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Research Article

Strong Convergence Theorems of the *CQ* Algorithm for *H*-Monotone Operators in Hilbert Spaces

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The aim of this paper is to show the strong convergence theorems of the CQ algorithm for H-monotone operators in Hilbert spaces by hybrid method in the mathematical programming. The main results extend and improve the corresponding results. Moreover, the assumption conditions of our results are weaker than those of the corresponding results.

1. Introduction

In this paper, we show a CQ algorithm for solving the inclusion problem in Hilbert space $\mathscr{H}$. The inclusion problem is finding the zero solutions of T; that is,

$$x \in \mathscr{H}, \quad \text{s.t. } 0 \in T(x). \tag{1}$$

This problem is closely related to many problems, such as variational inequalities, fixed points problem, and complementarity problem of mathematical programming, and it plays an important role in convex analysis and some partial differential equations. The inclusion problem (1) on monotone operator and maximal monotone operators is extensively investigated by so many researchers. See Tseng [1], Kamimura et al. [2–19], and so on.

In 2003, Fang and Huang [20] firstly introduced H-monotone operators and discussed some properties of this class of operators.

Motivated by Fang and Huang [20], very recently, we firstly consider the inclusion problem $0 \in T(x)$ of H-monotone operator for finding the solutions of it in a Hilbert space $\mathscr{H}$ [21].

In [21], we mainly presented strong and weak convergence theorems for Halpern type and Mann type algorithms, respectively, and the relations between maximal monotone operators and H-monotone operators are analyzed in detail. Simultaneously, we apply these results to the minimization problem for $T = \partial f$ and provide some numerical examples to support the theoretical findings. These results start a new branch of research for the inclusion problem $0 \in T(x)$, and we do further extending study for this subject.

Motivated by the main results of Nakajo and Takahashi [22], we propose a so-called CQ iteration algorithm as follows:

$$\begin{aligned}
&x_0 = x \in \mathscr{H},\\
&y_n = \alpha_n (x_n + f_n) + (1 - \alpha_n) J^T_{H,r_n} H (x_n + f_n),\\
&C_n = \{z \in H \mid \|y_n - z\| \le \|x_n + f_n - z\|\},\\
&Q_n = \{z \in H \mid \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{aligned} \tag{2}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, +\infty)$.

The aim of this paper is to establish the strong convergence theorems to approximate the zero point of H-monotone operator, namely, finding the $x \in \mathscr{H}$ such that $0 \in T(x)$.

2. Preliminaries

Definition 1. A multivalued operator $T : \mathscr{H} \to 2^{\mathscr{H}}$ is said to be

(i) monotone if

$$\langle x-y, u-v\rangle \geq 0, \quad \forall u, v \in \mathscr{H},\ x \in Tu,\ y \in Tv; \tag{3}$$

(ii) maximal monotone if T is monotone and $(I+\lambda T)(\mathscr{H}) = \mathscr{H}$ for all $\lambda > 0$, where I denotes the identity mapping on $\mathscr{H}$.

We note that the T is maximal monotone if and only if T is monotone and the graph

$$G(T) = \{(z, w) \in \mathscr{H} \times \mathscr{H} \mid w \in Tz\} \tag{4}$$

is not properly contained in the graph of any other monotone operator $T' : \mathscr{H} \to 2^{\mathscr{H}}$.

Definition 2 (see [20]). Let $H : \mathscr{H} \to \mathscr{H}$ be a single mapping and $T : \mathscr{H} \to 2^{\mathscr{H}}$ a multivalued mapping. T is said to be

(i) H-monotone if T is monotone and $(H+\lambda T)(\mathscr{H}) = \mathscr{H}$ holds for every $\lambda > 0$;

(ii) strongly H-monotone if T is strongly monotone and $(H+\lambda T)(\mathscr{H}) = \mathscr{H}$ holds for every $\lambda > 0$.

Definition 3. Let $T : \mathscr{H} \to \mathscr{H}$ be a single-valued operator. T is said to be

(i) strictly monotone if T is monotone and

$$\langle Tx - Ty, x-y\rangle = 0, \quad \text{iff } x = y; \tag{5}$$

(ii) strongly monotone if there exists some constant $r > 0$ such that

$$\langle Tx - Ty, x-y\rangle \geq r\|x-y\|^2, \quad \forall x, y \in \mathscr{H}; \tag{6}$$

(iii) Lipschitz continuous if there exists some constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x-y\|, \quad \forall x, y \in \mathscr{H}. \tag{7}$$

Remark 4. We note that if T is strongly monotone, then T is strictly monotone, but vice is not. If T is strongly monotone with constant r and Lipschitz continuous with constant s, then we have $r \leq s$. As $r = s > 0$, then T satisfies

$$\|Tx - Ty\| = r\|x-y\|. \tag{8}$$

Let $T : \mathscr{H} \to \mathscr{H}$ be a single-valued operator and let $H : \mathscr{H} \to \mathscr{H}$ be a strongly monotone and Lipschitz continuous operator with constant γ. Let $T : \mathscr{H} \to 2^{\mathscr{H}}$ be an H-monotone operator and the resolvent operator $J^T_{H,\rho} : \mathscr{H} \to \mathscr{H}$ is defined by

$$J^T_{H,\rho}(u) = (H+\rho T)^{-1}(u), \quad \forall u \in \mathscr{H}, \tag{9}$$

for each $\rho > 0$. We can define the following operators which are called *Yosida approximations*:

$$A_\rho = \frac{1}{\rho}\left(I - H \cdot J^T_{H,\rho}\right), \quad \forall \rho > 0. \tag{10}$$

We give some elementary properties of $J^T_{H,\rho}$ and A_ρ.

Lemma 5 (Proposition 4.1 in [21]). *Let $H : \mathscr{H} \to \mathscr{H}$ be a strongly monotone and Lipschitz continuous operator with constant γ and let $T : \mathscr{H} \to 2^{\mathscr{H}}$ be an H-monotone operator. Then the following properties hold:*

(i) $\|J^T_{H,\rho}(x) - J^T_{H,\rho}(y)\| \leq (1/\gamma)\|x-y\|$, *for all* $x, y \in R(H+\rho T)$;

(ii) $\|H\cdot J^T_{H,\rho}(x) - H\cdot J^T_{H,\rho}(y)\| \leq \|x-y\|$, *for all* $x, y \in \mathscr{H}$, *or* $\|J^T_{H,\rho}\cdot H(x) - J^T_{H,\rho}\cdot H(y)\| \leq \|x-y\|$, *for all* $x, y \in \mathscr{H}$;

(iii) A_ρ *is monotone and*

$$\left\|A_\rho x - A_\rho y\right\| \leq \frac{2}{\rho}\|x-y\|, \quad \forall x, y \in R(H+\rho T); \tag{11}$$

(iv) $A_\rho x \in TJ^T_{H,\rho}(x)$, *for all* $x \in R(H+\rho T)$.

Lemma 6 (Proposition 4.2 in [21]). *Consider $u \in T^{-1}0$ if and only if u satisfies the relation*

$$u = J^T_{H,\rho}(H(u)), \tag{12}$$

where $\rho > 0$ is a constant and $J^T_{H,\rho}$ is the resolvent operator defined by (9).

Lemma 7 (Proposition 2.1 in [20]). *Let $H : \mathscr{H} \to \mathscr{H}$ be a strictly monotone single-valued operator and $T : \mathscr{H} \to 2^{\mathscr{H}}$ an H-monotone operator. Then T is maximal monotone.*

Lemma 8 (see [23]). *Let E be a real Banach space. Then for all $x, y \in E$*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y). \tag{13}$$

3. Strong Convergence Theorems for *CQ* Algorithm

We consider the following algorithm, and the sequence $\{x_n\}$ is generated by

$$\begin{aligned}
&x_0 = x \in \mathscr{H},\\
&y_n = \alpha_n(x_n + f_n) + (1-\alpha_n) J^T_{H,r_n} H(x_n + f_n),\\
&C_n = \{z \in H : \|y_n - z\| \leq \|x_n + f_n - z\|\},\\
&Q_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{aligned} \tag{14}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, +\infty)$. Motivated by Nakajo and Takahashi [22] and Fang et al. [12, 13, 20], we get the following results.

Theorem 9. *Let $H : \mathscr{H} \to \mathscr{H}$ be a strongly monotone and Lipschitz continuous operator with constant γ. Let $T : \mathscr{H} \to 2^{\mathscr{H}}$ be an H-monotone operator; let $x \in \mathscr{H}$ and $\{x_n\}$ be a sequence defined by* (14), *where $\{\alpha_n\} \subset [0,1]$ and*

$\{r_n\} \subset (0,+\infty)$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\liminf_{n\to\infty} r_n > 0$, and $\lim_{n\to\infty}\|f_n\| = 0$. If $T^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to Px, where P is the metric projection of $\mathscr{H}$ onto $T^{-1}0$.

Proof. The proofs can be divided into three steps.

Step 1 ($\{x_n\}$ is well defined and $T^{-1}0 \subset C_n \cap Q_n$). Based on the definitions of C_n and Q_n, we can get that Q_n is a closed and convex set of $\mathscr{H}$ for every $n \in \mathbb{N}$.

And since the inequality

$$\|y_n - z\| \leq \|x_n + f_n - z\| \tag{15}$$

is equivalent to

$$\|y_n - x_n - f_n\|^2 + 2\langle y_n - x_n - f_n, x_n + f_n - z\rangle \leq 0, \tag{16}$$

hence, C_n is closed and convex, so is $C_n \cap Q_n$ for every $n \in \mathbb{N}$.

From Lemma 6, there exists $u \in T^{-1}0$ such that $u = J^T_{H,\rho}(H(u))$ for all $\rho > 0$, and based on Lemma 5, we know $J^T_{H,\rho} \cdot H$ is a nonexpansive mapping from $\mathscr{H}$ into itself.

For all $u \in T^{-1}0$, it follows that

$$\begin{aligned}
&\|y_n - u\| \\
&\quad = \left\|\alpha_n(x_n + f_n) + (1-\alpha_n) J^T_{H,r_n} H(x_n + f_n) - u\right\| \\
&\quad \leq \alpha_n \|x_n + f_n - u\| + (1-\alpha_n)\left\|J^T_{H,r_n} H(x_n + f_n) - u\right\| \\
&\quad \leq \|x_n + f_n - u\|.
\end{aligned} \tag{17}$$

Then $u \in C_n$ for each $n \in \mathbb{N}$. Therefore, $T^{-1}0 \subset C_n$ for every $n \in \mathbb{N}$.

Next, we show that $\{x_n\}$ is well defined and $T^{-1}0 \subset C_n \cap Q_n$ by mathematical induction.

For $n = 0$, we have $x_0 = x \in \mathscr{H}$ and $Q_0 = \mathscr{H}$. Hence, $T^{-1}0 \subset C_0 \cap Q_0$, because $T^{-1}0 \subset C_n$.

For $n = k$, suppose that $\{x_k\}$ is given and $T^{-1}0 \subset C_k \cap Q_k$ for all $k \in \mathbb{N}$. Then, there exists a unique $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x_0$, because $C_k \cap Q_k$ is closed and convex.

Based on the property of the projection operators, we can get that $x_{k+1} = P_{C_k \cap Q_k} x_0$ is equivalent to

$$\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0, \tag{18}$$

for all $z \in C_k \cap Q_k$. By the assumption $T^{-1}0 \subset C_k \cap Q_k$, we have $T^{-1}0 \subset Q_{k+1}$.

Hence,

$$T^{-1}0 \subset C_{k+1} \cap Q_{k+1}. \tag{19}$$

Step 2 ($\{x_n\}$ is bounded and $\|x_{n+1} - x_n\| \to 0$). By Lemma 5, we get that $J^T_{H,r_n} H$ is nonexpansive. And from Lemma 6, we have that $u \in T^{-1}0$ is equivalent to $u = J^T_{H,\rho}(H(u))$. So, $u \in T^{-1}0$ is a closed and convex subset of $\mathscr{H}$.

The rest of the proofs of this step can follow Lemmas 3.2 and 3.3 of [22].

Step 3 ($x_n \to z_0 = P_{T^{-1}0} x_0$). From $\lim_{n\to\infty}\|f_n\| = 0$, we get that $\{f_n\}$ is bounded. So, $\{y_n\}$ is bounded.

Now, we suppose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. Since $x_{n+1} \in C_n$, we have

$$\begin{aligned}
\|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\leq \|x_n + f_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\leq 2\|x_{n+1} - x_n\| + \|f_n\|.
\end{aligned} \tag{20}$$

From Step 2 $\|x_{n+1} - x_n\| \to 0$ and the assumption $\lim_{n\to\infty}\|f_n\| = 0$, we obtain

$$\|y_n - x_n\| \longrightarrow 0. \tag{21}$$

Hence,

$$y_{n_i} \rightharpoonup w_0. \tag{22}$$

And similarly, it follows from $x_{n+1} \in C_n$ and Step 2 that

$$\begin{aligned}
&\left\|J^T_{H,r_n} H(x_n + f_n) - x_n - f_n\right\| \\
&\quad = \frac{1}{1-\alpha_n}\|y_n - x_n - f_n\| \\
&\quad \leq \frac{1}{1-\alpha_n}\left(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n - f_n\|\right) \\
&\quad \leq \frac{2}{1-\alpha_n}\|x_{n+1} - x_n - f_n\| \\
&\quad \leq \frac{2}{1-\alpha_n}\left(\|x_{n+1} - x_n\| + \|f_n\|\right).
\end{aligned} \tag{23}$$

Therefore,

$$\left\|J^T_{H,r_n} H(x_n + f_n) - x_n - f_n\right\| \longrightarrow 0, \tag{24}$$

$$\begin{aligned}
&\left\|J^T_{H,r_n} H(x_n + f_n) - x_n\right\| \\
&\quad \leq \left\|J^T_{H,r_n} H(x_n + f_n) - x_n - f_n\right\| + \|f_n\|.
\end{aligned} \tag{25}$$

This implies that

$$\left\|J^T_{H,r_n} H(x_n + f_n) - x_n\right\| \longrightarrow 0. \tag{26}$$

Thus,

$$J^T_{H,r_{n_i}} H\left(x_{n_i} + f_{n_i}\right) \rightharpoonup w_0 \in \mathscr{H}. \tag{27}$$

Next, we show that $w_0 \in T^{-1}0$. Since T is monotone and $A_{r_n}(H(x_n + f_n)) \in T J^T_{H,r_n} H(x_n + f_n)$ due to $A_\rho(x) \in T J^T_{H,\rho}(x)$ for all $x \in \mathscr{H}$, we obtain

$$\left\langle z - J^T_{H,r_{n_i}} H\left(x_{n_i} + f_{n_i}\right), z' - A_{r_{n_i}}\left(H\left(x_{n_i} + f_{n_i}\right)\right)\right\rangle \geq 0, \tag{28}$$

for all $z' \in Tz$. Due to the assumption $\liminf_{n\to\infty} r_n > 0$ and inequality (24), we have

$$\left\| A_{r_n}\left(H\left(x_n+f_n\right)\right)\right\| = \left\| \frac{H\left(x_n+f_n\right) - HJ_{H,r_n}^{T} H\left(x_n+f_n\right)}{r_n} \right\| \leq \frac{\gamma}{r_n} \left\| J_{H,r_n}^{T} H\left(x_n+f_n\right) - x_n - f_n \right\|. \quad (29)$$

So, $A_{r_n}(H(x_n+f_n)) \to 0$ and

$$\left\langle z - w_0, z' \right\rangle \geq 0, \quad \forall z' \in Tz. \quad (30)$$

By Lemma 5, we know T is a maximal monotone operator; from the maximality property of T; we have

$$w_0 \in T^{-1}0. \quad (31)$$

If $z_0 = P_{T^{-1}0}x_0$, by the lower semicontinuity of the norm, we get

$$\begin{aligned} \left\|x_0 - z_0\right\| \leq \left\|x_0 - w_0\right\| &\leq \liminf_{i\to\infty} \left\|x_0 - x_{n_i}\right\| \\ &\leq \limsup_{i\to\infty} \left\|x_0 - x_{n_i}\right\| \\ &\leq \left\|x_0 - z_0\right\|. \end{aligned} \quad (32)$$

Therefore, we obtain

$$\lim_{i\to\infty} \left\|x_{n_i} - x_0\right\| = \left\|x_0 - w_0\right\| = \left\|x_0 - z_0\right\|. \quad (33)$$

Adding $x_{n_i} \rightharpoonup w_0$, we can get that $x_{n_i} \to w_0 = z_0$.

Thus, $x_n \to z_0 = P_{T^{-1}0}x_0$. □

Remark 10. In the main results, Theorem 9 of this paper, the assumption condition is $\liminf_{n\to\infty} r_n > 0$, and we can obtain the conclusion $A_{r_n}(H(x_n+f_n)) \to 0$. However, the conclusion $A_{r_n}(H(x_n+f_n)) \to 0$ is obtained by assuming the condition $\liminf_{n\to\infty} r_n = \infty$ that appeared in [2, 20].

It is worth noting that condition $\liminf_{n\to\infty} r_n > 0$ is weaker than condition
$\liminf_{n\to\infty} r_n = \infty$ by observing

$$\liminf_{n\to\infty} r_n = \infty \Longrightarrow \liminf_{n\to\infty} r_n > 0. \quad (34)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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