Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 916260, 6 pages
http://dx.doi.org/10.1155/2014/916260

Research Article

Eigenvalue Problem for Nonlinear Fractional Differential Equations with Integral Boundary Conditions

Guotao Wang,[1] Sanyang Liu,[1] and Lihong Zhang[2]

[1] *Department of Applied Mathematics, Xidian University, Xi'an, Shaanxi 710071, China*
[2] *School of Mathematics and Computer Science, Shanxi Normal University, Linfen, Shanxi 041004, China*

Correspondence should be addressed to Sanyang Liu; liusanyang@126.com

Received 26 November 2013; Accepted 13 February 2014; Published 24 March 2014

Academic Editor: Hossein Jafari

Copyright © 2014 Guotao Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By employing known Guo-Krasnoselskii fixed point theorem, we investigate the eigenvalue interval for the existence and nonexistence of at least one positive solution of nonlinear fractional differential equation with integral boundary conditions.

1. Introduction

Fractional calculus has been receiving more and more attention in view of its extensive applications in the mathematical modelling coming from physical and other applied sciences; see books [1–5]. Recently, the existence of solutions (or positive solutions) of nonlinear fractional differential equation has been investigated in many papers (see [6–28] and references cited therein). However, in terms of the eigenvalue problem of fractional differential equation, there are only a few results [29–33].

To the best of author's knowledge, no paper has considered the eigenvalue problem of the following nonlinear fractional differential equation with integral boundary conditions:

$$\begin{gathered} {}^{C}D^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \\ 0 < t < 1, \quad n < \alpha \le n+1, \quad n \ge 2, \quad n \in N. \\ u(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0, \\ u(1) = \xi \int_0^1 u(s)\, ds, \end{gathered} \tag{1}$$

where $0 < \xi < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

Our proof is based upon the properties of the Green function and Guo-Krasnoselskii's fixed point theorem given in [34]. Our purpose here is to give the eigenvalue interval for nonlinear fractional differential equation with integral boundary conditions. Moreover, according to the range of the eigenvalue λ, we establish some sufficient conditions for the existence and nonexistence of at least one positive solution of the problem (1).

2. Preliminaries

For the convenience of the readers, we first present some background materials.

Definition 1. For a function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad n = [\alpha] + 1, \tag{2}$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2. The Riemann-Liouville fractional integral of order α for a function f is defined as

$$ I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad \alpha > 0, \tag{3}$$

provided that such integral exists.

Lemma 3. *Let $\alpha > 0$; then*

$$I^{\alpha}\,{}^{C}D^{\alpha}u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}, \quad (4)$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 4 (see [34]). *Let E be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that*

(i) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 5. *Let $n < \alpha \leq n+1$, $n \geq 2$, $n \in N$, and $\xi \neq 2$. Assume $y \in C[0,1]$; then the unique solution of the problem*

$$\begin{gathered} {}^{C}D^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0, \\ u(1) = \xi \int_0^1 u(s)\,ds, \end{gathered} \quad (5)$$

is given by the expression

$$u(t) = \int_0^1 G(t,s)\,y(s)\,ds, \quad (6)$$

where

$$G(t,s) = \begin{cases} \dfrac{2t(1-s)^{\alpha-1}(\alpha - \xi + \xi s) - (2-\xi)\alpha(t-s)^{\alpha-1}}{(2-\xi)\Gamma(\alpha+1)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{2t(1-s)^{\alpha-1}(\alpha - \xi + \xi s)}{(2-\xi)\Gamma(\alpha+1)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (7)$$

Proof. It is well known that the equation ${}^{C}D^{\alpha}u(t) + y(t) = 0$ can be reduced to an equivalent integral equation:

$$u(t) = -I^{\alpha}y(t) - \sum_{i=0}^{n} b_i t^i = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \sum_{i=0}^{n} b_i t^i, \quad (8)$$

for some $b_i \in \mathbb{R}$ $(i = 0, 1, 2, \ldots, n)$.

By the conditions $u(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0$ and $u(1) = \xi \int_0^1 u(s)ds$, we can get that $b_0 = b_2 = b_3 = \cdots = b_n = 0$ and

$$b_1 = -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \xi \int_0^1 u(s)\,ds. \quad (9)$$

Hence, we have

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds \\ &+ \xi t \int_0^1 u(s)\,ds. \end{aligned} \quad (10)$$

Put $\int_0^1 u(s)ds = A$; then, from (10), we deduce that

$$\begin{aligned} A &= \int_0^1 u(t)\,dt \\ &= -\int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\,dt \\ &\quad + \iint_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\,dt + \int_0^1 \xi A t\,dt \\ &= -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)} y(s)\,ds + \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{1}{2}\xi A, \end{aligned} \quad (11)$$

which implies that

$$\begin{aligned} A = &-\frac{2}{2-\xi}\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)} y(s)\,ds \\ &+ \frac{1}{2-\xi}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds. \end{aligned} \quad (12)$$

Replacing this value in (10), we obtain the following expression for function $u(t)$:

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds \\ &+ t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \frac{2\xi}{2-\xi}\int_0^1 \frac{t(1-s)^{\alpha}}{\alpha\Gamma(\alpha)} y(s)\,ds \\ &+ \frac{\xi}{2-\xi}\int_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds \\ = &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds \\ &+ \int_0^1 \frac{2t(1-s)^{\alpha-1}(\alpha-\xi+\xi s)}{(2-\xi)\alpha\Gamma(\alpha)} y(s)\,ds \\ = &\int_0^t \Big(\big(2t(1-s)^{\alpha-1}(\alpha-\xi+\xi s) - (2-\xi)\alpha(t-s)^{\alpha-1}\big) \\ &\quad \times ((2-\xi)\Gamma(\alpha+1))^{-1} y(s)\,ds \Big) \\ &+ \int_t^1 \frac{2t(1-s)^{\alpha-1}(\alpha-\xi+\xi s)}{(2-\xi)\Gamma(\alpha+1)} y(s)\,ds \\ = &\int_0^1 G(t,s)\,y(s)\,ds. \end{aligned} \quad (13)$$

This completes the proof. □

Lemma 6. *Let G be the Green function, which is given by the expression (7). For $0 < \lambda < 2$, the following property holds:*

$$t\, G(1,s) \le G(t,s) \le \frac{2\alpha}{\xi(\alpha-2)} G(1,s), \quad \forall t, s \in (0,1). \tag{14}$$

The proof is similar to that of Lemma 2.4 in [7], so we omit it.

Consider the Banach space $X = C[0,1]$ with general norm

$$\|u\| = \sup_{t\in[0,1]} |u(t)|. \tag{15}$$

Define the cone $P = \{u \in X : u(t) \ge (\xi(\alpha-1)/2\alpha)t\|u\|\}$.

Suppose u is a solution of (1). It is clear from Lemma 5 that

$$u(t) = \lambda \int_0^1 G(t,s) f(s,u(s))\, ds, \quad \forall t \in [0,1]. \tag{16}$$

Define the operator $S_\lambda : P \to X$ as follows:

$$(S_\lambda u)(t) = \lambda \int_0^1 G(t,s) f(s,u(s))\, ds, \quad \forall t \in [0,1]. \tag{17}$$

Lemma 7. *$S_\lambda : P \to P$ is completely continuous.*

Proof. Since $0 < \xi < 2$, it is obvious that $G(t,s) \ge 0$. So we have

$$\begin{aligned}
\|S_\lambda u\| &= \sup_{t\in[0,1]} \lambda \int_0^1 G(t,s) f(s,u(s))\, ds \\
&\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s) f(s,u(s))\, ds, \\
(S_\lambda u)(t) &= \lambda \int_0^1 G(t,s) f(s,u(s))\, ds \\
&\ge \frac{\xi(\alpha-2)}{2\alpha} t\lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s) f(s,u(s))\, ds \\
&\ge \frac{\xi(\alpha-2)}{2\alpha} t \|S_\lambda u\|.
\end{aligned} \tag{18}$$

Therefore, $S_\lambda(P) \subset P$. The other proof is similar to that in [7], so we omit it. □

3. Main Result

For convenience, we list the denotation:

$$\begin{aligned}
F_0 &= \lim_{u\to 0^+} \sup_{t\in[0,1]} \frac{f(t,u(t))}{u}, \\
F_\infty &= \lim_{u\to +\infty} \sup_{t\in[0,1]} \frac{f(t,u(t))}{u}, \\
f_0 &= \lim_{u\to 0^+} \inf_{t\in[0,1]} \frac{f(t,u(t))}{u}, \\
f_\infty &= \lim_{u\to +\infty} \inf_{t\in[0,1]} \frac{f(t,u(t))}{u}.
\end{aligned} \tag{19}$$

Next, we will establish some sufficient conditions for the existence and nonexistence of positive solution for problem (1).

Theorem 8. *Let $l \in (0,1)$ be a constant. Then for each*

$$\begin{aligned}
\lambda \in \Bigg(&\left(\frac{\xi(\alpha-2) l f_\infty}{2\alpha} \int_0^1 sG(1,s)\, ds \right)^{-1}, \\
&\left(\frac{2\alpha F_0}{\xi(\alpha-2)} \int_0^1 G(1,s)\, ds \right)^{-1} \Bigg),
\end{aligned} \tag{20}$$

problem (1) has at least one positive solution.

Proof. First, for any $\varepsilon > 0$, from (20) we have

$$\begin{aligned}
&\left(\frac{\xi(\alpha-2) l (f_\infty - \varepsilon)}{2\alpha} \int_0^1 sG(1,s)\, ds \right)^{-1} \\
&\quad \le \lambda \le \left(\frac{2\alpha(F_0+\varepsilon)}{\xi(\alpha-2)} \int_0^1 G(1,s)\, ds \right)^{-1}.
\end{aligned} \tag{21}$$

On the one hand, by the definition of F_0, there exists $r_1 > 0$ such that, for any $u \in [0, r_1]$, we have

$$f(t,u) \le (F_0 + \varepsilon) u. \tag{22}$$

Choose $\Omega_1 = \{u \in X : \|u\| \le r_1\}$. For $u \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|S_\lambda u\| &= \sup_{t\in[0,1]} \lambda \int_0^1 G(t,s) f(s,u(s))\, ds \\
&\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s)(F_0+\varepsilon) u(s)\, ds \\
&\le \lambda \frac{2\alpha(F_0+\varepsilon)}{\xi(\alpha-2)} \int_0^1 G(1,s)\, ds \|u\| \le \|u\|.
\end{aligned} \tag{23}$$

On the other hand, by the definition of F_∞, there exists $r_2 > r_1$ such that, for any $u \in [r_2, +\infty)$, we have

$$f(t,u) \ge (f_\infty - \varepsilon) u. \tag{24}$$

Take $\Omega_2 = \{u \in X : \|u\| \le r_2\}$. For $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
\|S_\lambda u\| &\ge (S_\lambda u)(l) \ge \lambda \int_0^1 lG(1,s)(f_\infty - \varepsilon) u(s)\, ds \\
&\ge \lambda l \frac{\xi(\alpha-2) f_\infty}{2\alpha} \int_0^1 sG(1,s)\, ds \|u\| \ge \|u\|.
\end{aligned} \tag{25}$$

According to (23), (25), and Lemma 4, S_λ has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \le \|u\| \le r_2$, which is a positive solution of (1). □

Remark 9. If $F_0 = 0$ and $f_\infty = \infty$, then we can get

$$\begin{aligned}
\frac{2\alpha F_0}{\xi(\alpha-2)} \int_0^1 G(1,s)\, ds &= 0, \\
\frac{\xi(\alpha-2) l f_\infty}{2\alpha} \int_0^1 sG(1,s)\, ds &= +\infty.
\end{aligned} \tag{26}$$

Theorem 8 implies that, for $\lambda \in (0, +\infty)$, problem (1) has at least one positive solution.

Theorem 10. *Let $l \in (0,1)$ be a constant. Then for each*

$$\lambda \in \left(\left(\frac{\xi(\alpha-2) l f_0}{2\alpha} \int_0^1 sG(1,s)\,ds \right)^{-1}, \left(\frac{2\alpha F_\infty}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \right)^{-1} \right), \tag{27}$$

problem (1) *has at least one positive solution.*

Proof. First, it follows from (27) that, for any $\varepsilon > 0$,

$$\left(\frac{\xi(\alpha-2) l (f_0 - \varepsilon)}{2\alpha} \int_0^1 sG(1,s)\,ds \right)^{-1} \le \lambda \le \left(\frac{2\alpha (F_\infty + \varepsilon)}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \right)^{-1}. \tag{28}$$

By the definition of f_0, there exists $r_1 > 0$ such that, for any $u \in [0, r_1]$, we have

$$f(t,u) \ge (f_0 + \varepsilon) u. \tag{29}$$

Choose $\Omega_1 = \{u \in X : \|u\| \le r_1\}$. For $u \in P \cap \partial\Omega_1$, we have $\|u\| = r_1$. Similar to the proof in Theorem 8, it holds from (28) and (29) that

$$\begin{aligned} \|S_\lambda u\| &\ge (S_\lambda u)(l) \\ &\ge \lambda l \frac{\xi(\alpha-2) f_0}{2\alpha} \int_0^1 sG(1,s)\,ds \|u\| \ge \|u\|. \end{aligned} \tag{30}$$

Note $F_\infty = \lim_{u \to +\infty} \sup_{t\in[0,1]} f(t,u(t))/u$. There exists $r_3 > r_1$, such that

$$f(t,u) \le (F_\infty + \varepsilon) u, \quad u \in (r_3, +\infty). \tag{31}$$

We consider the problem on two cases. (I) Suppose f is bounded. There exists $M > 0$, such that $f(t,u(t)) \le M$, $\forall u \in (r_3, +\infty)$. Choose $r_4 = \max\{r_3, M\lambda(2\alpha/\xi(\alpha-2)) \int_0^1 G(1,s)ds\}$. Let $\Omega_2' = \{u \in X : \|u\| \le r_4\}$. For $u \in P \cap \partial\Omega_2'$, we have

$$\begin{aligned} \|S_\lambda u\| &= \sup_{t\in[0,1]} \lambda \int_0^1 G(t,s) f(s,u(s))\,ds \\ &\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s) f(s,u(s))\,ds \\ &\le \lambda M \frac{2\alpha}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \|u\| \le r_4 \\ &= \|u\|. \end{aligned} \tag{32}$$

(II) Suppose f is unbounded. There exists $r_5 > r_3$ such that

$$f(t,u(t)) \le u, \quad u \in (r_5, +\infty). \tag{33}$$

Let $\Omega_2'' = \{u \in X : \|u\| \le r_5\}$. For $u \in P \cap \partial\Omega_2''$, we have

$$\begin{aligned} \|S_\lambda u\| &\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s) f(s,u(s))\,ds \\ &\le \lambda \frac{2\alpha(F_\infty + \varepsilon)}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \|u\| \le \|u\|. \end{aligned} \tag{34}$$

Combining (I) and (II), take $\Omega_2 = \{u \in X : \|u\| \le r_2\}$; here, $r_2 \ge \max\{r_4, r_5\}$. Then for $u \in P \cap \partial\Omega_2$, we have

$$\|S_\lambda u\| \le \|u\|. \tag{35}$$

Hence, (30) and (42) together with Lemma 4 imply that S_λ has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \le \|u\| \le r_2$, which is a positive solution of (1). □

Theorem 11. *Assume $F_0 < +\infty$ and $F_\infty < +\infty$. Problem* (1) *has no positive solution provided*

$$\lambda < \left(\frac{2\alpha k}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \right)^{-1}, \tag{36}$$

where k is a constant defined in (38).

Proof. Since $F_0 < +\infty$ and $F_\infty < +\infty$, together with the definitions of F_0 and F_∞, there exist positive constants k_1, k_2, r_1, and r_2 satisfying $r_1 < r_2$ such that

$$\begin{aligned} f(t,u) &\le k_1 u, \qquad u \in [0, r_1], \\ f(t,u) &\le k_2 u, \qquad u \in [r_2, +\infty]. \end{aligned} \tag{37}$$

Take

$$k = \max\left\{ k_1, k_2, \sup_{(t,u)\in(0,1)\times(k_1,k_2)} \frac{f(t,u)}{u} \right\}. \tag{38}$$

It follows that $f(t,u) \le ku$ for any $u \in (0,+\infty)$. Suppose that $v(t)$ is a positive solution of (1). That is,

$$(S_\lambda v)(t) = v(t), \quad \forall t \in J. \tag{39}$$

In sequence,

$$\begin{aligned} \|v\| = \|S_\lambda v\| &= \sup_{t\in[0,1]} \lambda \int_0^1 G(t,s) f(s,v(s))\,ds \\ &\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s) f(s,v(s))\,ds \\ &\le \lambda k \frac{2\alpha}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \|v\| < \|v\|, \end{aligned} \tag{40}$$

which is a contradiction. Hence, (1) has no positive solution. □

Theorem 12. *Assume $f_0 > 0$ and $f_\infty > 0$. Problem* (1) *has no positive solution provided*

$$\lambda > \left(\frac{\xi k(\alpha-2)}{2\alpha} \int_0^1 s^2 G(1,s)\,ds \right)^{-1}, \tag{41}$$

where k is a constant defined in (43).

Proof. Since $f_0 > 0$ and $f_\infty > 0$, together with the definitions of f_0 and f_∞, there exist positive constants k_1, k_2, r_1, and r_2 satisfying $r_1 < r_2$ such that

$$\begin{aligned} f(t,u) &\geq k_1 u, \qquad u \in [0, r_1], \\ f(t,u) &\geq k_2 u, \qquad u \in [r_2, +\infty]. \end{aligned} \tag{42}$$

Take

$$k = \min\left\{k_1, k_2, \inf_{(t,u)\in(0,1)\times(k_1,k_2)} \frac{f(t,u)}{u}\right\}. \tag{43}$$

It follows that $f(t,u) \geq ku$ for any $u \in (0,+\infty)$. Suppose that $v(t)$ is a positive solution of (1). That is,

$$(S_\lambda v)(t) = v(t), \quad \forall t \in J. \tag{44}$$

In sequence,

$$\begin{aligned} \|v\| &\geq \lambda \int_0^1 sG(1,s) f(s, v(s))\,ds \\ &\geq \lambda k \frac{\xi(\alpha-2)}{2\alpha} \int_0^1 s^2 G(1,s)\,ds \|v\| > \|v\|, \end{aligned} \tag{45}$$

which is a contradiction. Hence, (1) has no positive solution. □

Example 13. Consider the fractional differential equation

$$\begin{gathered} {}^{C}D^{5/2} u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u(1) = \int_0^1 u(s)\,ds. \end{gathered} \tag{46}$$

In this example, take

$$f(t, u(t)) = \frac{(500u^2 + u)(7 - t^2)}{u + 7}. \tag{47}$$

Obviously, we have

$$\begin{aligned} F_0 &= \lim_{u\to 0^+} \sup_{t\in[0,1]} \frac{(500u^2 + u)(7 - t^2)}{u(u+7)} = 1, \\ f_\infty &= \lim_{u\to +\infty} \inf_{t\in[0,1]} \frac{(500u^2 + u)(7 - t^2)}{u(u+7)} = 3000. \end{aligned} \tag{48}$$

Since $\alpha = 5/2$ and $\xi = 1$, through a computation, we can get

$$\begin{aligned} &\int_0^1 G(1,s)\,ds \\ &\quad = \int_0^1 \frac{2t(1-s)^{\alpha-1}(\alpha - \xi + \xi s) - (2-\xi)\alpha(t-s)^{\alpha-1}}{(2-\xi)\Gamma(\alpha+1)} ds \\ &\quad = \int_0^1 \frac{2(1-s)^{3/2}(3/2 + s) - (5/2)(1-s)^{3/2}}{\Gamma(7/2)} ds \\ &\quad \leq \frac{1}{\Gamma(7/2)}, \\ &\int_0^1 sG(1,s)\,ds \\ &\quad = \int_0^1 \frac{2s(1-s)^{3/2}(3/2+s) - (5/2)s(1-s)^{3/2}}{\Gamma(7/2)} ds \\ &\quad = \int_0^1 \frac{s(1-s)^{3/2}}{2\Gamma(7/2)} ds \geq \frac{2}{35\Gamma(7/2)}. \end{aligned} \tag{49}$$

Choose $l = 2/3$; we have

$$\begin{aligned} &\left(\frac{\xi(\alpha-2) l f_\infty}{2\alpha} \int_0^1 sG(1,s)\,ds\right)^{-1} \\ &\quad \leq \frac{7\Gamma(7/2)}{80} < \frac{\Gamma(7/2)}{10} \leq \left(\frac{2\alpha F_0}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds\right)^{-1}. \end{aligned} \tag{50}$$

Theorem 8 implies that, for $\lambda \in (7\Gamma(7/2)/80, \Gamma(7/2)/10)$, the problem (46) has at least one positive solution.

Remark 14. In particular, if we take $f(t, u(t)) = u^2(1+t)$ in Example 13, then $F_0 = 0$ and $f_\infty = \infty$. Remark 9 implies that problem (46) has at least one positive solution for $\lambda \in (0, +\infty)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the NNSF of China (no. 61373174) and the Natural Science Foundation for Young Scientists of Shanxi Province, China (no. 2012021002-3).

References

[1] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[3] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, UK, 2009.

[4] J. Sabatier, O. P. Agrawal, and J. A. T. Machado, Eds., *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.

[5] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston, Mass, USA, 2012.

[6] D. Băleanu, O. G. Mustafa, and R. P. Agarwal, "On L^p-solutions for a class of sequential fractional differential equations," *Applied Mathematics and Computation*, vol. 218, no. 5, pp. 2074–2081, 2011.

[7] A. Cabada and G. Wang, "Positive solutions of nonlinear fractional differential equations with integral boundary value conditions," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 1, pp. 403–411, 2012.

[8] G. Wang, "Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments," *Journal of Computational and Applied Mathematics*, vol. 236, no. 9, pp. 2425–2430, 2012.

[9] G. Wang, R. P. Agarwal, and A. Cabada, "Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 25, no. 6, pp. 1019–1024, 2012.

[10] G. Wang, D. Baleanu, and L. Zhang, "Monotone iterative method for a class of nonlinear fractional differential equations," *Fractional Calculus and Applied Analysis*, vol. 15, no. 2, pp. 244–252, 2012.

[11] G. Wang, A. Cabada, and L. Zhang, "Integral boundary value problem for nonlinear differential equations 3of fractional order on an unbounded domain," *Journal of Integral Equations and Applications*. In press.

[12] G. Wang, S. Liu, and L. Zhang, "Neutral fractional integro-differential equation with nonlinear term depending on lower order derivative," *Journal of Computational and Applied Mathematics*, vol. 260, pp. 167–172, 2014.

[13] S. Liu, G. Wang, and L. Zhang, "Existence results for a coupled system of nonlinear neutral fractional differential equations," *Applied Mathematics Letters*, vol. 26, pp. 1120–1124, 2013.

[14] L. Zhang, B. Ahmad, G. Wang, R. P. Agarwal, M. Al-Yami, and W. Shammakh, "Nonlocal integrodifferential boundary value problem for nonlinear fractional differential equations on an unbounded domain," *Abstract and Applied Analysis*, vol. 2013, Article ID 813903, 5 pages, 2013.

[15] T. Jankowski, "Boundary problems for fractional differential equations," *Applied Mathematics Letters*, vol. 28, pp. 14–19, 2014.

[16] B. Ahmad and J. J. Nieto, "Sequential fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3046–3052, 2012.

[17] D. O'Regan and S. Staněk, "Fractional boundary value problems with singularities in space variables," *Nonlinear Dynamics*, vol. 71, no. 4, pp. 641–652, 2013.

[18] Y. Liu, B. Ahmad, and R. P. Agarwal, "Existence of solutions for a coupled system of nonlinear fractional differential equations with fractional boundary conditions on the half-line," *Advances in Difference Equations*, vol. 2013, article 46, 2013.

[19] L. Zhang, B. Ahmad, G. Wang, and R. P. Agarwal, "Nonlinear fractional integro-differential equations on unbounded domains in a Banach space," *Journal of Computational and Applied Mathematics*, vol. 249, pp. 51–56, 2013.

[20] M. Benchohra, A. Cabada, and D. Seba, "An existence result for nonlinear fractional differential equations on Banach spaces," *Boundary Value Problems*, vol. 2009, Article ID 628916, 2009.

[21] S. Zhang, "Positive solutions for boundary-value problems of nonlinear fractional differential equations," *Electronic Journal of Differential Equations*, vol. 2006, pp. 1–12, 2006.

[22] B. Ahmad, J. J. Nieto, A. Alsaedi, and M. El-Shahed, "A study of nonlinear Langevin equation involving two fractional orders in different intervals," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 2, pp. 599–606, 2012.

[23] M. Feng, X. Zhang, and W. Ge, "New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions," *Boundary Value Problems*, vol. 2011, Article ID 720702, 2011.

[24] H. A. H. Salem, "Fractional order boundary value problem with integral boundary conditions involving Pettis integral," *Acta Mathematica Scientia B*, vol. 31, no. 2, pp. 661–672, 2011.

[25] Y. Zhou, F. Jiao, and J. Li, "Existence and uniqueness for fractional neutral differential equations with infinite delay," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3249–3256, 2009.

[26] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 2, pp. 916–924, 2010.

[27] C. S. Goodrich, "Existence of a positive solution to systems of differential equations of fractional order," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1251–1268, 2011.

[28] X. Xu, D. Jiang, and C. Yuan, "Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4676–4688, 2009.

[29] W. Jiang, "Eigenvalue interval for multi-point boundary value problems of fractional differential equations," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4570–4575, 2013.

[30] G. Wang, S. K. Ntouyas, and L. Zhang, "Positive solutions of the three-point boundary value problem for fractional-order differential equations with an advanced argument," *Advances in Difference Equations*, vol. 2011, article 2, 2011.

[31] Z. Bai, "Eigenvalue intervals for a class of fractional boundary value problem," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3253–3257, 2012.

[32] X. Zhang, L. Liu, and Y. Wu, "The eigenvalue problem for a singular higher order fractional differential equation involving fractional derivatives," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8526–8536, 2012.

[33] S. Sun, Y. Zhao, Z. Han, and J. Liu, "Eigenvalue problem for a class of nonlinear fractional differential equations," *Annals of Functional Analysis*, vol. 4, no. 1, pp. 25–39, 2013.

[34] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, Boston, Mass, USA, 1988.