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Research Article

Oscillation Criteria for Certain Even Order Neutral Delay Differential Equations with Mixed Nonlinearities

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We establish some oscillation criteria for the following certain even order neutral delay differential equations with mixed nonlinearities: $\left(r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t)\right)' + q_0(t)|(x(\tau_0(t))|^{\alpha-1}x(\tau_0(t)) + q_1(t)\left|(x(\tau_1(t))\right|^{\beta-1}x(\tau_1(t)) + q_2(t)\left|(x(\tau_2(t))\right|^{\gamma-1}x(\tau_2(t)) = 0, t \geq t_0$, where $z(t) = x(t) + p(t)x(\sigma(t)), n$ is even integer, and $\gamma > \alpha > \beta > 0$. Our results generalize and improve some known results for oscillation of certain even order neutral delay differential equations with mixed nonlinearities.

1. Introduction

In this paper, we are concerned with oscillation behavior of the certain even order neutral delay differential equations with mixed nonlinearities:

$$\begin{aligned}
&\left(r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t)\right)' \\
&\quad + q_0(t)\left|x(\tau_0(t))\right|^{\alpha-1}x(\tau_0(t)) \\
&\quad + q_1(t)\left|x(\tau_1(t))\right|^{\beta-1}x(\tau_1(t)) \\
&\quad + q_2(t)\left|x(\tau_2(t))\right|^{\gamma-1}x(\tau_2(t)) = 0, \\
&\qquad t \geq t_0,
\end{aligned} \tag{1}$$

where $z(t) = x(t) + p(t)x(\sigma(t))$, n is even integer, and $\gamma > \alpha > \beta > 0$ are constants. $r, q_i \in C([t_0,\infty), R^+)$, $r'(t) \geq 0$, $p, \tau_i \in C([t_0,\infty), R)$ satisfy that $\tau_i(t) \leq t$, $i = 0,1,2$, and there exists a function $\sigma \in C([t_0,\infty), R)$, such that $\sigma(t) \leq t$, $\lim_{t\to\infty}\sigma(t) = \infty$. We assume that there exists a function $\tau \in C^1([t_0,\infty), R)$, such that $\tau(t) \leq \tau_i(t)$, $i = 0,1,2$, $\tau(t) \leq t$, $\tau'(t) > 0$, and $\lim_{t\to\infty}\tau(t) = \infty$.

We will consider the two cases

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}dt = \infty, \tag{2}$$

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}dt < \infty. \tag{3}$$

Recently, there have been a large number of papers devoted to the oscillation of the delay differential equations; see [1–8]. Furthermore, there have been a large number of works on the oscillation of the neutral differential equations, and we refer the readers to the articles [9–25].

Agarwal and Grace [3] studied the oscillation for functional differential equations of higher order,

$$\left(\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)f(x(g(t))) = 0, \tag{4}$$

and established some sufficient conditions for oscillation of (4).

Sun and Meng [6] examined the oscillation of (1), where $p(t) = 0$, $n = 2$.

Xu and Xia [7], by means of Riccati transformation technique, established some oscillation criteria for certain even order delay differential equations:

$$\left(\left|(x(t))^{(n-1)}\right|^{\alpha-1}(x(t))^{(n-1)}\right)' + F(t, x(g(t))) = 0. \tag{5}$$

In 2011, Zhang et al. [8] studied the oscillatory behavior of the following higher-order half-linear delay differential equation:

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)\,x^{\beta}(\tau(t)) = 0, \quad t \geq t_0, \tag{6}$$

where $\int_{t_0}^{\infty} r^{-(1/\alpha)}(t)dt < \infty$.

In 2013, Zhang et al. [23] improved those reported in [8].

Han et al. [9] studied the oscillation of second-order neutral differential equations:

$$\left(r(t)\,\psi(x(t))\left|Z'(t)\right|^{\alpha-1}Z'(t)\right)' + q(t)\,f(x(\sigma(t))) = 0, \tag{7}$$

where $Z(t) = x(t) + p(t)x(t-\tau)$ and $\alpha > 0$, $0 \leq p(t) < 1$. Some new oscillation criteria are established for the second-order nonlinear neutral delay differential equations:

$$\left[r(t)\left[x(t) + p(t)\,x(\tau(t))\right]'\right]' + q(t)\,f(x(\sigma(t))) = 0, \tag{8}$$

where $\int_{t_0}^{\infty} 1/r(t)dt < \infty$, $0 \leq p(t) \leq p_0 < +\infty$.

Meng and Xu [19], by using the Riccati transformation technique and inequalities, considered the oscillation for even order quasilinear neutral differential equations:

$$\begin{aligned} &\Big(r(t)\left|\left(x(t) + p(t)\,x(t-\sigma)\right)^{(n-1)}\right|^{\alpha-1} \\ &\quad\times\left(x(t) + p(t)\,x(t-\sigma)\right)^{(n-1)}\Big)' \\ &\quad + q(t)\,f(x(\sigma(t))) = 0, \end{aligned} \tag{9}$$

where $t \geq t_0$, $0 \leq p(t) < 1$.

In 2012, Sun et al. [22] considered the oscillation criteria for even order nonlinear neutral differential equations:

$$\left(r(t)\,z^{(n-1)}(t)\right)' + q(t)\,f(x(\sigma(t))) = 0, \tag{10}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $n \geq 2$ is even integer, and $0 \leq p(t) \leq p_0 < +\infty$. The results are obtained when $\int^{\infty} r^{-1}(t)dt = \infty$ or $\int^{\infty} r^{-1}(t)dt < \infty$. These criteria obtained in this paper extended and improved some known results in the literatures.

In 2013, Agarwal et al. [24] considered the oscillation criteria for even order neutral differential equations:

$$\left(x(t) + p(t)\,x(\tau(t))\right)^{n} + q(t)\,x(\sigma(t)) = 0. \tag{11}$$

Some new criteria are established that improve a number of related results reported in the literature and can be used in cases where known theorems fail to apply.

In 2014, Zhang et al. [25] study oscillation and asymptotic behavior of solutions to two classes of higher-order delay damped differential equations with p-Laplacian like operators:

$$\begin{aligned} &\left(a(t)\left|x^{(n-1)}(t)\right|^{\alpha-1}x^{(n-1)}(t)\right)' \\ &\quad + r(t)\left|x(t)\right|^{\alpha-1}x^{(n-1)}(t) \\ &\quad + q(t)\left|x(g(t))\right|^{\alpha-1}x(g(t)) = 0, \\ &\qquad t \geq t_0, \end{aligned} \tag{12}$$

where $\alpha > 0$. Some new criteria are presented that improve the related contributions to the subject.

Clearly, the equations (4)–(12) are special cases of (1). The purpose of this paper is to extend and improve the abovementioned oscillation theorems for certain even order neutral delay differential equations with mixed nonlinearities (1).

The paper is organized as follows. In the next section, we present some lemmas which will be used in the following results. In Sections 3 and 4, by developing Riccati transformations technique and inequalities, some sufficient conditions for oscillation of all solutions of (1) are established. In Section 5, we give an example to illustrate Theorem 11.

2. Lemmas

In this section, in order to prove our main results, we need the following lemmas.

Lemma 1 (see [5]). *Let $u \in C^n([t_0,\infty), R^+)$. If $u^{(n)}(t)$ is eventually of one sign for all large t, then there exist $t_x > t_1$, for some $t_1 > t_0$, and an integer l, $0 \leq l \leq n$, with $n+l$ even for $u^{(n)}(t) \geq 0$ or $n+l$ odd for $u^{(n)}(t) \leq 0$ such that $l > 0$ implies that $u^{(k)}(t) > 0$ for $t > t_x$, $k = 0, 1, \ldots, l-1$, and $l \leq n-1$ implies that $(-1)^{l+k}u^{(k)}(t) > 0$ for $t > t_x$, $k = l, l+1, \ldots, n-1$.*

Lemma 2 (see [1], Lemma 2.2.2). *If the function u is as in Lemma 1 and $u^{(n-1)}(t)u^{(n)}(t) \leq 0$ for $t > t_x$, then, for every λ, $0 < \lambda < 1$, there exists a constant $M > 0$ such that*

$$u(\lambda t) \geq M t^{n-1}\left|u^{(n-1)}(t)\right|, \tag{13}$$

for all large t.

Lemma 3 (see [1], Lemma 2.2.3). *If the function u is as in Lemma 1 and $u^{(n-1)}(t)u^{(n)}(t) \leq 0$ for $t > t_x$, $\lim_{t\to\infty} u(t) \neq 0$, then, for every λ, $0 < \lambda < 1$, such that*

$$u(t) \geq \frac{\lambda}{(n-1)!}t^{n-1}\left|u^{(n-1)}(t)\right|, \tag{14}$$

for all large t.

Lemma 4 (see [2, 5]). *Consider the half-linear differential equations*

$$\left(a(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + q(t)\left|x(t)\right|^{\alpha-1}x(t) = 0, \tag{15}$$

where $\alpha > 0$, $a, q \in C([t_0, \infty), R^+)$. *Then every solution of* (15) *is nonoscillatory if and only if there exist a real number* $T \geq t_0$ *and a function* $\nu \in C^1([t_0, \infty), R)$, *such that*

$$\nu'(t) + \alpha a^{-(1/\alpha)}(t) |\nu(t)|^{(\alpha+1)/\alpha} + q(t) \leq 0, \quad t \in [T, \infty). \tag{16}$$

Lemma 5. *If A and B are nonnegative constants, then*

$$A^\gamma - \gamma A B^{\gamma-1} + (\gamma - 1) B^\gamma \geq 0, \quad \gamma > 1, \tag{17}$$

and the equality holds if and only if $A = B$.

In the next section, by developing Riccati transformations technique and inequalities, some sufficient conditions for oscillation of all solutions of (1) are established.

3. Oscillation Criteria for an Oscillating Function p

In this section, we assume the following.

(H) p is an oscillating function, and $\lim_{t\to\infty} p(t) = 0$.

Lemma 6. *Assume that* (2) *holds. Furthermore, assume that* x *is an eventually positive solution of* (1), *which is bounded and does not converge to zero. Then there exists* $t_1 \geq t_0$, *such that*

$$z(t) > 0, \qquad z'(t) > 0, \qquad z^{(n-1)}(t) > 0, \qquad z^{(n)}(t) \leq 0, \quad \forall t \geq t_1. \tag{18}$$

Proof. Since x is an eventually positive solution of (1), there exists a constant $t_1 \geq t_0$, such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau_i(t)) > 0$, $i = 0, 1, 2$, for all $t \geq t_1$. Then, by (1), we have $(r(t)|z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t))' \leq 0$, $t \geq t_1$.

Furthermore, since x is a bounded solution and $\lim_{t\to\infty} x(t) \neq 0$, by (H) we know that $\lim_{t\to\infty} p(t)x(\sigma(t)) = 0$; then there exists $t_2 \geq t_1$, such that $z(t) = x(t) + p(t)x(\sigma(t)) > 0$, $t \geq t_2$. So z is eventually positive and bounded.

The rest of the proof is similar to that of Meng and Xu [19, Lemma 2.3], so it is omitted. □

Theorem 7. *Assume that (H) and* (2) *hold. Furthermore, assume that there exists a constant* λ, $0 < \lambda < 1$, *and, for every constant* $M > 0$, *assume that there exists a positive function* $\rho \in C^1([t_0, \infty), R)$, *for sufficiently large* $t_1 \geq t_0$, *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\rho(s) Q(s) - \frac{r(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda M \rho(s) \tau^{n-2}(s) \tau'(s))^\alpha} \right) ds = \infty, \tag{19}$$

where

$$Q(t) := \frac{1}{2^\alpha} \left\{ q_0(t) + [k_1 q_1(t)]^{1/k_1} [k_2 q_2(t)]^{1/k_2} \right\}, \quad k_1 := \frac{(\gamma - \beta)}{(\gamma - \alpha)}, \quad k_2 := \frac{(\gamma - \beta)}{(\alpha - \beta)}. \tag{20}$$

Then every bounded solution of (1) *is oscillatory or converges to zero.*

Proof. Suppose that (1) has a bounded nonoscillatory solution x. We may assume without loss of generality that there exists a number $t_1 \geq t_0$, such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$, for all $t \geq t_1$. Furthermore, we assume that $\lim_{t\to\infty} x(t) \neq 0$. Using the definition of z and Lemma 6, we have $z(t) > 0$, $z'(t) > 0$, $z^{(n-1)}(t) > 0$, and $z^{(n)}(t) \leq 0$, $t \geq t_1$. Hence there exists $t_2 \geq t_1$, such that

$$x(t) = z(t) - p(t)x(\sigma(t)) \geq \frac{z(t)}{2}, \quad t \geq t_2. \tag{21}$$

From (1) and the above inequality, we obtain

$$\left(r(t)\left(z^{(n-1)}(t)\right)^\alpha\right)' + \frac{1}{2^\alpha} q_0(t) z^\alpha(\tau(t)) + \frac{1}{2^\beta} q_1(t) z^\beta(\tau(t)) + \frac{1}{2^\gamma} q_2(t) z^\gamma(\tau(t)) \leq 0, \quad t \geq t_3 \geq t_2. \tag{22}$$

Because of $z'(t) > 0$, by Lemma 2, $z^{(n-1)}(t) > 0$, and $z^{(n)}(t) \leq 0$, there exists $t_4 \geq t_3$, and, for every $0 < \lambda < 1$, there exists a constant $M > 0$, we have

$$z'(\lambda\tau(t)) \geq M\tau^{n-2}(t) z^{(n-1)}\tau(t) \geq M\tau^{n-2}(t) z^{(n-1)}(t), \tag{23}$$

for $t \geq t_4$. We define the function ω by

$$\omega(t) = \rho(t) r(t) \left(\frac{z^{(n-1)}(t)}{z(\lambda\tau(t))}\right)^\alpha, \quad t \geq t_4. \tag{24}$$

Then $\omega(t) > 0$, $t \geq t_4$. Next differentiating (24), we get

$$\omega'(t) = \rho'(t) r(t) \left(\frac{z^{(n-1)}(t)}{z(\lambda\tau(t))}\right)^\alpha + \rho(t) \frac{\left(r(t)\left(z^{(n-1)}(t)\right)^\alpha\right)'}{z^\alpha(\lambda\tau(t))} - \alpha\lambda\rho(t) \frac{r(t)\left(z^{(n-1)}(t)\right)^\alpha \tau'(t) z'(\lambda\tau(t))}{z^{\alpha+1}(\lambda\tau(t))}. \tag{25}$$

So by (22) and (23), we obtain

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) \times \left(\frac{1}{2^\alpha} q_0(t) + \frac{1}{2^\beta} q_1(t) z^{\beta-\alpha}(\tau(t)) + \frac{1}{2^\gamma} q_2(t) z^{\gamma-\alpha}(\tau(t)) \right) \left(\frac{z(\tau(t))}{z(\lambda\tau(t))}\right)^\alpha - \frac{\alpha\lambda M \tau^{n-2}(t)\tau'(t)}{(\rho(t) r(t))^{1/\alpha}} \omega^{(\alpha+1)/\alpha}(t). \tag{26}$$

Let

$$a = k_1 \frac{1}{2^\beta} q_1(t) z^{\beta-\alpha}(\tau(t)), \qquad b = k_2 \frac{1}{2^\gamma} q_2(t) z^{\gamma-\alpha}(\tau(t)), \tag{27}$$

where k_1 and k_2 are defined as in Theorem 7. Using the inequality

$$\frac{|a|}{p} + \frac{|b|}{q} \geq |a|^{1/p}|b|^{1/q}, \quad p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{28}$$

we have

$$\begin{aligned} &\frac{1}{2^\beta} q_1(t) z^{\beta-\alpha}(\tau(t)) + \frac{1}{2^\gamma} q_2(t) z^{\gamma-\alpha}(\tau(t)) \\ &\quad \geq \left[k_1 \frac{1}{2^\beta} q_1(t) z^{\beta-\alpha}(\tau(t))\right]^{1/k_1} \left[k_2 \frac{1}{2^\gamma} q_2(t) z^{\gamma-\alpha}(\tau(t))\right]^{1/k_2} \\ &\quad = \frac{1}{2^\alpha} [k_1 q_1(t)]^{1/k_1} [k_2 q_2(t)]^{1/k_2}, \end{aligned} \tag{29}$$

so we get

$$\begin{aligned} \omega'(t) \leq{}& \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) Q(t) \\ &- \frac{\alpha\lambda M \tau^{n-2}(t) \tau'(t)}{(\rho(t) r(t))^{1/\alpha}} \omega^{(\alpha+1)/\alpha}(t). \end{aligned} \tag{30}$$

Let

$$\begin{aligned} A &= \left(\alpha\lambda M \frac{\tau^{n-2}(t)\tau'(t)}{(\rho(t) r(t))^{1/\alpha}}\right)^{1/\gamma} \omega(t), \\ B &= \left(\frac{\rho'(t)}{\gamma\rho(t)} \left(\alpha\lambda M \frac{\tau^{n-2}(t)\tau'(t)}{(\rho(t) r(t))^{1/\alpha}}\right)^{-(1/\gamma)}\right)^{1/(\gamma-1)}, \end{aligned} \tag{31}$$

where $\gamma = (\alpha+1)/\alpha > 1$. Applying the inequality in Lemma 5, we obtain

$$\begin{aligned} &\frac{\rho'(t)}{\rho(t)} \omega(t) - \alpha\lambda M \frac{\tau^{n-2}(t)\tau'(t)}{(\rho(t) r(t))^{1/\alpha}} \omega^\gamma(t) \\ &\quad \leq \frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda M \rho(t) \tau^{n-2}(t) \tau'(t))^\alpha}. \end{aligned} \tag{32}$$

Thus, by (30) and (32), we get

$$\omega'(t) \leq -\left(\rho(t) Q(t) - \frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda M \rho(t) \tau^{n-2}(t) \tau'(t))^\alpha}\right). \tag{33}$$

Integrating (33) from t_1 to t, we have

$$\begin{aligned} \omega(t) \leq{}& \omega(t_1) \\ &- \int_{t_1}^{t} \left(\rho(s) Q(s) - \frac{r(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda M \rho(s) \tau^{n-2}(s) \tau'(s))^\alpha}\right) ds. \end{aligned} \tag{34}$$

Let $t \to \infty$ in (34), which leads to a contradiction with (19). The proof is complete. □

Theorem 8. *Assume that (H) and* (2) *hold, and there exists a constant λ, $0 < \lambda < 1$, and, for every constant $M > 0$, such that*

$$\begin{aligned} &\left(\left(\frac{(r(t))^{1/\alpha}}{\lambda M \tau^{n-2}(t) \tau'(t)}\right)^\alpha |x'(t)|^{\alpha-1} x'(t)\right)' \\ &\quad + Q(t) |x(t)|^{\alpha-1} x(t) = 0, \end{aligned} \tag{35}$$

is oscillatory, where Q is defined as in Theorem 7. Then every bounded solution of (1) *is oscillatory or converges to zero.*

Proof. Suppose that (1) has a bounded nonoscillatory solution x. We may assume without loss of generality that there exists $t_1 \geq t_0$, such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$, for all $t \geq t_1$. Furthermore, we assume that $\lim_{t\to\infty} x(t) \neq 0$. We define ν by

$$\nu(t) = r(t) \left(\frac{z^{(n-1)}(t)}{z(\lambda\tau(t))}\right)^\alpha. \tag{36}$$

Proceeding as in the proof of Theorem 7, for every $0 < \lambda < 1$, there exists $M > 0$, and we have

$$\nu'(t) + Q(t) + \frac{\alpha\lambda M \tau^{n-2}(t) \tau'(t)}{(r(t))^{1/\alpha}} \nu^{(\alpha+1)/\alpha}(t) \leq 0. \tag{37}$$

That is,

$$\begin{aligned} &\nu'(t) + Q(t) \\ &\quad + \alpha\left[\left(\frac{(r(t))^{1/\alpha}}{\lambda M \tau^{n-2}(t) \tau'(t)}\right)^\alpha\right]^{-(1/\alpha)} \nu^{(\alpha+1)/\alpha}(t) \leq 0. \end{aligned} \tag{38}$$

Based on Lemma 4, we obtain that (35) is nonoscillatory, which leads to a contradiction. The proof is complete. □

Theorem 9. *Assume that (H) and* (3) *hold. Furthermore, assume that there exists a constant λ, $0 < \lambda < 1$, and, for every constant $M > 0$, assume that there exists a positive function $\rho \in C^1([t_0, \infty), R)$, such that* (19) *holds. If, for sufficiently large $t_1 \geq t_0$,*

$$\begin{aligned} \limsup_{t\to\infty} \int_{t_1}^{t} &\left[\left(\frac{\lambda}{(n-2)!}\right)^\alpha Q(s) \tau^{\alpha(n-2)}(s) \delta^\alpha(s)\right. \\ &\left. - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{r^{1/\alpha}(s) \delta(s)}\right] ds = \infty, \end{aligned} \tag{39}$$

where Q and h are defined as in Theorem 7, and $\delta(t) = \int_t^{\infty}(1/r^{1/\alpha}(s))ds$, then every bounded solution of (1) is oscillatory or converges to zero.

Proof. Suppose that (1) has a bounded nonoscillatory solution x. We may assume without loss of generality that there exists $t_1 \geq t_0$, such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$, for all $t \geq t_1$. Then it follows from (1) that

$$\left(r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t)\right)' \leq 0, \quad t \geq t_1. \tag{40}$$

Therefore, $r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is a nonincreasing function on $[t_1,\infty)$. Consequently, it is easy to conclude that there exist two possible cases of the sign of $z^{(n-1)}(t)$. Furthermore, we assume that $\lim_{t\to\infty}x(t) \neq 0$.

Case I. If $z^{(n-1)}(t) > 0$, for $t \geq t_1$, then we go back to the proof of Theorem 7, and we get a contradiction to (19), so we omit the details.

Case II. $z^{(n-1)}(t) < 0$, for $t \geq t_1$. Applying Lemma 1, we get $z^{(n-2)}(t) > 0$. Define the function v by

$$v(t) = \frac{r(t)\left(-z^{(n-1)}(t)\right)^{\alpha-1}z^{(n-1)}(t)}{\left(z^{(n-2)}(t)\right)^{\alpha}}, \quad t \geq t_1. \tag{41}$$

Then $v(t) < 0$ for $t \geq t_1$. Noting that $r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is nonincreasing, we obtain

$$r^{1/\alpha}(s)\,z^{(n-1)}(s) \leq r^{1/\alpha}(t)\,z^{(n-1)}(t), \quad s \geq t. \tag{42}$$

Dividing (42) by $r^{1/\alpha}(s)$ and integrating it from t to l $(l \geq t)$, we have

$$z^{(n-2)}(l) \leq z^{(n-2)}(t) + r^{1/\alpha}(t)\,z^{(n-1)}(t)\int_t^l \frac{1}{r^{1/\alpha}(s)}ds. \tag{43}$$

Letting $l \to \infty$ in the above inequality, we get

$$0 \leq z^{(n-2)}(t) + r^{1/\alpha}(t)\,z^{(n-1)}(t)\,\delta(t), \tag{44}$$

which implies that

$$-1 \leq \frac{r^{1/\alpha}(t)\,z^{(n-1)}(t)}{z^{(n-2)}(t)}\delta(t), \quad t \geq t_1, \tag{45}$$

where δ is defined as in Theorem 9. Hence, by (41), we obtain

$$-1 \leq v(t)\,\delta^{\alpha}(t) \leq 0, \quad t \geq t_1. \tag{46}$$

Differentiating (41), we have

$$\begin{aligned} v'(t) = {} & \frac{\left(r(t)\left(-z^{(n-1)}(t)\right)^{\alpha-1}z^{(n-1)}(t)\right)'}{\left(z^{(n-2)}(t)\right)^{\alpha}} \\ & - \frac{\alpha r(t)\left(-z^{(n-1)}(t)\right)^{\alpha-1}z^{(n-1)}(t)\,z^{(n-1)}(t)}{\left(z^{(n-2)}(t)\right)^{\alpha+1}}. \end{aligned} \tag{47}$$

From (22), we get

$$\begin{aligned} v'(t) \leq {} & \left[-\frac{1}{2^{\alpha}}q_0(t)\,z^{\alpha}(\tau(t)) - \frac{1}{2^{\beta}}q_1(t)\,z^{\beta}(\tau(t)) \right. \\ & \left. -\frac{1}{2^{\gamma}}q_2(t)\,z^{\gamma}(\tau(t))\right]\frac{1}{\left(z^{(n-2)}(t)\right)^{\alpha}} \\ & - \frac{\alpha r(t)\left(-z^{(n-1)}(t)\right)^{\alpha-1}z^{(n-1)}(t)\,z^{(n-1)}(t)}{\left(z^{(n-2)}(t)\right)^{\alpha+1}}. \end{aligned} \tag{48}$$

On the other hand, by $\lim_{t\to\infty}z(t) \neq 0$ and Lemma 3, we obtain

$$z(t) \geq \frac{\lambda}{(n-2)!}t^{n-2}z^{(n-2)}(t); \tag{49}$$

that is, because of $z^{(n-1)} < 0$,

$$\begin{aligned} z(\tau(t)) & \geq \frac{\lambda}{(n-2)!}\tau^{n-2}(t)\,z^{(n-2)}(\tau(t)) \\ & \geq \frac{\lambda}{(n-2)!}\tau^{n-2}(t)\,z^{(n-2)}(t), \end{aligned} \tag{50}$$

for every $0 < \lambda < 1$ and $t \geq t_1$. Then from (41), (48), and (50), we have

$$\begin{aligned} v'(t) \leq {} & \left[-\frac{1}{2^{\alpha}}q_0(t) - \frac{1}{2^{\beta}}q_1(t)\,z^{\beta-\alpha}(\tau(t)) \right. \\ & \left. -\frac{1}{2^{\gamma}}q_2(t)\,z^{\gamma-\alpha}(\tau(t))\right]\frac{z^{\alpha}(\tau(t))}{\left(z^{(n-2)}(t)\right)^{\alpha}} \\ & - \frac{\alpha r(t)\left(-z^{(n-1)}(t)\right)^{\alpha-1}z^{(n-1)}(t)\,z^{(n-1)}(t)}{\left(z^{(n-2)}(t)\right)^{\alpha+1}} \\ \leq {} & -\left(\frac{\lambda}{(n-1)!}\right)^{\alpha}Q(t)\,\tau^{\alpha(n-2)}(t) \\ & - \frac{\alpha}{r^{1/\alpha}(t)}(-v(t))^{(\alpha+1)/\alpha}, \end{aligned} \tag{51}$$

where Q is defined as in Theorem 7. Multiplying (51) by $\delta^{\alpha}(t)$ and integrating it from t_1 to t, we get

$$\begin{aligned} & \delta^{\alpha}(t)\,v(t) - \delta^{\alpha}(t_1)\,v(t_1) + \alpha\int_{t_1}^{t}\frac{\delta^{\alpha-1}(s)}{r^{1/\alpha}(s)}v(s)\,ds \\ & + \left(\frac{\lambda}{(n-2)!}\right)^{\alpha}\int_{t_1}^{t}Q(s)\,\tau^{\alpha(n-2)}(s)\,\delta^{\alpha}(s)\,ds \\ & + \alpha\int_{t_1}^{t}\frac{\delta^{\alpha}(s)}{r^{1/\alpha}(s)}(-v(s))^{(\alpha+1)/\alpha}ds \leq 0. \end{aligned} \tag{52}$$

Let

$$\begin{aligned} A & = \left(\frac{\alpha\delta^{\alpha}(s)}{r^{1/\alpha}(s)}\right)^{1/\gamma}(-v(s)), \\ B & = \left(\frac{\alpha^2}{\alpha+1}\frac{\delta^{\alpha-1}(s)}{r^{1/\alpha}(s)}\left(\frac{\alpha\delta^{\alpha}(s)}{r^{1/\alpha}(s)}\right)^{-(1/\gamma)}\right)^{1/(\gamma-1)}, \end{aligned} \tag{53}$$

where $\gamma = (\alpha+1)/\alpha > 1$. Applying the inequality in Lemma 5, we obtain

$$\frac{\alpha\delta^{\alpha-1}(s)}{r^{1/\alpha}(s)}(-v(s)) - \frac{\alpha\delta^{\alpha}(s)}{r^{1/\alpha}(s)}(-v(s))^{(\alpha+1)/\alpha} \leq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{r^{1/\alpha}(s)\,\delta(s)}. \tag{54}$$

Therefore, it follows from (52) that

$$\delta^{\alpha}(t)\,v(t) + \int_{t_1}^{t}\left[\left(\frac{\lambda}{(n-2)!}\right)^{\alpha} Q(s)\,\tau^{\alpha(n-2)}(s)\,\delta^{\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{r^{1/\alpha}(s)\,\delta(s)}\right]ds \leq \delta^{\alpha}(t_1)\,v(t_1). \tag{55}$$

From (39) and the above inequality, we get a contradiction to (46). The proof is complete. □

Theorem 10. *Assume that (H) and* (3) *hold. Furthermore, assume that there exists a constant λ, $0 < \lambda < 1$, and for every constant $M > 0$ such that* (35) *is oscillatory. If, for sufficiently large $t_1 \geq t_0$, one has* (39)*, where Q and h are defined as in Theorem 7 and δ is defined as in Theorem 9, then every bounded solution of* (1) *is oscillatory or converges to zero.*

4. Oscillation Criteria for $0 \leq p(t) < 1$

In this section, we assume that $0 \leq p(t) < 1$.

Theorem 11. *Assume that* (2) *holds, there exists a constant λ, $0 < \lambda < 1$, and, for every constant $M > 0$, there exists a positive function $\rho \in C^1([t_0,\infty), R)$, such that, for sufficiently large $t_1 \geq t_0$,*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\rho(s)\,\overline{Q(s)} - \frac{r(s)\left(\rho'(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\lambda M\rho(s)\,\tau^{n-2}(s)\,\tau'(s)\right)^{\alpha}}\right)ds = \infty, \tag{56}$$

where

$$\overline{Q(t)} = \xi^{\alpha}(t)\left\{q_0(t) + \left[k_1 q_1(t)\right]^{1/k_1}\left[k_2 q_2(t)\right]^{1/k_2}\right\},$$
$$\xi(t) = \min\left\{1 - p(\tau_0(t)),\, 1 - p(\tau_1(t)),\, 1 - p(\tau_2(t))\right\}, \tag{57}$$

and k_1 and k_2 are defined as in Theorem 7. Then every solution of (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution x. Without loss of generality, we may assume that there exists $t_1 \geq t_0$, such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$, for all $t \geq t_1$. Similar to the proof of Lemma 2.3 in [19], there exists $t_2 \geq t_1$, such that

$$z(t) > 0, \quad z'(t) > 0, \quad z^{(n-1)}(t) > 0, \quad z^{(n)}(t) \leq 0, \quad t \geq t_2. \tag{58}$$

From the definition of z, we have

$$x(t) = z(t) - p(t)\,x(\sigma(t)) \geq z(t) - p(t)\,z(\sigma(t)) \geq (1 - p(t))\,z(t), \quad t \geq t_3 \geq t_2. \tag{59}$$

Since $\lim_{t\to\infty}\tau(t) = \infty$, there exists $t_4 \geq t_3$, such that $\tau(t) \geq t_4$, $t \geq t_4$, so

$$x(\tau(t)) \geq (1 - p(\tau(t)))\,z(\tau(t)), \quad t \geq t_4. \tag{60}$$

From (1), (59), and (60), we get

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' + 1 - p(\tau_0(t))^{\alpha} q_0(t)\,z^{\alpha}(\tau(t)) + 1 - p(\tau_1(t))^{\beta} q_1(t)\,z^{\beta}(\tau(t)) + 1 - p(\tau_2(t))^{\gamma} q_2(t)\,z^{\gamma}(\tau(t)) \leq 0, \quad t \geq t_4. \tag{61}$$

For every $0 < \lambda < 1$, we define the function

$$\omega(t) = \rho(t)\,r(t)\left(\frac{z^{(n-1)}(t)}{z(\lambda\tau(t))}\right)^{\alpha}, \quad t \geq t_4. \tag{62}$$

Then $\omega(t) > 0$, $t \geq t_4$. Next differentiating (62), we obtain

$$\omega'(t) = \rho'(t)\,r(t)\left(\frac{z^{(n-1)}(t)}{z(\lambda\tau(t))}\right)^{\alpha} + \rho(t)\frac{\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'}{z^{\alpha}(\lambda\tau(t))} - \alpha\lambda\rho(t)\,r(t)\frac{\left(z^{(n-1)}(t)\right)^{\alpha} z'(\lambda\tau(t))}{z^{\alpha+1}(\lambda\tau(t))}. \tag{63}$$

From (23), (61), and (62), we have

$$\begin{aligned}\omega'(t) \leq{}& \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t) \times \left(\xi^{\alpha}(t)\,q_0(t) + \xi^{\beta}(t)\,q_1(t)\,z^{\beta-\alpha}(\tau(t)) + \xi^{\gamma}(t)\,q_2(t)\,z^{\gamma-\alpha}(\tau(t))\right)\left(\frac{z(\tau(t))}{z(\lambda\tau(t))}\right)^{\alpha} \\ &- \alpha\lambda M\frac{\tau^{n-2}(t)\,\tau'(t)}{\left(\rho(t)\,r(t)\right)^{1/\alpha}}\omega^{(\alpha+1)/\alpha}(t),\end{aligned} \tag{64}$$

where ξ is defined as in Theorem 11. Setting

$$a = k_1\xi^{\beta}(t)\,q_1(t)\,z^{\beta-\alpha}(\tau(t)), \quad b = k_2\xi^{\gamma}(t)\,q_2(t)\,z^{\gamma-\alpha}(\tau(t)), \tag{65}$$

by inequality (28), we get

$$\begin{aligned}\xi^{\beta}(t)\,q_1(t)\,z^{\beta-\alpha}(\tau(t)) &+ \xi^{\gamma}(t)\,q_2(t)\,z^{\gamma-\alpha}(\tau(t))\\ &\geq \left[k_1\xi^{\beta}(t)\,q_1(t)\,z^{\beta-\alpha}(\tau(t))\right]^{1/k_1}\\ &\quad\times\left[k_2\xi^{\gamma}(t)\,q_2(t)\,z^{\gamma-\alpha}(\tau(t))\right]^{1/k_2}\\ &= \xi^{\alpha}(t)\left[k_1q_1(t)\right]^{1/k_1}\left[k_2q_2(t)\right]^{1/k_2};\end{aligned} \tag{66}$$

hence,

$$\begin{aligned}\omega'(t) \leq{}& \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)\overline{Q(t)}\\ &- \alpha\lambda M\frac{\tau^{n-2}(t)\,\tau'(t)}{(\rho(t)\,r(t))^{1/\alpha}}\omega^{(\alpha+1)/\alpha}(t).\end{aligned} \tag{67}$$

Let

$$\begin{aligned}A &= \left(\alpha\lambda M\frac{\tau^{n-2}(t)\,\tau'(t)}{(\rho(t)\,r(t))^{1/\alpha}}\right)^{1/\gamma}\omega(t),\\ B &= \left(\frac{\rho'(t)}{\gamma\rho(t)}\left(\alpha\lambda M\frac{\tau^{n-2}(t)\,\tau'(t)}{(\rho(t)\,r(t))^{1/\alpha}}\right)^{-(1/\gamma)}\right)^{1/(\gamma-1)},\end{aligned} \tag{68}$$

where $\gamma = (\alpha+1)/\alpha > 1$. Applying the inequality in Lemma 5, we obtain

$$\begin{aligned}\frac{\rho'(t)}{\rho(t)}\omega(t) &- \alpha\lambda M\frac{\tau^{n-2}(t)\,\tau'(t)}{(\rho(t)\,r(t))^{1/\alpha}}\omega^{\gamma}(t)\\ &\leq \frac{r(t)\left(\rho'(t)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\lambda M\rho(t)\,\tau^{n-2}(t)\,\tau'(t)\right)^{\alpha}}.\end{aligned} \tag{69}$$

Thus, by (67) and (69), we get

$$\begin{aligned}\omega'(t) \leq -\Bigg(&\rho(t)\overline{Q(t)}\\ &-\frac{r(t)\left(\rho'(t)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\lambda M\rho(t)\,\tau^{n-2}(t)\,\tau'(t)\right)^{\alpha}}\Bigg).\end{aligned} \tag{70}$$

Integrating (70) from t_1 to t, we have

$$\begin{aligned}\omega(t) \leq{}& \omega(t_1)\\ &-\int_{t_1}^{t}\Bigg(\rho(s)\overline{Q(s)}\\ &-\frac{r(s)\left(\rho'(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\lambda M\rho(s)\,\tau^{n-2}(s)\,\tau'(s)\right)^{\alpha}}\Bigg)ds.\end{aligned} \tag{71}$$

Letting $t \to \infty$ in (71), we get a contradiction with (56). This completes the proof of Theorem 11. □

Remark 12. From Theorem 11, we can obtain different conditions for oscillation of all solutions of (1) with different choices of ρ.

Theorem 13. *Assume that* (3) *holds, assume that there exists a constant λ, $0 < \lambda < 1$, and, for every constant $M > 0$, there exists a positive function $\rho \in C^1([t_0,\infty), R)$, such that, for sufficiently large $t_1 \geq t_0$,* (56) *holds. If there exists a positive function $\eta \in C^1([t_0,\infty), R)$, $\eta'(t) \geq 0$, such that*

$$\begin{aligned}\int_{t_1}^{\infty}\Bigg(\frac{1}{\eta(t)\,r(t)}\int_{t_1}^{t}&\left(\frac{\lambda M}{(n-2)!}\tau^{n-2}(s)\,\delta(\tau(s))\right)^{\alpha}\\ &\times\eta(s)\overline{Q(s)}ds\Bigg)^{1/\alpha}dt = \infty,\end{aligned} \tag{72}$$

$$\begin{aligned}\int_{t_1}^{\infty}\Bigg(\frac{1}{\eta(t)\,r(t)}\int_{t_1}^{t}&\left(\frac{\lambda M}{(n-2)!}s^{n-2}\delta(s)\right)^{\alpha}\\ &\times\eta(s)\overline{Q(s)}ds\Bigg)^{1/\alpha}dt = \infty,\end{aligned} \tag{73}$$

where δ is defined as in Theorem 9 and $\overline{Q}$ is defined as in Theorem 11, then every solution of (1) *is oscillatory or converges to zero.*

Proof. Suppose that (1) has a nonoscillatory solution x. We may assume without loss of generality that there exists $t_1 \geq t_0$, such that $x(t) > 0$, $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$, and $z(t) > 0$, for all $t \geq t_1$. Furthermore, we assume that $\lim_{t\to\infty}x(t) \neq 0$. Similar to the proof of Theorem 9, we find that $r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is a nonincreasing function on $[t_1,\infty)$ and there exist two possible cases of the sign of $z^{(n-1)}(t)$.

Case I. If $z^{(n-1)}(t) > 0$, for $t \geq t_1$, then we go back to the proof of Theorem 11, and we get a contradiction to (56), so we omit the details.

Case II. $z^{(n-1)}(t) < 0$, for $t \geq t_1$. Applying Lemma 1, we get $z^{(n-2)}(t) > 0$, $z'(t) > 0$ or $z^{(n-2)}(t) > 0$, $z'(t) < 0$.

If $z^{(n-2)}(t) > 0$, $z'(t) > 0$. Since $r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is nonincreasing, we obtain

$$\begin{aligned}r(s)&\left|z^{(n-1)}(s)\right|^{\alpha-1}z^{(n-1)}(s)\\ &\leq r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t)\\ &\leq r(t_1)\left|z^{(n-1)}(t_1)\right|^{\alpha-1}z^{(n-1)}(t_1),\end{aligned} \tag{74}$$

for $s \geq t \geq t_1$; that is,

$$-z^{(n-1)}(s) \geq \left(\frac{r(t)}{r(s)}\right)^{1/\alpha}\left(-z^{(n-1)}(t)\right). \tag{75}$$

Integrating (75) from t to ∞, we get

$$z^{(n-2)}(t) \geq r^{1/\alpha}(t)\left(-z^{(n-1)}(t)\right)\int_t^\infty \frac{1}{r^{1/\alpha}(s)}ds \geq M\delta(t), \quad t \geq t_1, \tag{76}$$

where $M = r^{1/\alpha}(t_1)(-z^{(n-1)}(t_1))$. From (49) and the above inequality, we obtain

$$z(t) \geq M\frac{\lambda}{(n-2)!}t^{n-2}\delta(t). \tag{77}$$

Using (59) and (77) in (1) and noting that $z'(t) > 0$, we have

$$\begin{aligned} -&\left(r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t)\right)' \\ &= q_0(t)x^\alpha(\tau_0(t)) + q_1(t)x^\beta(\tau_1(t)) + q_2(t)x^\gamma(\tau_2(t)) \\ &\geq q_0(t)(1-p(\tau_0(t)))^\alpha z^\alpha(\tau_0(t)) \\ &\quad + q_1(t)(1-p(\tau_1(t)))^\beta z^\beta(\tau_1(t)) \\ &\quad + q_2(t)(1-p(\tau_2(t)))^\gamma z^\gamma(\tau_2(t)) \\ &\geq \xi^\alpha(t)q_0(t)z^\alpha(\tau(t)) + \xi^\beta(t)q_1(t)z^\beta(\tau(t)) \\ &\quad + \xi^\gamma(t)q_2(t)z^\gamma(\tau(t)). \end{aligned} \tag{78}$$

Setting

$$\begin{aligned} a &= k_1\xi^\beta(t)q_1(t)z^\beta(\tau(t)), \\ b &= k_2\xi^\gamma(t)q_2(t)z^\gamma(\tau(t)), \end{aligned} \tag{79}$$

by inequality (28), we get

$$\begin{aligned} &\xi^\beta(t)q_1(t)z^\beta(\tau(t)) + \xi^\gamma(t)q_2(t)z^\gamma(\tau(t)) \\ &\quad \geq \left[k_1\xi^\beta q_1(t)z^\beta(\tau(t))\right]^{1/k_1}\left[k_2\xi^\gamma q_2(t)z^\gamma(\tau(t))\right]^{1/k_2} \\ &\quad \geq \xi^\alpha(t)\left[k_1q_1(t)\right]^{1/k_1}\left[k_2q_2(t)\right]^{1/k_2}z^\alpha(\tau(t)). \end{aligned} \tag{80}$$

Therefore, combining (77), (78), and (80), we obtain

$$\begin{aligned} -&\left(r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t)\right)' \\ &\geq \overline{Q(t)}z^\alpha(\tau(t)) \geq \left(\frac{\lambda M}{(n-2)!}\tau^{n-2}(t)\delta(\tau(t))\right)^\alpha\overline{Q(t)}. \end{aligned} \tag{81}$$

Define the function u by

$$\begin{aligned} u(t) &= \eta(t)r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t) \\ &= -\eta(t)r(t)\left(-z^{(n-1)}(t)\right)^\alpha, \quad t \geq t_1. \end{aligned} \tag{82}$$

Then $u(t) < 0$. Differentiating $u(t)$ and from (81), we find that

$$\begin{aligned} u'(t) &= -\eta'(t)r(t)\left(-z^{(n-1)}(t)\right)^\alpha \\ &\quad - \eta(t)\left(r(t)\left(-z^{(n-1)}(t)\right)^\alpha\right)' \\ &\leq -\eta(t)\left(q_0(t)x^\alpha(\tau_0(t)) + q_1(t)x^\beta(\tau_1(t))\right. \\ &\qquad \left. + q_2(t)x^\gamma(\tau_2(t))\right) \\ &\leq -\left(\frac{\lambda M}{(n-2)!}\tau^{n-2}(t)\delta(\tau(t))\right)^\alpha\eta(t)\overline{Q(t)}. \end{aligned} \tag{83}$$

Integrating (83) from t_1 to t, we get

$$\begin{aligned} &u(t) - u(t_1) \\ &\quad \leq -\int_{t_1}^t\left(\frac{\lambda M}{(n-2)!}\tau^{n-2}(s)\delta(\tau(s))\right)^\alpha\eta(s)\overline{Q(s)}ds. \end{aligned} \tag{84}$$

Therefore,

$$\begin{aligned} &-\eta(t)r(t)\left(-z^{(n-1)}(t)\right)^\alpha \\ &\quad \leq -\int_{t_1}^t\left(\frac{\lambda M}{(n-2)!}\tau^{n-2}(s)\delta(\tau(s))\right)^\alpha\eta(s)\overline{Q(s)}ds; \end{aligned} \tag{85}$$

that is,

$$\begin{aligned} &z^{(n-1)}(t) \\ &\quad \leq -\left(\frac{1}{\eta(t)r(t)}\int_{t_1}^t\left(\frac{\lambda M}{(n-2)!}\tau^{n-2}(s)\delta(\tau(s))\right)^\alpha\right. \\ &\qquad \left. \times\eta(s)\overline{Q(s)}ds\right)^{1/\alpha}. \end{aligned} \tag{86}$$

Integrating the above inequality from t_1 to l $(l > t_1)$, we obtain

$$\begin{aligned} &z^{(n-2)}(l) - z^{(n-2)}(t_1) \\ &\quad \leq -\int_{t_1}^l\left(\frac{1}{\eta(t)r(t)}\int_{t_1}^t\left(\frac{\lambda M}{(n-2)!}\tau^{n-2}(s)\delta(\tau(s))\right)^\alpha\right. \\ &\qquad \left. \times\eta(s)\overline{Q(s)}ds\right)^{1/\alpha}dt. \end{aligned} \tag{87}$$

Letting $l \to \infty$ and using (73) in (97), we have $\lim_{l\to\infty}z^{(n-2)}(l) = -\infty$, which is a contradiction with the fact that $z^{(n-2)}(t) > 0$.

If $z^{(n-2)}(t) > 0$, $z'(t) < 0$. Because of $\lim_{t\to\infty}x(t) \neq 0$, $\lim_{t\to\infty}z(t) \neq 0$. By Lemma 3, we obtain (49). Proceeding as in the proof of the above, (77) holds. Using (59) and (77) in (1) and noting that $z'(t) < 0$, we have

$$\begin{aligned} &-\left(r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t)\right)' \\ &\quad= q_0(t)x^{\alpha}(\tau_0(t)) + q_1(t)x^{\beta}(\tau_1(t)) + q_2(t)x^{\gamma}(\tau_2(t)) \\ &\quad\geq q_0(t)(1-p(\tau_0(t)))^{\alpha}z^{\alpha}(\tau_0(t)) \\ &\qquad + q_1(t)(1-p(\tau_1(t)))^{\beta}z^{\beta}(\tau_1(t)) \\ &\qquad + q_2(t)(1-p(\tau_2(t)))^{\gamma}z^{\gamma}(\tau_2(t)) \\ &\quad\geq \xi^{\alpha}(t)q_0(t)z^{\alpha}(t) + \xi^{\beta}(t)q_1(t)z^{\beta}(t) \\ &\qquad + \xi^{\gamma}(t)q_2(t)z^{\gamma}(t). \end{aligned} \tag{88}$$

Setting

$$\begin{aligned} a_1 &= k_1\xi^{\beta}(t)q_1(t)z^{\beta}(t), \\ b_1 &= k_2\xi^{\gamma}(t)q_2(t)z^{\gamma}(t), \end{aligned} \tag{89}$$

by inequality (28), we get

$$\begin{aligned} &\xi^{\beta}(t)q_1(t)z^{\beta}(t) + \xi^{\gamma}(t)q_2(t)z^{\gamma}(t) \\ &\quad\geq \left[k_1\xi^{\beta}q_1(t)z^{\beta}(t)\right]^{1/k_1}\left[k_2\xi^{\gamma}q_2(t)z^{\gamma}(t)\right]^{1/k_2} \\ &\quad\geq \xi^{\alpha}(t)\left[k_1q_1(t)\right]^{1/k_1}\left[k_2q_2(t)\right]^{1/k_2}z^{\alpha}(t). \end{aligned} \tag{90}$$

Therefore, combining (77), (88), and (90), we obtain

$$\begin{aligned} &-\left(r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t)\right)' \\ &\quad\geq \overline{Q(t)}z^{\alpha}(t) \geq \left(\frac{\lambda M}{(n-2)!}t^{n-2}\delta(t)\right)^{\alpha}\overline{Q(t)}. \end{aligned} \tag{91}$$

Define the function u by

$$\begin{aligned} u(t) &= \eta(t)r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t) \\ &= -\eta(t)r(t)\left(-z^{(n-1)}(t)\right)^{\alpha}, \quad t \geq t_1. \end{aligned} \tag{92}$$

Then

$$\begin{aligned} u'(t) &= -\eta'(t)r(t)\left(-z^{(n-1)}(t)\right)^{\alpha} \\ &\quad - \eta(t)\left(r(t)\left(-z^{(n-1)}(t)\right)^{\alpha}\right)' \\ &\leq -\eta(t)\left(q_0(t)x^{\alpha}(\tau_0(t)) + q_1(t)x^{\beta}(\tau_1(t))\right. \\ &\qquad \left.+ q_2(t)x^{\gamma}(\tau_2(t))\right) \\ &\leq -\left(\frac{\lambda M}{(n-2)!}t^{n-2}\delta(t)\right)^{\alpha}\eta(t)\overline{Q(t)}. \end{aligned} \tag{93}$$

Integrating from t_1 to t, we get

$$u(t) - u(t_1) \leq -\int_{t_1}^{t}\left(\frac{\lambda M}{(n-2)!}s^{n-2}\delta(s)\right)^{\alpha}\eta(s)\overline{Q(s)}ds. \tag{94}$$

Therefore,

$$\begin{aligned} &-\eta(t)r(t)\left(-z^{(n-1)}(t)\right)^{\alpha} \\ &\quad\leq -\int_{t_1}^{t}\left(\frac{\lambda M}{(n-2)!}s^{n-2}\delta(s)\right)^{\alpha}\eta(s)\overline{Q(s)}ds; \end{aligned} \tag{95}$$

that is,

$$\begin{aligned} &z^{(n-1)}(t) \\ &\quad\leq -\left(\frac{1}{\eta(t)r(t)}\int_{t_1}^{t}\left(\frac{\lambda M}{(n-2)!}s^{n-2}\delta(s)\right)^{\alpha}\right. \\ &\qquad\qquad \left.\times\eta(s)\overline{Q(s)}ds\right)^{1/\alpha}. \end{aligned} \tag{96}$$

Integrating the above inequality from t_1 to l $(l > t_1)$, we obtain

$$\begin{aligned} &z^{(n-2)}(l) - z^{(n-2)}(t_1) \\ &\quad\leq -\int_{t_1}^{l}\left(\frac{1}{\eta(t)\mathrm{r}(t)}\int_{t_1}^{t}\left(\frac{\lambda M}{(n-2)!}s^{n-2}\delta(s)\right)^{\alpha}\right. \\ &\qquad\qquad \left.\times\eta(s)\overline{Q(s)}ds\right)^{1/\alpha}dt. \end{aligned} \tag{97}$$

Letting $l \to \infty$ and using (73) in (97), we have $\lim_{l\to\infty}z^{(n-2)}(l) = -\infty$, which is a contradiction with the fact that $z^{(n-2)}(t) > 0$. This completes the proof. □

5. Example

In this section, we will give an example to illustrate Theorem 11.

Example 1. Consider the even order neutral delay differential equations with mixed nonlinearities:

$$\begin{aligned} &\left(t^{\theta}\left|z^{(n-1)}(t)\right|^{1/2}z^{(n-1)}(t)\right)' + \frac{1}{t}\left|x(\tau_0(t))\right|^{1/2}x(\tau_0(t)) \\ &\quad + \frac{1}{t^{3/2}}\left|x(\tau_1(t))\right|^{-(1/2)}x(\tau_1(t)) \\ &\quad + \frac{1}{t^{3/4}}\left|x(\tau_2(t))\right|x(\tau_2(t)) = 0, \\ &\qquad\qquad t \geq t_0, \end{aligned} \tag{98}$$

where $z(t) = x(t) + 1/2x(\sigma(t))$ and n is even integer.

Set $r(t) = t^{\theta}$, $p(t) = 1/2$, $q_0(t) = 1/t$, $q_1(t) = b/t^{3/2}$, $q_2(t) = c/t^{3/4}$, $\gamma = 2$, $\alpha = 2/3$, $\beta = 1/2$, and $\theta \leq 2/3$. Then (2) holds, $k_1 = 3$, and $k_2 = 3/2$.

Take $\rho(t) = 1$. It is easy to show that

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\rho(s)\overline{Q(s)} - \frac{r(s)\left(\rho'(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(f(n,\lambda)\rho(s)\tau^{n-2}(s)\tau'(s)\right)^{\alpha}}\right)ds$$

$$= \limsup_{t\to\infty}\int_{t_1}^{t}\left(\frac{1}{2}\right)^{3/2} \times\left\{\frac{1}{t}+\left(3\cdot\frac{1}{t^{3/2}}\right)^{1/3}\left(\frac{3}{2}\cdot\frac{1}{t^{3/4}}\right)^{2/3}\right\}ds = \infty. \tag{99}$$

Hence, by Theorem 11, every solution of (98) is oscillatory.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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