Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 925010, 9 pages
http://dx.doi.org/10.1155/2014/925010

Research Article

Positive Solutions for the Eigenvalue Problem of Semipositone Fractional Order Differential Equation with Multipoint Boundary Conditions

Ge Dong

Department of Basic Teaching, Shanghai Jianqiao College, Shanghai 201319, China

Correspondence should be addressed to Ge Dong; shanghaidongge@163.com

Received 17 January 2014; Accepted 14 February 2014; Published 15 April 2014

Academic Editor: Xinguang Zhang

We study the existence of positive solution for the eigenvalue problem of semipositone fractional order differential equation with multipoint boundary conditions by using known Krasnosel'skii's fixed point theorem. Some sufficient conditions that guarantee the existence of at least one positive solution for eigenvalues $\lambda > 0$ sufficiently small and $\lambda > 0$ sufficiently large are established.

1. Introduction

In this paper, we study the existence of positive solutions to the following eigenvalue problem of semipositone fractional order differential equation with multipoint boundary conditions:

$$\begin{gathered} -\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(t) = \lambda f\left(t, x(t), \mathscr{D}_{\mathbf{t}}{}^{\gamma} x(t)\right), \quad t \in (0,1), \\ \mathscr{D}_{\mathbf{t}}{}^{\gamma} x(0) = 0, \qquad \mathscr{D}_{\mathbf{t}}{}^{\gamma+1} x(0) = 0, \\ \mathscr{D}_{\mathbf{t}}{}^{\gamma} x(1) = \sum_{j=1}^{m-2} a_j \mathscr{D}_{\mathbf{t}}{}^{\gamma} x\left(\xi_j\right), \end{gathered} \tag{1}$$

where $3 < \alpha \le 4$, $0 < \gamma \le \alpha - 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_j \in [0, +\infty)$ with $0 < \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1} < 1$, λ is a positive parameter, and $D_t^{\alpha}, D_t^{\gamma}$ are the standard Rieman-Liouville derivative. Throughout the paper, we assume that f is semipositone; that is, $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous and there exists $M > 0$, such that $f(t,x) \ge -M$, for any $(t,x) \in [0,1] \times [0,\infty)$.

The multipoint boundary value problems (BVPs for short) for ordinary differential equations arise in a variety of different applied mathematics and physics. Recently, Feng and Bai [1] investigated the existence of positive solutions for a semipositone second-order multipoint boundary value problem:

$$\begin{gathered} x''(t) + \lambda f(t, x(t)) = 0, \quad t \in (0,1), \\ x'(0) = \sum_{j=1}^{m-2} a_j x\left(\xi_j\right), \qquad x(1) = \sum_{j=1}^{m-2} b_j x\left(\xi_j\right). \end{gathered} \tag{2}$$

By using Krasnosel'skii's fixed point theorem, some sufficient conditions that guarantee the existence of at least one positive solution are obtained. In [2], a $(n-1,1)$-type conjugate boundary value problem for the nonlinear fractional differential equation,

$$\begin{gathered} \mathscr{D}_{\mathbf{t}}{}^{\alpha} x(t) + \lambda f(t, x(t)) = 0, \quad t \in (0,1), \\ x^{j}(0) = 0, \quad 0 \le j \le n-2, \quad x(1) = 0, \end{gathered} \tag{3}$$

is considered. Based on the nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorems, the existence of positive solution of the semipositone boundary value problems (3) for a sufficiently small $\lambda > 0$ was given. In recent paper [3], Zhang et al. established the existence of multiple positive solutions for a general higher order fractional differential equation with derivatives and a negatively Carathèodory perturbed term:

$$\begin{gathered}
-\mathscr{D}^{\alpha}x(t) \\
= p(t)\,f\left(t, x(t), \mathscr{D}^{\mu_1}x(t), \mathscr{D}^{\mu_2}x(t), \ldots, \mathscr{D}^{\mu_{n-1}}x(t)\right) \\
- g\left(t, x(t), \mathscr{D}^{\mu_1}x(t), \mathscr{D}^{\mu_2}x(t), \ldots, \mathscr{D}^{\mu_{n-1}}x(t)\right), \\
\mathscr{D}^{\mu_i}x(0) = 0, \quad 1 \le i \le n-1, \\
\mathscr{D}^{\mu_{n-1}+1}x(0) = 0, \qquad \mathscr{D}^{\mu_{n-1}}x(1) = \sum_{j=1}^{m-2} a_j \mathscr{D}^{\mu_{n-1}}x\left(\xi_j\right).
\end{gathered} \tag{4}$$

Some local and nonlocal growth conditions were adopted to guarantee the existence of at least two positive solutions for the higher order fractional differential equation (4). For the recent work in application, the reader is referred to [4–20].

Inspired by the above work, in this paper we study the existence of positive solutions to the semipositone BVP (1). Here we also emphasize that the main results of this paper contain not only the cases for $\lambda > 0$ sufficiently small, but also for $\lambda > 0$ sufficiently large, which is different from [2, 3].

2. Preliminaries and Lemmas

Definition 1 (see [21–24]). The fractional integral of order $\alpha > 0$ of a function $x : (a, +\infty) \to R$ is given by

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds, \tag{5}$$

provided that the right-hand side is pointwisely on $(a, +\infty)$.

Definition 2 (see [21–24]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function x: $(a, +\infty) \to R$ is given by

$$D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} x(s)\, ds, \tag{6}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, and $t > a$, provided that the right-hand side is defined on $(a, +\infty)$.

Lemma 3 (see [21–24]). *Assuming that $x \in L^1[0,1]$ with a fractional derivative of order $\alpha > 0$, then*

$$I^{\alpha}D_t^{\alpha}x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{7}$$

where $c_i \in R$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 4 (see [3]). *Suppose that $h \in L^1[0,1]$. Then the following boundary value problem*

$$\begin{gathered}
\mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}x(t) + h(t) = 0, \quad t \in (0,1), \\
x(0) = x'(0) = 0, \qquad x(1) = \sum_{j=1}^{m-2} a_j x\left(\xi_j\right)
\end{gathered} \tag{8}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s)\, h(s)\, ds, \tag{9}$$

where

$$G(t,s) = g(t,s) + \frac{t^{\alpha-\gamma-1}}{1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}} \sum_{j=1}^{m-2} a_j g\left(\xi_j, s\right) \tag{10}$$

is the Green function of the boundary value problem (8) *and*

$$g(t,s) = \begin{cases} \dfrac{(t(1-s))^{\alpha-\gamma-1} - (t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{(t(1-s))^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}, & 0 \le t \le s \le 1. \end{cases} \tag{11}$$

Lemma 5 (see [2]). *The function $g(t,s)$ in Lemma 4 has the following properties:*

(R1) $g(t,s) = g(1-s, 1-t)$, *for* $t, s \in [0,1]$;

(R2) $\Gamma(\alpha-\gamma)k(t)q(s) \le g(t,s) \le (\alpha-\gamma-1)q(s)$, *for* $t, s \in [0,1]$;

(R3) $\Gamma(\alpha-\gamma)k(t)q(s) \le g(t,s) \le (\alpha-\gamma-1)k(t)$, *for* $t, s \in [0,1]$, *where*

$$k(t) = \frac{t^{\alpha-\gamma-1}(1-t)}{\Gamma(\alpha-\gamma)}, \qquad q(t) = \frac{s(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}. \tag{12}$$

Lemma 6. *The following boundary value problem*

$$\begin{gathered}
\mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}x(t) + \lambda M = 0, \quad t \in (0,1), \\
x(0) = x'(0) = 0, \qquad x(1) = \sum_{j=1}^{m-2} a_j x\left(\xi_j\right)
\end{gathered} \tag{13}$$

has a unique solution w, which satisfies

$$\begin{gathered}
w(t) \le \frac{4\lambda M \sigma(t)}{1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}}, \\
\sigma(t) = \left(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right) k(t) \\
+ \sum_{j=1}^{m-2} a_j k\left(\xi_j\right) t^{\alpha-\gamma-1} \le t^{\alpha-\gamma-1}.
\end{gathered} \tag{14}$$

Proof. By Lemma 4, the unique solution of (13) is

$$w(t) = \lambda M \int_0^1 G(t,s)\, ds. \tag{15}$$

So

$$\begin{aligned}
w(t) &= \lambda M \int_0^1 G(t,s)\, ds \le \lambda M (\alpha-\gamma-1) \\
&\quad \times \int_0^1 \left(k(t) + \frac{\sum_{j=1}^{m-2} a_j k\left(\xi_j\right)}{1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}} t^{\alpha-\gamma-1}\right) ds \\
&\le \frac{4\lambda M \sigma(t)}{1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}},
\end{aligned} \tag{16}$$

and by $\alpha-\gamma \geq 2$, we have $\Gamma(\alpha-\gamma) \geq 1$, so

$$\begin{aligned}
\sigma(t) &= \left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right) k(t) + \sum_{j=1}^{m-2} a_j k(\xi_j) t^{\alpha-\gamma-1} \\
&= \frac{1}{\Gamma(\alpha-\gamma)}\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right) t^{\alpha-\gamma-1}(1-t) \\
&\quad + \frac{1}{\Gamma(\alpha-\gamma)} \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}(1-\xi_j) t^{\alpha-\gamma-1} \\
&\leq \left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right) t^{\alpha-\gamma-1} \\
&\quad + \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1} t^{\alpha-\gamma-1} = t^{\alpha-\gamma-1}.
\end{aligned} \tag{17}$$

□

The basic space used in this paper is $E = C([0,1];\mathbb{R})$, where $\mathbb{R}$ is the set of real numbers. Obviously, the space E is a Banach space if it is endowed with the norm as follows:

$$\|y\| = \max_{t\in[0,1]} |y(t)|, \tag{18}$$

for any $y \in E$. Let

$$P = \left\{ y \in E : y(t) \geq \frac{1}{8}\sigma(t)\|y\| \right\}, \tag{19}$$

and then P is a cone of E.

Now let $v(t) = \mathscr{D}_{\mathbf{t}}{}^{\gamma} x(t)$; then the boundary value problem (1) is equivalent to the following boundary value problem:

$$\begin{gathered}
-\mathscr{D}_{\mathbf{t}}{}^{\alpha-\gamma} v(t) = \lambda f(t, I^{\gamma} v(t), v(t)), \quad t \in (0,1), \\
v(0) = v'(0) = 0, \qquad v(1) = \sum_{j=1}^{m-2} a_j v(\xi_j).
\end{gathered} \tag{20}$$

Define a modified function $[\cdot]^*$ for any $\varphi \in C[0,1]$ by

$$[\varphi(t)]^* = \begin{cases} \varphi(t), & \varphi(t) \geq 0, \\ 0, & \varphi(t) < 0, \end{cases} \tag{21}$$

and consider

$$\begin{aligned}
&-\mathscr{D}_{\mathbf{t}}{}^{\alpha-\gamma} y(t) \\
&\quad = \lambda\left[f\left(t, I^{\gamma}[y(t)-w(t)]^*, [y(t)-w(t)]^*\right) + M\right], \\
&\qquad\qquad t \in (0,1), \\
&y(0) = y'(0) = 0, \qquad y(1) = \sum_{j=1}^{m-2} a_j y(\xi_j).
\end{aligned} \tag{22}$$

Lemma 7. *The BVP* (1) *and the BVP* (22) *are equivalent. Moreover, if y is a positive solution of the problem* (22) *and satisfies $y(t) \geq w(t)$, $t \in [0,1]$, then $I^{\gamma}[y(t)-w(t)]$ is a positive solution of the boundary value problem* (1).

Proof. Since y is a positive solution of the BVP (22) such that $y(t) \geq w(t)$ for any $t \in [0,1]$, we have

$$\begin{aligned}
&-\mathscr{D}_{\mathbf{t}}{}^{\alpha-\gamma} y(t) \\
&\quad = \lambda\left[f(t, I^{\gamma}[y(t)-w(t)], [y(t)-w(t)]) + M\right], \\
&\qquad\qquad t \in (0,1), \\
&y(0) = y'(0) = 0, \qquad y(1) = \sum_{j=1}^{m-2} a_j y(\xi_j).
\end{aligned} \tag{23}$$

Let $v = y - w$, and then we have

$$\begin{gathered}
\mathscr{D}_{\mathbf{t}}{}^{\alpha-\gamma} v(t) = \mathscr{D}_{\mathbf{t}}{}^{\alpha-\gamma} y(t) - \mathscr{D}_{\mathbf{t}}{}^{\alpha-\gamma} w(t), \\
w(0) = w'(0) = 0, \qquad w(1) = \sum_{j=1}^{m-2} a_j w(\xi_j).
\end{gathered} \tag{24}$$

Substitute (24) into (23), that is (20), which implies that $I^{\gamma}[y(t)-w(t)]$ is a positive solution of the BVP (1). □

It follows from Lemma 4 that the BVP (22) is equivalent to the integral equation

$$\begin{aligned}
y(t) = \lambda \int_0^1 G(t,s)\big[&f\left(s, I^{\gamma}[y(s)-w(s)]^*, [y(s)-w(s)]^*\right) \\
&+M\big].
\end{aligned} \tag{25}$$

Thus it is sufficient to find fixed points $y(t) \geq w(t)$, $t \in [0,1]$ for the mapping T defined by

$$\begin{aligned}
&(Ty)(t) \\
&\quad = \lambda \int_0^1 G(t,s)\big[f\left(s, I^{\gamma}[y(s)-w(s)]^*, [y(s)-w(s)]^*\right) \\
&\qquad\qquad +M\big].
\end{aligned} \tag{26}$$

Lemma 8. *$T : P \to P$ is a completely continuous operator.*

Proof. For any fixed $y \in P$, there exists a constant $L > 0$ such that $\|y\| \leq L$, and

$$\begin{aligned}
&0 \leq [y(s)-w(s)]^* \leq y(s) \leq \|y\| \leq L, \\
&0 \leq I^{\gamma}[y(s)-w(s)]^* = \int_0^t \frac{(t-s)^{\gamma-1}[y(s)-w(s)]^*}{\Gamma(\gamma)} ds \\
&\quad \leq \frac{L}{\Gamma(\gamma)}.
\end{aligned} \tag{27}$$

Take

$$N = \max_{[0,1]\times[0,L/\Gamma(\gamma)]\times[0,L]} f(t,u,v), \tag{28}$$

then

$$\begin{aligned}(Ty)(t) &= \lambda \int_0^1 G(t,s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]\\ &\le \lambda(M+N)\int_0^1 G(t,s)\,ds < +\infty.\end{aligned} \tag{29}$$

This implies that the operator $T : P \to E$ is bounded.

Next for any $y \in P$, by Lemma 5, we have

$$\begin{aligned}\|Ty\| &= \max_{0\le t\le 1}\left\{\lambda \int_0^1 G(t,s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\right.\\ &\qquad \left.\left.\left.[y(s)-w(s)]^*\right)+M\right]\right\}\\ &\le \lambda(\alpha-\gamma-1)\int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]\\ &\quad + \frac{\lambda}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}\\ &\quad \times \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j,s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]ds\\ &\le 4\lambda\left\{\int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]\\ &\quad + \frac{1}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}\\ &\quad \times \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j,s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.\left.[y(s)-w(s)]^*\right)+M\right]ds\right\}\\ &\le \frac{4\lambda}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}\\ &\quad \times\left\{\int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,[y(s)-w(s)]^*\right)\right.\right.\\ &\qquad \left.+M\right]+\sum_{j=1}^{m-2} a_j\int_0^1 g(\xi_j,s)\\ &\quad \times\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.\left.[y(s)-w(s)]^*\right)+M\right]ds\right\}.\end{aligned} \tag{30}$$

On the other hand, it follows from Lemma 5, $\Gamma(\alpha-\gamma) \ge 1$, and $\sigma(t) \le t^{\alpha-\gamma-1}$ that

$$\begin{aligned}(Ty)(t) &\ge \lambda\Gamma(\alpha-\gamma)k(t)\\ &\quad \times \int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]ds\\ &\quad + \frac{\lambda t^{\alpha-\gamma-1}}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}\\ &\quad \times \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j,s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]ds\\ &\ge \frac{1}{2}\lambda k(t)\int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]ds\\ &\quad + \frac{\lambda t^{\alpha-\gamma-1}}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}\\ &\quad \times \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j,s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]ds\\ &= \frac{1}{2}\left\{\lambda k(t)\int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\right.\\ &\qquad \left.\left.[y(s)-w(s)]^*\right)+M\right]ds\\ &\quad + \frac{\lambda t^{\alpha-\gamma-1}}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}\\ &\quad \times \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j,s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*,\right.\right.\\ &\qquad \left.\left.\left.[y(s)-w(s)]^*\right)+M\right]ds\right\}\\ &\quad + \frac{\lambda t^{\alpha-\gamma-1}}{2\left(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}\right)}\end{aligned}$$

$$\times \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j, s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)+M\right] ds$$
$$\geq \frac{1}{2}\lambda\left\{k(t)\int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)+M\right] ds + \frac{t^{\alpha-\gamma-1}\sum_{j=1}^{m-2} a_j k(\xi_j)}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}} \times \int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)+M\right] ds\right\}$$
$$+ \frac{\lambda t^{\alpha-\gamma-1}}{2\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right)} \times \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j, s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)+M\right] ds$$
$$= \frac{\lambda\sigma(t)}{2\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right)} \times \int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)+M\right] ds$$
$$+ \frac{\lambda\sigma(t)}{2\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right)} \times \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j, s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)+M\right] ds$$
$$= \frac{\lambda\sigma(t)}{2\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right)} \times \left(\int_0^1 q(s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)+M\right] ds + \sum_{j=1}^{m-2} a_j \int_0^1 g(\xi_j, s)\left[f\left(s, I^\gamma[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)+M\right] ds\right). \tag{31}$$

So, by (30) and (31), we have

$$(Ty)(t) \geq \frac{1}{8}\|Ty\|\sigma(t), \quad t \in [0,1], \tag{32}$$

which yields that $T(P) \subset P$.

At the end, using standard arguments, according to the Ascoli-Arzela Theorem, one can show that $T : P \to P$ is completely continuous. Thus $T : P \to P$ is a completely continuous operator. □

Lemma 9 (see [25]). *Let E be a real Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(1) $\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_2$, *or*

(2) $\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Result

Define

$$f^\infty = \limsup_{|x|+|y|\to\infty} \max_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|},$$
$$f_\infty = \liminf_{|x|+|y|\to\infty} \min_{t\in[1/4,3/4]} \frac{f(t,x,y)}{|x|+|y|}. \tag{33}$$

Theorem 10. *Suppose that*

$$f_\infty = \infty. \tag{34}$$

Then there exists a constant $\Lambda > 0$ such that, for any $\lambda \in (0, \Lambda]$, the BVP (1) has at least one positive solution.

Proof. Choosing $y \in P$ with $\|y\| = 1$, then

$$0 \leq [y(s)-w(s)]^* \leq y(s) \leq \|y\| \leq 1,$$
$$0 \leq I^\gamma[y(s)-w(s)]^* = \int_0^t \frac{(t-s)^{\gamma-1}[y(s)-w(s)]^*}{\Gamma(\gamma)} ds \leq \frac{1}{\Gamma(\gamma)}. \tag{35}$$

Let

$$N = \max_{(t,x,y)\in[0,1]\times[0,1/\Gamma(\gamma)]\times[0,1]} f(t,x,y),$$
$$\Lambda = \min\left\{\frac{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}}{34M}, \frac{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}}{4\sum_{j=1}^{m-2} a_j (M+N)}\right\}. \tag{36}$$

For any $y \in \partial B_1$, $B_1 = \{y \in P : \|y\| \le 1\}$, and $\lambda > 0$ sufficiently small such that $\lambda \in (0, \Lambda]$, we have

$$\begin{aligned}\|Ty\| &= \max_{0\le t\le 1}\left\{\lambda \int_0^1 G(t,s)\left[f\left(s, I^{\gamma}[y(s)-w(s)]^*,\right.\right.\right.\\ &\qquad\left.\left.\left.[y(s)-w(s)]^*\right)+M\right]\right\}\\ &\le \lambda(\alpha-\gamma-1)\\ &\quad\times\int_0^1 q(s)\left[f\left(s, I^{\gamma}[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)\right.\\ &\quad\left.+M\right] + \frac{\lambda(\alpha-\gamma-1)}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}\\ &\quad\times\sum_{j=1}^{m-2} a_j\int_0^1 q(s)\left[f\left(s, I^{\gamma}[y(s)-w(s)]^*,\right.\right.\\ &\qquad\left.\left.[y(s)-w(s)]^*\right)+M\right]ds\\ &\le \frac{4\lambda\sum_{j=1}^{m-2} a_j(M+N)}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}} \le 1 = \|y\|.\end{aligned} \tag{37}$$

Therefore,

$$\|Ty\| \le \|y\|, \quad y \in \partial B_1. \tag{38}$$

On the other hand, take

$$\epsilon = \left(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma}\right)\left(\frac{1}{4}\right)^{\alpha-\gamma}, \tag{39}$$

and choose a large enough $L > 0$ such that

$$\frac{\lambda L\epsilon}{64}\left(\frac{1}{4}\right)^{\alpha-\gamma}\int_{1/4}^{3/4} q(s)\,ds > 1. \tag{40}$$

By (33), we know that f is an unbounded continuous function. Therefore, for any $t \in [1/4, 3/4]$, there exists a constant $K > 0$ such that

$$f(t,x,y) \ge L(|x|+|y|), \quad \text{if } |x|+|y| > K. \tag{41}$$

Choosing

$$R > \max\left\{\frac{64\lambda M}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}, 1+K, \frac{16K}{\epsilon}\right\}, \tag{42}$$

then $R > K > 1$. Let $B_R = \{y \in P : \|y\| \le R\}$. Then for any $y \in \partial B_R$ and for any $t \in [1/4, 3/4]$, we have

$$\begin{aligned}y(t)-w(t) &\ge y(t) - \frac{4\lambda M\sigma(t)}{1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}}\\ &\ge y(t) - \frac{32\lambda M y(t)}{\left(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\gamma-1}\right)R}\\ &\ge \frac{1}{2}y(t) \ge \frac{1}{16}\sigma(t)R \ge \frac{1}{16}\epsilon R \ge K > 0.\end{aligned} \tag{43}$$

Consequently, for $s \in [1/4, 3/4]$, it follows from (43) that

$$\begin{aligned}&\left|I^{\gamma}[y(s)-w(s)]^*\right| + \left|[y(s)-w(s)]^*\right|\\ &\quad\ge \left|[y(s)-w(s)]^*\right| > K,\end{aligned} \tag{44}$$

and then by (41) and (44), for $s \in [1/4, 3/4]$, we get

$$\begin{aligned}&f\left(s, I^{\gamma}[y(s)-w(s)]^*, [y(s)-w(s)]^*\right)\\ &\quad\ge L\left(\left|I^{\gamma}[y(s)-w(s)]^*\right| + \left|[y(s)-w(s)]^*\right|\right)\\ &\quad\ge L\left|[y(s)-w(s)]\right| \ge \frac{1}{64}L\epsilon R.\end{aligned} \tag{45}$$

So for any $y \in \partial B_R$ and $t \in [0,1]$, by (45), we have

$$\begin{aligned}&\|Ty\|\\ &\ge \lambda\int_0^1 G\left(\frac{1}{4},s\right)\left[f\left(s, I^{\gamma}[y(s)-w(s)]^*,\right.\right.\\ &\qquad\left.\left.[y(s)-w(s)]^*\right)+M\right]ds\\ &\ge \lambda\int_0^1 G\left(\frac{1}{4},s\right)f\left(s, I^{\gamma}[y(s)-w(s)]^*,\right.\\ &\qquad\left.[y(s)-w(s)]^*\right)ds\\ &\ge \lambda\int_0^1 g\left(\frac{1}{4},s\right)f\left(s, I^{\gamma}[y(s)-w(s)]^*,\right.\\ &\qquad\left.[y(s)-w(s)]^*\right)ds\\ &\ge \lambda\Gamma(\alpha-\gamma)k\left(\frac{1}{4}\right)\\ &\quad\times\int_0^1 q(s)f\left(s, I^{\gamma}[y(s)-w(s)]^*,\right.\\ &\qquad\left.[y(s)-w(s)]^*\right)ds\\ &\ge \lambda\left(\frac{1}{4}\right)^{\alpha-\gamma}\int_{1/4}^{3/4} q(s)f\left(s, I^{\gamma}[y(s)-w(s)]^*,\right.\\ &\qquad\left.[y(s)-w(s)]^*\right)ds\\ &\ge \lambda\left(\frac{1}{4}\right)^{\alpha-\gamma}\int_{1/4}^{3/4} q(s)\frac{1}{64}L\epsilon R\,ds \ge R = \|y\|.\end{aligned} \tag{46}$$

Thus, we have

$$\|Ty\| \geq \|y\|, \quad y \in \partial B_R. \tag{47}$$

By Lemma 9, T has a fixed point y such that $1 \leq \|y\| \leq R$.

From

$$\lambda \leq \Lambda \leq \frac{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}}{32M}, \tag{48}$$

we have

$$\frac{32\lambda M}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}} \leq 1. \tag{49}$$

Thus

$$\begin{aligned} y(t) &\geq \frac{1}{8}\sigma(t)\|y\| \geq \frac{1}{8}\sigma(t) \\ &\geq \frac{4\lambda M\sigma(t)}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}} \geq w(t). \end{aligned} \tag{50}$$

By Lemma 7 and (50), the boundary value problem (1) has at least one positive solution. The proof of Theorem 10 is completed. □

Theorem 11. *Suppose that*

$$f^{\infty} = 0, \tag{51}$$

and there exist constants $\kappa \geq 0$ and $\theta > 0$ such that

$$\begin{aligned} &f(t,x,y) \geq \kappa(|x|+|y|), \\ &(t,x,y) \in \left[\frac{1}{4},\frac{3}{4}\right] \times \left[\frac{\theta}{4^{\gamma}\Gamma(\gamma+1)},\infty\right) \times [\theta,\infty). \end{aligned} \tag{52}$$

Then there exists a constant $\Lambda > 0$ such that, for any $\lambda \in [\Lambda,+\infty)$, the BVP (1) *has at least one positive solution.*

Proof. Choosing

$$\begin{aligned} \Lambda_1 &= \left[\frac{\kappa\epsilon}{4^{\alpha-\gamma+2}}\left(1+\frac{1}{4^{\gamma}\Gamma(\gamma+1)}\right)\int_{1/4}^{3/4} q(s)\,ds\right]^{-1}, \\ R_1 &= \max\left\{\frac{64\lambda M}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}}, \frac{16\theta}{\epsilon}\right\}, \end{aligned} \tag{53}$$

and let $B_{R_1} = \{y \in P : \|y\| \leq R_1\}$. Then for any $\in [\Lambda_1,\infty)$, $y \in \partial B_{R_1}$, and $t \in [1/4, 3/4]$, we have

$$\begin{aligned} y(t)-w(t) &\geq y(t) - \frac{4\lambda M\sigma(t)}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}} \\ &\geq y(t) - \frac{32\lambda M y(t)}{\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right)R_1} \\ &\geq \frac{1}{2}y(t) \geq \frac{1}{16}\sigma(t)R_1 \geq \frac{1}{16}\epsilon R_1 \geq \theta > 0, \end{aligned} \tag{54}$$

$$\begin{aligned} I^{\gamma}&[y(t)-w(t)] \\ &\geq I^{\gamma}\left(\frac{1}{16}\epsilon R_1\right) \geq \frac{\epsilon}{16\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\,ds R_1 \\ &\geq \frac{\epsilon R_1}{16\times 4^{\gamma}\Gamma(\gamma+1)} \geq \frac{\theta}{4^{\gamma}\Gamma(\gamma+1)} > 0, \end{aligned} \tag{55}$$

so for any $y \in \partial B_{R_1}$ and $t \in [0,1]$, by (52)–(55), we have

$$\begin{aligned} \|Ty\| &\geq \lambda\int_0^1 G\left(\frac{1}{4},s\right)\Big[f\big(s, I^{\gamma}[y(s)-w(s)]^{*}, \\ &\qquad [y(s)-w(s)]^{*}\big) + M\Big]\,ds \\ &\geq \lambda\int_0^1 G\left(\frac{1}{4},s\right) f\big(s, I^{\gamma}[y(s)-w(s)]^{*}, \\ &\qquad [y(s)-w(s)]^{*}\big)\,ds \\ &\geq \lambda\int_0^1 g\left(\frac{1}{4},s\right) f\big(s, I^{\gamma}[y(s)-w(s)]^{*}, \\ &\qquad [y(s)-w(s)]^{*}\big)\,ds \\ &\geq \lambda\Gamma(\alpha-\gamma)k\left(\frac{1}{4}\right)\int_0^1 q(s) f\big(s, I^{\gamma}[y(s)-w(s)]^{*}, \\ &\qquad [y(s)-w(s)]^{*}\big)\,ds \\ &\geq \lambda\left(\frac{1}{4}\right)^{\alpha-\gamma}\int_{1/4}^{3/4} q(s) f\big(s, I^{\gamma}[y(s)-w(s)]^{*}, \\ &\qquad [y(s)-w(s)]^{*}\big)\,ds \\ &\geq \frac{\lambda\kappa\epsilon}{4^{\alpha-\gamma+2}}\left(1+\frac{1}{4^{\gamma}\Gamma(\gamma+1)}\right)\int_{1/4}^{3/4} q(s)\,ds R_1 \geq R_1 \\ &= \|y\|. \end{aligned} \tag{56}$$

Thus, we have

$$\|Ty\| \geq \|y\|, \quad y \in \partial B_{R_1}. \tag{57}$$

According to (51), it is clear that

$$f^{\infty} = \limsup_{|x|+|y|\to\infty} \max_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|} = \limsup_{|x|+|y|\to\infty} \max_{t\in[0,1]} \frac{f(t,x,y)+M}{|x|+|y|} = 0. \quad (58)$$

Let us choose $\varepsilon > 0$ such that

$$\frac{4\lambda(\Gamma(\gamma+1)+1)\varepsilon}{\Gamma(\gamma+1)} < 1. \quad (59)$$

Then there exists a large enough $K > R_1$ such that

$$f(t,x,y) + M \le \varepsilon(|x|+|y|), \quad \text{for any } t \in [0,1],\ |x|+|y| > K. \quad (60)$$

Thus, by (60), if

$$\left|I^{\gamma}[y(s)-w(s)]^{*}\right| + \left|[y(s)-w(s)]^{*}\right| > K, \quad (61)$$

then

$$\begin{aligned} & f\left(t, I^{\gamma}[y(s)-w(s)]^{*}, [y(s)-w(s)]^{*}\right) + M \\ & \le \varepsilon\left(\left|I^{\gamma}[y(s)-w(s)]^{*}\right| + \left|[y(s)-w(s)]^{*}\right|\right) \\ & \le \varepsilon\left(\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} ds + 1\right)\|y\| \\ & \le \frac{\Gamma(\gamma+1)+1}{\Gamma(\gamma+1)}\varepsilon\|y\|, \\ & \text{for any } t \in [0,1],\ |x|+|y| > K. \end{aligned} \quad (62)$$

Now denote that

$$D = \max_{t\in[0,1],|x|+|y|\le K} f(t,x,y), \quad (63)$$

and choose

$$R_2 = \frac{4\lambda(D+M)}{1-4\lambda(\Gamma(\gamma+1)+1)\varepsilon/\Gamma(\gamma+1)} + K. \quad (64)$$

Then $R_2 > K > R_1$.

Next let $B_{R_2} = \{y \in P : \|y\| \le R_2\}$. Then for any $y \in \partial B_{R_1}$ and for any $t \in [0,1]$, we have

$$\begin{aligned} \|Ty\| &= \max_{t\in[0,1]} (Ty)(t) \\ &= \lambda \max_{t\in[0,1]} \int_0^1 G(t,s)\left[f\left(s, I^{\gamma}[y(s)-w(s)]^{*},\right.\right. \\ &\qquad \left.\left.[y(s)-w(s)]^{*}\right) + M\right] ds \\ &\le \lambda(\alpha-\gamma-1)\int_0^1 q(s)\left[f\left(s, I^{\gamma}[y(s)-w(s)]^{*},\right.\right. \\ &\qquad \left.\left.[y(s)-w(s)]^{*}\right) + M\right] ds \\ &\le \lambda(\alpha-\gamma-1) \\ &\quad \times\left(\max_{t\in[0,1],|x|+|y|\le K} f(t,x,y) + M\right)\int_0^1 q(s)\,ds \\ &\quad + \lambda(\alpha-\gamma-1)\int_0^1 q(s)\frac{\Gamma(\gamma+1)+1}{\Gamma(\gamma+1)}\varepsilon\|y\|\,ds \\ &\le 4\lambda(D+M)\int_0^1 q(s)\,ds \\ &\quad + 4\lambda\frac{\Gamma(\gamma+1)+1}{\Gamma(\gamma+1)}\varepsilon\int_0^1 q(s)\,ds R_2 \\ &\le \frac{4\lambda(D+M)}{\Gamma(\alpha-\gamma)} + \frac{4\lambda(\Gamma(\gamma+1)+1)\varepsilon}{\Gamma(\alpha-\gamma)\Gamma(\gamma+1)}R_2 \\ &\le 4\lambda(D+M) + \frac{4\lambda(\Gamma(\gamma+1)+1)\varepsilon}{\Gamma(\gamma+1)}R_2 \\ &\le R_2 = \|u\|, \end{aligned} \quad (65)$$

which implies that

$$\|Ty\| \le \|y\|, \quad y \in \partial B_{R_2}. \quad (66)$$

By Lemma 9, T has at least a fixed points $y \in (P \cap \overline{B_{R_2}}) \setminus B_{R_1}$ such that $R_1 \le \|y\| \le R_2$.

It follows from $R_1 \ge 64\lambda M/(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1})$ that

$$\begin{aligned} & y(t) - w(t) \\ & \ge y(t) - \frac{4\lambda M\sigma(t)}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}} \\ & \ge y(t) - \frac{32\lambda M y(t)}{\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\gamma-1}\right)R_1} \\ & \ge \frac{1}{2}y(t) \ge \frac{1}{16}\sigma(t)R_1 \ge 0. \end{aligned} \quad (67)$$

By Lemma 7 and (67), the boundary value problem (1) has at least one positive solution. The proof of Theorem 11 is completed. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors were supported financially by "Chen Guang" Project supported by Shanghai Municipal Education Development Foundation (10CGB25) and Shanghai Universities for Outstanding Young Teachers Scientific Research Selection and Training Special Fund (sjq08011).

References

[1] H. Feng and D. Bai, "Existence of positive solutions for semipositone multi-point boundary value problems," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2287–2292, 2011.

[2] C. Yuan, "Multiple positive solutions for $(n-1,1)$-type semipositone conjugate boundary value problems of nonlinear fractional differential equations," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 36, pp. 1–12, 2010.

[3] X. Zhang, L. Liu, and Y. Wu, "Existence results for multiple positive solutions of nonlinear higher order perturbed fractional differential equations with derivatives," *Applied Mathematics and Computation*, vol. 219, no. 4, pp. 1420–1433, 2012.

[4] C. S. Goodrich, "Positive solutions to boundary value problems with nonlinear boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 1, pp. 417–432, 2012.

[5] C. S. Goodrich, "Existence of a positive solution to systems of differential equations of fractional order," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1251–1268, 2011.

[6] X. Zhang, L. Liu, and Y. Wu, "The eigenvalue problem for a singular higher order fractional differential equation involving fractional derivatives," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8526–8536, 2012.

[7] C. Yang and J. Yan, "Positive solutions for third-order Sturm-Liouville boundary value problems with p-Laplacian," *Computers and Mathematics with Applications*, vol. 59, no. 6, pp. 2059–2066, 2010.

[8] J. Wang, H. Xiang, and Z. Liu, "Positive solutions for three-point boundary value problems of nonlinear fractional differential equations with p-Laplacian," *Far East Journal of Applied Mathematics*, vol. 37, no. 1, pp. 33–47, 2009.

[9] J. Wang and H. Xiang, "Upper and lower solutions method for a class of singular fractional boundary value problems with p-Laplacian operator," *Abstract and Applied Analysis*, vol. 2010, Article ID 971824, 12 pages, 2010.

[10] G. Chai, "Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator," *Boundary Value Problems*, vol. 2012, article 18, 2012.

[11] X. Zhang, L. Liu, and Y. Wu, "The uniqueness of positive solution for a singular fractional differential system involving derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 6, pp. 1400–1409, 2013.

[12] J. R. L. Webb, "Nonlocal conjugate type boundary value problems of higher order," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1933–1940, 2009.

[13] X. Zhang and Y. Han, "Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 555–560, 2012.

[14] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.

[15] X. Zhang, L. Liu, and Y. Wu, "Multiple positive solutions of a singular fractional differential equation with negatively perturbed term," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1263–1274, 2012.

[16] J. J. Nieto and J. Pimentel, "Positive solutions of a fractional thermostat model," *Boundary Value Problems*, vol. 2013, article 5, 2013.

[17] Y. Li and S. Lin, "Positive solution for the nonlinear Hadamard type fractional differential equation with p-Laplacian," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 951643, 10 pages, 2013.

[18] X. Zhang, L. Liu, Y. Wu, and Y. Lu, "The iterative solutions of nonlinear fractional differential equations," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4680–4691, 2013.

[19] A. A. M. Arafa, S. Z. Rida, and M. Khalil, "Fractional modeling dynamics of HIV and $CD4^{+}$ T-cells during primary infection," *Nonlinear Biomedical Physics*, vol. 6, no. 1, article 1, 2012.

[20] X. Zhang, L. Liu, B. Wiwatanapataphee, and Y. Wu, "Positive solutions of eigenvalue problems for a class of fractional differential equations with derivatives," *Abstract and Applied Analysis*, vol. 2012, Article ID 512127, 16 pages, 2012.

[21] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[22] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[23] J. Sabatier, O. P. Agrawal, and J. A. T. Machado, Eds., *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.

[24] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.

[25] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.