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Research Article

On Solvability Theorems of Second-Order Ordinary Differential Equations with Delay

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For each $x_0 \in [0, 2\pi)$ and $k \in \mathbf{N}$, we obtain some existence theorems of periodic solutions to the two-point boundary value problem $u''(x) + k^2 u(x - x_0) + g(x, u(x - x_0)) = h(x)$ in $(0, 2\pi)$ with $u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0$ when $g : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function which grows linearly in u as $|u| \to \infty$, and $h \in L^1(0, 2\pi)$ may satisfy a generalized Landesman-Lazer condition $(1 + \operatorname{sign}(\beta)) \int_0^{2\pi} h(x) v(x) dx < \int_{v(x)>0} g_\beta^+(x) |v(x)|^{1-\beta} dx + \int_{v(x)<0} g_\beta^-(x) |v(x)|^{1-\beta} dx$ for all $v \in N(L) \backslash \{0\}$. Here $N(L)$ denotes the subspace of $L^1(0, 2\pi)$ spanned by $\sin kx$ and $\cos kx$, $-1 < \beta \le 0$, $g_\beta^+(x) = \liminf_{u \to \infty} (g(x, u) u / |u|^{1-\beta})$, and $g_\beta^-(x) = \liminf_{u \to -\infty} (g(x, u) u / |u|^{1-\beta})$.

1. Introduction

Let $x_0 \in [0, 2\pi)$ and $k \in \mathbf{N}$ be fixed. We consider the following two-point boundary value problems:

$$u''(x) + k^2 u(x - x_0) + g(x, u(x - x_0)) = h(x) \quad \text{in } (0, 2\pi), \qquad (1)$$
$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0,$$

$$u''(x) + k^2 u(x) - g(x, u(x - x_0)) = -h(x) \quad \text{in } (0, 2\pi), \qquad (2)$$
$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0,$$

where $h \in L^1(0, 2\pi)$ is given and $g : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function; that is, $g(x, u)$ is continuous in $u \in \mathbf{R}$, for a.e. $x \in (0, 2\pi)$, is measurable in $x \in (0, 2\pi)$ for all $u \in \mathbf{R}$, and satisfies, for each $r > 0$, the fact that there exists an $a_r \in L^1(0, 2\pi)$ such that

$$|g(x, u)| \le a_r(x) \qquad (3)$$

for a.e. $x \in (0, 2\pi)$ and all $|u| \le r$. Concerning the growth condition of the nonlinear term g to (1) and (2), we assume that

(H) there exist constants $-1 < \beta \le 0$, $r_0 > 0$, and $a, b, c, d \in L^1(0, 2\pi)$, $a, b \ge 0$ and $a(x) \le 2k + 1$ for a.e. $x \in (0, 2\pi)$ with strict inequality on a positive measurable subset of $(0, 2\pi)$, such that for a.e. $x \in (0, 2\pi)$ and all $u \ge r_0$

$$c(x) |u|^{-\beta} \le g(x, u) \le a(x) |u| + b(x); \qquad (4)$$

and for a.e. $x \in (0, 2\pi)$ and all $u \le -r_0$

$$-a(x) |u| - b(x) \le g(x, u) \le d(x) |u|^{-\beta}; \qquad (5)$$

(G) there exist constants $-1 < \beta \le 0$, $r_0 > 0$, and $a, b, c, d \in L^1(0, 2\pi)$, $a, b \ge 0$ and $a(x) \le 2k - 1$ for a.e. $x \in (0, 2\pi)$ with strict inequality on a positive measurable subset of $(0, 2\pi)$, such that for a.e. $x \in (0, 2\pi)$ and all $u \ge r_0$

$$c(x) |u|^{-\beta} \le g(x, u) \le a(x) |u| + b(x); \qquad (6)$$

and for a.e. $x \in (0, 2\pi)$ and all $u \le -r_0$

$$-a(x) |u| - b(x) \le g(x, u) \le d(x) |u|^{-\beta}; \qquad (7)$$

respectively, and a generalized Landesman-Lazer condition

$$0 < \int_{v(x)>0} g_\beta^+(x)\, |v(x)|^{1-\beta}\, dx + \int_{v(x)<0} g_\beta^-(x)\, |v(x)|^{1-\beta}\, dx, \tag{8}$$

for all $v \in N(L)\backslash\{0\}$, may be satisfied. Here $N(L)$ denotes the subspace of $L^1(0,2\pi)$ spanned by $\sin kx$ and $\cos kx$, $\beta \in \mathbf{R}$, $g_\beta^+(x) = \liminf_{u\to\infty}(g(x,u)u/|u|^{1-\beta})$, and $g_\beta^-(x) = \liminf_{u\to-\infty}(g(x,u)u/|u|^{1-\beta})$. Under assumptions and either with or without the Landesman-Lazer condition

$$\int_0^{2\pi} h(x)\, v(x)\, dx < \int_{v(x)>0} g_0^+ |v(x)|\, dx + \int_{v(x)<0} g_0^- |v(x)|\, dx \tag{9}$$

for all $v \in N(L)\backslash\{0\}$, the solvability of the problem (1) has been extensively studied if the nonlinearity $g(x,u)$ has at most linear growth in u as $|u| \to \infty$ (see [1–13] for the case $x_0 = 0$ and [14–16] for the general case) or grows superlinearly in u in one of directions $u \to \infty$ and $u \to -\infty$ and may be bounded in the other (see [8, 17] for the case $x_0 = 0$ and [14] for the general case when $k = 0$). Based on the well-known Leray-Schauder continuation method (see [18, 19]), we obtain solvability theorems to (1) (resp., (2)) when $g(x,u)$ satisfies (H) (resp., (G)) and either (8) with $-1 < \beta < 0$ or (9) with $\beta = 0$ is satisfied, which extends the results of [15] for the nonresonance case, and has been established in [9] for the case $x_0 = 0$ and $g(x,u)$ grows sublinearly in u as $|u| \to \infty$ with $-1 < \beta \le 1$. Unfortunately, it is still unknown when $k \in \mathbf{N}$, $g(x,u)$ grows linearly in u as $|u| \to \infty$ and the assumption of (8) is replaced by

$$\int_0^{2\pi} h(x)\, v(x)\, dx = 0 < \int_{v(x)>0} g_\beta^+(x)\, |v(x)|^{1-\beta}\, dx + \int_{v(x)<0} g_\beta^-(x)\, |v(x)|^{1-\beta}\, dx \tag{10}$$

for all $v \in N(L)\backslash\{0\}$ with $\beta > 0$. In the following we will make use of real Banach spaces $L^p(0,2\pi)$, $C[0,2\pi]$ and Sobolev spaces $W^{2,1}(0,2\pi)$ and $H^1(0,2\pi)$. The norms of $L^p(0,2\pi)$, $C[0,2\pi]$ and $H^1(0,2\pi)$ are denoted by $\|u\|_{L^p}$, $\|u\|_C$ and $\|u\|_{H^1}$, respectively. By a solution of (1), we mean a periodic function $u : \mathbf{R} \to \mathbf{R}$ of period 2π which belongs to $W^{2,1}(0,2\pi)$ and satisfies the differential equation in (1) a.e. $x \in (0,2\pi)$.

2. Existence Theorems

For each $v \in W^{2,1}(0,2\pi)$ with $v(0) - v(2\pi) = v'(0) - v'(2\pi) = 0$ and $k \in \mathbf{N}$, we write $\overline{v} = \sum_{0\le j\le k} P_j v$, $\widetilde{v} = \sum_{j>k} P_j v$, and $v^\perp = \sum_{0\le j\ne k} P_j v$. Here $P_j v$ denotes the projection of v on the eigenspace of d^2/dx^2 spanned by $\sin jx$ and $\cos jx$ for $j \in \mathbf{N} \cup \{0\}$. Just as an application of [11, Lemma 2] or [1, Lemma 2.2], we can modify slightly the proof of [15, Lemma 1] to obtain the next lemma.

Lemma 1. *Let $k \in \mathbf{N} \cup \{0\}$ and Γ be a nonnegative $L^1(0,2\pi)$-function such that for a.e. $x \in (0,2\pi)$, $\Gamma(x) \le 2k+1$ with strict inequality on a positive measurable subset of $(0,2\pi)$. Then there exists a constant $K_1 > 0$ such that*

$$\int_0^{2\pi} (\overline{u}(x-x_0) - \widetilde{u}(x)) \cdot \left(u''(x) + k^2 u(x-x_0) + p(x)\, u(x-x_0)\right) dx \ge K_1 \left\|u^\perp\right\|_{H^1}^2 \tag{11}$$

whenever $p \in L^1(0,2\pi)$ with $0 \le p(x) \le \Gamma(x)$ for a.e. $x \in (0,2\pi)$ and $u \in W^{2,1}(0,2\pi)$ is a periodic function of period 2π with $u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0$.

Proof. Just as in [20, Lemma 1], we can modify slightly the proof of [11, Lemma 2] or [1, Lemma 2.2] to obtain the fact that there exists a constant $K_1 > 0$ such that

$$\int_0^{2\pi} \left(\widetilde{u}'(x)\right)^2 - \left(k^2 + p(x)\right)(\widetilde{u}(x))^2\, dx + \int_0^{2\pi} \left(k^2 + p(x)\right)(\overline{u}(x))^2 - \left(\overline{u}'(x)\right)^2 dx \ge 2K_1 \left\|u^\perp\right\|_{H^1}^2 \tag{12}$$

whenever $p \in L^1(0,2\pi)$ with $0 \le p(x) \le \Gamma(x)$ for a.e. $x \in (0,2\pi)$ and $u \in W^{2,1}(0,2\pi)$ with $u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0$. Let us extend $u(x)$ and $p(x)$ 2π periodically in x to all of $\mathbf{R}$ and then use the same notations for the periodic extensions as for the original functions. In this case, we have $\int_0^{2\pi}(\widetilde{u}'(x))^2 dx = \int_0^{2\pi}(\widetilde{u}'(x-x_0))^2 dx$ and

$$\begin{aligned}
&\int_0^{2\pi} \left[u''(x) + \left(k^2 + p(x)\right) u(x-x_0)\right] \cdot (\overline{u}(x-x_0) - \widetilde{u}(x))\, dx = \int_0^{2\pi} \left(\widetilde{u}'(x)\right)^2 dx \\
&\quad - \int_0^{2\pi} \overline{u}'(x)\, \overline{u}'(x-x_0)\, dx + \frac{1}{2}\int_0^{2\pi} \left(k^2 + p(x)\right) \cdot \left[(\overline{u}(x-x_0))^2 - (\widetilde{u}(x))^2 - (\widetilde{u}(x-x_0))^2\right] dx \\
&\quad + \frac{1}{2}\int_0^{2\pi} \left(k^2 + p(x)\right) \cdot \left[\overline{u}(x-x_0) + \widetilde{u}(x-x_0) - \widetilde{u}(x)\right]^2 dx
\end{aligned}$$

$$\begin{aligned}&\geq \int_0^{2\pi} \left(\tilde{u}'(x)\right)^2 dx - \frac{1}{2}\int_0^{2\pi} \left(\tilde{u}'(x)\right)^2 \\ &+ \left(\overline{u}'(x-x_0)\right)^2 dx + \frac{1}{2}\int_0^{2\pi}\left(k^2+p(x)\right) \\ &\cdot \left[(\overline{u}(x-x_0))^2 - (\tilde{u}(x))^2 - (\tilde{u}(x-x_0))^2\right] dx \\ &+ \frac{1}{2}\int_0^{2\pi}\left(k^2+p(x)\right) \\ &\cdot \left[\overline{u}(x-x_0)+\tilde{u}(x-x_0)-\tilde{u}(x)\right]^2 dx \geq \frac{1}{2} \\ &\cdot \int_0^{2\pi}\left(\tilde{u}'(x)\right)^2 \\ &- \left(k^2+p(x)\right)(\tilde{u}(x))^2 dx + \frac{1}{2} \\ &\cdot \int_0^{2\pi}\left(\tilde{u}'(x-x_0)\right)^2 - \left(k^2+p(x)\right) \\ &\cdot (\tilde{u}(x-x_0))^2 dx + \frac{1}{2}\int_0^{2\pi}\left(k^2+p(x)\right) \\ &\cdot (\overline{u}(x-x_0))^2 - \left(\overline{u}'(x-x_0)\right)^2 dx - \frac{1}{2} \\ &\cdot \int_0^{2\pi}\left(\overline{u}'(x)\right)^2 dx + \frac{1}{2}\int_0^{2\pi}\left(k^2+p(x)\right) \\ &\cdot \left[\overline{u}(x-x_0)+\tilde{u}(x-x_0)-\tilde{u}(x)\right]^2 dx.\end{aligned} \tag{13}$$

Since $\int_0^{2\pi}(\overline{u}(x))^2dx = \int_0^{2\pi}(\overline{u}(x-x_0))^2dx$, $p(x) \geq 0$ for a.e. $x \in (0, 2\pi)$, and $\int_0^{2\pi}\overline{v}(x)\tilde{w}(x) = 0$ for all $v, w \in W^{2,1}(0, 2\pi)$ with $v(0)-v(2\pi) = v'(0)-v'(2\pi) = 0$ and $w(0)-w(2\pi) = w'(0)-w'(2\pi) = 0$, we have $\int_0^{2\pi}(\tilde{u}'(x))^2 - (k^2+p(x))(\tilde{u}(x))^2dx \geq 0$ and

$$\begin{aligned}&-\frac{1}{2}\int_0^{2\pi}\left(\overline{u}'(x)\right)^2 dx + \frac{1}{2}\int_0^{2\pi}\left(k^2+p(x)\right) \\ &\cdot \left[\overline{u}(x-x_0)+\tilde{u}(x-x_0)-\tilde{u}(x)\right]^2 dx \geq -\frac{1}{2} \\ &\cdot \int_0^{2\pi}\left(\overline{u}'(x)\right)^2 dx + \frac{1}{2} \\ &\cdot \int_0^{2\pi} k^2\left[\overline{u}(x-x_0)+\tilde{u}(x-x_0)-\tilde{u}(x)\right]^2 dx \\ &= \frac{1}{2}\left[-\int_0^{2\pi}\left(\overline{u}'(x)\right)^2 dx \right. \\ &\left. + k^2\int_0^{2\pi}(\overline{u}(x-x_0))^2 dx\right] + \frac{1}{2} \\ &\cdot k^2\int_0^{2\pi}\left[\tilde{u}(x-x_0)-\tilde{u}(x)\right]^2 dx \geq 0.\end{aligned} \tag{14}$$

Combining (12) with (13), we have

$$\begin{aligned}&\int_0^{2\pi}\left(\overline{u}(x-x_0)-\tilde{u}(x)\right) \\ &\quad\cdot\left(u''(x)+k^2u(x-x_0)+p(x)u(x-x_0)\right)dx \\ &\quad\geq \frac{1}{2}\int_0^{2\pi}\left(\tilde{u}'(x-x_0)\right)^2 - \left(k^2+p(x)\right) \\ &\quad\cdot(\tilde{u}(x-x_0))^2 dx + \frac{1}{2}\int_0^{2\pi}\left(k^2+p(x)\right) \\ &\quad\cdot(\overline{u}(x-x_0))^2 - \left(\overline{u}'(x-x_0)\right)^2 dx \\ &\quad\geq K_1\left\|u^{\perp}\right\|^2_{H^1(x_0,x_0+2\pi)} = K_1\left\|u^{\perp}\right\|^2_{H^1}.\end{aligned} \tag{15}$$

□

Lemma 2. *Let $k \in \mathbf{N}$ and Γ be a nonnegative $L^1(0, 2\pi)$-function such that for a.e. $x \in (0, 2\pi)$, $\Gamma(x) \leq 2k-1$ with strict inequality on a positive measurable subset of $(0, 2\pi)$. Then there exists a constant $K_2 > 0$ such that*

$$\begin{aligned}&\int_0^{2\pi}\left(\overline{\overline{u}}(x-x_0)-\tilde{\tilde{u}}(x)\right) \\ &\quad\cdot\left(u''(x)+k^2u(x)-p(x)u(x-x_0)\right)dx \\ &\quad\geq K_2\left\|u^{\perp}\right\|^2_{H^1}\end{aligned} \tag{16}$$

whenever $p \in L^1(0, 2\pi)$ with $0 \leq p(x) \leq \Gamma(x)$ for a.e. $x \in (0, 2\pi)$ and $u \in W^{2,1}(0, 2\pi)$ is a periodic function of period 2π with $u(0)-u(2\pi) = u'(0)-u'(2\pi) = 0$. Here $\overline{\overline{v}} = \sum_{0\leq j<k} P_j v$ and $\tilde{\tilde{v}} = \sum_{j\geq k} P_j v$ for each $v \in W^{2,1}(0, 2\pi)$ with $v(0)-v(2\pi) = v'(0)-v'(2\pi) = 0$.

Theorem 3. *Let $k \in \mathbf{N} \cup \{0\}$ and $g : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ be a Caratheodory function satisfying (H). Then for each $h \in L^1(0, 2\pi)$ problem (1) has a solution u, provided that either (8) with $-1 < \beta < 0$ or (9) with $\beta = 0$ holds.*

Proof. Let $\alpha \in \mathbf{R}$ be fixed and $0 < \alpha < 2k+1$. We consider the boundary value problems

$$\begin{aligned}&u''(x)+k^2u(x-x_0)+(1-t)\alpha u(x-x_0) \\ &\quad + tg(x, u(x-x_0)) = th(x) \quad \text{in } (0, 2\pi), \\ &u(0)-u(2\pi) = u'(0)-u'(2\pi) = 0\end{aligned} \tag{17}$$

for $0 \leq t \leq 1$, which becomes the original problem when $t = 1$. Since $0 < \alpha < 2k+1$, we observe from Lemma 1 that (17) has only a trivial solution when $t = 0$. To apply the Leray-Schauder continuation method, it suffices to show that solutions to (17) for $0 < t < 1$ have an a priori bound in $H^1(0, 2\pi)$. To this end, let $\theta : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $0 \le \theta \le 1$, $\theta(u) = 0$ for $|u| \le r_0$, and $\theta(u) = 1$ for $|u| \ge 2r_0$. We define $e(x) = \max\{a_{r_0}(x), b(x), |c(x)|, |d(x)|\}$,

$$g_1(x,u) = \begin{cases} \min\left\{g(x,u) + e(x)|u|^{-\beta},\ a(x)u\right\}\theta(u) & \text{if } u \ge 0 \\ \max\left\{g(x,u) - e(x)|u|^{-\beta},\ a(x)u\right\}\theta(u) & \text{if } u \le 0, \end{cases} \quad (18)$$

and $g_2(x,u) = g(x,u) - g_1(x,u)$. Then $g_1, g_2 : (0,2\pi) \times \mathbf{R} \to \mathbf{R}$ are Caratheodory functions, such that for a.e. $x \in (0,2\pi)$ and $u \in \mathbf{R}$, $u \ne 0$

$$0 \le \frac{g_1(x,u)}{u} \le a(x), \quad (19)$$

$$\left|g_2(x,u)\right| \le e(x)|u|^{-\beta} + e(x). \quad (20)$$

If u is a possible solution to (17) for some $0 < t < 1$, then using (19), (20), and Lemma 1, we have

$$\begin{aligned} 0 &= \int_0^{2\pi} (\overline{u}(x) - \widetilde{u}(x - x_0))\left[u''(x) + k^2 u(x - x_0)\right. \\ &\quad + (1-t)\alpha u(x - x_0) + t g_1(x, u(x - x_0)) \\ &\quad \left.+ t g_2(x, u(x - x_0)) - t h(x)\right] dx \\ &\ge K_1 \left\|u^\perp\right\|_{H^1}^2 - \left(\|e\|_{L^1}\|u\|_C^{-\beta} + \|e\|_{L^1} + \|h\|_{L^1}\right)\left(\|\overline{u}\|_C\right. \\ &\quad \left.+ \|\widetilde{u}\|_C\right) \ge K_1 \left\|u^\perp\right\|_{H^1}^2 - C_1\left(\|u\|_C^{-\beta} + 1\right)\left(\|\overline{u}\|_{H^1}\right. \\ &\quad \left.+ \|\widetilde{u}\|_{H^1}\right), \end{aligned} \quad (21)$$

which implies that

$$\begin{aligned} \left\|u^\perp\right\|_{H^1}^2 &\le \frac{C_1}{K_1}\left(\|u\|_C^{-\beta} + 1\right)\left(\|\overline{u}\|_{H^1} + \|\widetilde{u}\|_{H^1}\right) \\ &\le C_2\left(\|u\|_{H^1} + \|u\|_{H^1}^{1-\beta}\right) \end{aligned} \quad (22)$$

for some constants $C_1, C_2 > 0$ independent of u. It remains to show that solutions to (17) for $0 < t < 1$ have an a priori bound in $H^1(0,2\pi)$. We argue by contradiction and suppose that there exists a sequence $\{u_n\}$ of periodic functions with period 2π and a corresponding sequence $\{t_n\}$ in $(0,1)$ such that u_n is a solution to (17) with $t = t_n$ and $\|u_n\|_{H^1} \ge n$ for all n. Let $v_n = u_n/\|u_n\|_{H^1}$; then $\|v_n\|_{H^1} = 1$ for all $n \in \mathbf{N}$, and by (22) we have $\|v_n^\perp\|_{H^1} \to 0$ as $n \to \infty$. Since $\|v_n\|_{H^1} = 1$ and $\|P_k v_n\|_{H^1} \le \|v_n\|_{H^1} + \|v_n^\perp\|_{H^1}$ for all $n \in \mathbf{N}$, we have a bounded sequence $\{P_k v_n\}$ in $H^1(0,2\pi)$. For simplicity, we may assume that v_n converges to v in $H^1(0,2\pi)$ for some $v \in N(L)$ with $\|v\|_{H^1} = 1$. In particular, $v_n \to v$ in $C[0,2\pi]$. Clearly, $v(\cdot - x_0) \in N(L)$ and $\|v(\cdot - x_0)\|_{H^1} = \|v\|_{H^1}$. It follows that $u_n(x) \to \infty$ for each $x \in \mathbf{R}$ with $v(x) > 0$, and $u_n(x) \to -\infty$ for each $x \in \mathbf{R}$ with $v(x) < 0$. Since $\int_0^{2\pi} u_n''(x) P_k u_n(x - x_0)dx + \int_0^{2\pi} k^2 u_n(x) P_k u_n(x - x_0)dx = 0$ and $\|P_k u_n(\cdot)\|_{L^2}^2 = \|P_k u_n(\cdot - x_0)\|_{L^2}^2$, we have

$$\begin{aligned} &\int_0^{2\pi} u_n''(x) P_k u_n(x - x_0)\,dx + \int_0^{2\pi} k^2 u_n(x - x_0) \\ &\quad \cdot P_k u_n(x - x_0)\,dx = \int_0^{2\pi} u_n''(x) \\ &\quad \cdot P_k u_n(x - x_0)\,dx + \int_0^{2\pi} k^2 u_n(x) \\ &\quad \cdot P_k u_n(x - x_0)\,dx \\ &\quad + \int_0^{2\pi} k^2\left[u_n(x - x_0) - u_n(x)\right] \\ &\quad \cdot P_k u_n(x - x_0)\,dx \\ &\quad = \int_0^{2\pi} k^2\left[u_n(x - x_0) - u_n(x)\right] \\ &\quad \cdot P_k u_n(x - x_0)\,dx \\ &\quad = \int_0^{2\pi} k^2\left[P_k u_n(x - x_0) - P_k u_n(x)\right] \\ &\quad \cdot P_k u_n(x - x_0)\,dx \ge 0. \end{aligned} \quad (23)$$

Multiplying each side of (17) by $P_k v_n(x - x_0)$, and then integrating them over $[0,2\pi]$ when $u = u_n$ and $t = t_n$, we get

$$\begin{aligned} &t_n \int_0^{2\pi} g(x, u_n(x - x_0)) P_k v_n(x - x_0)\,dx \\ &\quad \le (1 - t_n)\alpha \int_0^{2\pi} u_n(x - x_0) P_k v_n(x - x_0)\,dx \\ &\quad\quad + t_n \int_0^{2\pi} g(x, u_n(x - x_0)) P_k v_n(x - x_0)\,dx \\ &\quad \le t_n \int_0^{2\pi} h(x) P_k v_n(x - x_0)\,dx. \end{aligned} \quad (24)$$

By (19) and the assumption of $-1 < \beta \le 0$, we have

$$\begin{aligned} &g_1(x, u_n(x - x_0)) P_k v_n(x - x_0)\|u_n\|_{H^1}^\beta \\ &\quad = \frac{g_1(x, u_n(x - x_0))}{u_n(x - x_0)} u_n(x - x_0) P_k v_n(x - x_0) \\ &\quad \cdot \|u_n\|_{H^1}^\beta \ge \frac{g_1(x, u_n(x - x_0))}{u_n(x - x_0)} \\ &\quad \cdot \frac{-1}{2}\left[u_n(x - x_0) - P_k u_n(x - x_0)\right]^2 \|u_n\|_{H^1}^{\beta-1} \ge \frac{-1}{2} \\ &\quad \cdot a(x)\left[u_n^\perp(x - x_0)\right]^2 \|u_n\|_{H^1}^{\beta-1} \end{aligned} \quad (25)$$

for a.e. $x \in (0,2\pi)$. Combining (22) with (25), we get that $g_1(x, u_n(x - x_0)) P_k v_n(x - x_0)\|u_n\|_{H^1}^\beta$ is bounded from below by an $L^1(0,2\pi)$-function independent of n. By (20) and the assumption of $-1<\beta\le 0$, we have

$$\begin{aligned}&\left|g_2\left(x,u_n\left(x-x_0\right)\right)P_k v_n\left(x-x_0\right)\right|\left\|u_n\right\|_{H^1}^{\beta}\\&\quad\le\left[e\left(x\right)\left|u_n\left(x-x_0\right)\right|^{-\beta}+e\left(x\right)\right]\left|P_k v_n\left(x-x_0\right)\right|\\&\quad\cdot\left\|u_n\right\|_{H^1}^{\beta}\le\left[e\left(x\right)\left|v_n\left(x-x_0\right)\right|^{-\beta}+e\left(x\right)\right]\\&\quad\cdot\left|P_k v_n\left(x-x_0\right)\right|\end{aligned}\tag{26}$$

for a.e. $x\in(0,2\pi)$, In particular, $g_2(x,u_n(x-x_0))P_kv_n(x-x_0)\|u_n\|_{H^1}^{\beta}$ is bounded from below by an $L^1(0,2\pi)$-function independent of n, which implies that $g(x,u_n(x-x_0))P_kv_n(x-x_0)\|u_n\|_{H^1}^{\beta}$ is also so, $\int_{v(x-x_0)=0}\liminf_{n\to\infty}g(x,u_n(x-x_0))P_kv_n(x-x_0)\|u_n\|_{H^1}^{\beta}dx=0$, and

$$\begin{aligned}&g\left(x,u_n\left(x-x_0\right)\right)P_k v_n\left(x-x_0\right)\left\|u_n\right\|_{H^1}^{\beta}\\&\quad=\frac{g\left(x,u_n\left(x-x_0\right)\right)u_n\left(x-x_0\right)}{\left|u_n\left(x-x_0\right)\right|^{1-\beta}}\\&\quad\cdot\frac{P_k v_n\left(x-x_0\right)\operatorname{sgn}\left(u_n\left(x-x_0\right)\right)}{\left|v_n\left(x-x_0\right)\right|^{\beta}}\end{aligned}\tag{27}$$

for all $n\in\mathbf{N}$ with $u_n(x-x_0)\ne 0$. Here $\operatorname{sign}(w)=1$ if $w>0$, $\operatorname{sign}(w)=0$ if $w=0$, and $\operatorname{sign}(w)=-1$ if $w<0$. Applying Fatou's lemma to the integral $\int_0^{2\pi}g(x,u_n(x-x_0))P_kv_n(x-x_0)\|u_n\|_{H^1}^{\beta}dx$, we have

$$\begin{aligned}&\int_{v(x-x_0)>0}g_{\beta}^{+}\left(x\right)\left|v\left(x-x_0\right)\right|^{1-\beta}dx+\int_{v(x-x_0)<0}g_{\beta}^{-}\left(x\right)\\&\quad\cdot\left|v\left(x-x_0\right)\right|^{1-\beta}dx\\&\quad=\int_{v(x-x_0)>0}\liminf_{n\to\infty}\frac{g\left(x,u_n\left(x-x_0\right)\right)u_n\left(x-x_0\right)}{\left|u_n\left(x-x_0\right)\right|^{1-\beta}}\\&\quad\cdot\lim_{n\to\infty}\frac{P_k v_n\left(x-x_0\right)\operatorname{sgn}\left(u_n\left(x-x_0\right)\right)}{\left|v_n\left(x-x_0\right)\right|^{\beta}}dx\\&\quad+\int_{v(x-x_0)<0}\liminf_{n\to\infty}\frac{g\left(x,u_n\left(x-x_0\right)\right)u_n\left(x-x_0\right)}{\left|u_n\left(x-x_0\right)\right|^{1-\beta}}\\&\quad\cdot\lim_{n\to\infty}\frac{P_k v_n\left(x-x_0\right)\operatorname{sgn}\left(u_n\left(x-x_0\right)\right)}{\left|v_n\left(x-x_0\right)\right|^{\beta}}dx\\&\quad=\int_{v(x-x_0)>0}\liminf_{n\to\infty}g\left(x,u_n\left(x-x_0\right)\right)P_k v_n\left(x-x_0\right)\\&\quad\cdot\left\|u_n\right\|_{H^1}^{\beta}dx\\&\quad+\int_{v(x-x_0)<0}\liminf_{n\to\infty}g\left(x,u_n\left(x-x_0\right)\right)P_k v_n\left(x-x_0\right)\\&\quad\cdot\left\|u_n\right\|_{H^1}^{\beta}dx\\&\quad=\int_{v(x-x_0)>0}\liminf_{n\to\infty}g\left(x,u_n\left(x-x_0\right)\right)P_k v_n\left(x-x_0\right)\\&\quad\cdot\left\|u_n\right\|_{H^1}^{\beta}dx\\&\quad+\int_{v(x-x_0)<0}\liminf_{n\to\infty}g\left(x,u_n\left(x-x_0\right)\right)P_k v_n\left(x-x_0\right)\\&\quad\cdot\left\|u_n\right\|_{H^1}^{\beta}dx+\int_{v(x-x_0)=0}\lim_{n\to\infty}g\left(x,u_n\left(x-x_0\right)\right)\\&\quad\cdot P_k v_n\left(x-x_0\right)\left\|u_n\right\|_{H^1}^{\beta}dx\\&\quad=\int_0^{2\pi}\liminf_{n\to\infty}g\left(x,u_n\left(x-x_0\right)\right)P_k v_n\left(x-x_0\right)\\&\quad\cdot\left\|u_n\right\|_{H^1}^{\beta}dx\le\liminf_{n\to\infty}\int_0^{2\pi}g\left(x,u_n\left(x-x_0\right)\right)\\&\quad\cdot P_k v_n\left(x-x_0\right)\left\|u_n\right\|_{H^1}^{\beta}dx\le\left(1+\operatorname{sign}\left(\beta\right)\right)\\&\quad\cdot\int_0^{2\pi}h\left(x\right)v\left(x-x_0\right)dx,\end{aligned}\tag{28}$$

which is a contradiction when either (8) with $-1<\beta<0$ or (9) with $\beta=0$ is satisfied. Hence, the proof of this theorem is complete. □

By slightly modifying the proof of Theorem 3, we can apply Lemma 2 to obtain an existence theorem to (2) when condition (H) is replaced by (G) and either (8) with $-1<\beta<0$ or (9) with $\beta=0$ is satisfied, which has been established in [20] for the case $x_0=0$ when (9) with $\beta=0$ is satisfied and in [9] for the case $x_0=0$ when (8) with $\beta=-1$ is satisfied.

Theorem 4. *Let $k\in\mathbf{N}$ and $g:(0,2\pi)\times\mathbf{R}\to\mathbf{R}$ be a Caratheodory function satisfying (G). Then for each $h\in L^1(0,2\pi)$ problem (2) has a solution u, provided that either (8) with $-1<\beta<0$ or (9) with $\beta=0$ holds.*

Conflicts of Interest

There are no conflicts of interest involved.

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