor, Lichnerowicz-Poisson cohomology, decomposability, division property, isolated singularity.

1 Introduction

Let Π be an homogeneous Poisson tensor of degree k on $\mathbb{R}^n$. We attach to it the *homogeneous Lichnerowicz-Poisson cohomology complexes*: by definition, they are given, for every s,

$$\mathcal{V}_1^{(s-k+1)} \quad \xrightarrow{\partial_1^{s-k+1}(\Pi)} \quad \mathcal{V}_2^{(s)} \quad \xrightarrow{\partial_2^{s}(\Pi)} \quad \mathcal{V}_3^{(s+k-1)} \ldots$$

where $\mathcal{V}_r^{(s)}$ is the space of s-homogeneous r-vector fields on $\mathbb{R}^n$ (chosen to be $\{0\}$ for $s<0$), and the operators $\partial_r^{\ell}(\Pi)$ are defined by

$$\partial_r^{\ell}(\Pi)(A) = [\Pi, A],$$

for all homogeneous multi-vector field A. The associated second cohomology space is

$$H^{2,s}(\Pi) = \frac{\mathrm{Ker}\left(\partial_2^{s}(\Pi)\right)}{\mathrm{Im}\left(\partial_1^{s-k+1}(\Pi)\right)} .$$

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For any r-vector A on $\mathbb{R}^n$ we denote by DA its "curl" relatively to the volume $\Omega = dx_1 \wedge \cdots \wedge dx_n$: we recall that $D : \mathcal{V}_r^{(s)} \longrightarrow \mathcal{V}_{r-1}^{(s-1)}$ is the operator $(\Omega^\flat)^{-1} \circ d \circ \Omega^\flat$, where $\Omega^\flat$ is the contraction with Ω. We will use notations and sign conventions of [DZ05].
Our principal result is the following.

Theorem 1.1. *Let Π a k-homogeneous Poisson tensor on $\mathbb{R}^n$ with $k > 2$. We suppose that its maximal rank is 2 and that its curl $D\Pi$ has an isolated zero at the origin. Then we have*

$$H^{2,s}(\Pi) = 0$$

for any s different from 2, k and $2k-2$. For $s = k$ the cocycles which are not coboundary have the form $I \wedge V$, where V is a vector field. For $s = 2k-2$ cocycle which are not coboundary are multiples of Π.

The next section is dedicated to the proof of this theorem. It will use the following lemma.

Lemma 1.2. *Under the hypothesis of Theorem 1.1 we have*

$$\Pi = \frac{1}{2-n-k} I \wedge X$$

with $X = D\Pi$.

In the last section we will try to show that a consequence of Theorem 1.1 should be the following conjecture.

Conjecture 1.3. *Let Π a Poisson structure satisfying Theorem 1.1 hypothesis. Every Poisson structure Π', near enough Π in the C^{2k} compact open topology, is decomposable (i.e. $\Pi' = U \wedge V$) on a neighborhood of a point m which can be chosen as near as we want from the origin.*

If true, this result is a little surprising because, when dimension of the ambient space is big, it is very easy to perturb a decomposable bi-vector in a non decomposable one.

2 Proof of Theorem 1.1

Definition 2.1. *We say that an analytic vector field V on $\mathbb{R}^n$ has the analytic division property if for any analytic p-vector A on $\mathbb{R}^n$, with $p < n$, the relation*

$$V \wedge A = 0$$

implies

$$A = B \wedge V$$

for some analytic $(p-1)$-vector B.

Lemma 2.2. *Any polynomial vector field Z which have an isolated zero at the origin has the analytic division property.*

See section 2 of [M76] for a proof of this last lemma. We will use also the following evident lemma.

Lemma 2.3. *For any s-homogeneous r-vector A on* $\mathbb{R}^n$ *we have*

$$[I,A] = (s-r)A.$$

Lemma 2.4. *Koszul formula. We have the formula*

$$[A,B] = (-1)^b D(A \wedge B) - DA \wedge B - (-1)^b A \wedge DB\ , \tag{2.1}$$

for any r-vector (r arbitrary) A and any b-vector B.

See [DZ05], Formula (2.91).

Lemma 2.5. *Camacho Lins Neto Lemma. Let Z be an homogeneous vector field of degree greater or equal than 2 on* $\mathbb{R}^n$ *which has an isolated zero at the origin. If L is a linear vector field on* $\mathbb{R}^n$ *such that*

$$[Z,L] = 0\ , \tag{2.2}$$

then L vanishes identically.

See [CN82], Lemma 1.

Proof of Lemma 1.2. Koszul formula gives

$$0 = [\Pi,\Pi] = D(\Pi \wedge \Pi) - 2D\Pi \wedge \Pi.$$

The rank condition gives $\Pi \wedge \Pi = 0$ and so we have

$$X \wedge \Pi = 0$$

($X = D\Pi$). By the division property we get

$$\Pi = U \wedge X,$$

for a linear vector field U. But Koszul formula gives

$$X = D(U \wedge X) = -[U,X] - D(U)X$$

so

$$(1 + D(U))X = [X,U].$$

But, by Lemma 2.3, we have

$$(2-k)X = [X,I]$$

so we get

$$[X,U] = \frac{1+D(U)}{2-k}[X,I]$$

which can be rewritten

$$[X,L] = 0$$

with the linear vector field

$$L = U - \frac{1 + D(U)}{2-k} I.$$

Now Camacho Lins Neto Lemma implies

$$L = 0,$$

so

$$U = \frac{1 + D(U)}{2-k} I.$$

Now we apply the operator D to the members of the last equation to get

$$D(U) = \frac{n}{2-k-n}.$$

Putting this in the last expression of U we fall on

$$U = \frac{1}{2-k-n} I$$

which proves Lemma 1.2 □

Proof of Theorem 1.1.

To simplify notations we will write $\Pi = I \wedge Z$ with

$$Z = \frac{1}{2-n-k} X.$$

A- We suppose that s *is different from 2,* k *and* $2k-2$*.*

Let A a s-homogeneous 2-vector; it is a cocycle if we have relation

$$0 = [A, \Pi] = [A, I \wedge Z] = [A, I] \wedge Z - I \wedge [A, Z] \ . \tag{2.3}$$

So, using Lemma 2.3, this is equivalent to

$$(2-s) A \wedge Z = I \wedge [A, Z] \ . \tag{2.4}$$

We apply the curl operator D to the two members of cocycle Equation (2.4) to get

$$(2-s) D(A \wedge Z) = D(I \wedge [A, Z]) \ . \tag{2.5}$$

Using Koszul formula two times this becomes

$$(2-s)\{-[A, Z] - DA \wedge Z\} = [I, [A, Z]] + DI \wedge [A, Z] + I \wedge D[A, Z] \ , \tag{2.6}$$

which leads to

$$[A, Z] = \frac{2-s}{2-k-n} DA \wedge Z + \frac{1}{2-k-n} I \wedge D[A, Z] \ . \tag{2.7}$$

When we replace $[A, Z]$ in (2.4) according to the above formula we get

$$A \wedge Z = \frac{1}{2-k-n} I \wedge DA \wedge Z \ ; \tag{2.8}$$

which can be written in the form

$$\{A + \frac{1}{k+n-2} I \wedge DA\} \wedge Z \ = 0 \ . \tag{2.9}$$

Using division property of Z and checking homogeneity degrees this gives

$$A + \frac{1}{k+n-2} I \wedge DA = U \wedge Z. \tag{2.10}$$

where $U \equiv 0$ if $s < k-1$ and is a $(s-k+1)$-homogeneous vector field for $s \geq k-1$.

We apply the operator D to the two members of (2.10) to get

$$DA + \frac{1}{k+n-2} D(I \wedge DA) = D(U \wedge Z) \ . \tag{2.11}$$

But we have the Koszul formula

$$[I, DA] = -D(I \wedge DA) - DI \wedge DA \ , \tag{2.12}$$

which leads to

$$DA + \frac{1}{k+n-2}(-[I, DA] - nDA) = D(U \wedge Z) \ , \tag{2.13}$$

so to

$$DA(1 - \frac{1}{k+n-2}(s-2+n)) = D(U \wedge Z) \ , \tag{2.14}$$

and finally to

$$\frac{k-s}{k+n-2} DA = D(U \wedge Z) \ . \tag{2.15}$$

When we put Result (2.15) in Equation (2.10) we get

$$A = U \wedge Z + \frac{1}{s-k} I \wedge D(U \wedge Z) \ . \tag{2.16}$$

When it doesn't vanish, U can be put on the form

$$U = \lambda I + U_0 \ , \tag{2.17}$$

where λ is a $(s-k)$-homogeneous function and U_0 is a $(s-k+1)$-homogeneous vector field such that $DU_0 = 0$. So the cocycle A decomposes in

$$A = A_1 + A_0 \tag{2.18}$$

where

$$A_1 = \lambda I \wedge Z + \frac{1}{s-k} I \wedge D(\lambda I \wedge Z) \tag{2.19}$$

and

$$A_0 = U_0 \wedge Z + \frac{1}{s-k} I \wedge D(U_0 \wedge Z) \ . \tag{2.20}$$

But Koszul formula gives

$$[I, \lambda I \wedge Z] = -DI \wedge \lambda I \wedge Z - I \wedge D(\lambda I \wedge Z) \tag{2.21}$$

and this leads to

$$I \wedge D(\lambda I \wedge Z) = (-s-n+2)\lambda\Pi \ . \tag{2.22}$$

This gives

$$A_1 = \frac{-k-n+2}{s-k}\lambda\Pi \ . \tag{2.23}$$

Now, for any $(s-k)$-homogeneous function μ, we have

$$[\mu I, \Pi] = [\mu I, I] \wedge Z + I \wedge [\mu I, Z] = -I(\mu) I \wedge Z + I \wedge \mu [I, Z]$$

$$= (2k-s-2)\mu\Pi. \tag{2.24}$$

and this shows that (for s different from $2k-2$) A_1 is a coboundary.

Now A_0 can be rewritten as

$$A_0 = \frac{1}{k-s}\{(k-s)U_0 \wedge Z - I \wedge D(U_0 \wedge Z)\} = \frac{1}{k-s}\{[U_0, I] \wedge Z + I \wedge [U_0, Z]\} \ , \tag{2.25}$$

so we have

$$A_0 = \frac{1}{k-s}[U_0, I \wedge Z] = [\frac{1}{k-s}U_0, \Pi] \tag{2.26}$$

which shows that A_0 is also a coboundary.

B- We suppose $s = k$.

In that case we can perform the same calculations as in the case A until Formula (2.15): we get that U is a linear vector field and this last formula gives that $D(U \wedge Z)$ vanishes. So Koszul formula gives

$$[U, Z] + D(U)Z = 0 \ , \tag{2.27}$$

where $D(U)$ is a constant. But we have also

$$D(U)Z = [\frac{D(U)}{k-2}I, Z] \ ; \tag{2.28}$$

So we get

$$[U - \frac{D(U)}{k-2}I, Z] = 0 \ , \tag{2.29}$$

which, by Camacho Lins Neto Lemma, gives

$$U = \frac{D(U)}{k-2}I \ . \tag{2.30}$$

When we put this in Formula (2.10) we get

$$A = I \wedge V \ , \tag{2.31}$$

proving the theorem in that case.

C- We suppose $s = 2k-2$.

In that case all the calculations of A- go through. The only difference is that the cocycles A_1 $(= \frac{-k-n+2}{s-k}\lambda\Pi)$ are not always coboundaries. But all other cocycles are coboundaries. So we have proved the Theorem 1.1 □

3 Decomposability of certain Poisson tensors

Theorem 1.1 says nothing concerning $H^{2,2}(\Pi)$, the case where $s = 2$. But we conjecture that it vanishes. At least we know that it vanishes in all precise examples we have computed. For example we prove this result in [D07] for the particular case $\Pi = I \wedge X^{(k-1)}$ with $X^{(k-1)} = \sum_{i=1}^{n} x_i^{k-1}\partial/\partial x_i$ ($k > 3$, $n > 2$).

Also it is proven in [DW06] and [D07] that, when $H^{2,s}(\Pi)$ vanishes for $s = 0, \ldots, k-1$ then the origin is a stable singular point. More precisely this means that: for any neighborhood U of the origin in $\mathbb{R}^n$, there is a neighborhood W of Π in the C^{2k}-topology such that, any Poisson vector Π' in W vanishes at order $k-1$ at a point m in U.

Now let $\Pi^{'(k)}$ the k-order part of Π' at m. Up to a shrinking of W we can suppose that the curl X' of $\Pi^{'(k)}$ is as near as we want from the curl X of Π. A consequence is that we can suppose that X' has, like X, an isolated zero at the origin (which corresponds now to m). This is an evident exercise based on the fact that any homogeneous vector field which vanishes at a point different from the origin vanishes also on a line which passes by the origin.

The most conjectural part of this section is that, shrinking a little more W, $\Pi^{'(k)}$ should have at most rank 2. A first clue for this is the last part of Theorem 1.1 where we find that k-cocycles which are not coboundary have the form $I \wedge V$; this gives the intuition that Poisson deformations of Π must have the form $I \wedge Y$. Another clue is based on the following, easy to prove, lemma.

Lemma 3.1. *Any $k-$homogeneous 2-vector A on $\mathbb{R}^n$ has a unique decomposition*

$$A = A_0 + \frac{1}{2-n-k} I \wedge A_1 \;, \tag{3.1}$$

where A_0 has a null curl and A_1 is the curl of A. Moreover A is a Poisson tensor if and only if we have the equation

$$[A_0, A_0] = A_0 \wedge \frac{2k-4}{2-n-k} A_1 \;. \tag{3.2}$$

Now we could probably use this lemma like this: if $\Pi^{'(k)}$ is sufficiently near Π then its "zero curl" part $\Pi_0^{'(k)}$ is near zero ($\Pi_0 \equiv 0$). But, because $\Pi^{'(k)}$ has a curl $\Pi_1^{'(k)}$ which vanishes only at the origin, Equation (3.2) should give

$$\| \Pi_0^{'(k)} \|^2 \geq a \| \Pi_0^{'(k)} \| \;, \tag{3.3}$$

for a good norm $\| \;\; \|$ and some strictly positive constant a. This must implies $\Pi_0^{'(k)} = 0$ and so $\Pi^{'(k)} = I \wedge V$.

Under the above (conjectural) assumption, we can apply Lemma 1.2 and then Theorem 1.1 to $\Pi^{'(k)}$. This last theorem for $s > k$ and classical techniques (see for example [DZ05] Proposition 2.2.1) show that, for any $s > 2k-2$ there is a local polynomial diffeomorphism ϕ which changes Π' into $(1+f)\Pi^{'(k)}$+terms of order more than s, near m, f being an homogeneous $(k-2)$-polynomial. This can be used to show step by step that $\Pi' \wedge \Pi'$ vanishes at any order at m. So, at least in the analytic case, it vanishes near m. This should prove that Π' has maximal rank 2 near m. Finally it should be easy to improve a little this to get $\Pi' \wedge I = 0$ and so $\Pi' = I \wedge V$ near m.

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