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Research Article

Convergence of Solutions to a Certain Vector Differential Equation of Third Order

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We give some sufficient conditions to guarantee convergence of solutions to a nonlinear vector differential equation of third order. We prove a new result on the convergence of solutions. An example is given to illustrate the theoretical analysis made in this paper. Our result improves and generalizes some earlier results in the literature.

1. Introduction

This paper is concerned with the following nonlinear vector differential equation of third order:

$$\dddot{X} + F\left(\ddot{X}\right) + G\left(\dot{X}\right) + H\left(X\right) = P\left(t, X, \dot{X}, \ddot{X}\right), \tag{1}$$

where $X \in \mathfrak{R}^n$ and $F, G, H : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ and $P : \mathfrak{R} \times \mathfrak{R}^n \times \mathfrak{R}^n \times \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ are continuous functions in their respective arguments.

It should be noted that, in 2005, Afuwape and Omeike [1] considered the following nonlinear vector differential equation of third order:

$$\dddot{X} + A\ddot{X} + G\left(\dot{X}\right) + H\left(X\right) = P\left(t, X, \dot{X}, \ddot{X}\right), \tag{2}$$

where A is real symmetric $n \times n$-matrix. The author established a new result on the convergence of solutions of (2) under different conditions on the function P. For some related papers on the convergence of solutions to certain vector differential equations of third order, the readers can referee to the papers of Afuwape [2], Afuwape and Omeike [3], and Olutimo [4]. Further, it is worth mentioning that in a sequence of results Afuwape [2, 5, 6], Afuwape and Omeike [3], Afuwape and Ukpera [7], Ezeilo [8], Ezeilo and Tejumola [9, 10], Meng [11], Olutimo [4], Reissig et al. [12], Tiryaki [13], Tunç [14–16], Tunç and Ateş [17], C. Tunç and E. Tunç [18], and Tunç and Karakas [19] investigated the qualitative behaviors of solutions, stability, boundedness, uniform boundedness and existence of periodic solutions, and so on, except convergence of solutions, for some kind of vector differential equations of third order.

The Lyapunov direct method was used with the aid of suitable differentiable auxiliary functions throughout the mentioned papers. However, to the best of our knowledge, till now, the convergence of the solutions to (1) has not been discussed in the literature. Thus, it is worthwhile to study the topic for (1). It should be noted that the result to be established here is different from that in Afuwape [2], Afuwape and Omeike [1, 3], Olutimo [4], and the above mentioned papers. This paper is an extension and generalization of the result of Afuwape and Omeike [3]. It may be useful for the researchers working on the qualitative behaviors of solutions (see, also, Tunç and Gözen [20]).

It should be noted that throughout the paper R^n will denote the real Euclidean space of n-vectors and $\|X\|$ will denote the norm of the vector X in R^n.

Definition 1. Any two solutions $X_1(t)$, $X_2(t)$ of (1) in R^n will be said to converge to each other if

$$\left\|X_2(t) - X_1(t)\right\| \longrightarrow 0, \qquad \left\|\dot{X}_2(t) - \dot{X}_1(t)\right\| \longrightarrow 0,$$
$$\left\|\ddot{X}_2(t) - \ddot{X}_1(t)\right\| \longrightarrow 0 \tag{3}$$

as $t \rightarrow \infty$.

2. Main Result

The main result of this paper is the following theorem.

Theorem 2. *We assume that there are positive constants δ_g, δ_h, δ_f, Δ_g, Δ_h, Δ_f, and Δ_1 such that the following conditions hold:*

(i) *the Jacobian matrices $J_g(Y) = \partial g_i/\partial y_j$, $J_h(X) = \partial h_i/\partial x_j$, and $J_f(Z) = \partial f_i/\partial z_j$ exist and are symmetric and their eigenvalues satisfy*

$$\begin{aligned} 0 < \delta_g \le \lambda_i\left(J_g(Y)\right) \le \Delta_g, \\ 0 < \delta_h \le \lambda_i\left(J_h(X)\right) \le \Delta_h, \\ 0 < \delta_f \le \lambda_i\left(J_f(Z)\right) \le \Delta_f, \quad (i = 1, 2, \ldots, n), \end{aligned} \tag{4}$$

for all X, Y, Z in R^n;

(ii) *$P(t, X, Y, Z)$ satisfies*

$$\begin{aligned} &\left\| P\left(t, X_2, Y_2, Z_2\right) - P\left(t, X_1, Y_1, Z_1\right) \right\| \\ &\quad \le \Delta_1 \left\{ \left\| X_2 - X_1 \right\|^2 + \left\| Y_2 - Y_1 \right\|^2 + \left\| Z_2 - Z_1 \right\|^2 \right\}^{1/2} \end{aligned} \tag{5}$$

for any X_i, Y_i, Z_i, $(i = 1, 2)$, in R^n.

If

$$\begin{aligned} &\Delta_1 < \varepsilon, \\ &\Delta_h \le \min\left\{3^{-1}\beta\left(1-\beta\right)\delta_g^2;\right. \\ &\quad \left. 6^{-1}\alpha\left(1-\beta\right)\delta_g\delta_f(1+\alpha)^{-2}\right\} = k\delta_g\delta_f, \end{aligned} \tag{6}$$

then any two solutions $X_1(t)$, $X_2(t)$ of (1) *necessarily converge, where α, ε, k, β are some positive constants with $0 < \beta < 1$ and $k(< 1)$,*

$$k = \min\left\{3^{-1}\beta\left(1-\beta\right)\delta_g\delta_f^{-1}; 6^{-1}\alpha\left(1-\beta\right)(1+\alpha)^{-2}\right\}. \tag{7}$$

Remark 3. The mentioned theorem itself still holds valid with (5) replaced by the much weaker condition

$$\begin{aligned} &\left\| P\left(t, X_2, Y_2, Z_2\right) - P\left(t, X_1, Y_1, Z_1\right) \right\| \\ &\quad \le \phi(t) \left\{ \left\| X_2 - X_1 \right\|^2 + \left\| Y_2 - Y_1 \right\|^2 + \left\| Z_2 - Z_1 \right\|^2 \right\}^{1/2} \end{aligned} \tag{8}$$

for arbitrary t any X_i, Y_i, Z_i, $(i = 1, 2)$, in R^n, where it is assumed that $\int_0^t \phi^{\nu}(s)ds \le \Delta_1 t$ for $1 \le \nu \le 2$.

The following lemma is needed in our later analysis.

Lemma 4. *Let A be a real symmetric $n \times n$-matrix and*

$$\overline{a} \ge \lambda_i(A) \ge a > 0, \quad (i = 1, 2, \ldots, n), \tag{9}$$

where $\overline{a}$ and a are constants.

Then

$$\begin{aligned} \overline{a}\langle X, X\rangle \ge \langle AX, X\rangle \ge a\langle X, X\rangle, \\ \overline{a}^2\langle X, X\rangle \ge \langle AX, AX\rangle \ge a^2\langle X, X\rangle. \end{aligned} \tag{10}$$

Proof (see Afuwape [5]). Our main tool in the proof of our result is the continuous function $V = V(X, Y, Z)$ defined for any triple vectors X, Y, Z in R^n, by

$$\begin{aligned} 2V = &\left\langle \beta\left(1-\beta\right)\delta_g^2 X, X\right\rangle + \left\langle \beta\delta_g Y, Y\right\rangle \\ &+ \left\langle \alpha\delta_g Y, Y\right\rangle + \langle \alpha Z, Z\rangle \\ &+ \left\langle Z + Y + \left(1-\beta\right)\delta_g X, Z + Y + \left(1-\beta\right)\delta_g X\right\rangle. \end{aligned} \tag{11}$$

This function can be rearranged as

$$\begin{aligned} 2V = &\beta\left(1-\beta\right)\delta_g^2\|X\|^2 + \beta\delta_g\|Y\|^2 + \alpha\delta_g\|Y\|^2 \\ &+ \alpha\|Z\|^2 + \left\| Z + Y + \left(1-\beta\right)\delta_g X \right\|^2, \end{aligned} \tag{12}$$

where $0 < \beta < 1$ and $\alpha > 0$

The following result is immediate from the estimate (11). □

Lemma 5. *Assume that all the conditions on the vectors $F(Z)$, $H(X)$, and $G(Y)$ in the theorem hold. Then, there exist positive constants δ_1 and δ_2 such that*

$$\begin{aligned} 2V(X, Y, Z) \ge \delta_1\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right), \\ 2V(X, Y, Z) \le \delta_2\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) \end{aligned} \tag{13}$$

for arbitrary X, Y, Z in R^n.

Proof. Let

$$\begin{aligned} \delta_1 &= \min\left\{\beta\left(1-\beta\right)\delta_g^2, \delta_g\left(\beta+\alpha\right), \alpha\right\}, \\ \delta_2 &= \max\left\{\delta_g\left(1-\beta\right)\left(1+\delta_g\right), \delta_g\left(\beta+\alpha\right) + 1\right. \\ &\quad \left. + \delta_g\left(1-\beta\right), 1 + \alpha + \delta_g\left(1-\beta\right)\right\}. \end{aligned} \tag{14}$$

Then the proof can be easily completed by using Lemma 4. Therefore, we omit the details of the proof. □

Proof of the Theorem. Let X in R^n be any solution of (1). For such a solution, let $\dot{X}$ and $\ddot{X}$ be denoted, respectively, by Y and Z. Then, we can rewrite (1) in the following equivalent system form:

$$\begin{aligned} \dot{X} = Y, \qquad \dot{Y} = Z, \\ \dot{Z} = -F(Z) - G(Y) - H(X) + P(t, X, Y, Z). \end{aligned} \tag{15}$$

Let $X_1(t)$, $X_2(t)$ in R^n be any solution of (1), define $W = W(t)$ by

$$W(t) = V\left(X_2(t) - X_1(t), Y_2(t) - Y_1(t), Z_2(t) - Z_1(t)\right), \tag{16}$$

where V is the function defined in (11) with X, Y, Z replaced by $X_2 - X_1$, $Y_2 - Y_1$ and $Z_2 - Z_1$, respectively.

By Lemma 5, it follows that there exist $\delta_3 > 0$ and $\delta_4 > 0$ such that

$$\delta_3 \left(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right) \leq W \leq \delta_4 \left(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right). \tag{17}$$

When we differentiate the function $W(t)$ with respect to t along the system (15), it follows, after simplification, that

$$\dot{W}(t) = -W_1 - W_2 - W_3 - W_4 - W_5 + W_6, \tag{18}$$

where

$$\begin{aligned}
W_1 &= \frac{1}{2}\delta_g (1-\beta) \langle X_2 - X_1, H(X_2) - H(X_1) \rangle \\
&\quad + \frac{1}{2}\beta\delta_g \langle Y_2 - Y_1, Y_2 - Y_1 \rangle \\
&\quad + \frac{1}{2}\alpha \langle Z_2 - Z_1, F(Z_2) - F(Z_1) \rangle, \\
W_2 &= \frac{1}{6}\delta_g (1-\beta) \langle X_2 - X_1, H(X_2) - H(X_1) \rangle \\
&\quad + \langle Y_2 - Y_1, H(X_2) - H(X_1) \rangle \\
&\quad + \frac{1}{2}\beta\delta_g \langle Y_2 - Y_1, Y_2 - Y_1 \rangle \\
&\quad + \langle Z_2 - Z_1, F(Z_2) - F(Z_1) \rangle, \\
W_3 &= \frac{1}{6}\delta_g (1-\beta) \langle X_2 - X_1, H(X_2) - H(X_1) \rangle \\
&\quad + \frac{1}{4}\alpha \langle F(Z_2) - F(Z_1), Z_2 - Z_1 \rangle \\
&\quad + \langle (1+\alpha)(Z_2 - Z_1), H(X_2) - H(X_1) \rangle, \\
W_4 &= \frac{1}{6}\delta_g (1-\beta) \langle X_2 - X_1, H(X_2) - H(X_1) \rangle \\
&\quad + \Big\langle \delta_g (1-\beta)(X_2 - X_1), G(Y_2) - G(Y_1) \\
&\qquad - \delta_g (Y_2 - Y_1) \Big\rangle \\
&\quad + \frac{1}{2} \Big\langle Y_2 - Y_1, G(Y_2) - G(Y_1) - \delta_g (Y_2 - Y_1) \Big\rangle \\
&\quad + \langle F(Z_2) - F(Z_1) - (Z_2 - Z_1), Y_2 - Y_1 \rangle \\
&\quad + \Big\langle F(Z_2) - F(Z_1) - (Z_2 - Z_1), \\
&\qquad (1-\beta)\delta_g (X_2 - X_1) \Big\rangle, \\
W_5 &= \frac{1}{4}\alpha \langle F(Z_2) - F(Z_1), Z_2 - Z_1 \rangle \\
&\quad + \Big\langle (1+\alpha)(Z_2 - Z_1), G(Y_2) - G(Y_1) - \delta_g (Y_2 - Y_1) \Big\rangle \\
&\quad + \frac{1}{2} \Big\langle Y_2 - Y_1, G(Y_2) - G(Y_1) - \delta_g (Y_2 - Y_1) \Big\rangle, \\
W_6 &= \Big\langle \delta_g (1-\beta)(X_2 - X_1) + Y_2 - Y_1 + (1+\alpha)(Z_2 - Z_1), \\
&\qquad P(t, X_2, Y_2, Z_2) - P(t, X_1, Y_1, Z_1) \Big\rangle \\
&\quad + \langle Z_2 - Z_1, Z_2 - Z_1 \rangle.
\end{aligned} \tag{19}$$

Note that the existence of the following estimates is clear (see Afuwape and Omeike [1]):

$$\begin{aligned}
H(X_2) - H(X_1) &= \int_0^1 J_h(\xi)(X_2 - X_1)\, ds, \\
G(Y_2) - G(Y_1) &= \int_0^1 J_g(\tau)(Y_2 - Y_1)\, dt, \\
F(Z_2) - F(Z_1) &= \int_0^1 J_f(\eta)(Z_2 - Z_1)\, d\mu,
\end{aligned} \tag{20}$$

where $\xi = sX_2 + (1-s)X_1$, $0 \leq s \leq 1$, $\tau = tY_2 + (1-t)Y_1$, $0 \leq t \leq 1$, $\eta = \mu Z_2 + (1-\mu)Z_1$, $0 \leq \mu \leq 1$.

Subject to the assumptions, it can be easily obtained that

$$W_j \geq 0, \quad (j = 3, 4, 5). \tag{21}$$

In view of the assumptions of the theorem, it is also clear that

$$\begin{aligned}
&\langle Y_2 - Y_1, H(X_2) - H(X_1) \rangle \\
&\quad = \left\| k_1 (Y_2 - Y_1) + 2^{-1} k_1^{-1} (H(X_2) - H(X_1)) \right\|^2 \\
&\qquad - \left\langle k_1^2 (Y_2 - Y_1), Y_2 - Y_1 \right\rangle \\
&\qquad - \left\langle 4^{-1} k_1^{-2} (H(X_2) - H(X_1)), H(X_2) - H(X_1) \right\rangle, \\
&\langle Z_2 - Z_1, F(Z_2) - F(Z_1) \rangle \geq \delta_f \langle Z_2 - Z_1, Z_2 - Z_1 \rangle.
\end{aligned} \tag{22}$$

Hence,

$$\begin{aligned}
W_2 &\geq \left\| k_1 (Y_2 - Y_1) + 2^{-1} k_1^{-1} (H(X_2) - H(X_1)) \right\|^2 \\
&\quad + \left\langle Y_2 - Y_1, \left(2^{-1}\beta\delta_g - k_1^2\right)(Y_2 - Y_1) \right\rangle \\
&\quad + \Big\langle H(X_2) - H(X_1), \left(6^{-1}\delta_g (1-\beta)\right)(X_2 - X_1) \\
&\qquad - 4^{-1} k_1^{-2} (H(X_2) - H(X_1)) \Big\rangle \\
&\quad + \delta_f \langle Z_2 - Z_1, Z_2 - Z_1 \rangle.
\end{aligned} \tag{23}$$

Using the estimate $0 < \delta_h \leq \lambda_i(J_h(X)) \leq \Delta_h$, it follows that

$$\begin{aligned}
W_2 &\geq \left\| k_1 (Y_2 - Y_1) + 2^{-1} k_1^{-1} (H(X_2) - H(X_1)) \right\|^2 \\
&\quad + \left\langle Y_2 - Y_1, \left(2^{-1}\beta\delta_g - k_1^2\right)(Y_2 - Y_1) \right\rangle \\
&\quad + \Big\langle H(X_2) - H(X_1), \left(6^{-1}\delta_g (1-\beta)\right)(X_2 - X_1) \\
&\qquad - 4^{-1} k_1^{-2} \delta_h (X_2 - X_1) \Big\rangle + \delta_f \langle Z_2 - Z_1, Z_2 - Z_1 \rangle
\end{aligned}$$

$$\begin{aligned}
&\geq \left\| k_1 (Y_2 - Y_1) + 2^{-1} k_1^{-1} (H(X_2) - H(X_1)) \right\|^2 \\
&\quad + \left\langle Y_2 - Y_1, \left(2^{-1}\beta\delta_g - k_1^2\right)(Y_2 - Y_1) \right\rangle \\
&\quad + \left\langle \delta_h (X_2 - X_1), 6^{-1}\delta_g (1-\beta)(X_2 - X_1) \right\rangle \\
&\quad - \left\langle \Delta_h (X_2 - X_1), -4^{-1} k_1^{-2} \delta_h (X_2 - X_1) \right\rangle \\
&\quad + \delta_f \langle Z_2 - Z_1, Z_2 - Z_1 \rangle \\
&= \left\| k_1 (Y_2 - Y_1) + 2^{-1} k_1^{-1} (H(X_2) - H(X_1)) \right\|^2 \\
&\quad + \left(\frac{1}{2}\beta\delta_g - k_1^2\right) \|Y_2 - Y_1\|^2 \\
&\quad + \left(\frac{1}{6}\delta_h\delta_g (1-\beta) - \frac{1}{4k_1^2}\Delta_h\delta_h\right) \|X_2 - X_1\|^2 \\
&\quad + \delta_f \|Z_2 - Z_1\|^2.
\end{aligned} \tag{24}$$

Then

$$W_2 \geq 0 \quad \forall X, Y, Z \text{ in } R^n \tag{25}$$

if $k_1^2 \leq (1/2)\beta\delta_g$ with $\Delta_h \leq 3^{-1}\beta\delta_g^2(1-\beta)$.

Further, since

$$\begin{aligned}
&\left\langle 2^{-1}(1-\beta)\delta_g (X_2 - X_1), H(X_2) - H(X_1) \right\rangle \\
&\qquad \geq \frac{1}{2}(1-\beta)\delta_g\delta_h \|X_2 - X_1\|^2,
\end{aligned} \tag{26}$$

$$\langle F(Z_2) - F(Z_1), Z_2 - Z_1 \rangle \geq \delta_f \|Z_2 - Z_1\|^2,$$

then

$$\begin{aligned}
W_1 &\geq \frac{1}{2}(1-\beta)\delta_g\delta_h \|X_2 - X_1\|^2 + \frac{1}{2}\beta\delta_g \|Y_2 - Y_1\|^2 \\
&\quad + \frac{1}{2}\alpha\delta_f \|Z_2 - Z_1\|^2 \\
&\geq 2\delta_5 \left(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right),
\end{aligned} \tag{27}$$

where $\delta_5 = (1/4)\min\{(1-\beta)\delta_g\delta_h, \beta\delta_g, \alpha\delta_f\}$.

Moreover, it is obvious that

$$\begin{aligned}
|W_6| \leq \; &\{(1-\beta)\delta_g \|X_2 - X_1\| + \|Y_2 - Y_1\| \\
&+ (\alpha + 1)\|Z_2 - Z_1\| \} \|\theta\|,
\end{aligned} \tag{28}$$

where $\theta = P(t, X_2, Y_2, Z_2) - P(t, X_1, Y_1, Z_1)$.

Hence,

$$\begin{aligned}
|W_6| \leq \delta_6 &\left\{ \|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 \right. \\
&\left. + \|Z_2 - Z_1\|^2 \right\}^{1/2} \|\theta\|.
\end{aligned} \tag{29}$$

Using the assumption (5), we get

$$|W_6| \leq \delta_6 \Delta_1 \left\{ \|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right\}, \tag{30}$$

so that

$$\begin{aligned}
\dot{W}(t) \leq \; &-(2\delta_5 - \delta_6\Delta_1) \\
&\times \left\{ \|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right\}.
\end{aligned} \tag{31}$$

There exists a constant $\delta_7 > 0$ such that

$$\dot{W}(t) \leq -\delta_7 \left\{ \|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2 \right\}, \tag{32}$$

provided that $\Delta_1 < \varepsilon$, where ε is a sufficiently small positive constant.

In view of (17), the last estimate implies that

$$\dot{W}(t) \leq -\delta_8 W(t) \tag{33}$$

for some positive constant δ_8.

The conclusion of the theorem is immediate if, provided that $\Delta_1 < \varepsilon$, on integrating $\dot{W}(t)$ in (33) between t_0 and t, we have

$$W(t) \leq W(t_0) \exp[-\delta_8 (t - t_0)], \quad t \geq t_0, \tag{34}$$

which implies that

$$W(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{35}$$

By (17), this shows that

$$\begin{aligned}
&\|X_2(t) - X_1(t)\| \longrightarrow 0, \qquad \|Y_2(t) - Y_1(t)\| \longrightarrow 0, \\
&\qquad \|X_2(t) - X_1(t)\| \longrightarrow 0, \quad \text{as } t \longrightarrow \infty.
\end{aligned} \tag{36}$$

This completes the proof of the theorem. □

Example 6. Let us consider (1),

$$\dddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad X \in R^2 \tag{37}$$

with

$$\begin{gathered}
F = \begin{pmatrix} \ddot{x}_1 + \arctan \ddot{x}_1 \\ \ddot{x}_2 + \arctan \ddot{x}_2 \end{pmatrix}, \qquad G = \begin{pmatrix} \tan^{-1}\dot{x}_1 + 0.00004\dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, \\
H = \begin{pmatrix} 0.001\tan^{-1}x_1 + 0.0001x_1 \\ 0.0001x_2 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\
P(t) = \begin{pmatrix} e^{-t} \\ \sin t \end{pmatrix},
\end{gathered} \tag{38}$$

where e^{-t}, $\sin t$ are bounded continuous functions on $[0, \infty)$.

Then, it can be easily seen that

$$J_f(\ddot{X}) = \begin{pmatrix} 1 + \dfrac{1}{1+\ddot{x}_1^2} & 0 \\ 0 & 1 + \dfrac{1}{1+\ddot{x}_2^2} \end{pmatrix},$$

$$\lambda_1(J_f) = 1 + \frac{1}{1+\ddot{x}_1^2}, \qquad \lambda_2(J_f) = 1 + \frac{1}{1+\ddot{x}_2^2},$$

$$J_g(\dot{X}) = \begin{pmatrix} \dfrac{1}{1+\dot{x}_1^2} + 0.00004 & 0 \\ 0 & 1 \end{pmatrix}, \tag{39}$$

$$\lambda_1(J_g) = 1, \qquad \lambda_2(J_g) = \frac{1}{1+\dot{x}_1^2} + 0.00004,$$

$$J_h(X) = \begin{pmatrix} \dfrac{0.001}{1+x_1^2} + 0.0001 & 0 \\ 0 & 0.0001 \end{pmatrix},$$

$$\lambda_1(J_h) = \frac{0.001}{1+x_1^2} + 0.0001, \qquad \lambda_2(J_h) = 0.00001.$$

Thus, $\delta_f = 1$, $\Delta_f = 2$, $\delta_g = 1$, $\Delta_g = 1.00004$, $\delta_h = 0.0001$, and $\Delta_h = 0.0011$.

Let us choose

$$\begin{aligned} \alpha &= 3, \\ \beta &= \frac{1}{2}\text{in}\left(\Delta_h \le \min\left\{3^{-1}\beta(1-\beta)\delta_g^2;\right.\right. \\ &\qquad\left.\left. 6^{-1}\alpha(1-\beta)\delta_g\delta_f(\alpha+1)^{-2}\right\}\right. \\ &\left.= k\delta_g\delta_f\right). \end{aligned} \tag{40}$$

Then,

$$k = \frac{1}{64} < 1. \tag{41}$$

Since $0.0011 < 1/64$, then all the conditions of Theorem 2 hold. Therefore, all solutions of the equation considered converge (see, also, [1]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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