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Research Article

Qualitative Properties of Nonnegative Solutions for a Doubly Nonlinear Problem with Variable Exponents

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We consider the Dirichlet initial boundary value problem $\partial_t u^{m(x)} - \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u) = a(x,t)u^{q(x,t)}$, where the exponents $p(x,t) > 1$, $q(x,t) > 0$, and $m(x) > 0$ are given functions. We assume that $a(x,t)$ is a bounded function. The aim of this paper is to deal with some qualitative properties of the solutions. Firstly, we prove that if $\operatorname{ess\,sup} p(x,t) - 1 < \operatorname{ess\,inf} m(x)$, then any weak solution will be extinct in finite time when the initial data is small enough. Otherwise, when $\operatorname{ess\,sup} m(x) < \operatorname{ess\,inf} p(x,t) - 1$, we get the positivity of solutions for large t. In the second part, we investigate the property of propagation from the initial data. For this purpose, we give a precise estimation of the support of the solution under the conditions that $\operatorname{ess\,sup} m(x) < \operatorname{ess\,inf} p(x,t) - 1$ and either $q(x,t) = m(x)$ or $a(x,t) \le 0$ a.e. Finally, we give a uniform localization of the support of solutions for all $t > 0$, in the case where $a(x,t) < a_1 < 0$ a.e. and $\operatorname{ess\,sup} q(x,t) < \operatorname{ess\,inf} p(x,t) - 1$.

1. Introduction

This paper is devoted to studying qualitative properties of nonnegative weak solutions for the following doubly nonlinear parabolic problem with variable exponents

$$\mathscr{P}: \begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \Delta_{p(x,t)} u = a(x,t)|u|^{q(x,t)-1}u & \text{in } Q_T = \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ b(x,u(x,0)) = b(x,u_0(x)) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \ge 1$, with smooth boundary $\partial\Omega$, $b(x,u) = |u|^{m(x)-1}u$ and $\Delta_{p(x,t)}u$ is defined as

$$\Delta_{p(x,t)} u = \operatorname{div}\left(|\nabla u|^{p(x,t)-2}\nabla u\right). \tag{2}$$

The exponents p, q, m and the coefficient a are given measurable functions. It will be assumed throughout the paper that these functions satisfy some specific conditions.

Problems of this form appear in various applications; for instance in models for gas or fluid flow in porous media ([1, 2]) and for the spread of certain biological populations ([3]). Our motivation to study problem $\mathscr{P}$ with variable exponents is the fact that it is considered as a model of an important class of non-Newtonian fluids which are well known as electrorheological fluids, see ([4]). It appears also as a model in image restoration ([5]) and in elasticity ([6]).

It is well known that solutions of problems such as $\mathscr{P}$ exhibit various qualitative properties, which reflect natural phenomena, according to certain conditions on $p(x,t)$, $q(x,t)$, $m(x)$, $a(x,t)$, and u_0, (see for example [7–13] and the references therein). Among the phenomena that interest us in this work is the finite speed of propagation, which means that if $\rho_0 > 0$ is such that $\operatorname{supp}(u_0) \subset B(x_0,\rho_0)$, then $\operatorname{supp}(u(x,t)) \subset B(x_0,\rho(t))$, for any $t \in (0,T)$, where $\rho(t)$ is a positive function which depends on ρ_0, (i.e., solutions with compact support). This property has various physical meanings; for instance, in the study of turbulent filtration of gas through porous media, a solution with compact support means that gas will remain confined to a bounded region of space, (see [14]).

The phenomenon of finite speed of propagation was investigated by Kalashnikov in [15]. He considered, for $N = 1$, the equation $\partial b(u)/dt - \Delta u = 0$ in $\mathbb{R} \times (0,\infty)$ and, under specific conditions, proved that if the initial condition u_0 has a compact support, then the condition $\int_{0^+}(1/b(s))ds < +\infty$ is necessary and sufficient for solutions to have compact support. This result was extended by Dìaz for $N \geq 1$, in [16]. Later, in [17] Dìaz and Hernández considered the doubly nonlinear problem with absorption term $\partial b(u)/\partial t - \Delta_p u + |u|^{q-1}u = 0$, in $\mathbb{R}^N \times (0,\infty)$, where $b(u) = |u|^{m-1}u$. Under the assumption that u_0 has a compact support and $0 < q < p-1$, they proved that any solution has a compact support for all $t > 0$. This result was obtained by the construction of a local uniform super-solution. Let us recall that the finite speed of propagation phenomenon has been studied by many authors in the last decades, (see [18–21]).

Besides, extinction and nonextinction are also important properties for solutions of evolution equations that have attracted many authors in the last few decades. Most of them focused on equations with constant exponents of nonlinearity, (see [22–26]). For example, Hong et al., dealt in [27] with the homogeneous equation $u_t - \Delta_p u^m = 0$, in $\Omega \times (0,\infty)$, where $p > 1$ and $m > 0$. They proved that the condition $1 < p < 1 + 1/m$ is necessary and sufficient for extinction to occur. Moreover, Zhou and Mu ([28]) studied the extinction behavior of weak solutions for the equation with source term $u_t - \Delta_p u^m = \lambda u^q$, in $\Omega \times (0,\infty)$, where $p > 1$, $m, q, \lambda > 0$ and $m(p-1) < 1$. They proved that $q = m(p-1)$ is a critical extinction exponent.

Otherwise, it is worth noting that problem $\mathscr{P}$ has been treated by Antontsev and Shamarev in several papers. In [29, 30], they proved the existence of weak and strong solutions. Moreover, under certain regularity hypotheses on $m(x)$, $p(x,t)$, and under the sign condition $a(x,t) \leq 0$ a.e, they studied properties of finite speed of propagation and extinction in finite time in [9, 10]. Their results were established by using the local energy method. Here, we shall use the so-called method of sub- and supersolutions to extend some of the results in [9, 10]. To the best of our knowledge, there are few results concerning the study of qualitative properties for parabolic equations with variable exponents by using this method. Furthermore, we shall also extend to the parabolic case some of the results by Zhang et al. in [31], where radial sub- and supersolutions for some elliptic problems with variable exponents are constructed, and some of the results by Chung and Park in [22] and by Yuan et al. in [27], to variable exponents case. In fact, we shall exploit their arguments in our parabolic problem setting with less conditions on the exponents $p(x,t)$, $q(x,t)$, and $m(x)$ and the coefficient $a(x,t)$.

The present paper is organized as follows. In Section 2, we introduce some basic facts about the variable exponents spaces. In Section 3, we give assumptions and general definitions; then, we establish a comparison principle which ensures the uniqueness of solutions. In Section 4, we investigate the extinction and nonextinction properties for the solution of $\mathscr{P}$. Finally in Section 5, we study the property of finite speed of propagation.

2. Preliminaries

In this section we give some elementary results for the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and Sobolev spaces $W^{1,p(x)}(\Omega)$, where Ω is a bounded set of $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary. For more details, see ([11, 32, 33]).

$$C_+\left(\overline{\Omega}\right) = \left\{p(x) : \overline{\Omega} \longrightarrow [p^-, p^+] \subset (1,\infty) ; p \text{ is a continous function}\right\}, \tag{3}$$

where

$$p^+ = \sup_{x\in\overline{\Omega}} p(x), \qquad p^- = \inf_{x\in\overline{\Omega}} p(x). \tag{4}$$

For any $p(x) \in C_+(\Omega)$, we introduce the variable exponent Lebesgue space as follows:

$$L^{p(x)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R}, u \text{ is a measurable function}, \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}, \tag{5}$$

endowed with the Luxemburg norm

$$\|u\|_{p(x),\Omega} = \inf\left\{\lambda : \lambda > 0, \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}. \tag{6}$$

Proposition 1 (see [11, 32, 33]).

(i) *The space $L^{p(x)}(\Omega)$ is a separable and reflexive Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. Moreover, for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|u\|_{p(x),\Omega}\|v\|_{q(x),\Omega}. \tag{7}$$

(ii) *Let $p_1, p_2 \in C_+(\overline{\Omega})$ be given such that $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$ then $L^{p_2(x)}(\Omega)$ is continuously embedded into $L^{p_1(x)}(\Omega)$.*

Proposition 2 (see [11, 32, 33]). *Let*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(x)}(\Omega). \tag{8}$$

Then, we have

(i) $\|u\|_{p(x),\Omega} < 1 (= 1; > 1) \Longleftrightarrow \rho(u) < 1 (= 1; > 1)$

(ii) $\min(\|u\|^{p^+}_{p(x),\Omega}, \|u\|^{p^-}_{p(x),\Omega}) \leq \rho(u) \leq \max(\|u\|^{p^+}_{p(x),\Omega}, \|u\|^{p^-}_{p(x),\Omega})$

(iii) $\|u\|_{p(x),\Omega} \longrightarrow 0 \Longleftrightarrow \rho(u) \longrightarrow 0$; $\|u\|_{p(x),\Omega} \longrightarrow \infty \Longleftrightarrow \rho(u) \longrightarrow \infty$.

Now, we define the variable Sobolev space $W^{1,p(x)}(\Omega)$ as follows:

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u|^{p(x)} \in L^1(\Omega)\right\}, \tag{9}$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x),\Omega} + \|\nabla u\|_{p(x),\Omega}. \tag{10}$$

We say that $p \in C_+(\Omega)$ satisfies the log-Hölder condition in Ω if

$$\begin{gathered} \forall x, y \in \Omega, \\ |x - y| < 1, \\ |p(x) - p(y)| \leq \omega(|x - y|, \end{gathered} \tag{11}$$

where ω satisfies

$$\limsup_{\tau \longrightarrow 0^+} \omega(\tau) \ln\left(\frac{1}{\tau}\right) < \infty. \tag{12}$$

Proposition 3 (see [11, 32, 33]).

(i) *$W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space.*

(ii) *If $p(x)$ satisfies the log-Hölder condition (11), then the space $C^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$. Moreover, we can define the Sobolev space with zero boundary values, $W_0^{1,p(x)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$, with respect to the* norm $\|.\|_{W^{1,p(x)}(\Omega)}$.

Next, let $m(x) > 0$ and $p(x,t) > 1$ be given functions. For $T > 0$ fixed, we denote $Q_T = \Omega \times (0,T)$. Let $m \in C^0(\overline{\Omega})$, we assume that $p(x,t) \in C_+(Q_T)$ satisfies the following log-Hölder condition in Q_T,

$$\begin{gathered} \forall (x,t), (y,\tau) \in Q_T, \\ \text{such that } |(x,t) - (y,\tau)| = \sqrt{|x-y|^2 + |t-\tau|^2} < 1, \end{gathered} \tag{13}$$

we have

$$|p(x,t) - p(y,\tau)| \leq \omega(|(x,t) - (y,\tau)|, \tag{14}$$

where ω satisfies

$$\limsup_{\tau \longrightarrow 0^+} \omega(\tau) \ln\left(\frac{1}{\tau}\right) < \infty. \tag{15}$$

For every fixed $t \in [0,T]$, we introduce the following Banach.

$$\begin{aligned} &V_t(\Omega) \\ &= \left\{u \in L^{m(x)+1}(\Omega) \cap W_0^{1,1}(\Omega) : |\nabla u| \in L^{p(x,t)}(\Omega)\right\}, \end{aligned} \tag{16}$$

endowed with the norm

$$\|u\|_{V_t(\Omega)} = \|u\|_{m(x)+1,\Omega} + \|\nabla u\|_{p(x,t),\Omega}. \tag{17}$$

We denote by $W(Q_T)$ the following Banach space,

$$\begin{aligned} W(Q_T) = \{&u \in L^{m(x)+1}(Q_T) : |\nabla u|^{p(x,t)} \\ &\in L^1(Q_T),\ u(.,t) \in V_t(\Omega) \text{ a.e. } t \in (0,T)\}, \end{aligned} \tag{18}$$

endowed with the norm

$$\|u\|_{W(Q_T)} = \|u\|_{m(x)+1,Q_T} + \|\nabla u\|_{p(x,t),Q_T}. \tag{19}$$

We denote by $W'(Q_T)$ the dual of $W(Q_T)$.

3. Assumptions and Results

Throughout this paper we assume that the coefficients and the exponents of nonlinearity satisfy the following conditions,

$$\begin{aligned} &\text{there exist positive constants } p^\pm, q^\pm, \\ &\quad m^\pm \text{ and } a_1 \text{ such that, for any } (x,t) \text{ in } Q_T \\ &1 < p^- \leq p(x,t) \leq p^+, \\ &0 < q^- \leq q(x,t) \leq q^+, \\ &0 < m^- \leq m(x) \leq m^+, \\ &|a(x,t)| \leq a_1, \end{aligned} \tag{20}$$

and the initial data u_0 satisfies

$$u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \text{ a.e. in } \Omega. \tag{21}$$

Now, let us state the definition of weak solutions for the problem $\mathscr{P}$.

Definition 4. We say that $u(x,t)$ is a super-(sub)solution of $\mathscr{P}$ on Q_T if

(1) $u \in L^\infty(Q_T) \cap W(Q_T)$ and $(\partial/\partial t)b(x,u) \in W'(Q_T)$.

(2) for every nonnegative test function $\phi \in W(Q_T)$ and $(\partial/\partial t)\phi \in W'(Q_T)$, we have

$$\begin{aligned} &\int_{Q_T} \phi \frac{\partial}{\partial t} b(x,u) + |\nabla u|^{p(x,t)-2} \nabla u \nabla \phi\,dx\,dt \\ &\quad \geq (\leq) \int_{Q_T} a(x,t)\,|u|^{q(x,t)-1} u\phi\,dx\,dt. \end{aligned} \tag{22}$$

(3) $b(x,u(.,0)) \geq (\leq) b(x,u_0)$ a.e. in Ω, and $u \geq (\leq) 0$ on $\partial\Omega\times(0,T)$.

A function u is a weak solution of $\mathscr{P}$ if it is simultaneously a supersolution and a subsolution.

The following result concerning the local existence of weak solutions of problem $\mathscr{P}$ is established in [29].

Theorem 5. *Let $m \in C^0(\Omega)$, $p(x,t)$ satisfies the log-Hölder condition in Q_T (14), and let conditions (20) and (21) be fulfilled. Moreover, we assume that*

$$\left|\nabla\left(\frac{1}{m(x)}\right)\right| \in L^{\beta}(\Omega), \quad \text{with some } \beta > 1, \tag{23}$$

and the exponents m, p satisfy one of the following conditions

(1) *p is independent of t, and $m(x) > 0$ in Ω,*

(2) *$p(x,t) > 1$, $m(x) > 1$, and $|\nabla(1/m(x))| \in L^{p(x,t)}(Q_T)$,*

(3) *$p(x,t) > 1$, $m(x) > 0$, $|\nabla(1/m(x))| \in L^{p(x,t)}(Q_T)$, and*

$$1 > \frac{1}{p(x,t)} + m(x) \quad \text{in } Q_T. \tag{24}$$

Then, the problem $\mathscr{P}$ has at least one nonnegative weak solution in Q_{T^*}, with

$$T^* = \sup\left\{\theta : \|u(t)\|_{\infty,\Omega} < \infty,\ \forall t \in (0,\theta)\right\}. \tag{25}$$

Moreover, for small τ the solution satisfies the estimate

$$\|u\|_{\infty,\Omega} \leq \left\|u_0\right\|_{\infty,\Omega} e^{At}, \quad t \in [0,\tau]. \tag{26}$$

with a constant A depending only on the data.

The following comparison principle is essential to prove uniqueness and qualitative properties of nonnegative solutions.

Proposition 6. *Let u (respectively v) be a subsolution (respectively supersolution) of $\mathscr{P}$, with the initial datum u_0 (respectively v_0), satisfying (21). We assume that $(\partial/\partial t)b(x,u)$, $(\partial/\partial t)b(x,v) \in L^1(Q_T)$, and that conditions (20) are fulfilled. If either $a(x,t) \leq 0$ a.e. in Q_T, or $m^+ \leq q^-$, then we have $u \leq v$ a.e. in Q_T.*

Remark 7. Note that the comparison principle is true for weak solutions u with $(\partial/\partial t)b(x,u) \in L^1(Q_T) \cap W'(Q_T)$ and recall that in the papers [29, 30], the authors gave some conditions on the data of problem $\mathscr{P}$ in order to ensure that this class of solutions is nonempty.

Proof. We consider the test function $\phi_\eta = \operatorname{sign}_\eta(u-v)$, where

$$\operatorname{sign}_\eta(s) = \begin{cases} 1, & \text{if } s > \eta, \\ \dfrac{s}{\eta}, & \text{if } |s| \leq \eta, \\ 0, & \text{if } s < -\eta, \end{cases} \tag{27}$$

and $\eta > 0$ is small. It is easy to see that

$$\phi_\eta(s) \longrightarrow \operatorname{sign}^+(s) \quad \text{as } \eta \longrightarrow 0, \tag{28}$$

where $\operatorname{sign}_+(s) = 1$, if $s > 0$, and $\operatorname{sign}_+(s) = 0$, if $s \leq 0$. Moreover, we claim that for all $u, v \in W(Q_t)$ the function $\phi_\eta(u-v) \in W(Q_T)$. Indeed, we observe that for all $s \in \mathbb{R}$, $|\phi_\eta(s)| \leq 1$. Then, by Proposition 2

$$\left\|\phi_\eta(u-v)\right\|_{m(x)+1,Q_T} < C(T). \tag{29}$$

On the other hand, we have

$$\begin{aligned} &\int_{Q_T} \left|\nabla\phi_\eta(u-v)\right|^{p(x,t)} dx\,dt \\ &= \int_0^T \int_{\{|u-v|\leq\eta\}} \left|\nabla\phi_\eta(u-v)\right|^{p(x,t)} dx\,dt \\ &\quad + \int_0^T \int_{\{u-v>\eta\}} \left|\nabla\phi_\eta(u-v)\right|^{p(x,t)} dx\,dt \\ &\quad + \int_0^T \int_{\{u-v<-\eta\}} \left|\nabla\phi_\eta(u-v)\right|^{p(x,t)} dx\,dt \\ &= \int_0^T \int_{\{|u-v|\leq\eta\}} \left|\frac{\nabla(u-v)}{\eta}\right|^{p(x,t)} dx\,dt \\ &\leq C(\eta)\left(\int_{Q_T} |\nabla u|^{p(x,t)} + |\nabla u|^{p(x,t)}\right) dx < \infty. \end{aligned} \tag{30}$$

Hence, from Proposition 2 we get

$$\left\|\nabla\phi_\eta(u-v)\right\|_{p(x,t),Q_T} < \infty. \tag{31}$$

Therefore, combining (29) and (31) we deduce the claim. On the other hand, from Definition 4, we obtain

$$\begin{aligned} &\int_{Q_T} \frac{\partial}{\partial t}(b(x,u) - b(x,v))\,\phi_\eta dx\,dt \\ &\quad + \int_{Q_T} \left(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\right)\nabla\phi_\eta dx\,dt \\ &\leq \int_{Q_T} a(x,t)\left(|u|^{q(x,t)-1}u - |v|^{q(x,t)-1}v\right)\phi_\eta dx\,dt. \end{aligned} \tag{32}$$

Due to a monotonicity argument, we have

$$\int_{Q_T} \left(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\right)\nabla\phi_\eta dx\,dt \geq 0, \tag{33}$$

then

$$\begin{aligned} &\int_{Q_T} \frac{\partial}{\partial t}(b(x,u) - b(x,v))\,\phi_\eta dx\,dt \\ &\leq \int_{Q_T} a(x,t)\left(|u|^{q(x,t)-1}u - |v|^{q(x,t)-1}v\right)\phi_\eta dx\,dt. \end{aligned} \tag{34}$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{\eta\longrightarrow 0}\int_{Q_T}\frac{\partial}{\partial t}(b(x,u)-b(x,v))\phi_\eta dx\,dt = \int_{Q_T}\frac{\partial}{\partial t}(b(x,u)-b(x,v))_+ dx\,dt. \quad (35)$$

Now, we can write

$$\begin{aligned}&\int_{Q_T} a(x,t)\left(|u|^{q(x,t)-1}u-|v|^{q(x,t)-1}v\right)\phi_\eta dx\,dt\\ &= \int_{Q_T} a(x,t)\\ &\cdot\int_0^1\frac{\partial}{\partial s}\left(|su+(1-s)v|^{q(x,t)-1}(su+(1-s)v)\right)\\ &\cdot\phi_\eta ds\,dx\,dt = \int_{Q_T} a(x,t)\int_0^1 q(x,t)\\ &\cdot\left(|su+(1-s)v|^{q(x,t)-1}(u-v)\right)\phi_\eta ds\,dx\,dt.\end{aligned} \quad (36)$$

Then, from (34) and (35), by letting $\eta \longrightarrow 0$, we obtain

$$\begin{aligned}&\int_{Q_T}\frac{\partial}{\partial t}(b(x,u)-b(x,v))_+ dx\,dt \le \int_{Q_T} a(x,t)\\ &\cdot\int_0^1 q(x,t)|su+(1-s)v|^{q(x,t)-1}\\ &\cdot(u-v)_+ ds\,dx\,dt.\end{aligned} \quad (37)$$

Hence, if $a(x,t)\le 0$ a.e. in Q_T, it follows that

$$\int_{Q_T}\frac{\partial}{\partial t}(b(x,u)-b(x,v))_+ dx\,dt \le 0. \quad (38)$$

Then, by Gronwall's lemma we deduce the desired result. Now, we continue the proof without any sign condition on $a(x,t)$. From (37), by using $m^+ \le q^-$ and the Lebesgue's dominated convergence theorem it follows that

$$\begin{aligned}\int_{Q_T}\frac{\partial}{\partial t}(b(x,u)-b(x,v))_+ dx\,dt &\le a_1 q^+\int_{Q_T}\int_0^1 |su+(1-s)v|^{q(x,t)-m(x)}|su+(1-s)v|^{m(x)-1}(u-v)_+ ds\,dx\,dt\\ &\le \int_{Q_T}\left(\|u\|_{\infty,Q_T}+\|v\|_{\infty,Q_T}\right)^{q(x,t)-m(x)}\left[\int_0^1 |su+(1-s)v|^{m(x)-1}(u-v)_+ ds\right]dx\,dt\\ &\le C\lim_{\eta\longrightarrow 0}\int_{Q_T}\int_0^1 |su+(1-s)v|^{m(x)-1}(u-v)\phi_\eta ds\,dx\,dt\\ &\le C\lim_{\eta\longrightarrow 0}\int_{Q_T}\frac{1}{m(x)}\int_0^1\left[\frac{\partial}{\partial s}|su+(1-s)v|^{m(x)-1}(su+(1-s)v)\,ds\right]\phi_\eta dx\,dt\\ &\le \frac{C}{m^-}\int_{Q_T}(b(x,u)-b(x,v))_+ dx\,dt,\end{aligned} \quad (39)$$

where C is depending on the supnorms of u and v. Hence we deduce from Gronwall's lemma that

$$\int_\Omega (b(x,u)-b(x,v))_+ dx \le e^{CT}\int_\Omega (b(x,u(x,0))-b(x,v(x,0)))_+ dx, \quad (40)$$

which allows us to conclude the result. □

Definition 8. We call $u(x,t)$ a strong solution of $\mathscr{P}$, if u is a weak solution and satisfies

$$\frac{\partial}{\partial t}b(x,u)\in L^1(Q_T). \quad (41)$$

4. Finite Time Extinction and Nonextinction

This section is devoted to studying extinction and positivity properties for nonnegative solutions of problem $\mathscr{P}$, without any sign condition on the coefficient $a(x,t)$, and according to the ranges of $p(x,t)$, $q(x,t)$, and $m(x)$. The proof of the results is based on the construction of suitable sub- and supersolutions and on the use of the preceding comparison principle given in Proposition 6.

4.1. Finite Time Extinction.

We state and prove our main extinction result.

Theorem 9. *Let u be a strong solution of $\mathscr{P}$. Assume that $m^+ < q^-$, $p^+ - 1 < m^-$, $\sup_{Q_T}|\nabla p(x,t)| < \infty$ and $\|u_0\|_\infty$ is small enough. Then, there exists a finite time T_1 such that for all $t \ge T_1$*

$$u(x,t)=0,\quad a.e.\ x\in\Omega. \quad (42)$$

Proof. We consider the following function

$$v(x,t)=k(T_1-t)_+^\alpha \ln(l+x_1+\cdots+x_n) \quad (43)$$

where

$$l = \sup_{(x_1,\dots,x_n)\in\Omega} \{|x_1| + \cdots + |x_n|\} + 2, \quad (44)$$

$$T_1 = \left(\frac{\|u_0\|_\infty}{k \ln(2)} \right)^{1/\alpha}, \quad (45)$$

and

$$\alpha = \frac{1}{m^- - p^+ + 1}, \quad (46)$$

where $k > 0$ will be specified later. Our goal is to prove that v is a supersolution of $\mathscr{P}$ and by comparison principle, we can thus deduce the result. Firstly, we shall show that

$$v \in L^\infty(Q_T) \cap W(Q_T), \qquad \frac{\partial}{\partial t} v^{m(x)} \in W'(Q_T), \qquad \forall T > 0. \quad (47)$$

For all $x \in \Omega$ and $t > 0$, we have

$$k(T_1 - t)^\alpha \ln(l + x_1 + \cdots + x_n) \le kT_1^\alpha \ln(2l) < \infty, \quad (48)$$

and

$$|\nabla u| = \frac{k(T_1 - t)^\alpha N^{1/2}}{l + x_1 + \cdots + x_n} \le \frac{KT_1^\alpha N^{1/2}}{2} < \infty, \quad (49)$$

which implies that $v \in L^\infty(Q_T) \cap W(Q_T)$. Moreover, we have

$$\frac{\partial}{\partial t} v^{m(x)} = -\alpha m(x) (k \ln(l + x_1 + \cdots + x_n))^{m(x)} \cdot (T_1 - t)_+^{\alpha m(x) - 1}, \quad (50)$$

then

$$\left| \frac{\partial}{\partial t} v^{m(x)} \right| \le \alpha m^+ k^{m^-} \max\left\{ T_1^{\alpha m^+ - 1}, T_1^{\alpha m^- - 1} \right\} (\ln(2l))^{m^+}, \quad (51)$$

and hence $(\partial/\partial t) v^{m(x)} \in L^\infty(Q_T)$. Due to the embedding

$$L^\infty(Q_T) = \left(L^1(Q_T)\right)' \hookrightarrow W'(Q_T), \quad (52)$$

we get that $(\partial/\partial t) v^{m(x)} \in W'(Q_T)$.

On the other hand, it is clear that $v(x,0) \ge \|u\|_\infty \ge u_0(x)$, for a.e. $x \in \Omega$, and $v(x,t) \ge 0$, for all $x \in \partial\Omega$, $t \ge 0$. Next, we prove that

$$\frac{\partial}{\partial t} v^{m(x)} - \Delta_{p(x,t)} v \ge a(x,t) v^{q(x,t)}, \quad \text{in } Q_T. \quad (53)$$

Since $|a(x,t)| \le a_1$, it suffices to prove that

$$\frac{\partial}{\partial t} v^{m(x)} - \Delta_{p(x,t)} v \ge a_1 v^{q(x,t)}, \quad \text{in } Q_T. \quad (54)$$

By simple calculations, we obtain

$$\begin{aligned}
-\Delta_{p(x,t)} v &= \frac{1}{N^{1/2}} \left(\frac{k(T_1 - t)_+^\alpha N^{1/2}}{l + x_1 + \cdots + x_n} \right)^{p(x,t)-1} \\
&\quad \cdot \left[\frac{(p(x,t) - 1) N}{l + x_1 + \cdots + x_n} - \sum_{i=1}^N \frac{\partial p}{\partial x_i} \ln\left(\frac{k(T_1 - t)_+^\alpha N^{1/2}}{l + x_1 + \cdots + x_n} \right) \right] \\
&\ge \frac{1}{N^{1/2}} \left(\frac{k(T_1 - t)_+^\alpha N^{1/2}}{l + x_1 + \cdots + x_n} \right)^{p(x,t)-1} \\
&\quad \cdot \left[\frac{(p^- - 1) N}{l + x_1 + \cdots + x_n} - \sup_{Q_T} |\nabla p| \frac{k(T_1 - t)_+^\alpha N}{l + x_1 + \cdots + x_n} \right] \\
&\ge \frac{\left(k(T_1 - t)_+^\alpha\right)^{p(x,t)-1} N^{p(x,t)/2}}{(l + x_1 + \cdots + x_n)^{p(x,t)}} \left[(p^- - 1) - \sup_{Q_T} |\nabla p| k(T_1)^\alpha \right] \\
&= \frac{\left(k(T_1 - t)_+^\alpha\right)^{p(x,t)-1} N^{p(x,t)/2}}{(l + x_1 + \cdots + x_n)^{p(x,t)}} \left[(p^- - 1) - \sup_{Q_T} |\nabla p| \frac{\|u_0\|_\infty}{\ln(2)} \right].
\end{aligned} \quad (55)$$

We set

$$S = (p^- - 1) - \sup_{Q_T} |\nabla p| \frac{\|u_0\|_\infty}{\ln(2)}. \quad (56)$$

If $\sup_{Q_T} |\nabla p| = 0$, then $S = p^- - 1 > 0$. Otherwise, since $\|u_0\|_\infty$ is small enough, then we can assume that $\|u_0\|_\infty < (p^- - 1)\ln(2)/\sup_{Q_T} |\nabla p|$, to deduce that $S > 0$.

Now, we are looking for conditions on k to get (54). Thanks to (50) and (55), it is sufficient to have

$$\alpha m(x) (k \ln(l + x_1 + \cdots + x_n))^{m(x)} (T_1 - t)_+^{\alpha m(x) - 1} \ge a_1 \left(k(T_1 - t)_+^\alpha \ln(l + x_1 + \cdots + x_n) \right)^{q(x,t)}, \quad (57)$$

and

$$\frac{\left(k(T_1 - t)_+^\alpha\right)^{p(x,t)-1} N^{p(x,t)/2}}{(l + x_1 + \cdots + x_n)^{p(x,t)}} S \ge 2\alpha m(x) \cdot (k \ln(l + x_1 + \cdots + x_n))^{m(x)} (T_1 - t)_+^{\alpha m(x) - 1}. \quad (58)$$

As $\|u_0\|_\infty$ is small enough, we can assume also that $\|u_0\|_\infty \le k\ln(2)$. Then it yields $T_1 \le 1$, which implies that $(T_1 - t)_+ \le 1$.

Since $\alpha = 1/(m^- - p^+ + 1)$ and $q^- > m^+$, thus (57) and (58) reduce to

$$(T_1 - t)_+^{\alpha m(x)-1} \left(\alpha m(x) \left(k \ln(l + x_1 + \cdots + x_n)\right)^{m(x)} - a_1 \left(k \ln(l + x_1 + \cdots + x_n)\right)^{q(x,t)}\right) \geq 0, \tag{59}$$

and

$$(T_1 - t)_+^{\alpha(p(x,t)-1)} \left(\frac{k^{p(x,t)-1} N^{p(x,t)/2}}{(l + x_1 + \cdots + x_n)^{p(x,t)}} S - 2\alpha m(x) \left(k \ln(l + x_1 + \cdots + x_n)\right)^{m(x)}\right) \geq 0, \tag{60}$$

which are satisfied if

$$k^{q(x,t)-m(x)} \leq \frac{\alpha m^- (\ln(2))^{m^+}}{a_1 (\ln(2l))^{q^+}}, \tag{61}$$

and

$$k^{m(x)-p(x,t)+1} \leq \frac{S N^{p^-/2}}{2\alpha m^+ (2l)^{p^+} (\ln(2l))^{m^+}}. \tag{62}$$

By setting $E = SN^{p^-/2}/2\alpha m^+ (2l)^{p^+} (\ln(2l))^{m^+}$ and $F = \alpha m^- (\ln(2))^{m^+}/a_1(\ln(2l))^{q^+}$, we can choose k such that

$$k = \min\left\{(E)^{1/(m^- - p^+ + 1)}, (E)^{1/(m^+ - p^- + 1)}, (F)^{1/(q^- - m^+)}, (F)^{1/(q^+ - m^-)}\right\}. \tag{63}$$

Therefore, we get the desired result. □

Next, we will mention an extinction result where there is no condition between the ranges of $p(x,t)$ and $m(x)$.

Proposition 10. *Let u be a strong solution of $\mathscr{P}$. Assume that $m^- > q^+$ and $a(x,t) \leq -c < 0$, $\sup_{Q_T} |\nabla p(x,t)| < \infty$ and $\|u_0\|_\infty$ is small enough. Then, there exists a finite time T_1 such that for all $t \geq T_1$*

$$u(x,t) = 0, \quad a.e.\ x \in \Omega. \tag{64}$$

Proof. We consider the same supersolution $v(x,t)$ as in the proof of Theorem 9 but we choose here $k = 1$, which means

$$v(x,t) = (T_1 - t)_+^{\alpha} \ln(l + x_1 + \cdots + x_n) \tag{65}$$

where

$$l = \sup_{(x_1,\ldots,x_n)\in\Omega} \{|x_1| + \cdots + |x_n|\} + 2, \tag{66}$$

$$T_1 = \left(\frac{\|u_0\|_\infty}{\ln(2)}\right)^{1/\alpha}, \tag{67}$$

and

$$\alpha = \frac{1}{m^- - q^+}. \tag{68}$$

We have already shown in Theorem 9 that $-\Delta_{p(x,t)} v \geq 0$. We claim that

$$\frac{\partial}{\partial t} v^{m(x)} \geq a(x,t) u^{q(x,t)}, \tag{69}$$

by using the same lines as in the proof of Theorem 9. Since $q^+ < m^-$, it is therefore sufficient to have

$$(T_1 - t)_+^{\alpha q(x,t)} \left(c \left(\ln(l + x_1 + \cdots + x_n)\right)^{q(x,t)} - \alpha m(x) \left(\ln(l + x_1 + \cdots + x_n)\right)^{m(x)}\right) \geq 0, \tag{70}$$

which is satisfied if we choose

$$c = \frac{\alpha m^+ (\ln(2l))^{m^+}}{(\ln(2))^{q^-}}. \tag{71}$$

Consequently, by the comparison principle we deduce the extinction of solution in finite time. □

4.2. Nonextinction of Solutions. The following theorem deals with the positivity of weak solutions.

Theorem 11. *Let u be a strong solution of $\mathscr{P}$ and u_0 not identically zero. Assume that $m^+ < q^-$, $m^+ < p^- - 1$, and $\sup_{Q_T} |\nabla p(x,t)| < \infty$. Then, there exists a finite time T_p such that for all $t \geq T_p$*

$$u(x,t) > 0, \quad a.e.\ x \in \Omega. \tag{72}$$

The method of proof is inspired from [27], where the constant exponents case is treated. However, some difficulties arise in the construction of subsolutions due to the fact that the exponents are variable. The proof of this theorem is divided into two lemmas. In the first lemma, we show by using a comparison function that the support of weak solution is nondecreasing with respect to time. In the second lemma, we show that the solution is positive locally in Ω; then, by a finite covering argument, we deduce the result.

Lemma 12. *Let u be a strong solution of $\mathscr{P}$. Assume that $m^+ < q^-$ and the initial condition u_0 is nontrivial. Then*

$$\operatorname{supp} u(.,s) \subset \operatorname{supp} u(.,t), \tag{73}$$

for all $0 < s < t$.

The proof of Lemma 12 follows the same lines as that of lemma 4.2 in [22], where the constant exponents case is studied. For completeness, we shall give it here.

Proof. The argument used here is based on a comparison function with which we show that the support of solution is increasing. For that we consider ω an arbitrary set which is a nonzero measure subset of Ω such that $\inf_{x\in\omega} u_0 \neq 0$. We divide the proof in two cases, firstly we treat the case where

$m^- \geq 1$ and then the case where $m^+ < 1$. If $m^- \geq 1$, we consider the following function:

$$v_1(x,t) = \begin{cases} \left(\delta^{m^+-q^-} + (q^- - m^+) a_1 t\right)^{1/(m^+-q^-)}, & \text{in } \omega \times (0,\infty), \\ 0 & \text{on } \partial\omega \times (0;\infty), \end{cases} \quad (74)$$

where $\delta = \min\{\inf_{x\in\omega} u_0, 1\}$. It is clear that $v_1 \in L^\infty(Q_T) \cap W(Q_T)$, and it is easy to verify that

$$\frac{\partial}{\partial t} v_1^{m(x)} \in L^\infty(Q_T) \hookrightarrow W'(Q_T). \quad (75)$$

On the other hand, by direct calculations we get

$$\frac{\partial}{\partial t} v_1^{m^+} = -a_1 m^+ v_1^{q^-}. \quad (76)$$

Hence

$$v_1^{m^+-1} \frac{\partial v_1}{\partial t} = -a_1 v_1^{q^-}. \quad (77)$$

Since $m(x) \geq 1$ and $v_1(x,t) \leq 1$ for all $x \in \omega$, $t > 0$, it follows that

$$\frac{\partial}{\partial t} v_1^{m(x)} - \Delta_{p(x,t)} v_1 - a(x,t) v_1^{q(x,t)} \leq 0, \quad x \in \omega,\ t > 0. \quad (78)$$

Moreover, from the definition of v_1, we have $v_1(x,0) \leq u_0$, almost everywhere in ω, and $v_1(x,t) = 0$ for all, $x \in \partial\omega$, $t > 0$. Thus, by comparison principle we conclude that for any arbitrary ω where $\inf_{x\in\omega} u_0 > 0$, the weak solution of $\mathscr{P}$ satisfies

$$u(x,t) > 0 \quad \text{a.e. } x \in \omega, \text{ and all } t > 0; \quad (79)$$

and the result follows in this case.

If $m^+ < 1$, we consider the following function:

$$v_2(x,t) = \begin{cases} \left(\delta^{m^+-q^-} + \dfrac{(q^- - m^+)}{m^-} a_1 t\right)^{1/(m^+-q^-)}, & \text{in } \omega \times (0,\infty) \\ 0, & \text{on } \partial\omega \times (0;\infty) \end{cases} \quad (80)$$

where $\delta = \min\{\inf_{x\in\omega} u_0, 1\}$. By the same argument used previously we obtain that $v_2 \in L^\infty(Q_T) \cap W(Q_T)$, and $(\partial/\partial t) v_2^{m(x)} \in W'(Q_T)$. Moreover, by direct calculations we get

$$\frac{\partial}{\partial t} v_2^{m^+} = -\frac{a_1 m^+}{m^-} v_2^{q^-}, \quad (81)$$

since $m^+ < 1$ and $v_2(x,t) \leq 1$ for all $x \in \omega$, $t > 0$. Hence

$$\frac{\partial}{\partial t} v_2^{m(x)} + a_1 v_2^{q(x,t)} \leq 0, \quad (82)$$

Therefore, we have

$$\frac{\partial}{\partial t} v_2^{m(x)} - \Delta_{p(x,t)} v_2 - a(x,t) v_2^{q(x,t)} \leq 0, \quad x \in \omega,\ t > 0. \quad (83)$$

Thus, by the same argument used previously we deduce our result. □

Lemma 13. *Under the same assumptions of Theorem 11, let the initial condition satisfies* $\inf_{x\in B_r(x_0)} u_0 \neq 0$, *for some* $0 < r < R \leq 1/2$. *Then, there exists* $T_p > 0$, *such that for any* $t \geq T_p$,

$$u(x,t) > 0, \quad \text{a.e. } x \in B_R(x_0), \quad (84)$$

where $B_R(x_0) = \{x \in \Omega : |x - x_0| < R\}$.

Proof. We consider the following function:

$$\psi(x,t) = krS^{-\alpha/\lambda}(t) H^{(p^--1)/\gamma}(x,t), \quad (85)$$

where

$$H(x,t) = \left[1 - \left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{p^-/(p^--1)}\right]_+, \qquad S(t) = r^{\lambda/\alpha} + \lambda\sigma k^\beta r^\gamma t, \quad 0 \leq t \leq T_p, \quad (86)$$

and

$$T_p = \frac{R^\lambda - r^{\lambda/\alpha}}{\lambda\sigma k^\beta r^\gamma}, \quad (87)$$

where k and α are positive constants small and large enough, respectively, γ is a positive constant such that

$$\gamma \leq \frac{p^- - 1}{p^+ - 1}(p^- - 1 - m^+), \quad (88)$$

and λ, γ, β are positive constants and will be determined later. By direct calculations, we get

$$|\nabla\psi| = \frac{krp^-}{\gamma} S^{-(\alpha+1)/\lambda}(t) H^{(p^--1)/\gamma-1}(x,t) \cdot \left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{1/(p^--1)}, \quad (89)$$

and

$$\begin{aligned} \frac{\partial}{\partial t}\psi^{m(x)} &= -m(x)\sigma\alpha k^{\beta+m(x)} r^{\gamma+m(x)} S^{-\alpha m(x)/\lambda-1}(t) \\ &\quad \cdot H^{m(x)((p^--1)/\gamma)}(x,t) + m(x)\frac{\sigma p^-}{\gamma} \\ &\quad \cdot k^{\beta+m(x)} r^{\gamma+m(x)} S^{-\alpha m(x)/\lambda-1}(t) \\ &\quad \cdot H^{m(x)((p^--1)/\gamma)-1}(x,t) \\ &\quad \cdot \left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{p^-/(p^--1)}. \end{aligned} \quad (90)$$

We can see easily that $H(x,t) \leq 1$ a.e. in $B_R(x_0) \times (0,T_p)$. Then, by using (88) we obtain

$$\psi \in L^{\infty}\left(B_R(x_0) \times \left(0, T_p\right)\right) \cap W\left(B_R(x_0) \times \left(0, T_p\right)\right), \tag{91}$$

and

$$\frac{\partial}{\partial t}\psi^{m(x)} \in L^{\infty}\left(B_R(x_0) \times \left(0, T_p\right)\right) \hookrightarrow W'\left(B_R(x_0) \times \left(0, T_p\right)\right). \tag{92}$$

Moreover, from the definition of ψ we have

$$\psi(x,0) = kH^{(p^- -1)/\gamma}(x,0) \leq k \leq u_0(x) \quad \text{a.e. } x \in B_R(x_0). \tag{93}$$

Since for all $t \in [0, T_p]$,

$$S^{1/\lambda}(t) \leq S^{1/\lambda}\left(T_p\right) = R, \tag{94}$$

then $H(x,t) = 0$ on $\partial B_R(x_0) \times [0,T_p]$, which implies that $\psi(x,t) = 0$ on $\partial B_R(x_0) \times [0,T_p]$. Now, let us show that

$$\frac{\partial}{\partial t}\psi^{m(x)} - \Delta_{p(x,t)}\psi - a(x,t)\psi^{q(x,t)} \leq 0, \quad \text{in } B_R(x_0) \times \left(0, T_p\right). \tag{95}$$

To do so, it suffices to show that

$$\frac{\partial}{\partial t}\psi^{m(x)} - \Delta_{p(x,t)}\psi + a_1\psi^{q(x,t)} \leq 0, \quad \text{in } B_R(x_0) \times \left(0, T_p\right). \tag{96}$$

Using (89), we obtain by simple calculations

$$\begin{aligned} -\Delta_{p(x,t)}\psi = {} & \frac{|\nabla\psi|^{p(x,t)-1}}{|x-x_0|}\left(N - 1 + \frac{p(x,t)-1}{p^- - 1}\right) \\ & + |\nabla\psi|^{p(x,t)-1}\ln\left(|\nabla\psi|\right) \\ & \cdot \sum_{i=1}^{N}\frac{\partial p(x,t)}{\partial x_i}\frac{x_i - x_{0,i}}{|x-x_0|} - |\nabla\psi|^{p(x,t)-1} \\ & \cdot \left(\frac{p^- - 1}{\gamma} - 1\right)\left(\frac{p^-}{p^- - 1}\right)H^{-1}(x,t) \\ & \cdot \left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{1/(p^- -1)}\left(\frac{p(x,t)-1}{S^{1/\lambda}(t)}\right). \end{aligned} \tag{97}$$

We set $C_1 = (N-1+(p^+-1)/(p^- -1))$, $\mu = \sup_{Q_{T_p}}|\nabla p(x,t)|$, and

$$G(x,t) = m(x)\sigma\alpha k^{\beta+m(x)}r^{\gamma+m(x)}S^{-\alpha m(x)/\lambda - 1}(t) \cdot H^{m(x)((p^- -1)/\gamma)}(x,t). \tag{98}$$

To get (96), by (90) and (97) it suffices to have

$$G(x,t)\left(\alpha - 3\frac{p^-}{\gamma}\left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{p^-/(p^- -1)}H^{-1}(x,t)\right) \geq 0, \tag{99}$$

$$G(x,t) \geq 3\frac{C_1}{|x-x_0|}\left[\frac{krp^-}{\gamma}S^{-(\alpha+1)/\lambda}(t) \cdot H^{(p^- -1)/\gamma - 1}(x,t)\left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{1/(p^- -1)}\right]^{p(x,t)-1}, \tag{100}$$

$$G(x,t) \geq 3a_1\left[krS^{-\alpha/\lambda}(t)H^{(p^- -1)/\gamma}(x,t)\right]^{q(x,t)} \tag{101}$$

and

$$\left(\frac{p^- - 1}{\gamma} - 1\right)\left(\frac{p^-}{S^{1/\lambda}(t)}\right)H^{-1}(x,t) \cdot \left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{1/(p^- -1)} \geq \mu\frac{krp^-}{\gamma}S^{-(\alpha+1)/\lambda}(t) \cdot H^{(p^- -1)/\gamma - 1}(x,t)\left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{1/(p^- -1)} \tag{102}$$

Now, our goal is to choose the constants γ, λ, and β in order to verify each of the inequalities above. Firstly, let us show the inequality (99). Since α is large enough and $R < 1/2$, we have

$$\alpha > \frac{\ln(r)}{\ln\left(2^{(p^- -1)/p^-}R\right)}, \tag{103}$$

from which we get

$$\left(\frac{1}{2}\right)^{(p^- -1)/p^-} > \frac{R}{r^{1/\alpha}} > \frac{|x-x_0|}{S^{1/\lambda}(0)}. \tag{104}$$

Then, it yields

$$H(x,t) \geq \left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{p^-/(p^- -1)}, \quad \text{in } B_R(x_0) \times \left(0, T_p\right). \tag{105}$$

Therefore, we have

$$3\alpha - \frac{p^-}{\gamma}\left(\frac{|x-x_0|}{S^{1/\lambda}(t)}\right)^{p^-/(p^- -1)}H^{-1}(x,t) \geq 3\alpha - \frac{p^-}{\gamma} \geq 0, \tag{106}$$

so (99) is satisfied. Next, to get (100) and (101), we use the fact that $s(t) \leq 1$ and $H(x,t) \leq 1$ a.e. in $B_R(x_0)\times(0,T_p)$. By setting

$$\begin{aligned}\gamma &= \min\left\{\frac{p^- - 1}{p^+ - 1}\left(p^- - 1 - m^+\right), q^- - m^+\right\},\\ \beta &= \min\left\{p^- - 1 - m^+, q^- - m^+\right\}\\ \lambda &= \max\left\{\left(\left(\alpha + \frac{p^-}{p^- - 1}\right)\left(p^+ - 1\right) - \alpha m^-\right),\right.\\ &\qquad \left.\alpha\left(q^+ - m^-\right)\right\},\end{aligned} \tag{107}$$

and using α large enough, so that

$$\alpha > \max\left\{Z\frac{3C_1}{\sigma m^-}, \frac{3a_1}{m^-\sigma}\right\}, \tag{108}$$

where

$$Z = \max\left\{\left(\frac{p^-}{\gamma}\right)^{p^+-1}, \left(\frac{p^-}{\gamma}\right)^{p^--1}\right\}, \tag{109}$$

we obtain the inequalities (100) and (101). Finally, to get the inequality (102), it suffices to have

$$p^-\left(\frac{p^- - 1}{\gamma} - 1\right)S^{\alpha/\lambda}(0) \geq \mu\frac{krp^-}{\gamma}, \tag{110}$$

which reduces to

$$\left(\frac{p^- - 1}{\gamma} - 1\right) \geq \mu\frac{k}{\gamma}. \tag{111}$$

Since k is small enough, the last inequality holds, which means that the inequality (102) is satisfied and allows us to deduce inequality (96). Then, we obtain by comparison principle, for each $t \in [0, T_p]$,

$$u(x,t) \geq \psi(x,t) \quad \text{a.e. } x \in B_R(x_0). \tag{112}$$

Therefore, the result follows from Lemma 12. □

Proof of Theorem 11. The proof is similar to that of Theorem 1.2, in [27], and we omit the details here. □

5. Finite Speed of Propagation Property

In this section we shall give precise estimates for the of support of the solution of $\mathcal{P}$, depending on the size of the support of u_0. Let us emphasize that each estimation is obtained under a sign condition on $a(x,t)$ and depending on the range of the exponents $p(x,t)$, $q(x,t)$, and $m(x)$. As in [21], the proof is based on the construction of local supersolutions and on the use of the comparison principle.

Concerning the construction of supersolutions, we shall proceed as in [31].

Note that under some conditions on the data, if $a(x,t)$ is positive, then the solutions will blow up in finite time (see [10]). For that it requires to construct a supersolution defined locally in time, which means in $(0,T)$ for any $T \leq T^*$, where T^* is the maximal existence time. We denote $\Omega_R = \overline{\Omega} \cap B(0,R)$.

Theorem 14. *Let u be a strong solution of $\mathcal{P}$. Let $0 < R < +\infty$ be such that $\operatorname{supp} u_0 \subset \Omega_R$. We assume $m^+ < p^- - 1$, $q(x,t) = m(x)$, and $\sup_{Q_T}|\nabla p(x,t)| < \infty$. Then, for any $t \in (0,T)$, there exists a unique compactly supported solution of $\mathcal{P}$ such that*

$$\operatorname{supp} u(t,.) \subset \left\{x \in \overline{\Omega} : |x| \leq \rho(t)\right\}, \tag{113}$$

where $\rho(t)$ will be specified in the proof below.

Proof. The idea of the proof is to construct a suitable supersolution $\overline{u}$ with compact support which is not necessarily defined in the whole Ω. Then, by the comparison principle, we deduce directly that $\{(x,t) \in Q_T : \overline{u} = 0\} \subset \{(x,t) \in Q_T : u = 0\}$, and the result follows. For all $t \in (0,T)$, we define $\overline{u}(x,t)$ as follows

$$\overline{u}(r,t) = \begin{cases} K(T)\,e^{(m^-+1)\lambda t}\left(c(t) + k(R-r)\right)^{\alpha}, & R \leq r < R + \dfrac{c(t)}{k} = \rho(t)\\ 0, & r \geq R + \dfrac{c(t)}{k}\end{cases} \tag{114}$$

where $r = |x|$,

$$\alpha = \frac{p^-}{p^- - 1 - m^+}, \tag{115}$$

$$M = \max\left\{1, \|u_0\|_\infty\right\},$$

$$\lambda = \frac{a_1}{(m^-)^2}, \tag{116}$$

$$c(t) = \left(\frac{M}{K(T)}\right)^{1/\alpha} e^{\lambda t((p^+-1-m^-)/\alpha)}, \tag{117}$$

and

$$K(T) = \left[\frac{m^-\lambda e^{-(m^-+1)\lambda T(p^+-1-m^-)}}{\left(k(\alpha-1)\left(p^+-1\right) + \epsilon^{n-1}\right)k^{p^--1}\alpha^{p^+-1}}\right]^{1/(p^+-1-m^-)}, \tag{118}$$

with sufficiently small $\epsilon > 0$.

Firstly, we denote $Q_T^R = (\Omega \setminus \Omega_R) \times (0,T)$. It is clear that $\overline{u} \in C^1(Q_T^R)$ and by direct calculations, we have

$$\frac{\partial}{\partial t}\overline{u}^{m(x)} = \left(m^- + 1\right) m(x)\,\lambda \overline{u}^{m(x)} + \alpha m(x) \cdot \left(K(T)\, e^{(m^-+1)\lambda t}\right)^{m(x)} \cdot \left(c(t) + k(R-r)\right)^{\alpha m(x)-1} c'(t). \quad (119)$$

Thus, it follows that

$$\frac{\partial}{\partial t}\overline{u}^{m(x)} \in L^{\infty}\left(Q_T^R\right) \subset W'\left(Q_T^R\right). \quad (120)$$

Moreover, from the definition of $\overline{u}$ we have

$$\overline{u}(R,0) = M \geq \left\|u_0\right\|_{\infty} \geq u_0. \quad (121)$$

Next, we will show that

$$\frac{\partial}{\partial t}\overline{u}^{m(x)} - \Delta_{p(x,t)}\overline{u} \geq a(x,t)\,\overline{u}^{m(x)} \quad \text{on } Q_T^R. \quad (122)$$

Since $|a(x,t)| \leq a_1$, it suffices to show that

$$\frac{\partial \overline{u}^{m(x)}}{\partial t} - \Delta_{p(x,t)}\overline{u} \geq a_1 \overline{u}^{m(x)} \quad \text{on } Q_T^R. \quad (123)$$

Using the hypothesis $p^+ > m^- + 1$, we have $c'(t) > 0$ for all $t \in (0,T)$. Then, from (119) and (116) we obtain

$$\frac{\partial}{\partial t}\overline{u}^{m(x)} \geq \left(m^- + 1\right) m(x)\,\lambda \overline{u}^{m(x)} \geq a_1 \overline{u}^{m(x)} + m(x)\,\lambda \overline{u}^{m(x)}. \quad (124)$$

Due to the last inequality, it remains to prove that

$$-\Delta_{p(x,t)}\overline{u} + m(x)\,\lambda \overline{u}^{m(x)} \geq 0. \quad \text{on } Q_T^R. \quad (125)$$

We have

$$\frac{\partial \overline{u}}{\partial x_i} = -\alpha k K(T)\, e^{(m^-+1)\lambda t}\left(c(t) + k(R-r)\right)^{\alpha-1} \frac{x_i}{|x|}, \quad (126)$$

and

$$|\nabla \overline{u}| = \alpha k K(T)\, e^{(m^-+1)\lambda t}\left(c(t) + k(R-r)\right)^{\alpha-1}. \quad (127)$$

We set

$$\nu_T(t) = \alpha k K(T)\, e^{(m^-+1)\lambda t}. \quad (128)$$

Then

$$\Delta_{p(x,t)}\overline{u} = \left(c(t) + k(R-r)\right)^{(\alpha-1)(p(x,t)-1)-1} \cdot \left(\nu_T(t)\right)^{p(x,t)-1} \cdot \left[k(\alpha-1)\left(p(x,t)-1\right) - h(x,t)\right], \quad (129)$$

where

$$\begin{aligned} h(x,t) &= \ln\left(c(t) + k(R-r)\right)\left(c(t) + k(R-r)\right) \cdot (\alpha-1)\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|} + \left[\frac{(N-1)}{|x|} + \sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln\left(\nu_T(t)\right)\right]\left(c(t) + k(R-r)\right) \\ &= n\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln\left(\left(c(t) + k(R-r)\right)^{1/n}\right) \cdot \left(c(t) + k(R-r)\right)^{1/n}\left(c(t) + k(R-r)\right)^{1-1/n} \\ &\quad \cdot (\alpha-1) + \left(\frac{(N-1)}{|x|} + \sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln\left(\nu_T\right)\right)\left(c(t) + k(R-r)\right)^{1/n}\left(c(t) + k(R-r)\right)^{1-1/n}, \end{aligned} \quad (130)$$

for all $n \geq 3$. By straightforward considerations, (boundedness of different functions), there exist positive constants $A, B \geq 1$ such that, for all $t \in (0,T)$ and $R \leq r < \rho(t)$,

$$\left|n\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln\left(\left(c(t) + k(R-r)\right)^{1/n}\right) \cdot \left(c(t) + k(R-r)\right)^{1/n}(\alpha-1)\right| \leq A$$
$$\left|\left(\frac{(N-1)}{|x|} + \sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln\left(\nu_T(t)\right)\right)\left(c(t) + k(R-r)\right)^{1/n}\right| \leq B. \quad (131)$$

Thus

$$|h(x,t)| \leq (A+B)\left(c(t) + k(R-r)\right)^{1-1/n} \leq \left((A+B)\left(c(t) + k(R-r)\right)^{1/n}\right)^{n-1}. \quad (132)$$

In this case we have $k(r-R) \leq c(t)$, and from (118), we can choose k small enough, so that $c(t)$ is also small for all $t \in (0,T)$, whence we have

$$(A+B)\left(c(t) + k(R-r)\right)^{1/n} \leq \epsilon, \quad (133)$$

for sufficiently small $\epsilon > 0$, which implies that

$$\Delta_{p(x,t)}\overline{u} \leq \left(c(t) + k(R-r)\right)^{(\alpha-1)(p(x)-1)-1} \cdot \left(\nu_T(t)\right)^{p(x,t)-1} \cdot \left(k(\alpha-1)\left(p(x,t)-1\right) + \epsilon^{n-1}\right). \quad (134)$$

Set $W(x,t) = k(\alpha-1)(p(x,t)-1) + \epsilon^{n-1}$. Now, to get (125) it suffices to have

$$\begin{aligned}&(c(t) + k(R-r))^{(\alpha-1)(p(r,t)-1)-1}(\nu_T(t))^{p(x,t)-1} W(r,t)\\&\quad\le (c(t)+k(R-r))^{\alpha m(x)} m(x)\\&\quad\cdot \lambda K^{m(x)} e^{(m^- +1)\lambda m(x)t}.\end{aligned} \tag{135}$$

Since $c(t) + k(R-r)$ is small enough then, from the value of α and $K(T)$, we obtain that

$$\begin{aligned}&(c(t)+k(R-r))^{(\alpha-1)(p(x,t)-1)-1}\\&\quad\le (c(t)+k(R-r))^{\alpha m(x)},\end{aligned} \tag{136}$$

and

$$\begin{aligned}&\left(\alpha k K(T) e^{(m^-+1)\lambda t}\right)^{p(x,t)-1} W(x,t)\\&\quad\le m(x)\lambda K(T)^{m(x)} e^{(m^-+1)\lambda m(x)t},\end{aligned} \tag{137}$$

which implies (125). Therefore we deduce the desired result. □

Theorem 15. *Let u be a strong solution of $\mathcal{P}$. Let $0 < R < +\infty$ be such that $\operatorname{supp} u_0 \subset \Omega_R$. We assume $m^+ < p^- - 1$, $\sup_{Q_T}|\nabla p(x,t)| < \infty$, and $a(x,t) \le 0$. Then, for any $t \in (0,T)$, there exists a unique compactly supported solution of $\mathcal{P}$ such that*

$$\operatorname{supp} u(t,.) \subset \left\{x \in \overline{\Omega} : |x| \le \rho(t)\right\}, \tag{138}$$

where $\rho(t)$ is specified in the proof above.

Proof. We consider the same supersolution $\overline{u}(r,t)$ as in the proof of Theorem 14 and we just need to prove that

$$\frac{\partial \overline{u}^{m(x)}}{\partial t} - \Delta_{p(x,t)}\overline{u} \ge a(x,t)\overline{u}^{q(x,t)} \quad \text{on } Q_T^R, \tag{139}$$

where $Q_T^R = (\Omega \setminus \Omega_R) \times (0,T)$. Since $a(x,t) \le 0$, it is sufficient to prove that

$$\frac{\partial \overline{u}^{m(x)}}{\partial t} - \Delta_{p(x,t)}\overline{u} \ge 0 \quad \text{on } Q_T^R. \tag{140}$$

Combining the same lines as in the previous theorem and the comparison principle in Proposition 6, we conclude the result. □

Finally, we state the following result on uniform localization of the support of solution.

Theorem 16. *Let u be a strong solution of $\mathcal{P}$. Let $0 < R < +\infty$ be such that $\operatorname{supp} u_0 \subset \Omega_R$. We assume $q^+ < p^- - 1$, $\sup_{Q_T}|\nabla p(x,t)| < \infty$, and $a(x,t) \le a_2 < 0$. Then for any $t \in (0,\infty)$, there exists a unique compactly supported solution of $\mathcal{P}$ such that*

$$\operatorname{supp} u(t,.) \subset \left\{x \in \overline{\Omega} : |x| \le R_1\right\} \tag{141}$$

where R_1 is positive constants determined in the proof below.

Proof. In order to get the desired estimation of the support of the solution, we define a suitable local supersolution associated with the stationary problem related to $\mathcal{P}$. Let $M = \max\{\|u_0\|_{L^\infty(\Omega)}, 1\}$. We define $\widehat{u}$ as follows:

$$\widehat{u}(r) = \begin{cases} K(R_1 - r)^\alpha & R \le r < R_1 \\ 0 & r \ge R_1,\end{cases} \tag{142}$$

where $r = |x|$, and

$$K = \frac{M}{(R_1 - R)^\alpha}, \tag{143}$$

where $R_1 - R \le 1$ and $\alpha > 1$ is a constant that will be determined hereafter. It is easy to verify that $\widehat{u} \in C^1(Q_T^R)$ for any $T > 0$, where $Q_T^R = (\Omega \setminus \Omega_R) \times (0,T)$. Concerning initial condition, we have $\widehat{u}(R,0) = \widehat{u}(R) \ge M \ge u_0$. Now, let us show that

$$\frac{\partial b(x,\widehat{u})}{\partial t} - \Delta_{p(x,t)}\widehat{u} \ge a(x,t)\widehat{u}^{q(x,t)}, \quad (x,t) \in Q_T^R. \tag{144}$$

We have

$$\frac{\partial \widehat{u}}{\partial x_i} = -\alpha K (R_1 - r)^{\alpha-1}\frac{x_i}{r}, \tag{145}$$

and

$$|\nabla \widehat{u}| = \alpha K (R_1 - r)^{\alpha-1}. \tag{146}$$

Then

$$\begin{aligned}\Delta_{p(x,t)}\widehat{u} &= (\alpha K)^{p(x,t)-1}(R_1 - r)^{(\alpha-1)(p(x,t)-1)-1}\\&\quad\cdot\left[(\alpha-1)(p(x,t)-1) - H(x,t)\right],\end{aligned} \tag{147}$$

with

$$\begin{aligned}&H(x,t)\\&\quad= (\alpha-1)(R_1-r)\ln(R_1-r)\sum_{i=1}^N \frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\\&\quad-\left(\frac{N-1}{r} + \sum_{i=1}^N \frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln(K\alpha)\right)(R_1 - r).\end{aligned} \tag{148}$$

By straightforward considerations, there exists a constant $A \ge 1$ such that, for every r with $R \le r < R_1$, we have

$$\left|(\alpha-1)(p(x,t)-1) - H(x,t)\right| \le A. \tag{149}$$

Hence, we get

$$\Delta_{p(x,t)}\widehat{u} \le A(\alpha K)^{p(x,t)-1}(R_1 - r)^{(\alpha-1)(p(x,t)-1)-1}. \tag{150}$$

Now, in order to obtain the inequality (144), we need to show that

$$\Delta_{p(x,t)}\widehat{u} \le -a(x,t)(\widehat{u}(r))^{q(x,t)}. \tag{151}$$

To do so, we just need to get

$$A(\alpha K)^{p(x,t)-1}(R_1-r)^{(\alpha-1)(p(x,t)-1)-1} \leq -a_2(K)^{q(x,t)}(R_1-r)^{\alpha q(x,t)}, \tag{152}$$

Due to the fact that $R_1 - r \leq R_1 - R \leq 1$, (152) holds if

$$(\alpha-1)(p(x,t)-1)-1-\alpha q(x,t) \geq 0, \quad A\alpha^{p(x,t)-1}K^{p(x,t)-1-q(x,t)} \leq -a_2. \tag{153}$$

Then, it is suffices to take

$$\alpha = \frac{p^-}{p^- - 1 - q^+}, \quad a_2 = -A\alpha^{p^+-1}K^{p^+-1-q^-}, \tag{154}$$

to complete the proof. □

Remark 17. As in [20, 21], we can assume in this section that Ω is a set of $\mathbb{R}^{\mathbb{N}}$, not necessarily bounded.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] S. N. Antontsev and S. I. Shmarev, "A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions," *Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal*, vol. 60, no. 3, pp. 515–545, 2005.

[2] D. G. Aronson, *The Porous Medium Equation*, F. Primicerio, Ed., vol. 1224 of *Nonlinear Diffusion Problems*, Springer, Berlin, Germany, 1986.

[3] M. E. Gurtin and R. C. MacCamy, "On the diffusion of biological populations," *Mathematical Biosciences*, vol. 33, no. 1-2, pp. 35–49, 1977.

[4] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, vol. 1748 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[5] Y. M. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM Journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.

[6] V. V. Zhikov, "Density of smooth functions in sobolev-orlicz spaces," *Journal of Mathematical Sciences*, vol. 132, no. 3, pp. 285–294, 2006.

[7] G. Akagi, "Doubly nonlinear parabolic equations involving variable exponents," *Discrete and Continuous Dynamical Systems - Series S*, vol. 7, no. 1, pp. 1–16, 2014.

[8] G. Akagi and K. Matsuura, "Nonlinear diffusion equations driven by the p(.)-Laplacian," *Nonlinear Differential Equations and Applications NoDEA*, vol. 20, no. 1, pp. 37–64, 2013.

[9] S. N. Antontsev and S. I. Shmarev, "Doubly degenerate parabolic equations with variable nonlinearity II: Blow-up and extinction in a finite time," *Nonlinear Analysis*, vol. 95, pp. 483–498, 2014.

[10] S. Antontsev and S. Shmarev, "On the localization of solutions of doubly nonlinear parabolic equations with nonstandard growth in filtration theory," *Applicable Analysis: An International Journal*, vol. 95, no. 10, pp. 2162–2180, 2016.

[11] S. Antontsev and S. Shmarev, *Evolution PDEs with Nonstandard Growth Conditions: Existence, Uniqueness, Localization, Blow-Up*, vol. 4 of *Atlantis Studies in Differential Equations*, Atlantis Press, 2015.

[12] B. Guo and W. Gao, "Study of weak solutions for parabolic equations with nonstandard growth conditions," *Journal of Mathematical Analysis and Applications*, vol. 374, no. 2, pp. 374–384, 2011.

[13] B. Guo and W. Gao, "Existence and asymptotic behavior of solutions for nonlinear parabolic equations with variable exponent of nonlinearity," *Acta Mathematica Scientia B*, vol. 32, no. 3, pp. 1053–1062, 2012.

[14] M. Bertsch, R. Kersner, and L. A. Peletier, "Positivity versus localization in degenerate diffusion equations," *Nonlinear Analysis Journal*, vol. 9, no. 9, pp. 987–1008, 1985.

[15] A. S. Kalashnikov, "Some problems of the qualitative theory of second-order nonlinear degenerate parabolic equations," *Uspekhi Matematicheskikh Nauk*, vol. 254, no. 2, pp. 135–176, 1987.

[16] J. I. Díaz, "Solutions with compact support for some degenerate parabolic problems," *Nonlinear Analysis*, vol. 3, no. 6, pp. 831–847, 1979.

[17] J. I. Díaz and J. Hernandez, "Qualitative properties of free boundaries for some nonlinear degenerate parabolic equations," in *Nonlinear Parabolic Equations: Qualitative Properties of Solutions*, L. Boccardo and A. Teseieds, Eds., vol. 149 of *Pitman Research Notes*, pp. 85–93, Longman, 1987.

[18] S. Antontsev, J. I. Díaz, and S. Shmarev, *Energy Methods for Free Boundary Problems: Applications to Non-Linear PDEs and Fluid Mechanics*, vol. 48 of *Progress in Nonlinear Differential Equations and their Applications*, Birkhäuser, Boston, Massachusetts, Mass, USA, 2002.

[19] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries*, vol. 106 of *Pitman (Advanced Publishing Program)*, London, UK, 1985.

[20] J. I. Díaz and M. A. Herrero, "Proprietés de support compact pour certaines équations elliptiques et paraboliques non linéaires," *Comptes Rendus de l'Académie des Sciences*, vol. 286, no. 19, pp. 815–817, 1978.

[21] J. I. Díaz and M. A. Herrero, "Estimates on the support of the solutions of some nonlinear elliptic and parabolic problems," *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, vol. 89, no. 3-4, pp. 249–258, 1981.

[22] S.-Y. Chung and J.-H. Park, "A complete characterization of nonlinear absorption for the evolution p-Laplacian equations to have positive or extinctive solutions," *Computers and Mathematics with Applications*, vol. 71, no. 8, pp. 1624–1635, 2016.

[23] J. I. Díaz, "Qualitative study of nonlinear parabolic equations: an introduction," *Extracta Mathematicae*, vol. 16, no. 3, pp. 303–341, 2001.

[24] Z. B. Fang and G. Li, "Extinction and decay estimates of solutions for a class of doubly degenerate equations," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1795–1802, 2012.

[25] C. Jin, J. Yin, and Y. Ke, "Critical extinction and blow-up exponents for fast diffusive polytropic filtration equation with sources," *Proceedings of the Edinburgh Mathematical Society*, vol. 52, no. 2, pp. 419–444, 2009.

[26] Y. G. Gu, "Necessary and sufficient conditions for extinction of solutions to parabolic equations," *Acta Mathematica Sinica*, vol. 37, no. 1, pp. 73–79, 1994.

[27] H. J. Yuan, S. Z. Lian, C. L. Cao, W. J. Gao, and X. J. Xu, "Extinction and positivity for a doubly nonlinear degenerate parabolic equation," *Acta Mathematica Sinica*, vol. 23, no. 10, pp. 1751–1756, 2007.

[28] J. Zhou and C. Mu, "Critical blow-up and extinction exponents for non-Newton polytropic filtration equation with source," *Bulletin of the Korean Mathematical Society*, vol. 46, no. 6, pp. 1159–1173, 2009.

[29] S. Antontsev and S. Shmarev, "Existence and uniqueness for doubly nonlinear parabolic equations with nonstandard growth conditions," *Differential Equations and Applications*, vol. 4, no. 1, pp. 67–94, 2012.

[30] S. Antontsev and S. Shmarev, "Doubly degenerate parabolic equations with variable nonlinearity I: Existence of bounded strong solutions," *Advances in Differential Equations*, vol. 17, no. 11-12, pp. 1181–1212, 2012.

[31] Q. Zhang, X. Liu, and Z. Qiu, "On the boundary blow-up solutions of $p(x)$-Laplacian equations with singular coefficient," *Nonlinear Analysis*, vol. 70, no. 11, pp. 4053–4070, 2009.

[32] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, Germany, 2011.

[33] X. L. Fan and Q.-H. Zhang, "On the spaces $L^{p(x)}$ and $W^{m,p(x)}$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.