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Research Article

Portfolio Selection with Liability and Affine Interest Rate in the HARA Utility Framework

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This paper studied an asset and liability management problem with stochastic interest rate, where interest rate is assumed to be governed by an affine interest rate model, while liability process is driven by the drifted Brownian motion. The investors wish to look for an optimal investment strategy to maximize the expected utility of the terminal surplus under hyperbolic absolute risk aversion (HARA) utility function, which consists of power utility, exponential utility, and logarithm utility as special cases. By applying dynamic programming principle and Legendre transform, the explicit solutions for HARA utility are achieved successfully and some special cases are also discussed. Finally, a numerical example is provided to illustrate our results.

1. Introduction

In practice of investments, it is all well known that some investors or financial institutions are often confronted with liability factor. It is very clear that considering liability factor in the portfolio selection problems will be more practical. In recent years, portfolio selection problems with liability have inspired literally hundreds of extensions and applications. For example, Chiu and Li [1] supposed liability process to be driven by a geometric Brownian motion and studied the efficient strategy and the efficient frontier under mean-variance criterion. Xie et al. [2] studied a mean-variance model with random liability, which is assumed to be governed by the drifted Brownian motion. Zeng and Li [3] investigated a mean-variance model with a jump diffusion liability process. Chen et al. [4] studied the asset and liability management (ALM) problems with regime switching, which described the effect of the changes in macroeconomic conditions. Chiu and Wong [5] investigated an ALM problem under the assumption that risky assets were cointegrated. Those models studied the ALM problems with exogenous liabilities, which cannot be controlled. Along another line, the ALM problems with endogenous liabilities have been paid more and more attention nowadays. Being different from exogenous liabilities, endogenous liabilities can be controlled by various financial instruments and investors' decisions. The interested readers can refer to the works of Leippold et al. [6] and Yao et al. [7].

However, the above mentioned research results were generally achieved under the assumption of constant interest rate. As a matter of fact, interest rate is always changing with time and can be delineated by some term structure models, for example, the Vasicek model [8] or the CIR model [9]. Therefore, some scholars began to be concerned with the portfolio selection problems with stochastic interest rate. For example, Korn and Kraft [10] investigated the portfolio selection problems with stochastic interest rates and proved the verification theorem. Deelstra et al. [11] studied the optimal investment strategies in the affine interest rate environment. Gao [12] applied dynamic programming principle and Legendre transform to study the DC pension problems with affine interest rate. Liu [13] and Chang and Rong [14] considered the optimal investment and consumption strategy with stochastic interest rate and stochastic volatility. These models were all studied in the utility maximization framework and the optimal investment strategies were established under power utility or logarithm utility.

Power utility, logarithm utility, and exponential utility are all special cases of HARA utility, which used to be studied by Grasselli [15], Çanakoğlu and Özekici [16], and Jung and Kim [17]. Due to the complexity of HARA utility, there is little work on HARA utility in the existing literature. In recent years, some scholars found that Legendre transform is an effective method in solving sophisticated portfolio section problems. One can refer to the works of Jonsson and Sircar [18], Gao [12], and Jung and Kim [17].

This paper introduces liability process into a continuous-time dynamic portfolio selection problem and assumes interest rate to be driven by affine interest rate model. The objective of the investor is to look for an optimal investment strategy to maximize the expected utility of the terminal surplus. Considering the generality of HARA utility, we take HARA utility for our analysis. Applying dynamic programming principle, we obtain the Hamilton-Jacobi-Bellman (HJB) equation for the value function. Due to the nonlinearity of HARA utility, it is very difficult for us to conjecture the form of the solution for the HJB equation. Therefore, we apply Legendre transform to change the nonlinear HJB equation into a linear dual one. What is more, the form of the solution for the dual equation is linear and is easy to construct. Finally, we establish the explicit expression of the optimal investment strategy for HARA utility. Some special cases are also discussed. There are three highlights: (i) we study an asset and liability management problem under the affine interest rate model; (ii) interest rate is supposed to be governed by the affine interest rate model, while liability is assumed to be driven by the drifted Brownian motion; (iii) the optimal investment strategies for HARA utility are obtained explicitly.

This paper proceeds as follows. In Section 2, an asset and liability management problem with affine interest rate is formulated. In Section 3, we use dynamic programming principle to obtain the HJB equation and apply Legendre transform to change the HJB equation into a linear dual one. In Section 4, we derive the optimal investment strategy for HARA utility explicitly. Section 5 discusses some special cases of the optimal policies. Section 6 provides a numerical example to illustrate our results. Some interesting results are concluded in Section 7.

2. Model and Assumptions

In this paper, we assume that the financial market consists of three assets: one risk-free asset, one risky asset, and one zero-coupon bond. $[0, T]$ represents certain investment horizon, where $T < \infty$. An investor is confronted with two key sources of risk, that is, interest rate risk and stock information risk, which are described by two one-dimensional standard and adapted independent Brownian motion $W_r(t)$ and $W_S(t)$. $W_r(t)$ and $W_S(t)$ are all defined on complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

The price process of the risk-free asset (i.e., cash), denoted by $S_0(t)$, is given by

$$dS_0(t) = r(t) S_0(t)\, dt, \quad S_0(0) = 1, \tag{1}$$

where $r(t)$ is the risk-free interest rate. In this paper, $r(t)$ is supposed to be stochastic process and is governed by

$$\begin{aligned} dr(t) &= (a - br(t))\, dt - \sqrt{k_1 r(t) + k_2} dW_r(t), \\ r(0) &= r_0 > 0, \end{aligned} \tag{2}$$

where the parameters a, b, k_1, and k_2 are positive real constants.

Remark 1. Notice that (2) consists of the Vasicek model (resp., the CIR model) as special cases, when k_1 (resp., k_2) is equal to zero. Under these dynamics, the term structure of interest rate is affine.

The price process of the risky asset (i.e., the stock), denoted by $S_1(t)$, can be described by the following stochastic differential equation (SDE) (referring to Deelstra et al. [11]):

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= r(t)\, dt + \sigma_1 \left(dW_S(t) + \lambda_1 dt\right) \\ &\quad + \sigma_2 \sqrt{k_1 r(t) + k_2} \\ &\quad \times \left(dW_r(t) + \lambda_2 \sqrt{k_1 r(t) + k_2} dt\right), \\ S_1(0) &= 1, \end{aligned} \tag{3}$$

where λ_1, λ_2 (resp., σ_1, σ_2) are constants (resp., positive constants).

The third asset is one zero-coupon bond with maturity T, whose price process is denoted by $B(t, T)$. Then $B(t, T)$ evolves (referring to Deelstra et al. [11])

$$\begin{aligned} \frac{dB(t,T)}{B(t,T)} &= r(t)\, dt + \sigma_B(t, r(t)) \\ &\quad \times \left(dW_r(t) + \lambda_2 \sqrt{k_1 r(t) + k_2} dt\right), \\ B(T,T) &= 1, \end{aligned} \tag{4}$$

where $\sigma_B(t, r(t)) = h(t)\sqrt{k_1 r(t) + k_2}$ and $h(t)$ is given by

$$\begin{aligned} h(t) &= \frac{2\left(e^{m(T-t)} - 1\right)}{m - (b - k_1\lambda_2) + e^{m(T-t)}\left(m + b - k_1\lambda_2\right)}, \\ m &= \sqrt{\left(b - k_1\lambda_2\right)^2 + 2k_1}. \end{aligned} \tag{5}$$

Remark 2. Equations (3) and (4) have been investigated by Deelstra et al. [11] and Gao [12], but their results were only achieved under power utility or logarithm utility. From (3), we can see that the price process of the stock is affected by uncertainty of interest rate, which is consistent with reality. In addition, we find that the price process of zero-coupon bond and interest rate is driven by the same Brownian motion. It implies that the zero-coupon bond is reduced to the risk-free asset when interest rate is reduced to a constant.

Assume that the liability process satisfies the following SDE:

$$dL(t) = u\,dt + v\,dW_S(t), \quad L(0) = l_0 > 0, \tag{6}$$

where u and v are positive constants.

Assume that the initial capital of an investor is given by w_0 and the initial liability is denoted by l_0; then the initial surplus is $w_0 - l_0$, which is denoted by x_0. The investor wishes to invest the surplus in the financial assets to maximize the expected utility of the terminal surplus. The amount invested in the stock and zero-coupon bond is denoted by $\pi_1(t)$ and $\pi_2(t)$, respectively. Therefore, the surplus process $X(t)$ at time t under the strategy $\pi(t) = (\pi_1(t), \pi_2(t))$ satisfies the following SDE:

$$\begin{aligned} dX(t) = \Big(& X(t) r(t) + \pi_1(t) \sigma_1 \lambda_1 \\ & + \pi_1(t) \sigma_2 \lambda_2 (k_1 r(t) + k_2) \\ & + \pi_2(t) \sigma_B \lambda_2 \sqrt{k_1 r(t) + k_2} - u \Big) dt \\ & + (\pi_1(t) \sigma_1 - v) dW_S(t) \\ & + \left(\pi_1(t) \sigma_2 \sqrt{k_1 r(t) + k_2} + \pi_2(t) \sigma_B \right) dW_r(t), \end{aligned} \tag{7}$$

with the initial surplus $X(0) = x_0 > 0$.

Definition 3 (admissible strategy). An investment strategy $\pi(t) = (\pi_1(t), \pi_2(t))$ is said to be admissible if the following conditions are satisfied:

(i) $\pi_1(t)$ and $\pi_2(t)$ are all $\mathcal{F}_t$-measurable;

(ii) $E(\int_0^T ((\pi_1(t)\sigma_1 - v)^2 + (\pi_1(t)\sigma_2 \sqrt{k_1 r(t) + k_2} + \pi_2(t)\sigma_B)^2) dt) < +\infty$;

(iii) SDE (7) has a unique solution for all $\pi(t) = (\pi_1(t), \pi_2(t))$.

Assume that the set of all admissible strategies is denoted by Π. The investor wishes to look for an optimal strategy to maximize the expected utility of the terminal surplus; that is,

$$\underset{\pi(t) \in \Pi}{\text{Maximize}}\ \mathbb{E}(U(X(T))), \tag{8}$$

where $U(\cdot)$ represents utility function, which is strictly concave and satisfies the Inada conditions: $\dot{U}(0) = +\infty$ and $\dot{U}(+\infty) = 0$.

In this paper, we choose hyperbolic absolute risk aversion (HARA) utility function for our analysis. The HARA utility function with parameters η, p, and q is given by

$$U(x) = U(\eta, p, q, x) = \frac{1-p}{qp} \left(\frac{q}{1-p} x + \eta \right)^p, \quad q > 0,\ p < 1,\ p \neq 0. \tag{9}$$

As a matter of fact, HARA utility function recovers power utility and exponential utility as special cases.

(i) If we choose $\eta = 0$ and $q = 1 - p$, then we have

$$U(0, p, 1-p, x) = \frac{x^p}{p} = U_{\text{power}}(x). \tag{10}$$

(ii) If we choose $\eta = 1$ and $p \to -\infty$, then we have

$$U(1, p, q, x) = -\frac{e^{-qx}}{q} = U_{\exp}(x). \tag{11}$$

3. HJB Equation and Legendre Transform

According to dynamic programming principle, we define the value function as

$$V(t, r, x) = \sup_{\pi(t) \in \Pi} \mathbb{E}(U(X(T)) \mid X(t) = x, r(t) = r) \tag{12}$$

with boundary condition given by $V(T, r, x) = U(x)$.

By applying the principle of optimality, we obtain the following proposition.

Proposition 4. *If the value function $V(t, r, x) \in C^{1,2,2}([0, T] \times \mathbb{R} \times \mathbb{R})$, then $V(t, r, x)$ satisfies the following HJB equation:*

$$\sup_{\pi(t) \in \Pi} \mathscr{A} V(t, r, x) = 0, \tag{13}$$

where $\mathscr{A}$ is a variational operator, and by letting $\sigma_r = \sqrt{k_1 r(t) + k_2}$ we obtain

$$\begin{aligned} \mathscr{A} V(t, r, x) = {} & V_t + \left(rx + \pi_1(t) \sigma_1 \lambda_1 + \pi_1(t) \sigma_2 \lambda_2 \sigma_r^2 + \pi_2(t) \sigma_B \lambda_2 \sigma_r - u \right) V_x \\ & + \frac{1}{2} \left((\pi_1(t) \sigma_1 - v)^2 + (\pi_1(t) \sigma_2 \sigma_r + \pi_2(t) \sigma_B)^2 \right) V_{xx} \\ & + (a - br) V_r + \frac{1}{2} \sigma_r^2 V_{rr} \\ & - \left(\pi_1(t) \sigma_2 \sigma_r^2 + \pi_2(t) \sigma_B \sigma_r \right) V_{rx}. \end{aligned} \tag{14}$$

Here, V_t, V_x, V_{xx}, V_r, V_{rr}, and V_{rx} represent first-order and second-order partial derivatives with respect to the variables t, r, and x.

Proof. The proof is standard. The interested readers can refer to the work of Korn and Kraft [10].

Assume that $H(t, r, x)$ is a solution of the HJB equation (13); then we get

$$\begin{aligned} \pi_1(t) &= -\frac{\lambda_1}{\sigma_1} \cdot \frac{H_x}{H_{xx}} + \frac{v}{\sigma_1}, \\ \pi_2(t) &= \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_2 \lambda_1 - \sigma_1 \lambda_2}{\sigma_1} \cdot \frac{H_x}{H_{xx}} + \frac{\sigma_r}{\sigma_B} \cdot \frac{H_{rx}}{H_{xx}} - \frac{\sigma_2}{\sigma_1} \cdot \frac{\sigma_r}{\sigma_B} v. \end{aligned} \tag{15}$$

Putting (15) into (13), we derive

$$\begin{aligned} & H_t + (rx + \lambda_1 v - u) H_x + (a - br) H_r + \frac{1}{2} \sigma_r^2 H_{rr} \\ & - \frac{1}{2} \left(\lambda_1^2 + \lambda_2^2 \sigma_r^2 \right) \frac{H_x^2}{H_{xx}} + \lambda_2 \sigma_r^2 \frac{H_x H_{rx}}{H_{xx}} - \frac{1}{2} \sigma_r^2 \frac{H_{rx}^2}{H_{xx}} = 0, \end{aligned} \tag{16}$$

with boundary condition given by $H(T, r, x) = ((1-p)/qp)((q/(1-p))x + \eta)^p$.

Noting that (16) is a nonlinear second-order partial differential equation and it is not easy for us to conjecture the structure of a solution to (16) for HARA utility, we introduce the following Legendre transform to change (16) into a linear second-order partial differential equation such that we can obtain the explicit solution to (16). □

Definition 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function. For all $z > 0$, Legendre transform can be defined as follows:

$$L(z) = \max_x \{f(x) - zx\}, \tag{17}$$

and then the function $L(z)$ is called the Legendre dual function of $f(x)$ (cf. [12, 17, 18]).

If $f(x)$ is strictly concave, the maximum in (17) will be attained at just one point, which we denoted by x_0. We can attain at the unique solution by the first-order condition

$$\frac{df(x)}{dx} - z = 0. \tag{18}$$

So we have

$$L(z) = f(x_0) - zx_0. \tag{19}$$

Following Jonsson and Sircar [18], Gao [12], and Jung and Kim [17], Legendre transform can be defined by

$$\widehat{H}(t,r,z) = \sup_{x>0} \{H(t,r,x) - zx\}, \tag{20}$$

where $z > 0$ denotes the dual variable to x. The value of x where this optimum is attained is denoted by $g(t,r,z)$, so we have

$$g(t,r,z) = \inf_{x>0} \left\{x \mid H(t,r,x) \geqslant zx + \widehat{H}(t,r,z)\right\}. \tag{21}$$

The relationship between $\widehat{H}(t,r,z)$ and $g(t,r,z)$ is given by

$$g(t,r,z) = -\widehat{H}_z(t,r,z). \tag{22}$$

Hence, we can choose either one of two functions $g(t,r,z)$ and $\widehat{H}(t,r,z)$ as the dual function of $H(t,r,x)$. In this paper, we choose $g(t,r,z)$. Moreover, we have

$$H_x = z, \qquad \widehat{H}(t,r,z) = H(t,r,g) - zg, \qquad g(t,r,z) = x. \tag{23}$$

Differentiating (23) with respect to t, r, and x, we get

$$\begin{gathered} H_t = \widehat{H}_t, \qquad H_x = z, \qquad H_{xx} = -\frac{1}{\widehat{H}_{zz}}, \\ H_r = \widehat{H}_r, \qquad H_{rr} = \widehat{H}_{rr} - \frac{\widehat{H}_{rz}^2}{\widehat{H}_{zz}}, \qquad H_{xr} = -\frac{\widehat{H}_{rz}}{\widehat{H}_{zz}}. \end{gathered} \tag{24}$$

Notice that $H(T,r,x) = U(x)$; then at the terminal time T, we can define

$$\begin{gathered} \widehat{H}(T,r,z) = \sup_{x>0} \{U(x) - zx\}, \\ g(T,r,z) = \inf_{x>0} \left\{x \mid U(x) \geqslant zx + \widehat{H}(T,r,z)\right\}. \end{gathered} \tag{25}$$

So we have $g(T,r,z) = (\dot{U})^{-1}(z)$, where $(\dot{U})^{-1}(z)$ is taken as the inverse of marginal utility.

Substituting (24) back into (16) yields

$$\begin{aligned} &\widehat{H}_t + rzg + (\lambda_1 \nu - u) z + (a - br) \widehat{H}_r + \frac{1}{2}\sigma_r^2 \widehat{H}_{rr} \\ &\quad + \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 \sigma_r^2\right) z^2 \widehat{H}_{zz} + \lambda_2 \sigma_r^2 z \widehat{H}_{rz} = 0. \end{aligned} \tag{26}$$

Differentiating (26) with respect to z and using (22), we obtain

$$\begin{aligned} &g_t - rg + \left(\lambda_1^2 + \lambda_2^2 \sigma_r^2 - r\right) z g_z \\ &\quad + \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 \sigma_r^2\right) z^2 g_{zz} + \left(a - br + \lambda_2 \sigma_r^2\right) g_r \\ &\quad + \frac{1}{2}\sigma_r^2 g_{rr} + \lambda_2 \sigma_r^2 z g_{rz} + u - \lambda_1 \nu = 0, \end{aligned} \tag{27}$$

with boundary condition given by $g(T,r,z) = (\dot{U})^{-1}(z)$.

Under HARA utility function (9), we have

$$g(T,r,z) = \frac{1-p}{q}\left(z^{1/(p-1)} - \eta\right). \tag{28}$$

4. The Optimal Portfolio

Assume that a solution of (27) is conjectured as follows:

$$g(t,r,z) = \frac{1-p}{q} z^{1/(p-1)} f(t,r) - \frac{1-p}{q}\eta h(t,r) + J(t,r), \tag{29}$$

with boundary conditions given by $f(T,r) = 1$, $h(T,r) = 1$, and $J(T,r) = 0$.

Further, the partial derivatives of $g(t,r,z)$ with respect to t, r, and z are as follows:

$$\begin{aligned} g_t &= \frac{1-p}{q} z^{1/(p-1)} f_t - \frac{1-p}{q}\eta h_t + J_t, \\ g_z &= -\frac{1}{q} z^{(2-p)/(p-1)} f, \\ g_r &= \frac{1-p}{q} z^{1/(p-1)} f_r - \frac{1-p}{q}\eta h_r + J_r, \\ g_{rz} &= -\frac{1}{q} z^{(2-p)/(p-1)} f_r, \\ g_{rr} &= \frac{1-p}{q} z^{1/(p-1)} f_{rr} - \frac{1-p}{q}\eta h_{rr} + J_{rr}, \\ g_{zz} &= -\frac{1}{q} \cdot \frac{2-p}{p-1} \cdot z^{(3-2p)/(p-1)} f. \end{aligned} \tag{30}$$

Plugging (30) into (27), after some simple calculations, we derive

$$\begin{aligned} &\frac{1-p}{q} z^{1/(p-1)} \left(f_t + \frac{p}{2(1-p)^2}\left(\lambda_1^2 + \lambda_2^2 \sigma_r^2\right) f + \frac{p}{1-p} r f \right. \\ &\quad \left. + \left(a - br - \frac{p}{1-p}\lambda_2 \sigma_r^2\right) f_r + \frac{1}{2}\sigma_r^2 f_{rr} \right) \end{aligned}$$

$$\begin{aligned}
&-\frac{1-p}{q}\eta\left(h_t-rh+\left(a-br+\lambda_2\sigma_r^2\right)h_r+\frac{1}{2}\sigma_r^2h_{rr}\right)\\
&+J_t-rJ+\left(a-br+\lambda_2\sigma_r^2\right)J_r\\
&+\frac{1}{2}\sigma_r^2J_{rr}+u-\lambda_1v=0.
\end{aligned}\tag{31}$$

Equation (31) can be decomposed into the following three equations:

$$\begin{aligned}
&f_t+\frac{p}{2(1-p)^2}\left(\lambda_1^2+\lambda_2^2\sigma_r^2\right)f+\frac{p}{1-p}rf\\
&\quad+\left(a-br-\frac{p}{1-p}\lambda_2\sigma_r^2\right)f_r+\frac{1}{2}\sigma_r^2f_{rr}=0,\\
&\qquad f(T,r)=1;
\end{aligned}\tag{32}$$

$$\begin{aligned}
&J_t-rJ+\left(a-br+\lambda_2\sigma_r^2\right)J_r+\frac{1}{2}\sigma_r^2J_{rr}+u-\lambda_1v=0,\\
&\qquad J(T,r)=0;
\end{aligned}\tag{33}$$

$$\begin{aligned}
&h_t-rh+\left(a-br+\lambda_2\sigma_r^2\right)h_r+\frac{1}{2}\sigma_r^2h_{rr}=0,\\
&\qquad h(T,r)=1.
\end{aligned}\tag{34}$$

Lemma 6. *Assume that a solution of (32) is conjectured as $f(t,r)=e^{D_1(t)+D_2(t)r}$, with boundary conditions given by $D_1(T)=0$ and $D_2(T)=0$; then under the condition of $p<\min\{1,b^2/(2k_1+(b-\lambda_2k_1)^2)\}$ and $p\neq 0$, $D_1(t)$ and $D_2(t)$ are given by (44) and (42), respectively.*

Proof. Putting $f(t,r)=e^{D_1(t)+D_2(t)r}$ into (32) and taking $\sigma_r=\sqrt{k_1r(t)+k_2}$ into consideration, we obtain

$$\begin{aligned}
e^{D_1(t)+D_2(t)r}\Bigg(&\dot{D}_1(t)+\frac{p}{2(1-p)^2}\left(\lambda_1^2+\lambda_2^2k_2\right)\\
&+\left(a-\frac{p}{1-p}\lambda_2k_2\right)D_2(t)+\frac{1}{2}k_2D_2^2(t)\\
&+r\Bigg(\dot{D}_2(t)+\frac{p}{2(1-p)^2}\lambda_2^2k_1+\frac{p}{1-p}\\
&-\left(b+\frac{p}{1-p}\lambda_2k_1\right)D_2(t)\\
&+\frac{1}{2}k_1D_2^2(t)\Bigg)\Bigg)=0.
\end{aligned}\tag{35}$$

Comparing the coefficients yields

$$\begin{aligned}
&\dot{D}_2(t)+\frac{p}{2(1-p)^2}\lambda_2^2k_1+\frac{p}{1-p}\\
&-\left(b+\frac{p}{1-p}\lambda_2k_1\right)D_2(t)+\frac{1}{2}k_1D_2^2(t)=0,
\end{aligned}\tag{36}$$

$$\begin{aligned}
&\dot{D}_1(t)+\frac{p}{2(1-p)^2}\left(\lambda_1^2+\lambda_2^2k_2\right)\\
&+\left(a-\frac{p}{1-p}\lambda_2k_2\right)D_2(t)+\frac{1}{2}k_2D_2^2(t)=0.
\end{aligned}\tag{37}$$

For the quadratic equation

$$\begin{aligned}
&-\frac{1}{2}k_1D_2^2(t)+\left(b+\frac{p}{1-p}\lambda_2k_1\right)D_2(t)\\
&-\left(\frac{p}{2(1-p)^2}\lambda_2^2k_1+\frac{p}{1-p}\right)=0,
\end{aligned}\tag{38}$$

it is easy to calculate its discriminant, which is given by

$$\Delta_1=\frac{p}{1-p}\left(\frac{1}{p}b^2-2k_1-\left(b-\lambda_2k_1\right)^2\right).\tag{39}$$

(i) If $0<p<1$, then when $p<b^2/(2k_1+(b-\lambda_2k_1)^2)$, that is, $0<p<\min\{1,b^2/(2k_1+(b-\lambda_2k_1)^2)\}$, we have $\Delta_1>0$.

(ii) If $p<0$, we find that $\Delta_1>0$ always holds.

Hence, under the condition of $p<\min\{1,b^2/(2k_1+(b-\lambda_2k_1)^2)\}$ and $p\neq 0$, we have $\Delta_1>0$. In addition, two different roots of (38) are given by

$$m_{1,2}=\frac{b+(p/(1-p))\lambda_2k_1}{k_1}\pm\frac{\sqrt{\Delta_1}}{k_1}.\tag{40}$$

Further, (36) can be rewritten as

$$\begin{aligned}
&\frac{1}{m_1-m_2}\int_t^T\left(\frac{1}{D_2(s)-m_1}-\frac{1}{D_2(s)-m_2}\right)dD_2(s)\\
&=-\frac{1}{2}k_1(T-t).
\end{aligned}\tag{41}$$

After easy calculations, we have

$$D_2(t)=\frac{m_1m_2\left(1-\exp\{-(1/2)k_1(m_1-m_2)(T-t)\}\right)}{m_1-m_2\cdot\exp\{-(1/2)k_1(m_1-m_2)(T-t)\}}.\tag{42}$$

By using (37) × k_1− (36) × k_2, we get

$$\begin{aligned}
\dot{D}_1(t)=&\frac{k_2}{k_1}\dot{D}_2(t)+\frac{ak_1+bk_2}{k_1}D_2(t)\\
&+\frac{p}{2(1-p)^2}\cdot\frac{\lambda_1^2}{k_1}-\frac{p}{1-p}\cdot\frac{k_2}{k_1}.
\end{aligned}\tag{43}$$

Integrating (43) from t to T, we derive

$$\begin{aligned}
D_1(t)=&\frac{k_2}{k_1}D_2(t)-\frac{ak_1+bk_2}{k_1}\int_t^TD_2(s)\,ds\\
&-\left(\frac{p}{2(1-p)^2}\cdot\frac{\lambda_1^2}{k_1}-\frac{p}{1-p}\cdot\frac{k_2}{k_1}\right)(T-t).
\end{aligned}\tag{44}$$

As a result, Lemma 6 is completed. □

Lemma 7. *Assume that a solution of* (33) *is given by the structure* $J(t,r) = (u - \lambda_1 v)\int_t^T \widehat{J}(s,r)ds$*; then* $\widehat{J}(t,r)$ *satisfies the following PDE:*

$$\widehat{J}_t - r\widehat{J} + \left(a - br + \lambda_2\sigma_r^2\right)\widehat{J}_r + \frac{1}{2}\sigma_r^2\widehat{J}_{rr} = 0, \quad \widehat{J}(T,r) = 1. \tag{45}$$

Proof. Introducing the following variational operator on any function $J(t,r)$:

$$\nabla J(t,r) = -rJ + \left(a - br + \lambda_2\sigma_r^2\right)J_r + \frac{1}{2}\sigma_r^2 J_{rr}, \tag{46}$$

(33) can be rewritten as

$$\frac{\partial J(t,r)}{\partial t} + \nabla J(t,r) + u - \lambda_1 v = 0, \quad J(T,r) = 0. \tag{47}$$

On the other hand, we have

$$\begin{aligned}\frac{\partial J(t,r)}{\partial t} &= -\left(u - \lambda_1 v\right)\widehat{J}(t,r)\\ &= \left(u - \lambda_1 v\right)\left(\int_t^T \frac{\partial \widehat{J}(s,r)}{\partial s}ds - \widehat{J}(T,r)\right),\\ \nabla J(t,r) &= \left(u - \lambda_1 v\right)\int_t^T \nabla\widehat{J}(s,r)\,ds.\end{aligned} \tag{48}$$

Putting (48) into (47), we get

$$\left(u - \lambda_1 v\right)\left(\int_t^T\left(\frac{\partial \widehat{J}(s,r)}{\partial s} + \nabla\widehat{J}(s,r)\right)ds - \widehat{J}(T,r) + 1\right) = 0. \tag{49}$$

Therefore, we obtain

$$\frac{\partial \widehat{J}(s,r)}{\partial s} + \nabla\widehat{J}(s,r) = 0, \quad \widehat{J}(T,r) = 1. \tag{50}$$

That is, (45) holds. □

Lemma 8. *Assume that a solution of* (45) *is of the form* $\widehat{J}(t,r) = e^{D_3(t)+D_4(t)r}$*, with terminal conditions* $D_3(T) = 0$ *and* $D_4(T) = 0$*; then* $D_4(t)$ *and* $D_3(t)$ *are given by* (56) *and* (58)*, respectively.*

Proof. Plugging $\widehat{J}(t,r) = e^{D_3(t)+D_4(t)r}$ into (45), we derive

$$\begin{aligned}e^{D_3(t)+D_4(t)r}&\left(\dot{D}_3(t) + \left(\lambda_2k_2 + a\right)D_4(t) + \frac{1}{2}k_2D_4^2(t)\right.\\ &+ r\left(\dot{D}_4(t) - 1 + \left(\lambda_2k_1 - b\right)D_4(t)\right.\\ &\left.\left. + \frac{1}{2}k_1D_4^2(t)\right)\right) = 0.\end{aligned} \tag{51}$$

Comparing the coefficients on both sides of (51), we get the following two equations:

$$\begin{gathered}\dot{D}_4(t) - 1 + \left(\lambda_2k_1 - b\right)D_4(t) + \frac{1}{2}k_1D_4^2(t) = 0,\\ D_4(T) = 0;\end{gathered} \tag{52}$$

$$\begin{gathered}\dot{D}_3(t) + \left(\lambda_2k_2 + a\right)D_4(t) + \frac{1}{2}k_2D_4^2(t) = 0,\\ D_3(T) = 0.\end{gathered} \tag{53}$$

For (52), we find that the discriminant of the quadratic equation

$$-\frac{1}{2}k_1D_4^2(t) - \left(\lambda_2k_1 - b\right)D_4(t) + 1 = 0 \tag{54}$$

is given by $\Delta_2 = (\lambda_2k_1 - b)^2 + 2k_1 > 0$. Therefore, two different roots of (54) are given by

$$m_{3,4} = \frac{b - \lambda_2k_1}{k_1} \pm \frac{\sqrt{\Delta_2}}{k_1}. \tag{55}$$

Using the same approach as (36), we obtain

$$D_4(t) = \frac{m_3m_4\left(1 - \exp\left\{-(1/2)k_1\left(m_3 - m_4\right)(T - t)\right\}\right)}{m_3 - m_4\cdot\exp\left\{-(1/2)k_1\left(m_3 - m_4\right)(T - t)\right\}}. \tag{56}$$

By applying (53) × k_1 − (52) × k_2, we get

$$\dot{D}_3(t) = \frac{k_2}{k_1}\dot{D}_4(t) - \frac{ak_1 + bk_2}{k_1}D_4(t) - k_2. \tag{57}$$

Further, after easy integration, we obtain

$$\begin{aligned}&D_3(t)\\ &= \frac{k_2}{k_1}D_4(t) + k_2(T - t) + \frac{ak_1 + bk_2}{k_1}\\ &\quad\times\left(m_4(T - t) + \frac{2}{k_1}\right.\\ &\quad\left.\times\ln\frac{m_3 - m_4}{m_3 - m_4\cdot\exp\left\{-(1/2)k_1\left(m_3 - m_4\right)(T - t)\right\}}\right).\end{aligned} \tag{58}$$

Therefore, the proof is completed. □

Lemma 9. *Suppose that* $h(t,r) = e^{D_5(t)+D_6(t)r}$ *is a solution of* (34)*, with boundary conditions* $D_5(T) = 0$ *and* $D_6(T) = 0$*; then one has* $D_5(t) = D_3(t)$ *and* $D_6(t) = D_4(t)$*.*

Proof. Investigating (34), we find that (34) and (45) have the same solutions. Hence, the conclusion holds. □

Applying (24), (29), (30), and Lemmas 6–9, we have

$$\begin{aligned}\frac{H_x}{H_{xx}} &= -z\widehat{H}_{zz} = zg_z = -\frac{1}{q}z^{1/(p-1)}f\\ &= -\frac{1}{1-p}\left(g + \frac{1-p}{q}\eta h - J\right),\end{aligned}$$

$$\begin{aligned}\frac{H_{rx}}{H_{xx}} &= \widehat{H}_{rz} = -g_r = -\left(\frac{1-p}{q}z^{1/(p-1)}f_r - \frac{1-p}{q}\eta h_r + J_r\right)\\ &= -\frac{1-p}{q}z^{1/(p-1)}D_2(t)f + \frac{1-p}{q}\eta D_6(t)h - J_r\\ &= -D_2(t)\left(g + \frac{1-p}{q}\eta h - J\right) + \frac{1-p}{q}\eta D_6(t)h - J_r.\end{aligned} \tag{59}$$

Namely, we obtain

$$\begin{aligned}\frac{H_x}{H_{xx}} &= -\frac{1}{1-p}(X(t) - J(t,r)) - \frac{1}{q}\eta h(t,r),\\ \frac{H_{rx}}{H_{xx}} &= -D_2(t)(X(t) - J(t,r))\\ &\quad - \frac{1-p}{q}\eta h(t,r)(D_2(t) - D_6(t)) - \frac{\partial J(t,r)}{\partial r}.\end{aligned} \tag{60}$$

Meantime, considering (29) and letting $g(t,r,z) = x$, we derive

$$z = \left(\frac{q}{1-p}(x - J(t,r)) + \eta h(t,r)\right)^{p-1} f^{1-p}(t,r). \tag{61}$$

By using $H_x = z$, we obtain the following optimal value function:

$$\begin{aligned}&H^*(t,r,x)\\ &= \frac{1-p}{qp}\left(\frac{q}{1-p}(x - J(t,r)) + \eta h(t,r)\right)^{p} f^{1-p}(t,r).\end{aligned} \tag{62}$$

In conclusion, we can summarize the above results in the following theorem.

Theorem 10. *For HARA utility* (9), *if a solution to the HJB equation* (16) *is given by* $H(t,r,x)$, *then, under the condition of* $p < \min\{1, b^2/(2k_1 + (b - \lambda_2 k_1)^2)\}$ *and* $p \neq 0$, *the optimal investment strategies for the problem* (8) *are given by*

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\left(\frac{1}{1-p}(X(t) - J(t,r)) + \frac{1}{q}\eta h(t,r)\right) + \frac{\nu}{\sigma_1}, \tag{63}$$

$$\begin{aligned}&\pi_2^*(t)\\ &= \frac{\sigma_r}{\sigma_B}\cdot\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1}\\ &\quad\cdot\left(\frac{1}{1-p}(X(t) - J(t,r)) + \frac{1}{q}\eta h(t,r)\right)\\ &\quad - \frac{\sigma_r}{\sigma_B}\cdot\left(D_2(t)(X(t) - J(t,r)) + \frac{1-p}{q}\eta h(t,r)\right.\\ &\qquad\left.\times(D_2(t) - D_6(t)) + \frac{\partial J(t,r)}{\partial r}\right) - \frac{\sigma_2}{\sigma_1}\cdot\frac{\sigma_r}{\sigma_B}\nu,\end{aligned} \tag{64}$$

where $J(t,r) = (u - \lambda_1\nu)\int_t^T e^{D_3(s)+D_4(s)r}ds$, $h(t,r) = e^{D_5(t)+D_6(t)r}$, *and* $D_2(t)$ *are given by Lemmas 8, 9, and 6, respectively. Moreover,* $\partial J(t,r)/\partial r$ *in* (64) *is given by*

$$\frac{\partial J(t,r)}{\partial r} = (u - \lambda_1\nu)\int_t^T D_4(s)e^{D_3(s)+D_4(s)r}ds. \tag{65}$$

The following theorem verifies that the optimal strategy given by Theorem 10 is optimal for the problem (8).

Theorem 11 (verification theorem). *If* $H(t,r,x)$ *is a solution to* (13), *that is,* $H(t,r,x)$ *satisfies*

$$\sup_{\pi(t)\in\Pi}\mathscr{A}H(t,r,x) = 0, \tag{66}$$

then for all admissible strategies $\pi(t) = (\pi_1(t), \pi_2(t)) \in \Pi$, *one has* $V(t,r,x) \leqslant H(t,r,x)$; *if* $\pi^*(t) = (\pi_1^*(t), \pi_2^*(t))$ *satisfies*

$$\pi^*(t) \in \arg\sup_{\pi(t)\in\Pi}\mathscr{A}H(t,r,x), \tag{67}$$

then one has $V(t,r,x) = H(t,r,x)$, *and it implies that* $\pi^*(t) = (\pi_1^*(t), \pi_2^*(t)) \in \Pi$ *is the optimal investment strategy of the problem* (8).

Proof. Considering $H(t,r,x) \in C^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ and using Itô formula from t to T for $H(t,r,x)$, we obtain

$$\begin{aligned}&H(T,r(T),X(T))\\ &= H(t,r,x) + \int_t^T \mathscr{A}^{\pi(t)}H(s,r(s),X(s))\,ds\\ &\quad + \int_t^T (\pi_1(s)\sigma_1 - \nu)H_x(s,r(s),X(s))\,dW_S(s)\\ &\quad + \int_t^T \left(\pi_1(s)\sigma_2\sqrt{k_1 r(s) + k_2} + \pi_2(s)\sigma_B\right)\\ &\qquad\times H_x(s,r(s),X(s))\,dW_r(s)\\ &\quad - \int_t^T H_r(s,r(s),X(s))\sqrt{k_1 r(s) + k_2}\,dW_r(s).\end{aligned} \tag{68}$$

For $H(t,r,x)$ is a solution to (13); that is, we have $\sup_{\pi(t)\in\Pi}\mathscr{A}^{\pi(t)}H(s,r(s),X(s)) = 0$. So we obtain

$$\begin{aligned}&H(T,r(T),X(T))\\ &\leqslant H(t,r,x)\\ &\quad + \int_t^T (\pi_1(s)\sigma_1 - \nu)H_x(s,r(s),X(s))\,dW_S(s)\\ &\quad + \int_t^T \left(\pi_1(s)\sigma_2\sqrt{k_1 r(s) + k_2} + \pi_2(s)\sigma_B\right)\\ &\qquad\times H_x(s,r(s),X(s))\,dW_r(s)\\ &\quad - \int_t^T H_r(s,r(s),X(s))\sqrt{k_1 r(s) + k_2}\,dW_r(s).\end{aligned} \tag{69}$$

The last three terms in (69) are local martingales and their expected values are equal to zero. Thus, we get

$$\mathbb{E}\left(H\left(T, r\left(T\right), X\left(T\right)\right) \mid r\left(t\right) = r, X\left(t\right) = x\right) \leqslant H\left(t, r, x\right). \tag{70}$$

Maximizing (70) for all admissible strategies $\pi(t) = (\pi_1(t), \pi_2(t)) \in \Pi$, we derive

$$V\left(t, r, x\right) \leqslant H\left(t, r, x\right). \tag{71}$$

When $\pi(t) = \pi^*(t)$, all inequalities become equalities; that is, $H(t,r,x) = V(t,r,x)$, and $\pi^*(t) = (\pi_1^*(t), \pi_2^*(t))$ is the optimal investment strategy of the problem (8).

The proof is completed. □

Remark 12. Some interesting conclusions can be seen from (63) and (64). (i) The optimal amount $\pi_1^*(t)$ invested in the stock is affected by the parameters a, b, k_1, k_2, λ_1, σ_1, λ_2, u, and v but does not depend on the parameter σ_2. In fact, σ_2 has an influence on the dynamics of stock price, which can be observed from (3). (ii) The optimal amount $\pi_2^*(t)$ invested in the zero-coupon bond depends on all the model parameters $a, b, k_1, k_2, \lambda_1, \sigma_1, \lambda_2, u, v$, and σ_2; however, the price dynamics of zero-coupon bond is only impacted by b, k_1, k_2, and λ_2.

According to Theorem 10, we derive the following three corollaries.

Corollary 13. *If $\eta = 0$ and $q = 1 - p$, HARA utility* (9) *is degenerated into power utility* (10). *Therefore, under the condition of $p < \min\{1, b^2/(2k_1 + (b - \lambda_2 k_1)^2)\}$ and $p \neq 0$, the optimal investment strategies for power utility are given by*

$$\pi_1^*\left(t\right) = \frac{\lambda_1}{\sigma_1} \cdot \frac{1}{1-p}\left(X\left(t\right) - J\left(t, r\right)\right) + \frac{v}{\sigma_1}, \tag{72}$$

$$\begin{aligned} \pi_2^*\left(t\right) = {} & \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot \frac{1}{1-p}\left(X\left(t\right) - J\left(t, r\right)\right) \\ & - \frac{\sigma_r}{\sigma_B} \cdot \left(D_2\left(t\right)\left(X\left(t\right) - J\left(t, r\right)\right) + \frac{\partial J\left(t, r\right)}{\partial r}\right) \\ & - \frac{\sigma_2}{\sigma_1} \cdot \frac{\sigma_r}{\sigma_B} v, \end{aligned} \tag{73}$$

where $J(t,r) = (u - \lambda_1 v)\int_t^T e^{D_3(s)+D_4(s)r} ds$ and $D_2(t)$ are given by Lemmas 8 and 6, respectively. Moreover, $\partial J(t,r)/\partial r$ in (73) *is still given by*

$$\frac{\partial J\left(t, r\right)}{\partial r} = \left(u - \lambda_1 v\right)\int_t^T D_4\left(s\right) e^{D_3(s)+D_4(s)r} ds. \tag{74}$$

Proof. It is very easy to derive that

$$\begin{aligned} \frac{H_x}{H_{xx}} &= -\frac{1}{1-p}\left(X\left(t\right) - J\left(t, r\right)\right), \\ \frac{H_{rx}}{H_{xx}} &= -D_2\left(t\right)\left(X\left(t\right) - J\left(t, r\right)\right) - \frac{\partial J\left(t, r\right)}{\partial r}. \end{aligned} \tag{75}$$

Therefore, (72) and (73) hold. □

Corollary 14. *If $\eta = 1$ and $p \to -\infty$, HARA utility* (9) *is reduced to exponential utility* (11); *then the optimal policies for exponential utility are determined by*

$$\pi_1^*\left(t\right) = \frac{\lambda_1}{\sigma_1} \cdot \frac{1}{q} h\left(t, r\right) + \frac{v}{\sigma_1}, \tag{76}$$

$$\begin{aligned} & \pi_2^*\left(t\right) \\ & = \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot \frac{1}{q} h\left(t, r\right) - \frac{\sigma_2}{\sigma_1} \cdot \frac{\sigma_r}{\sigma_B} v \\ & \quad - \frac{\sigma_r}{\sigma_B} \cdot \left(D_4\left(t\right)\left(X\left(t\right) - J\left(t, r\right)\right) + \frac{\partial J\left(t, r\right)}{\partial r}\right. \\ & \quad + \frac{1}{q} h\left(t, r\right)\int_t^T \left(-\frac{1}{2}\lambda_2^2 k_1 + 1 - \lambda_2 k_1 D_4\left(s\right)\right) \\ & \quad \left. \cdot \exp\left\{\int_t^s \left(k_1 D_4\left(z\right) - b + \lambda_2 k_1\right) dz\right\} ds\right), \end{aligned} \tag{77}$$

where $J(t,r) = (u - \lambda_1 v)\int_t^T e^{D_3(s)+D_4(s)r} ds$ and $h(t,r) = e^{D_5(t)+D_6(t)r}$ are given by Lemmas 8 and 9, respectively. Moreover, $\partial J(t,r)/\partial r$ in (77) *is still given by*

$$\frac{\partial J\left(t, r\right)}{\partial r} = \left(u - \lambda_1 v\right)\int_t^T D_4\left(s\right) e^{D_3(s)+D_4(s)r} ds. \tag{78}$$

Proof. When $p \to -\infty$, then we have $D_2(t) \to D_4(t)$ and $D_6(t) = D_4(t)$. Moreover, we derive that

$$\begin{aligned} & \lim_{p \to -\infty} \frac{H_x}{H_{xx}} \\ & = \lim_{p \to -\infty}\left(-\frac{1}{1-p}\left(X\left(t\right) - J\left(t, r\right)\right) - \frac{1}{q}\eta h\left(t, r\right)\right) \\ & = -\frac{1}{q}\eta h\left(t, r\right), \\ & \lim_{p \to -\infty} \frac{H_{rx}}{H_{xx}} \\ & = \lim_{p \to -\infty}\left(-D_2\left(t\right)\left(X\left(t\right) - J\left(t, r\right)\right) - \frac{1-p}{q}\eta h\left(t, r\right)\right. \\ & \qquad \left. \times \left(D_2\left(t\right) - D_6\left(t\right)\right) - \frac{\partial J\left(t, r\right)}{\partial r}\right). \end{aligned} \tag{79}$$

On the other hand, we have

$$\begin{aligned} & \lim_{p \to -\infty} \frac{1-p}{q}\eta h\left(t, r\right)\left(D_2\left(t\right) - D_6\left(t\right)\right) \\ & = \frac{1}{q}\eta h\left(t, r\right) \lim_{p \to -\infty} \frac{D_2\left(t\right) - D_6\left(t\right)}{1/\left(1-p\right)}. \quad \left(\frac{0}{0}\right) \end{aligned} \tag{80}$$

Further, we obtain $\partial D_6(t)/\partial p = \partial D_4(t)/\partial p = 0$ and

$$\frac{\partial D_2(t)}{\partial p} = \int_t^T \left(\frac{1+p}{2(1-p)^3}\lambda_2^2 k_1 + \frac{1}{(1-p)^2}\left(1-\lambda_2 k_1 D_2(s)\right) \right) \times \exp\left\{ \int_t^s \left(k_1 D_2(z) - b - \frac{p}{1-p}\lambda_2 k_1 \right) dz \right\} ds. \tag{81}$$

Using L'Hopital's rule, (80) is equal to

$$\frac{1}{q}\eta h(t,r) \int_t^T \left(-\frac{1}{2}\lambda_2^2 k_1 + 1 - \lambda_2 k_1 D_4(s) \right) \times \exp\left\{ \int_t^s \left(k_1 D_4(z) - b + \lambda_2 k_1 \right) dz \right\} ds. \tag{82}$$

Let $\eta = 1$; we obtain that (76) and (77) hold. □

Corollary 15. *If $\eta = 0$, $p \to 0$, and $q \to 1$, HARA utility is reduced to logarithm utility $U_{\log}(x) = \ln x$; then the optimal policies for logarithm utility are given by*

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot (X(t) - J(t,r)) + \frac{\nu}{\sigma_1}, \tag{83}$$

$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot (X(t) - J(t,r)) - \frac{\sigma_r}{\sigma_B} \cdot \frac{\partial J(t,r)}{\partial r} - \frac{\sigma_2}{\sigma_1} \cdot \frac{\sigma_r}{\sigma_B}\nu, \tag{84}$$

where $J(t,r) = (u-\lambda_1\nu)\int_t^T e^{D_3(s)+D_4(s)r} ds$ is given by Lemma 8. Moreover, $\partial J(t,r)/\partial r$ in (83) is still given by

$$\frac{\partial J(t,r)}{\partial r} = (u - \lambda_1 \nu) \int_t^T D_4(s) e^{D_3(s)+D_4(s)r} ds. \tag{85}$$

Proof. If $p \to 0$, then we have $D_2(t) \to 0$. Therefore, we derive

$$\frac{H_x}{H_{xx}} = -(X(t) - J(t,r)), \qquad \frac{H_{rx}}{H_{xx}} = -\frac{\partial J(t,r)}{\partial r}. \tag{86}$$

Therefore, (83) and (84) hold. □

5. Special Cases

In this section, we give several special cases for HARA utility, power utility, exponential utility, and logarithm utility, respectively.

In our model, if we do not consider liability factor, that is, $u = \nu = 0$, then we have $J(t,r) = 0$. Therefore, we obtain the following special cases.

Special Case 1. Under HARA utility (9), if there is no liability and p satisfies the conditions $p < \min\{1, b^2/(2k_1 + (b - \lambda_2 k_1)^2)\}$ and $p \neq 0$, then the optimal policies for the problem (8) are given by

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot \left(\frac{1}{1-p} X(t) + \frac{1}{q}\eta h(t,r) \right),$$

$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot \left(\frac{1}{1-p} X(t) + \frac{1}{q}\eta h(t,r) \right) - \frac{\sigma_r}{\sigma_B} \cdot \left(D_2(t) X(t) + \frac{1-p}{q}\eta h(t,r) \left(D_2(t) - D_6(t)\right) \right), \tag{87}$$

where $h(t,r) = e^{D_5(t)+D_6(t)r}$ and $D_2(t)$ are given by Lemmas 9 and 6, respectively.

Special Case 2. If utility function is given by $U_{\text{power}}(x) = x^p/p$, $p < 1$ and $p \neq 0$, and p satisfies the conditions $p < \min\{1, b^2/(2k_1 + (b - \lambda_2 k_1)^2)\}$ and $p \neq 0$, then the optimal policies for the problem (8) under the condition of no liability are given by

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot \frac{1}{1-p} X(t),$$

$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot \frac{1}{1-p} X(t) - \frac{\sigma_r}{\sigma_B} \cdot D_2(t) X(t), \tag{88}$$

where $D_2(t)$ is given by Lemma 6.

Special Case 3. If utility function is given by $U_{\exp}(x) = -e^{-qx}/q$, $q > 0$, then the optimal policies for the problem (8) under the condition of no liability are

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot \frac{1}{q} h(t,r),$$

$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot \frac{1}{q} h(t,r) - \frac{\sigma_r}{\sigma_B} \cdot D_4(t) X(t),$$

$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot \frac{1}{q} h(t,r) - \frac{\sigma_r}{\sigma_B} \cdot \left(D_4(t) X(t) + \frac{1}{q} h(t,r) \int_t^T \left(-\frac{1}{2}\lambda_2^2 k_1 + 1 - \lambda_2 k_1 D_4(s) \right) \cdot \exp\left\{ \int_t^s \left(k_1 D_4(z) - b + \lambda_2 k_1 \right) dz \right\} ds \right), \tag{89}$$

where $D_4(t)$ and $h(t,r) = e^{D_5(t)+D_6(t)r}$ are given by Lemmas 8 and 9, respectively.

Special Case 4. If utility function is given by $U_{\log}(x) = \ln x$, then the optimal policies for the problem (8) with no liability are

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot X(t),$$

$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot X(t) - \frac{\sigma_r}{\sigma_B} \cdot D_2(t) X(t), \tag{90}$$

where $D_2(t)$ are given by Lemma 6.

On the other hand, if we consider the problem (8) in the constant interest rate environments, that is, $a = b = k_1 = k_2 = 0$, then the price dynamics of zero-coupon bond is degenerated into that of risk-free asset. As a result, the optimal amount invested in the zero-coupon bond is zero; that is, $\pi_2^*(t) = 0$. And it leads to some new expressions:

$$\begin{gathered} D_1(t) = \frac{p}{2(1-p)^2}\lambda_1^2(T-t), \\ D_2(t) = \frac{p}{1-p}(T-t), \\ D_3(t) = 0, \qquad D_4(t) = -(T-t). \end{gathered} \tag{91}$$

Therefore, we can derive the optimal policy of the problem (8) with liability in the following special cases.

Special Case 5. Under constant interest rate model, if utility function is given by HARA function (9), then the optimal policy of the problem (8) with liability is given by

$$\begin{aligned} \pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot \Bigg(\frac{1}{1-p} \left(X(t) - (u - \lambda_1 v) \int_t^T e^{-(T-s)r} ds \right) \\ + \frac{1}{q} \eta e^{-(T-t)r} \Bigg) + \frac{v}{\sigma_1}. \end{aligned} \tag{92}$$

Special Case 6. If utility function is given by $U_{\text{power}}(x) = x^p/p$, $p < 1$ and $p \neq 0$, then the optimal policy of the problem (8) with liability and constant interest rate is

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot \frac{1}{1-p} \left(X(t) - (u - \lambda_1 v) \int_t^T e^{-(T-s)r} ds \right) + \frac{v}{\sigma_1}. \tag{93}$$

Special Case 7. If utility function is given by $U_{\exp}(x) = -e^{-qx}/q$, $q > 0$, then the optimal policy of the problem (8) with liability and constant interest rate is

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot \frac{1}{q} e^{-(T-t)r} + \frac{v}{\sigma_1}. \tag{94}$$

Special Case 8. If utility function is given by $U_{\log}(x) = \ln x$, then the optimal policy of the problem (8) with liability and constant interest rate is

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1} \cdot \left(X(t) - (u - \lambda_1 v) \int_t^T e^{-(T-s)r} ds \right) + \frac{v}{\sigma_1}. \tag{95}$$

6. Numerical Analysis

In this section, we provide a numerical example to illustrate the effect of market parameters on the optimal investment strategy and compare our results with those in the existing literature. Throughout this section, unless otherwise stated, the basic parameters are given by $a = 0.18$, $b = 0.23$, $k_1 = 0.7$, $k_2 = 0.9$, $r(0) = 0.05$, $\lambda_1 = 0.6$, $\lambda_2 = 0.8$, $\sigma_1 = 1.9$, $\sigma_2 = 1.7$, $u = 0.6$, $v = 0.6$, $p = -0.2$, $q = 2$, $\eta = 0.02$, $t = 0$, $T = 1$, and $x_0 = 100$.

Table 1: The optimal policy with liability and affine interest rate.

	HARA utility	Power utility	Exponential utility	Logarithm utility
Stock	26.6064	26.6049	0.390775	31.8627
Bond	37.8308	37.8343	99.4978	24.7162
Cash	35.5629	35.5608	0.111445	43.4211

Table 2: The optimal policy only with affine interest rate.

	HARA utility	Power utility	Exponential utility	Logarithm utility
Stock	26.3173	26.3158	0.0749854	31.5789
Bond	38.3303	38.3338	100.06	25.2023
Cash	35.3525	35.3504	−0.134829	43.2187

Table 3: The optimal policy with liability and constant interest rate.

	HARA utility	Power utility	Exponential utility	Logarithm utility
Stock	26.573	26.57	0.465984	31.8208
Cash	73.427	73.43	99.534	68.1792

By applying the above conclusions obtained, some optimal investment policies are calculated in Tables 1, 2, and 3. From Tables 1–3, we draw some conclusions as follows.

(a1) The optimal policy under HARA utility is roughly equivalent to that under power utility but is markedly different from that under exponential utility or logarithm utility.

(a2) In the stochastic interest rate environments, the optimal amount invested in the stock and zero-coupon bond under HARA utility is less than that under exponential utility but is more than that under logarithm utility. This situation in the constant interest rate environments is contrary to that in the stochastic interest rate environments.

(a3) The optimal amount with liability and affine interest rate invested in the risky assets (including a stock and a zero-coupon bond) is less than that only with affine interest rate but is more than that with liability and constant interest rate.

In order to illustrate the impact of model parameters on the optimal investment strategy with liability and affine interest rate, we depict the following graphs. In Figures 1 and 2, the optimal amount invested in the stock is depicted by the dashed line and is denoted by $\pi_1^*(t)$; the optimal amount invested in the zero-coupon bond is depicted by the orange line and is denoted by $\pi_2^*(t)$; the optimal amount invested in the cash is depicted by the thick line and is denoted by $\pi_0^*(t)$.

Form Figures 1 and 2, we can draw some conclusions as follows.

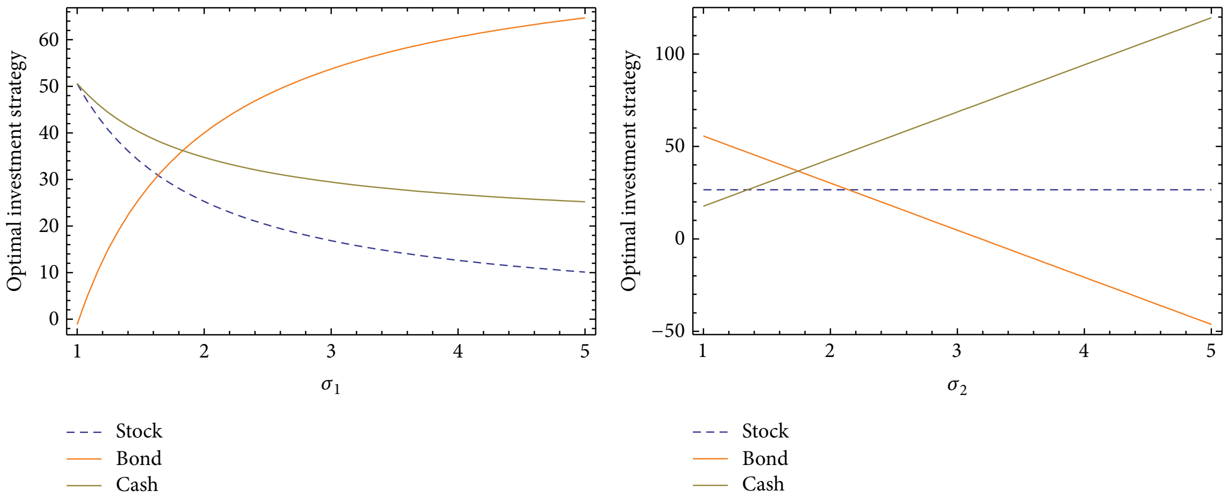

Figure 1: The effect of σ_1 and σ_2 on the optimal strategy.

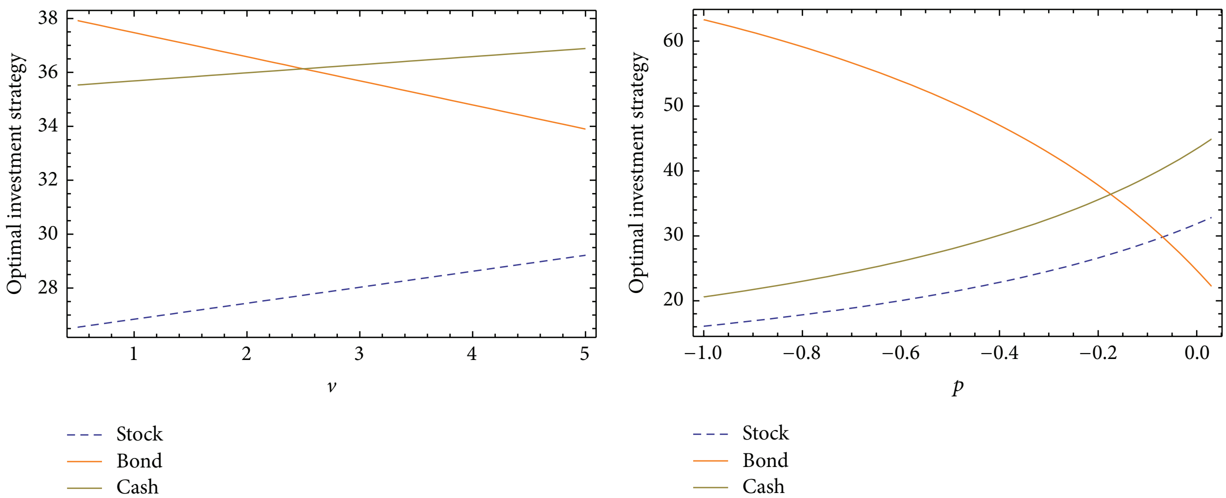

Figure 2: The effect of v and p on the optimal strategy.

(b1) $\pi_0^*(t)$ and $\pi_1^*(t)$ decrease with respect to the parameter σ_1, while $\pi_2^*(t)$ increases with respect to σ_1. In fact, as the value of σ_1 is increasing, the volatility risk of the stock is also increasing. It leads to that the investors would invest less money in the stock. In order to hedge interest rate risk, the investors need to invest much more money in the zero-coupon bond and less money in the cash.

(b2) $\pi_1^*(t)$ does not depend on the parameter σ_2, while $\pi_2^*(t)$ decreases and $\pi_0^*(t)$ increases with respect to σ_2. It shows that the investors need to invest less money in the zero-coupon bond and more money in the cash when the value of σ_2 increases.

(b3) $\pi_0^*(t)$ and $\pi_1^*(t)$ increase with respect to the parameter v, while $\pi_2^*(t)$ decreases with respect to v. As a matter of fact, as the value of v becomes larger, the volatility of liability is increasing. This means that the investors need to invest more money in the stock and cash in order to hedge the risk that resulted from liability.

(b4) $\pi_0^*(t)$ and $\pi_1^*(t)$ increase with respect to the parameter p, while $\pi_2^*(t)$ decreases with respect to p. This shows that the larger the value of p, the more the amount invested in the stock and cash. It leads to the fact that less money is invested in the zero-coupon bond.

7. Conclusions

This paper investigates the optimal investment strategy for an ALM problem in the HARA utility framework, where interest rate is supposed to be driven by an affine interest rate model, while liability process follows Brownian motion with drift. By applying dynamic programming principle and Legendre transform, we obtain the explicit expressions of the optimal investment strategies. Some special cases are discussed. Finally, we illustrate the impact of model parameters on the optimal policy by providing a numerical example. Some interesting conclusions are found as follows: (i) although the parameter σ_2 has an influence on the dynamics of stock price, the optimal amount invested in the stock does not depend on the value of σ_2; (ii) the optimal amount under HARA utility is roughly equivalent to that under power utility but is markedly different from that under exponential utility and logarithm utility.

As far as we know, there is little work on the ALM problems with stochastic interest rate in the existing literature. However, our work has also some limits: (i) we only consider the liability process driven by the drifted Brownian motion, which is the simplest stochastic process; (ii) we study an ALM problem in the utility framework and do not consider it in a continuous-time mean-variance framework. We would leave those problems for future research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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