

Research Article

Iterative Scheme with Errors for Common Zeros of Finite Accretive Mappings and Nonlinear Elliptic Systems

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We present a new iterative scheme with errors to solve the problems of finding common zeros of finite m -accretive mappings in a real Banach space. Strong convergence theorems are established, which extend the corresponding works given by some authors. Moreover, the relationship between zeros of m -accretive mappings and one kind of nonlinear elliptic systems is investigated, from which we can see that some restrictions imposed on the iterative scheme are valid and the solution of one kind of nonlinear elliptic systems can be approximated by a suitably defined iterative sequence.

1. Introduction and Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E . We use “ \rightarrow ” and “ \dashrightarrow ” to denote strong and weak convergence, respectively. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f \rangle$.

Let C be a nonempty, closed, and convex subset of E and let Q be a mapping of E onto C . Then Q is said to be sunny [1] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$.

A mapping Q of E into E is said to be a retraction [1] if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q .

A mapping $T : C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We use $F(T)$ to denote the fixed point set of T ; that is, $F(T) := \{x \in C : Tx = x\}$. A mapping $T : E \supset D(T) \rightarrow R(T) \subset E$ is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \dashrightarrow x \in D(T)$ and $Tx_n \rightarrow p$, it follows that $Tx = p$.

A subset C of E is said to be a sunny nonexpansive retract of E [2] if there exists a sunny nonexpansive retraction of E onto C and it is called a nonexpansive retract of E if there exists a nonexpansive retraction of E onto C . If E is reduced to a Hilbert space H , then the metric projection P_C is a sunny nonexpansive retraction from H to any closed and convex subset C of H . But this is not true in a general Banach space. We note that if E is smooth and Q is a retraction of C onto

$F(T)$, then Q is sunny and nonexpansive if and only if for all $x \in C, z \in F(T), \langle Qx - x, J(Qx - z) \rangle \leq 0$ [3].

We use J to denote the normalized duality mapping from E to 2^{E^*} which is defined by

$$Jx := \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E. \quad (1)$$

It is well known that J is single-valued if E^* is strictly convex. Moreover, $J(cx) = cJx$, for all $x \in E$ and $c \in \mathbb{R}^1$. We call that J is weakly sequentially continuous if $\{x_n\}$ is a sequence in E which converges weakly to x it follows that $\{Jx_n\}$ converges in weak^* to Jx .

A mapping $A : E \supset D(A) \rightarrow R(A) \subset E$ is called accretive if $\langle Ax - Ay, J(x - y) \rangle \geq 0$, for all $x, y \in D(A)$ and it is called m -accretive if $R(I + \lambda A) = E$, for all $\lambda > 0$. Let $A^{-1}0$ denote the set of zeros of A ; that is, $A^{-1}0 := \{x \in D(A) : Ax = 0\}$. We denote by J_r^A (for $r > 0$) the resolvent of A ; that is, $J_r^A := (I + rA)^{-1}$. Then J_r^A is nonexpansive and $F(J_r^A) = A^{-1}0$.

Interest in accretive mappings, which is an important class of nonlinear operators, stems mainly from their firm connection with equations of evolution. It is well known that many physically significant problems can be modelled by initial value problems of the form

$$x'(t) + Ax(t) = 0, \quad x(0) = x_0, \quad (2)$$

where A is an accretive mapping. Typical examples where such evolution equations occur can be found in the heat, wave, or Schrodinger equations. If $x(t)$ is dependent on t , then (2) is reduced to

$$Au = 0, \quad (3)$$

whose solutions correspond to the equilibrium of the system (2). Consequently, within the past 40 years or so, considerable research efforts have been devoted to methods for finding approximate solutions of (3). An early fundamental result of accretive operators, due to Browder [4]. One classical method for studying the problem $0 \in Ax$, where A is an m -accretive mapping, is the following so-called proximal method (c.f. [5]):

$$x_0 \in H, \quad x_{n+1} \approx J_{r_n}^A x_n, \quad n \geq 0, \quad (4)$$

where $J_{r_n}^A := (I + r_n A)^{-1}$. It was shown that the sequence generated by (4) converges weakly or strongly to a zero point of A under some conditions.

Recall that the following normal Mann iterative scheme to approximate the fixed point of a nonexpansive mapping $T : C \rightarrow C$ was introduced by Mann [6]:

$$x_0 \in C, \quad x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \quad n \geq 0. \quad (5)$$

It was proved that, under some conditions, the sequence $\{x_n\}$ produced by (5) converges weakly to a point in $F(T)$.

Later, many mathematicians try to combine the ideas of proximal method and Mann iterative method to approximate the zeros of m -accretive mappings; see, for example, [7–14] and the references therein.

In particular, in 2007, Qin and Su [7] presented the following iterative scheme:

$$\begin{aligned} x_1 &\in C, \\ y_n &= \beta_n x_n + (1 - \beta_n) J_{r_n}^A x_n, \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) y_n. \end{aligned} \quad (6)$$

And they showed that $\{x_n\}$ generated by the above scheme converges strongly to a zero of A .

Motivated by iterative schemes (4) and (5), Zegeye and Shahzad extended their discussion to the case of finite m -accretive mappings. They presented in [15] the following iterative scheme:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n) S_r x_n, \quad n \geq 0, \quad (7)$$

where $S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \dots + a_l J_{A_l}$ with $J_{A_i} = (I + A_i)^{-1}$ and $\sum_{i=0}^l a_i = 1$. If $\bigcap_{i=1}^l A_i^{-1}(0) \neq \emptyset$, they proved that $\{x_n\}$ generated by (7) converges strongly to the common zeros of A_i ($i = 1, 2, \dots, l$) under some conditions.

The work in [15] was then extended to the following one presented by Hu and Liu in [16]:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \vartheta_n S_{r_n} x_n, \quad n \geq 0, \quad (8)$$

where $S_{r_n} = a_0 I + a_1 J_{r_n}^{A_1} + a_2 J_{r_n}^{A_2} + \dots + a_l J_{r_n}^{A_l}$ with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$ and $\sum_{i=0}^l a_i = 1$. $\{\alpha_n\}, \{\beta_n\}, \{\vartheta_n\} \subset (0, 1)$, and $\alpha_n + \beta_n + \vartheta_n = 1$. If $\bigcap_{i=1}^l A_i^{-1}(0) \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to the common zeros of A_i ($i = 1, 2, \dots, l$) under some conditions.

In 2009, Yao et al. presented the following iterative scheme in the frame of Hilbert space in [17]:

$$\begin{aligned} x_1 &\in C, \\ y_n &= P_C [(1 - \alpha_n) x_n], \\ x_{n+1} &= (1 - \beta_n) x_n + \beta_n T y_n, \quad n \geq 1, \end{aligned} \quad (9)$$

where $T : C \rightarrow C$ is a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0, 1)$ satisfying

- (a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Then $\{x_n\}$ constructed by (9) converges strongly to a fixed point of T .

Motivated by the work in [15, 17], Shehu and Ezeora presented the following result in [2].

Theorem 1. *Let E be a real uniformly smooth and uniformly convex Banach space, and let C be a nonempty, closed, and convex sunny nonexpansive retract of E , where Q_C is the sunny nonexpansive retraction of E onto C . Suppose the duality mapping $J : E \rightarrow E^*$ is weakly sequentially continuous. For each $i = 1, 2, \dots, N$, let $A_i : C \rightarrow E$ be an m -accretive mapping such that $\bigcap_{i=1}^N A_i^{-1} 0 \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfy (a) and (b). Let $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} x_1 &\in C, \\ y_n &= Q_C [(1 - \alpha_n) x_n], \\ x_{n+1} &= (1 - \beta_n) x_n + \beta_n S_N y_n, \quad n \geq 1, \end{aligned} \quad (10)$$

where $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \dots + a_N J_{A_N}$ with $J_{A_i} = (I + A_i)^{-1}$, for $i = 1, 2, \dots, N$, $0 < a_k < 1$, for $k = 0, 1, 2, \dots, N$, and $\sum_{k=0}^N a_k = 1$. Then $\{x_n\}$ converges strongly to the common zero of A_i , where $i = 1, 2, \dots, N$.

Inspired by the work in [2], we present the following iterative scheme with errors:

$$\begin{aligned} x_1 &\in C, \\ u_n &= Q_C [(1 - \alpha_n) (x_n + e_n)], \\ v_n &= (1 - \beta_n) x_n + \beta_n S_n u_n, \\ x_{n+1} &= \gamma_n x_n + (1 - \gamma_n) S_n v_n, \quad n \geq 1, \end{aligned} \quad (A)$$

where $\{e_n\} \subset E$ is the error sequence and $\{A_i\}_{i=1}^N$ is a finite family of m -accretive mappings. $S_n := a_0 I + a_1 J_{r_{n,1}}^{A_1} + a_2 J_{r_{n,2}}^{A_2} + \dots + a_N J_{r_{n,N}}^{A_N}$, $J_{r_{n,i}}^{A_i} = (I + r_{n,i} A_i)^{-1}$, for $i = 1, 2, \dots, N$; $\sum_{k=0}^N a_k =$

$1, 0 < a_k < 1$, for $k = 0, 1, 2, \dots, N$. More details of iterative scheme (A) will be presented in Section 2. And, some strong convergence theorems are obtained.

Note that there are some differences between our work and Shehu and Ezeora's in [2] in the following aspects.

- (i) S_n in iterative scheme (A) is different from S_N in (10) since the former is changing with n and the latter is not, which results in A_i in scheme (A) having different coefficient r_{ni} for each different i .
- (ii) The idea of three-step iteration is employed in our paper.
- (iii) The error sequence $\{e_n\}$ is considered in the iterative scheme (A).
- (iv) Recall that, in [2], Lemma 8 is a key tool to prove the convergence of $\{x_n\}$ generated by (10). In particular, to obtain the main result, they imposed an additional condition on the function β in Lemma 8 that $\beta(t) \leq t / \max\{1, 2r_1\}$, where $r_1 > 0$ is a constant satisfying some conditions. One question arises: how to show the convergence of the iterative sequence $\{x_n\}$ if β does not satisfy this additional condition? To answer the question, we will use Lemma 4 instead of Lemma 8.

In Section 3, we will discuss the relationship between zero point of finite m -accretive mappings and the solution of one kind of nonlinear elliptic systems involving (p_1, p_2, \dots, p_N) -Laplacian operators. The discussion helps us not only to see that the topic of constructing iterative schemes to approximate zeros of m -accretive mappings is meaningful but also to see that the solution of (p_1, p_2, \dots, p_N) -Laplacian elliptic systems can be obtained by an iterative scheme.

Next, we list some results we need in sequel.

Lemma 2 (see [18]). *Let E be a real uniformly convex Banach space, let C be a nonempty, closed, and convex subset of E , and $T : C \rightarrow C$ is a nonexpansive mapping such that $F(T) \neq \emptyset$; then, $I - T$ is demiclosed at zero.*

Lemma 3 (see [16]). *Let E be a strictly convex Banach space which has a uniformly Gâteaux differential norm, and let C be a nonempty, closed, and convex subset of E . Let $\{A_i\}_{i=1}^N$ be a finite family of accretive mappings with $\bigcap_{i=1}^N A_i^{-1}0 \neq \emptyset$, satisfying the following range conditions:*

$$\overline{D(A_i)} \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \dots, N. \tag{11}$$

Let a_0, a_1, \dots, a_N be real numbers in $(0, 1)$ such that $\sum_{i=0}^N a_i = 1$ and $S_n = a_0I + a_1J_{r_n}^{A_1} + a_2J_{r_n}^{A_2} + \dots + a_NJ_{r_n}^{A_N}$, where $J_{r_n}^{A_i} = (I + r_nA_i)^{-1}$ and $r_n > 0$; then, S_n is nonexpansive and $F(S_n) = \bigcap_{i=1}^N A_i^{-1}0$.

Lemma 4 (see [13]). *In a real Banach space E , the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \quad \forall x, y \in E, \tag{12}$$

where $j(x + y) \in J(x + y)$.

Lemma 5 (see [19]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - c_n)a_n + b_nc_n, \quad \forall n \geq 1, \tag{13}$$

where $\{c_n\} \subset (0, 1)$ such that (i) $c_n \rightarrow 0$ and $\sum_{n=1}^{\infty} c_n = +\infty$ and (ii) either $\limsup_{n \rightarrow \infty} b_n \leq 0$ or $\sum_{n=1}^{\infty} |b_nc_n| < +\infty$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 6 (see [20]). *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space E such that $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$ for $n \geq 1$. Suppose $\{\beta_n\} \subset (0, 1)$ satisfying $0 < \liminf_{n \rightarrow +\infty} \beta_n \leq \limsup_{n \rightarrow +\infty} \beta_n < 1$. If $\limsup_{n \rightarrow +\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n \rightarrow +\infty} \|y_n - x_n\| = 0$.*

Lemma 7 (see [21]). *Let E be a Banach space and let A be an m -accretive mapping. For $\lambda > 0$, $\mu > 0$, and $x \in E$, one has*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x \right), \tag{14}$$

where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.

Lemma 8 (see [22]). *Let E be a real uniformly smooth Banach space; then there exists a nondecreasing continuous function $\beta : [0, +\infty) \rightarrow [0, +\infty)$ with $\lim_{t \rightarrow 0^+} \beta(t) = 0$ and $\beta(ct) \leq c\beta(t)$ for $c \geq 1$, such that, for all $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, Jx \rangle + \max\{\|x\|, 1\} \|y\| \beta(\|y\|). \tag{15}$$

2. Strong Convergence Theorem

Lemma 9 (see [2]). *Let E be a real uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny nonexpansive retract of E , and let Q_C be the sunny nonexpansive retraction of E onto C . Let $T : C \rightarrow C$ be nonexpansive with $F(T) \neq \emptyset$. Suppose that the duality mapping $J : E \rightarrow E^*$ is weakly sequentially continuous. If for each $t \in (0, 1)$ we define $T_t : C \rightarrow C$ by*

$$T_t x := TQ_C[(1 - t)x], \tag{16}$$

then T_t has a fixed point z_t , which is convergent strongly to the fixed point of T , as $t \rightarrow 0$. That is, $\lim_{t \rightarrow 0} z_t = z_0 \in F(T)$.

Lemma 10. *Let E be a strictly convex Banach space and let C be a nonempty, closed, and convex subset of E . Let $A_i : C \rightarrow E$ ($i = 1, 2, \dots, N$) be a finite family of m -accretive mappings such that $\bigcap_{i=1}^N A_i^{-1}0 \neq \emptyset$.*

Let a_0, a_1, \dots, a_N be real numbers in $(0, 1)$ such that $\sum_{i=0}^N a_i = 1$ and $S_n = a_0I + a_1J_{r_{n1}}^{A_1} + a_2J_{r_{n2}}^{A_2} + \dots + a_NJ_{r_{nN}}^{A_N}$, where $J_{r_{ni}}^{A_i} = (I + r_{ni}A_i)^{-1}$ and $r_{ni} > 0$, for $i = 1, 2, \dots, N$, and $n \geq 1$; then, $S_n : C \rightarrow C$ is nonexpansive and $F(S_n) = \bigcap_{i=1}^N A_i^{-1}0$, for $n \geq 1$.

Proof. The main idea of the proof is essentially from that in [15] or Lemma 3. For the sake of completeness, we present the proof in the following.

It is easy to check that $S_n : C \rightarrow C$ is nonexpansive and $\bigcap_{i=1}^N A_i^{-1}0 \subset F(S_n)$.

On the other hand, for all $p \in F(S_n)$, then $p = S_n p = a_0 p + a_1 J_{r_{n,1}}^{A_1} p + a_2 J_{r_{n,2}}^{A_2} p + \dots + a_N J_{r_{n,N}}^{A_N} p$.

For all $q \in \bigcap_{i=1}^N A_i^{-1}0 \subset F(S_n)$, then

$$\begin{aligned} \|p - q\| &\leq a_0 \|p - q\| + a_1 \|J_{r_{n,1}}^{A_1} p - q\| + \dots + a_N \|J_{r_{n,N}}^{A_N} p - q\| \\ &\leq (a_0 + a_1 + \dots + a_{N-1}) \|p - q\| + a_N \|J_{r_{n,N}}^{A_N} p - q\| \\ &= (1 - a_N) \|p - q\| + a_N \|J_{r_{n,N}}^{A_N} p - q\| \leq \|p - q\|. \end{aligned} \tag{17}$$

Therefore, $\|p - q\| = (1 - a_N) \|p - q\| + a_N \|J_{r_{n,N}}^{A_N} p - q\|$, which implies that $\|p - q\| = \|J_{r_{n,N}}^{A_N} p - q\|$. Similarly, $\|p - q\| = \|J_{r_{n,1}}^{A_1} p - q\| = \dots = \|J_{r_{n,N}}^{A_N} p - q\|$.

Then $\|p - q\| = \|(a_1 / \sum_{i=1}^N a_i)(J_{r_{n,1}}^{A_1} p - q) + (a_2 / \sum_{i=1}^N a_i)(J_{r_{n,2}}^{A_2} p - q) + \dots + (a_N / \sum_{i=1}^N a_i)(J_{r_{n,N}}^{A_N} p - q)\|$, which implies from the strictly convexity of E that $p - q = J_{r_{n,1}}^{A_1} p - q = J_{r_{n,2}}^{A_2} p - q = \dots = J_{r_{n,N}}^{A_N} p - q$.

Therefore, $J_{r_{n,i}}^{A_i} p = p$, for $i = 1, 2, \dots, N$. And then $p \in \bigcap_{i=1}^N A_i^{-1}0$, which completes the proof. \square

Lemma 11. Let E, C , and S_n be the same as those in Lemma 10. Suppose $\bigcap_{i=1}^N A_i^{-1}0 \neq \emptyset$. Then $S_n^2 : C \rightarrow C$ is nonexpansive and $F(S_n^2) = \bigcap_{i=1}^N A_i^{-1}0$.

Proof. From Lemma 10, we have $F(S_n) = \bigcap_{i=1}^N A_i^{-1}0$. It is easy to check that $S_n^2 : C \rightarrow C$ is nonexpansive. So, it suffices to show that $F(S_n^2) \subset F(S_n)$ since $F(S_n) \subset F(S_n^2)$ is trivial.

For all $p \in F(S_n^2)$, then $p = S_n^2 p$.

For all $q \in F(S_n) \subset F(S_n^2)$, then $q = S_n^2 q$. Now,

$$\begin{aligned} \|p - q\| &\leq \|S_n p - S_n q\| = \|S_n p - q\| \\ &\leq a_0 \|p - q\| + a_1 \|J_{r_{n,1}}^{A_1} p - q\| + \dots + a_N \|J_{r_{n,N}}^{A_N} p - q\| \\ &= (1 - a_N) \|p - q\| + a_N \|J_{r_{n,N}}^{A_N} p - q\| \leq \|p - q\|. \end{aligned} \tag{18}$$

Therefore, $\|p - q\| = (1 - a_N) \|p - q\| + a_N \|J_{r_{n,N}}^{A_N} p - q\|$, which implies that $\|p - q\| = \|J_{r_{n,N}}^{A_N} p - q\|$. Similarly, $\|p - q\| = \|J_{r_{n,1}}^{A_1} p - q\| = \dots = \|J_{r_{n,N}}^{A_N} p - q\|$.

Then repeating the discussion in Lemma 10, we know that $J_{r_{n,i}}^{A_i} p = p$, for $i = 1, 2, \dots, N$. And then $p \in F(S_n)$, which completes the proof. \square

Theorem 12. Let E be a real uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny nonexpansive retract of E , where Q_C is the sunny nonexpansive retraction of E onto C . Let $A_i : C \rightarrow E$ be m -accretive mappings, where $i = 1, 2, \dots, N$. Suppose that the duality mapping $J : E \rightarrow E^*$ is weakly sequentially continuous

and $D := \bigcap_{i=1}^N A_i^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be generated by the iterative scheme (A), where $S_n := a_0 I + a_1 J_{r_{n,1}}^{A_1} + a_2 J_{r_{n,2}}^{A_2} + \dots + a_N J_{r_{n,N}}^{A_N}$, and $J_{r_{n,i}}^{A_i} = (I + r_{n,i} A_i)^{-1}$, for $i = 1, 2, \dots, N$, $0 < a_k < 1$, for $k = 0, 1, 2, \dots, N$, $\sum_{k=0}^N a_k = 1$. Suppose that $\{e_n\} \subset E$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0, 1)$ and $\{r_{n,i}\} \subset (0, +\infty)$ satisfying the following conditions:

- (i) $\alpha_n \rightarrow 0, \beta_n \rightarrow 0$, as $n \rightarrow \infty$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = +\infty$;
- (iii) $0 < \liminf_{n \rightarrow +\infty} \gamma_n \leq \limsup_{n \rightarrow +\infty} \gamma_n < 1$;
- (iv) $\sum_{n=1}^{\infty} |r_{n+1,i} - r_{n,i}| < +\infty$ and $r_{n,i} \geq \varepsilon > 0$, for $n \geq 1$ and $i = 1, 2, \dots, N$;
- (v) $\|e_n\| / \alpha_n \rightarrow 0$, as $n \rightarrow +\infty$, and $\sum_{n=1}^{\infty} \|e_n\| < +\infty$.

Then $\{x_n\}$ converges strongly to a point $p_0 \in D$.

Proof. We will split the proof into five steps.

Step 1. $\{x_n\}, \{u_n\}, \{S_n u_n\}, \{v_n\}$, and $\{S_n x_n\}$ are all bounded.

We will first show that

$$\forall p \in D, \quad \|x_{n+1} - p\| \leq M_1 + \sum_{i=1}^n \|e_i\|, \tag{19}$$

where $M_1 = \max\{\|x_1 - p\|, \|p\|\}$.

By using the induction method, we see that, for $n = 1$, $\forall p \in D$,

$$\begin{aligned} \|x_2 - p\| &\leq \gamma_1 \|x_1 - p\| + (1 - \gamma_1) \|S_1 v_1 - p\| \\ &\leq \gamma_1 \|x_1 - p\| + (1 - \gamma_1) \|v_1 - p\| \\ &\leq \gamma_1 \|x_1 - p\| + (1 - \gamma_1) (1 - \beta_1) \|x_1 - p\| \\ &\quad + \beta_1 (1 - \gamma_1) \|u_1 - p\| \\ &\leq \gamma_1 \|x_1 - p\| + (1 - \gamma_1) (1 - \beta_1) \|x_1 - p\| \\ &\quad + \beta_1 (1 - \gamma_1) \|(1 - \alpha_1)(x_1 + e_1) - p\| \\ &\leq [1 - \alpha_1 \beta_1 (1 - \gamma_1)] \|x_1 - p\| + \alpha_1 \beta_1 (1 - \gamma_1) \|p\| \\ &\quad + (1 - \alpha_1) \beta_1 (1 - \gamma_1) \|e_1\| \leq M_1 + \|e_1\|. \end{aligned} \tag{20}$$

Suppose that (19) is true for $n = k$. Then, for $n = k + 1$,

$$\begin{aligned} \|x_{k+2} - p\| &\leq \gamma_{k+1} \|x_{k+1} - p\| + (1 - \gamma_{k+1}) \|v_{k+1} - p\| \\ &\leq \gamma_{k+1} \|x_{k+1} - p\| + (1 - \gamma_{k+1}) \\ &\quad \times [(1 - \beta_{k+1}) \|x_{k+1} - p\| + \beta_{k+1} \|u_{k+1} - p\|] \end{aligned}$$

$$\begin{aligned}
 &\leq \gamma_{k+1} \|x_{k+1} - p\| + (1 - \gamma_{k+1}) \\
 &\quad \times [(1 - \beta_{k+1}) \|x_{k+1} - p\| + \beta_{k+1} \\
 &\quad \quad \times \|(1 - \alpha_{k+1})(x_{k+1} + e_{k+1}) - p\|] \\
 &\leq [1 - \alpha_{k+1}\beta_{k+1}(1 - \gamma_{k+1})] \|x_{k+1} - p\| \\
 &\quad + \alpha_{k+1}\beta_{k+1}(1 - \gamma_{k+1}) \|p\| \\
 &\quad + \beta_{k+1}(1 - \alpha_{k+1})(1 - \gamma_{k+1}) \|e_{k+1}\| \\
 &\leq M_1 + [1 - \alpha_{k+1}\beta_{k+1}(1 - \gamma_{k+1})] \\
 &\quad \times \sum_{i=1}^k \|e_i\| + (1 - \alpha_{k+1})\beta_{k+1}(1 - \gamma_{k+1}) \|e_{k+1}\| \\
 &\leq M_1 + \sum_{i=1}^{k+1} \|e_i\|. \tag{21}
 \end{aligned}$$

Thus (19) is true for all $n \in N$. Since $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, then (19) ensures that $\{x_n\}$ is bounded.

For all $p \in D$, from $\|u_n - p\| \leq \|(1 - \alpha_n)(x_n + e_n) - p\| \leq \|x_n\| + \|e_n\| + \|p\|$, we see that $\{u_n\}$ is bounded.

Since $\|S_n u_n\| \leq \|S_n u_n - S_n p\| + \|p\| \leq \|u_n - p\| + \|p\|$, then $\{S_n u_n\}$ is bounded. Since both $\{S_n u_n\}$ and $\{x_n\}$ are bounded, then $\{v_n\}$ is bounded. Similarly, $\{S_n x_n\}$, $\{J_{r_{n,i}}^{A_i} u_n\}$, and $\{J_{r_{n,i}}^{A_i} v_n\}$ are all bounded, for $i = 1, 2, \dots, N$.

Then we set $M_2 = \sup\{\|u_n\|, \|J_{r_{n,i}}^{A_i} u_n\|, \|v_n\|, \|J_{r_{n,i}}^{A_i} v_n\| : n \geq 1, i = 1, 2, \dots, N\}$.

Step 2. $\lim_{n \rightarrow \infty} \|x_n - S_n v_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.
In fact,

$$\begin{aligned}
 &\|S_{n+1} v_{n+1} - S_n v_n\| \\
 &\leq a_0 \|v_{n+1} - v_n\| + \sum_{i=1}^N a_i \|J_{r_{n+1,i}}^{A_i} v_{n+1} - J_{r_{n,i}}^{A_i} v_n\|. \tag{22}
 \end{aligned}$$

Next, we discuss $\|J_{r_{n+1,i}}^{A_i} v_{n+1} - J_{r_{n,i}}^{A_i} v_n\|$.

If $r_{n,i} \leq r_{n+1,i}$, then, by using Lemma 7, we have

$$\begin{aligned}
 &\|J_{r_{n+1,i}}^{A_i} v_{n+1} - J_{r_{n,i}}^{A_i} v_n\| \\
 &= \left\| J_{r_{n,i}}^{A_i} \left(\frac{r_{n,i}}{r_{n+1,i}} v_{n+1} + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right) J_{r_{n+1,i}}^{A_i} v_{n+1} \right) - J_{r_{n,i}}^{A_i} v_n \right\| \\
 &\leq \left\| \frac{r_{n,i}}{r_{n+1,i}} v_{n+1} + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right) J_{r_{n+1,i}}^{A_i} v_{n+1} - v_n \right\| \\
 &\leq \frac{r_{n,i}}{r_{n+1,i}} \|v_{n+1} - v_n\| + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right) \|J_{r_{n+1,i}}^{A_i} v_{n+1} - v_n\| \\
 &\leq \|v_{n+1} - v_n\| + \frac{r_{n+1,i} - r_{n,i}}{\varepsilon} \|J_{r_{n+1,i}}^{A_i} v_{n+1} - v_n\|. \tag{23}
 \end{aligned}$$

If $r_{n+1,i} \leq r_{n,i}$, then imitating the proof of (23), we have

$$\begin{aligned}
 &\|J_{r_{n+1,i}}^{A_i} v_{n+1} - J_{r_{n,i}}^{A_i} v_n\| \\
 &\leq \|v_{n+1} - v_n\| + \frac{r_{n,i} - r_{n+1,i}}{\varepsilon} \|J_{r_{n+1,i}}^{A_i} v_{n+1} - v_n\|. \tag{24}
 \end{aligned}$$

Combining (23) and (24), we have

$$\begin{aligned}
 &\|J_{r_{n+1,i}}^{A_i} v_{n+1} - J_{r_{n,i}}^{A_i} v_n\| \\
 &\leq \|v_{n+1} - v_n\| + \frac{|r_{n,i} - r_{n+1,i}|}{\varepsilon} \|J_{r_{n+1,i}}^{A_i} v_{n+1} - v_n\| \\
 &\leq \|v_{n+1} - v_n\| + \frac{2|r_{n,i} - r_{n+1,i}|}{\varepsilon} M_2. \tag{25}
 \end{aligned}$$

Putting (25) into (22), we have

$$\|S_{n+1} v_{n+1} - S_n v_n\| \leq \|v_{n+1} - v_n\| + \frac{2M_2}{\varepsilon} \sum_{i=1}^N |r_{n,i} - r_{n+1,i}|. \tag{26}$$

Similarly, we have

$$\|S_{n+1} u_{n+1} - S_n u_n\| \leq \|u_{n+1} - u_n\| + \frac{2M_2}{\varepsilon} \sum_{i=1}^N |r_{n,i} - r_{n+1,i}|. \tag{27}$$

Therefore, from (26) and (27), we have

$$\begin{aligned}
 &\|S_{n+1} v_{n+1} - S_n v_n\| \\
 &\leq \|v_{n+1} - v_n\| + \frac{2M_2}{\varepsilon} \sum_{i=1}^N |r_{n,i} - r_{n+1,i}| \\
 &\leq \|x_{n+1} - x_n\| + \beta_n \|x_n\| + \beta_{n+1} \|x_{n+1}\| \\
 &\quad + |\beta_{n+1} - \beta_n| \|S_{n+1} u_{n+1}\| + \beta_n \|S_{n+1} u_{n+1} - S_n u_n\| \\
 &\quad + \frac{2M_2}{\varepsilon} \sum_{i=1}^N |r_{n,i} - r_{n+1,i}| \\
 &\leq \|x_{n+1} - x_n\| + \beta_n \|x_n\| + \beta_{n+1} \|x_{n+1}\| \\
 &\quad + |\beta_{n+1} - \beta_n| \|S_{n+1} u_{n+1}\| + \beta_n \|u_{n+1} - u_n\| \\
 &\quad + \frac{4M_2}{\varepsilon} \sum_{i=1}^N |r_{n,i} - r_{n+1,i}| \\
 &\leq \|x_{n+1} - x_n\| + \beta_n \|x_n\| + \beta_{n+1} \|x_{n+1}\| \\
 &\quad + |\beta_{n+1} - \beta_n| \|S_{n+1} u_{n+1}\| \\
 &\quad + \beta_n \|(1 - \alpha_{n+1})(x_{n+1} + e_{n+1}) - (1 - \alpha_n)(x_n + e_n)\| \\
 &\quad + \frac{4M_2}{\varepsilon} \sum_{i=1}^N |r_{n,i} - r_{n+1,i}|
 \end{aligned}$$

$$\begin{aligned}
&\leq (1 + \beta_n) \|x_{n+1} - x_n\| + (\beta_n + \alpha_n \beta_n) \|x_n\| \\
&\quad + (\beta_{n+1} + \alpha_{n+1} \beta_n) \|x_{n+1}\| + |\beta_{n+1} - \beta_n| \|S_{n+1} u_{n+1}\| \\
&\quad + \beta_n \|e_{n+1} - e_n\| + \beta_n \|\alpha_{n+1} e_{n+1} - \alpha_n e_n\| \\
&\quad + \frac{4M_2}{\varepsilon} \sum_{i=1}^N |r_{n,i} - r_{n+1,i}|.
\end{aligned} \tag{28}$$

From Step 1, we know that $\limsup_{n \rightarrow +\infty} (\|S_{n+1} v_{n+1} - S_n v_n\| - \|x_{n+1} - x_n\|) \leq 0$. Using Lemma 6, we have from (28) that $\lim_{n \rightarrow \infty} \|x_n - S_n v_n\| = 0$ and then $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \gamma_n) \|S_n v_n - x_n\| = 0$.

Step 3. $\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - S_n^2 x_n\| = 0$.

In fact,

$$\begin{aligned}
\|x_n - S_n x_n\| &\leq \|x_n - S_n v_n\| + \|S_n v_n - S_n x_n\| \\
&\leq \|x_n - S_n v_n\| + \|v_n - x_n\| \\
&\leq \|x_n - S_n v_n\| + \beta_n \|x_n - S_n u_n\|.
\end{aligned} \tag{29}$$

Noticing the results of Steps 1 and 2 and $\beta_n \rightarrow 0$, we have $\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0$.

Since $\|x_n - S_n^2 x_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - S_n^2 x_n\| \leq 2\|x_n - S_n x_n\|$, then $\|x_n - S_n^2 x_n\| \rightarrow 0$, as $n \rightarrow \infty$.

Step 4. $\limsup_{n \rightarrow +\infty} \langle p_0, J(p_0 - x_n) \rangle \leq 0$, where p_0 is an element in D .

From Lemma 11, we know that $S_n^2 : C \rightarrow C$ is nonexpansive and $F(S_n^2) = D$. Then Lemma 9 implies that there exists $z_t \in C$ such that $z_t = S_n^2 Q_C[(1-t)z_t]$ for $t \in (0, 1)$. Moreover, $z_t \rightarrow p_0 \in D$, as $t \rightarrow 0$.

Since $\|z_t - p_0\| \leq \|(1-t)z_t - p_0\| \leq (1-t)\|z_t - p_0\| + \|p_0\|$, then $\{z_t\}$ is bounded. Let $M_3 = \sup\{\|z_t - x_n\| : n \geq 1, t > 0\}$. Then from Step 1, we know that M_3 is a positive constant. Using Lemma 4, we have

$$\begin{aligned}
&\|z_t - x_n\|^2 \\
&= \|z_t - S_n^2 x_n + S_n^2 x_n - x_n\|^2 \\
&\leq \|z_t - S_n^2 x_n\|^2 + 2 \langle S_n^2 x_n - x_n, J(z_t - x_n) \rangle \\
&\leq \|z_t - S_n^2 x_n\|^2 + 2 \|S_n^2 x_n - x_n\| \|z_t - x_n\| \\
&\leq \|(1-t)z_t - x_n\|^2 + 2 \|S_n^2 x_n - x_n\| \|z_t - x_n\| \\
&\leq \|z_t - x_n\|^2 - 2t \langle z_t, J[(1-t)z_t - x_n] \rangle \\
&\quad + 2M_3 \|S_n^2 x_n - x_n\|.
\end{aligned} \tag{30}$$

So $\langle z_t, J[(1-t)z_t - x_n] \rangle \leq (M_3/t) \|S_n^2 x_n - x_n\|$, which implies that $\lim_{t \rightarrow 0} \limsup_{n \rightarrow +\infty} \langle z_t, J[(1-t)z_t - x_n] \rangle \leq 0$ in view of Step 3.

Since $\{x_n\}$ is bounded and J is uniformly continuous on each bounded subset of E , then $\langle p_0, J(p_0 - x_n) - J[(1-t)z_t - x_n] \rangle \rightarrow 0$, as $t \rightarrow 0$.

Moreover, noticing the fact that

$$\begin{aligned}
&\langle p_0, J(p_0 - x_n) \rangle \\
&= \langle p_0, J(p_0 - x_n) - J[(1-t)z_t - x_n] \rangle \\
&\quad + \langle p_0 - z_t, J[(1-t)z_t - x_n] \rangle \\
&\quad + \langle z_t, J[(1-t)z_t - x_n] \rangle,
\end{aligned} \tag{31}$$

we have $\limsup_{n \rightarrow +\infty} \langle p_0, J(p_0 - x_n) \rangle \leq 0$.

Since $\langle p_0, J[p_0 - x_n - (1 - \alpha_n)e_n + \alpha_n x_n] \rangle = \langle p_0, J[p_0 - x_n - (1 - \alpha_n)e_n + \alpha_n x_n] - J(p_0 - x_n) \rangle + \langle p_0, J(p_0 - x_n) \rangle$ and J is uniformly continuous on each bounded subset of E , then

$$\limsup_{n \rightarrow +\infty} \langle p_0, J[p_0 - x_n - (1 - \alpha_n)e_n + \alpha_n x_n] \rangle \leq 0. \tag{32}$$

Step 5. $x_n \rightarrow p_0$, as $n \rightarrow +\infty$, where $p_0 \in D$ is the same as that in Step 4.

Let $M_4 = \sup\{\|(1 - \alpha_n)(x_n + e_n) - p_0\| : n \geq 1\}$. By using Lemma 4 again, we have

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\leq \gamma_n \|x_n - p_0\|^2 + (1 - \gamma_n) \|v_n - p_0\|^2 \\
&\leq \gamma_n \|x_n - p_0\|^2 + (1 - \gamma_n) (1 - \beta_n) \|x_n - p_0\|^2 \\
&\quad + (1 - \gamma_n) \beta_n \|u_n - p_0\|^2 \\
&= (1 - \beta_n + \beta_n \gamma_n) \|x_n - p_0\|^2 + (1 - \gamma_n) \beta_n \|u_n - p_0\|^2 \\
&\leq (1 - \beta_n + \beta_n \gamma_n) \|x_n - p_0\|^2 \\
&\quad + (1 - \gamma_n) \beta_n \|(1 - \alpha_n)(x_n + e_n) - p_0\|^2 \\
&\leq (1 - \beta_n + \beta_n \gamma_n) \|x_n - p_0\|^2 + (1 - \gamma_n) \beta_n (1 - \alpha_n) \\
&\quad \times \|x_n - p_0\|^2 + 2(1 - \gamma_n) \beta_n (1 - \alpha_n) \\
&\quad \times \langle e_n, J[(1 - \alpha_n)(x_n + e_n) - p_0] \rangle + 2\alpha_n \beta_n (1 - \gamma_n) \\
&\quad \times \langle p_0, J[p_0 - x_n - (1 - \alpha_n)e_n + \alpha_n x_n] \rangle \\
&\leq [(1 - \alpha_n) \beta_n (1 - \gamma_n)] \|x_n - p_0\|^2 + 2(1 - \gamma_n) \\
&\quad \times (1 - \alpha_n) \beta_n M_4 \|e_n\| + 2\alpha_n \beta_n (1 - \gamma_n) \\
&\quad \times \langle p_0, J[p_0 - x_n - (1 - \alpha_n)e_n + \alpha_n x_n] \rangle.
\end{aligned} \tag{33}$$

Let $c_n = (1 - \gamma_n) \alpha_n \beta_n$; then (33) reduces to $\|x_{n+1} - p_0\|^2 \leq (1 - c_n) \|x_n - p_0\|^2 + 2c_n \{\langle p_0, J[p_0 - x_n - (1 - \alpha_n)e_n + \alpha_n x_n] \rangle + (1 - \alpha_n) M_4 (\|e_n\|/\alpha_n)\}$.

From (32), (33), and the assumptions, by using Lemma 5, we know that $x_n \rightarrow p_0$, as $n \rightarrow +\infty$.

This completes the proof. \square

If, in Theorem 12, $C = E$, then we have the following theorem.

Theorem 13. Let E and D be the same as those in Theorem 12. Suppose that the duality mapping $J : E \rightarrow E^*$ is weakly sequentially continuous. Let $A_i : E \rightarrow E$ ($i = 1, 2, \dots, N$) be a finite family of m -accretive mappings. Let $\{e_n\} \subset E$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$, and $\{r_{n,i}\} \subset (0, +\infty)$ satisfy the conditions presented in Theorem 12.

Let $\{x_n\}$ be generated by the following scheme:

$$\begin{aligned} x_1 &\in E, \\ u_n &= (1 - \alpha_n)(x_n + e_n), \\ v_n &= (1 - \beta_n)x_n + \beta_n S_n u_n, \\ x_{n+1} &= \gamma_n x_n + (1 - \gamma_n) S_n v_n \quad n \geq 1. \end{aligned} \tag{B}$$

Then $\{x_n\}$ converges strongly to a point $p_0 \in D$, where S_n is the same as that in Theorem 12.

3. Nonlinear Elliptic Systems with (p_1, p_2, \dots, p_N) -Laplacian

In what follows in this paper, unless otherwise stated, we will assume that $(2N/N + 1) < p_i < +\infty$, $(1/p_i) + (1/p'_i) = 1$. We use $\|\cdot\|_{p_i}$, $\|\cdot\|_{p'_i}$, and $\|\cdot\|_{1,p_i,\Omega}$ to denote the norms of spaces $L^{p_i}(\Omega)$, $L^{p'_i}(\Omega)$, and $W^{1,p_i}(\Omega)$, respectively, where $i = 1, 2, \dots, N$.

Now, we will examine the following nonlinear elliptic systems:

$$\begin{aligned} -\operatorname{div} \left[(C_i(x) + |\nabla u|^2)^{(p_i-2)/2} \nabla u \right] &= f_i(x), \quad \text{a.e. } x \in \Omega, \\ -\left\langle \vartheta, (C_i(x) + |\nabla u|^2)^{(p_i-2)/2} \nabla u \right\rangle &= 0, \quad \text{a.e. } x \in \Gamma, \\ i &= 1, 2, \dots, N. \end{aligned} \tag{C}$$

In (C), Ω is a bounded conical domain of a Euclidean space R^N with its boundary $\Gamma \in C^1$, (c.f. [23]). $f_i(x) \in L^{p'_i}(\Omega)$ is a given function and $0 \leq C_i(x) \in L^{p_i}(\Omega)$, for $i = 1, 2, \dots, N$. ϑ denotes the exterior normal derivative of Γ .

Lemma 14 (see [24]). Define the mapping $B_{p_i} : W^{1,p_i}(\Omega) \rightarrow (W^{1,p_i}(\Omega))^*$ by

$$(v, B_{p_i} u) = \int_{\Omega} \left\langle (C_i(x) + |\nabla u|^2)^{(p_i-2)/2} \nabla u, \nabla v \right\rangle dx, \tag{34}$$

for any $u, v \in W^{1,p_i}(\Omega)$. Then, B_{p_i} is everywhere defined, strictly monotone, hemicontinuous, coercive, and maximal monotone, for $i = 1, 2, \dots, N$.

Lemma 15 (see [24]). Define the mapping $A_i : L^2(\Omega) \rightarrow L^2(\Omega)$ by

$$\begin{aligned} D(A_i) &= \{u(x) \in L^2(\Omega) : \text{there exists } g_i(x) \in L^2(\Omega) \\ &\text{such that } g_i(x) = B_{p_i} u\} \end{aligned} \tag{35}$$

for any $u \in D(A_i)$, $A_i u = g_i(x)$. Then, A_i is m -accretive, for $i = 1, 2, \dots, N$.

Lemma 16 (see [24]). For $i = 1, 2, \dots, N$, define the mapping $A_{p_i} : L^{p_i}(\Omega) \rightarrow L^{p_i}(\Omega)$ in the following way:

- (i) if $p_i \geq 2$, $D(A_{p_i}) = \{u(x) \in L^{p_i}(\Omega) : \text{there exists } h_i(x) \in L^{p_i}(\Omega) \text{ such that } h_i(x) = B_{p_i} u\}$, then for any $u \in D(A_{p_i})$, $A_{p_i} u = h_i(x)$,
- (ii) if $2N/(N + 1) < p_i < 2$, we define $A_{p_i} : L^{p_i}(\Omega) \rightarrow L^{p_i}(\Omega)$ as the L^{p_i} -closure of $A_i : L^2(\Omega) \rightarrow L^2(\Omega)$ defined in (i) above.

Then, A_{p_i} is m -accretive, for $i = 1, 2, \dots, N$.

Lemma 17 (see [25]). Let Ω be a bounded conical domain in R^N . If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$; if $0 < mp \leq N$ and $q_0 = Np/(N - mp)$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $1 \leq q < q_0$.

Lemma 18 (see [26]). Let X_0 denote the closed subspace of all constant functions in $W^{1,p}(\Omega)$. Let X be the quotient space $W^{1,p}(\Omega)/X_0$. For $u \in W^{1,p}(\Omega)$, define the mapping $P : W^{1,p}(\Omega) \rightarrow X_0$ by $Pu = (1/\operatorname{meas}(\Omega)) \int_{\Omega} u dx$. Then, there is a constant $C > 0$, such that $\forall u \in W^{1,p}(\Omega)$,

$$\|u - Pu\|_p \leq C \|\nabla u\|_{(L^p(\Omega))^N}. \tag{36}$$

Theorem 19. For $i = 1, 2, \dots, N$, one has $A_{p_i}^{-1}0 = \{u \in L^{p_i}(\Omega) : u(x) \equiv \text{Constant on } \Omega\}$.

Proof. (i) $p_i \geq 2$.

Let $u(x) \in A_{p_i}^{-1}0$; then $0 = (u, B_{p_i} u) = \int_{\Omega} (C_i(x) + |\nabla u|^2)^{(p_i-2)/2} |\nabla u|^2 dx \geq \int_{\Omega} |\nabla u|^{p_i} dx \geq 0$, which implies that $u(x) \equiv \text{Constant}$. That is, $A_{p_i}^{-1}0 \subset \{u \in L^{p_i}(\Omega) : u(x) \equiv \text{Constant}\}$.

On the other hand, suppose $u(x) \equiv \text{Constant}$. Then $0 = (v, B_{p_i} u)$, for all $v \in W^{1,p_i}(\Omega)$. Then $u \in A_{p_i}^{-1}0$. Therefore, the result holds.

(ii) $2N/(N + 1) < p_i < 2$.

Suppose $u \in L^{p_i}(\Omega)$ and $u(x) \equiv \text{Constant}$. Then let $u_n \equiv u$, and we can easily see that $u \in A_{p_i}^{-1}0$ in view of the definition of A_{p_i} .

On the other hand, let $u \in A_{p_i}^{-1}0$. Then there exist $\{u_n\}$ and $\{f_n\}$ in $L^2(\Omega)$ such that $u_n \rightarrow u$, $f_n \rightarrow 0$ in $L^{p_i}(\Omega)$ and $A_i u_n = f_n$. Now, define the following functions:

$$\begin{aligned} \chi(t) &= \begin{cases} |t|^{p_i-1} \operatorname{sgnt}, & \text{if } |t| \geq 1 \\ t, & \text{if } |t| < 1, \end{cases} \\ \xi(t) &= \begin{cases} |t|^{2-(2/p_i)} \operatorname{sgnt}, & \text{if } |t| \geq 1 \\ t, & \text{if } |t| < 1. \end{cases} \end{aligned} \tag{37}$$

Then for $u \in L^2(\Omega)$, the function $t \in R \rightarrow \int_{\Omega} \xi(u + t) dx \in R$ is continuous and $\lim_{t \rightarrow \pm\infty} \int_{\Omega} \xi(u + t) dx = \pm\infty$. Therefore, there exists $t_u \in R$ such that $\int_{\Omega} \xi(u + t_u) dx = 0$.

So, for $u_n \in L^2(\Omega)$, we may assume that there exist $t_n \in R$ such that $\int_{\Omega} \xi(u_n + t_n) dx = 0$ and $A_i u_n = f_n$, for $n \geq 1$. Let $v_n = u_n + t_n$; then $A_i v_n = A_i u_n = f_n$, for $n \geq 1$.

Now, compute the following:

$$\begin{aligned} & \|f_n\|_{p_i} \left(\int_{|v_n| \leq 1} |v_n|^{p'_i} dx + \int_{|v_n| \geq 1} |v_n|^2 dx \right)^{(p_i-1)/p_i} \\ & \geq \|f_n\|_{p_i} \left(\int_{|v_n| \leq 1} |v_n|^{p'_i} dx + \int_{|v_n| \geq 1} |v_n|^{p_i} dx \right)^{(p_i-1)/p_i} \\ & \leq \|f_n\|_{p_i} \|\chi(v_n)\|_{p_i'} \geq (\chi(v_n), f_n) \\ & \geq (\chi(v_n), A_i v_n) \geq \int_{\Omega} |\nabla v_n|^{p_i} \chi'(v_n) dx \\ & \geq \text{Const.} \int_{\Omega} |\nabla(\xi(v_n))|^{p_i} dx. \end{aligned} \tag{38}$$

Using Lemma 18,

$$\int_{\Omega} |\nabla(\xi(v_n))|^{p_i} dx \geq \text{Const.} \|\xi(v_n)\|_{1,p_i,\Omega}^{p_i}. \tag{39}$$

Then Lemma 17 implies that

$$\begin{aligned} & \|\xi(v_n)\|_{1,p_i,\Omega}^{p_i} \\ & \geq \text{Const.} \|\xi(v_n)\|_{p_i'}^{p_i} \\ & = \text{Const.} \left(\int_{|v_n| \leq 1} |v_n|^{p'_i} dx + \int_{|v_n| \geq 1} |v_n|^{(2-(2/p_i))p'_i} dx \right)^{p_i/p'_i} \\ & = \text{Const.} \left(\int_{|v_n| \leq 1} |v_n|^{p'_i} dx + \int_{|v_n| \geq 1} |v_n|^2 dx \right)^{p_i/p'_i}. \end{aligned} \tag{40}$$

From (38), (39), and (40), we know that $\|\xi(v_n)\|_{1,p_i,\Omega}^{p_i} \leq \text{Const.} \|f_n\|_{p_i} \rightarrow 0$, as $n \rightarrow \infty$. Then $\xi(v_n) \rightarrow 0$ in $L^{p'_i}(\Omega)$. Since Nemytskiy mapping $u \in L^{p'_i}(\Omega) \rightarrow \xi^{-1}(u) \in L^{p_i}(\Omega)$ is continuous, then $v_n \rightarrow 0$ in $L^{p_i}(\Omega)$. And then $u(x) \equiv \text{Constant}$.

This completes the proof. □

Remark 20. Theorem 19 helps us to see the assumption that $\bigcap_{i=1}^N A_{p_i}^{-1} 0 \neq \emptyset$ in Theorems 12 and 13 are valid.

Remark 21. If we set $C_{p_i} = A_{p_i} - f_i(x)$, then C_{p_i} is also m -accretive. And then iterative scheme (A) can be used to approximate the element in $\bigcap_{i=1}^N C_{p_i}^{-1} 0$, which is just the solution of nonlinear elliptic system (C).

Remark 22. If $C_i(x) \equiv 0$, then (C) is reduced to the following nonlinear elliptic systems with (p_1, p_2, \dots, p_N) -Laplacian:

$$\begin{aligned} & -\Delta_{p_i} u = f_i(x), \quad \text{a.e. } x \in \Omega, \\ & -\langle \vartheta, |\nabla u|^{p_i-2} \nabla u \rangle = 0, \quad \text{a.e. } x \in \Gamma, \end{aligned} \tag{D}$$

$i = 1, 2, \dots, N.$

Moreover, the solution of (D) is the zero of finitely many suitably defined m -accretive mappings, which can be approximated by iterative scheme (A).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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