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Research Article

Sharp Inequalities for Trigonometric Functions

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We establish several sharp inequalities for trigonometric functions and present their corresponding inequalities for bivariate means.

1. Introduction

A bivariate real value function $M:(0,\infty)\times(0,\infty)\mapsto(0,\infty)$ is said to be a mean if

$$\min(x,y)\le M(x,y)\le\max(x,y), \tag{1}$$

for all $x,y>0$. M is said to be homogeneous if

$$M(\lambda x,\lambda y)=\lambda M(x,y), \tag{2}$$

for any $\lambda,x,y>0$.

Remark 1 (see [1]). Let $M(x,y)$ be a homogeneous bivariate mean of two positive real numbers x and y. Then

$$M(x,y)=\sqrt{xy}M\left(e^{t},e^{-t}\right), \tag{3}$$

where $t=(1/2)\ln(x/y)$.

By this remark, almost all of the inequalities for homogeneous symmetric bivariate means can be transformed equivalently into the corresponding inequalities for hyperbolic functions and vice versa. More specifically, let $L(x,y)$, $I(x,y)$, and $A_r(x,y)$ be the logarithmic, identric, and rth power means of two distinct positive real numbers x and y given by

$$L=L(x,y)=\frac{x-y}{\ln x-\ln y},$$
$$I=I(x,y)=e^{-1}\left(\frac{x^x}{y^y}\right)^{1/(x-y)},$$
$$A_r=A_r(x,y)=\left(\frac{x^r+y^r}{2}\right)^{1/r}$$
$$\text{if } r\ne 0,\ A_0(x,y)=G=\sqrt{xy}, \tag{4}$$

respectively. Then, for $x>y>0$, we have

$$L\left(e^{t},e^{-t}\right)=\frac{\sinh t}{t},\qquad I\left(e^{t},e^{-t}\right)=e^{t\coth t-1},$$
$$A_p\left(e^{t},e^{-t}\right)=\cosh^{1/p}(pt),\qquad G\left(e^{t},e^{-t}\right)=1, \tag{5}$$

where $t=(1/2)\ln(x/y)>0$. By Remark 1, we can derive some inequalities for hyperbolic functions from certain known inequalities for bivariate means mentioned previously. For example,

$$A_{2/3}<I<A_{\ln 2}$$
$$\Longrightarrow\left(\cosh\frac{2t}{3}\right)^{3/2}<e^{t\coth t-1}<(\cosh(t\ln 2))^{1/\ln 2} \tag{6}$$

(see [2, 3]); consider

$$A_{2/3}<I<\sqrt{8}e^{-1}A_{2/3}$$
$$\Longrightarrow\left(\cosh\frac{2t}{3}\right)^{3/2}<e^{t\coth t-1}<\sqrt{8}e^{-1}\left(\cosh\frac{2t}{3}\right)^{3/2} \tag{7}$$

(see [4, 5]); consider that

$$A_p^{2/(3p)}G^{1-2/(3p)} < I < A_q^{2/(3q)}G^{1-2/(3q)} \Longrightarrow (\cosh pt)^{2/(3p^2)} < e^{t\coth t-1} < (\cosh qt)^{2/(3q^2)} \tag{8}$$

(see [1]) holds for $t > 0$ if and only if $p \geq 2/3$ and $0 < q \leq q_0 = \sqrt{10}/5$; consider

$$\sqrt{AG} < \sqrt{LI} < \frac{L+I}{2} < \frac{A+G}{2} \Longrightarrow \sqrt{\cosh t} < \sqrt{\frac{\sinh t}{t}e^{t\coth t-1}} < \frac{\sinh t/t + e^{t\coth t-1}}{2} < \frac{\cosh t+1}{2} \tag{9}$$

(see [6]); consider

$$\frac{1}{3} < \frac{I-L}{A-G} < \frac{2}{e} < \frac{I+L}{A+G} < 1 \Longrightarrow \frac{1}{3} < \frac{e^{t\cosh t/\sinh t-1} - \sinh t/t}{\cosh t-1} < \frac{2}{e} < \frac{e^{t\cosh t/\sinh t-1} + \sinh t/t}{1+\cosh t} < 1 \tag{10}$$

(see [7], (3.9), and (3.10)); if $0 < p \leq 6/5$, then the double inequality

$$\lambda_p A^p + (1-\lambda_p)G^p < I^p < \mu_p A^p + (1-\mu_p)G^p \Longrightarrow \lambda_p \cosh^p t + (1-\lambda_p) < e^{pt\coth t-p} < \mu_p\cosh^p t + (1-\mu_p) \tag{11}$$

(see [8]) holds if and only if $\lambda_p \leq 2/3$ and $\mu_p \geq (2/e)^p$; if $p \geq 2$, then inequality (11) holds if and only if $\lambda_p \leq (2/e)^p$ and $\mu_p \geq 2/3$; consider that

$$\left(\frac{2}{3}A^p + \frac{1}{3}G^p\right)^{1/p} < I < \left(\frac{2}{3}A^q + \frac{1}{3}G^q\right)^{1/q} \Longrightarrow \left(\frac{2}{3}\cosh^p t + \frac{1}{3}\right)^{1/p} < e^{t\coth t-1} < \left(\frac{2}{3}\cosh^q t + \frac{1}{3}\right)^{1/q} \tag{12}$$

(see [9]) holds if and only if $p \leq 6/5$ and $q \geq (\log 3 - \log 2)/(1-\log 2)$.

The main purpose of this paper is to find the sharp bounds for the functions $e^{t\cot t-1}$ ($t \in (0,\pi/2)$), which include the corresponding trigonometric version of the inequalities listed above. As applications, their corresponding inequalities for bivariate means are presented.

2. Lemmas

Lemma 2 (see [10, Theorem 1.25], [11, Remark 1]). *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b); let $g' \neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}. \tag{13}$$

If f'/g' is one-to-one, then the monotonicity in the conclusion is strict.

Lemma 3 (see [12]). *Let a_n and b_n ($n = 0,1,2,\ldots$) be real numbers and let the power series $A(t) = \sum_{n=1}^{\infty} a_n t^n$ and $B(t) = \sum_{n=1}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $a_n, b_n > 0$, for $n = 1,2,\ldots$, and a_n/b_n is (strictly) increasing (decreasing), for $n = 1,2,\ldots$, then the function $A(t)/B(t)$ is also (strictly) increasing (decreasing) on $(0,R)$.*

Lemma 4 (see [13, pages 227–229]). *One has*

$$\frac{1}{\sin t} = \frac{1}{t} + \sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|t^{2n-1}, \quad 0<|t|<\pi, \tag{14}$$

$$\cot t = \frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1}, \quad 0<|t|<\pi, \tag{15}$$

$$\tan t = \sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|t^{2n-1}, \quad |t|<\frac{\pi}{2}, \tag{16}$$

$$\frac{1}{\sin^2 t} = \frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}}{(2n)!}|B_{2n}|t^{2n-2}, \quad 0<|t|<\pi, \tag{17}$$

where B_n is the Bernoulli number.

Lemma 5. *For every $t \in (0,\pi/2)$, $p \in (0,1]$, the function F_p defined by*

$$F_p(t) = \frac{t\cot t - 1}{\ln(\cos pt)} \tag{18}$$

is increasing if $p \in (0,1/2]$ and decreasing if $p \in [\sqrt{10}/5, 1]$. Consequently, for $p \in (0,1/2]$, one has

$$\frac{2}{3p^2} < \frac{t\cot t-1}{\ln(\cos pt)} < -\frac{1}{\ln(\cos(\pi p/2))}. \tag{19}$$

It is reversed if $p \in [\sqrt{10}/5, 1]$.

Proof. For $t \in (0,\pi/2)$, we define $f_1(t) = t\cot t - 1$ and $f_2(t) = \ln(\cos pt)$, where $p \in (0,1]$. Note that $f_1(0^+) = f_2(0^+) = 0$, and $F_p(t)$ can be written as

$$F_p(t) = \frac{f_1(t) - f_1(0^+)}{f_2(t) - f_2(0^+)}. \tag{20}$$

Differentiation and using (14) and (15) yield

$$\begin{aligned}\frac{f_1'(t)}{f_2'(t)} &= \frac{t/\sin^2 t - \cot t}{p\tan pt}\\ &= \left(t\left(\frac{1}{t^2}+\sum_{n=1}^{\infty}\frac{(2n-1)\,2^{2n}}{(2n)!}\left|B_{2n}\right|t^{2n-2}\right)\right.\\ &\quad\left.-\left(\frac{1}{t}-\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}\left|B_{2n}\right|t^{2n-1}\right)\right)\\ &\quad\times\left(\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}p^{2n}\left|B_{2n}\right|t^{2n-1}\right)^{-1}\\ &= \frac{\sum_{n=1}^{\infty}\left(2^{2n}/(2n)!\right)2n\left|B_{2n}\right|t^{2n-1}}{\sum_{n=1}^{\infty}\left((2^{2n}-1)/(2n)!\right)2^{2n}p^{2n}\left|B_{2n}\right|t^{2n-1}}\\ &:= \frac{\sum_{n=1}^{\infty}a_n t^{2n-1}}{\sum_{n=1}^{\infty}b_n t^{2n-1}},\end{aligned} \tag{21}$$

where

$$a_n = \frac{2^{2n}}{(2n)!}2n\left|B_{2n}\right|, \qquad b_n = \frac{2^{2n}-1}{(2n)!}2^{2n}p^{2n}\left|B_{2n}\right|. \tag{22}$$

Clearly, if the monotonicity of a_n/b_n is proved, then by Lemma 3 we can get the monotonicity of f_1'/f_2', and then the monotonicity of the function F_p easily follows from Lemma 2. For this purpose, since $a_n, b_n > 0$, for $n \in \mathbb{N}$, we only need to show that b_n/a_n is decreasing if $p \in (0, 1/2]$ and increasing if $p \in [\sqrt{10}/5, 1]$. Indeed, an elementary computation yields

$$\begin{aligned}\frac{b_{n+1}}{a_{n+1}}-\frac{b_n}{a_n} &= \frac{1}{2n+2}p^{2n+2}\left(2^{2n+2}-1\right)-\frac{1}{2n}p^{2n}\left(2^{2n}-1\right)\\ &= \frac{4^{n+1}-1}{2n+2}p^{2n}\left(p^2-\frac{n+1}{n}\frac{4^n-1}{4^{n+1}-1}\right).\end{aligned} \tag{23}$$

It is easy to obtain that, for $n \in \mathbb{N}$,

$$\frac{b_{n+1}}{a_{n+1}}-\frac{b_n}{a_n}\begin{cases}\geq 0 & \text{if } p^2 > \max_{n\in\mathbb{N}}\left(\dfrac{n+1}{n}\dfrac{4^n-1}{4^{n+1}-1}\right)=\dfrac{2}{5},\\ \leq 0 & \text{if } p^2 \leq \min_{n\in\mathbb{N}}\left(\dfrac{n+1}{n}\dfrac{4^n-1}{4^{n+1}-1}\right)=\dfrac{1}{4},\end{cases} \tag{24}$$

which proves the monotonicity of a_n/b_n.

Making use of the monotonicity of F_p and the facts that

$$F_p\left(0^+\right)=\frac{2}{3p^2}, \qquad F_p\left(\frac{\pi^-}{2}\right)=-\frac{1}{\ln\left(\cos\left(\pi p/2\right)\right)}, \tag{25}$$

we get inequality (19) and its reverse immediately. □

Lemma 6. *For every $t \in (0, \pi/2)$, $p \in (0, 1]$, the function G_p defined by*

$$G_p(t) = \frac{\ln\left(\sin t/t\right)+t\cot t-1}{\ln\cos pt} \tag{26}$$

is increasing if $p \in (0, 1/2]$ and decreasing if $p \in [1/\sqrt{3}, 1]$. Consequently, for $p \in (0, 1/2]$, one has

$$\frac{1}{p^2} < \frac{\ln\left(\sinh t/t\right)+t\cot t-1}{\ln\left(\cos pt\right)} < \frac{\ln 2-\ln\pi-1}{\ln\left(\cos\left(\pi p/2\right)\right)}. \tag{27}$$

It is reversed if $p \in [1/\sqrt{3}, 1]$.

Proof. We define $g_1(t) = \ln(\sin t/t) + t\cot t - 1$ and $g_2(t) = \ln(\cos pt)$, where $p \in (0, 1]$. Note that $g_1(0^+) = g_2(0^+) = 0$, and $G_p(t)$ can be written as

$$G_p(t) = \frac{g_1(t)-g_1\left(0^+\right)}{g_2(t)-g_2\left(0^+\right)}. \tag{28}$$

Differentiating and using (14) and (15) yield

$$\begin{aligned}\frac{g_1'(t)}{g_2'(t)} &= \frac{2\left(\cos t/\sin t\right)-1/t-t\left(1/\sin^2 t\right)}{-p\tan\left(pt\right)}\\ &= \left(2\left(\frac{1}{t}-\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}\left|B_{2n}\right|t^{2n-1}\right)-\frac{1}{t}\right.\\ &\quad\left.-t\left(\frac{1}{t^2}+\sum_{n=1}^{\infty}\frac{(2n-1)\,2^{2n}}{(2n)!}\left|B_{2n}\right|t^{2n-2}\right)\right)\\ &\quad\times\left(-p\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}\left|B_{2n}\right|p^{2n-1}t^{2n-1}\right)^{-1}\\ &= \frac{\sum_{n=1}^{\infty}\left(2^{2n}/(2n)!\right)(2n+1)\left|B_{2n}\right|t^{2n-1}}{\sum_{n=1}^{\infty}\left((2^{2n}-1)/(2n)!\right)2^{2n}p^{2n}\left|B_{2n}\right|t^{2n-1}}\\ &:= \frac{\sum_{n=1}^{\infty}c_n t^{2n-1}}{\sum_{n=1}^{\infty}d_n t^{2n-1}},\end{aligned} \tag{29}$$

where

$$c_n = \frac{2^{2n}}{(2n)!}(2n+1)\left|B_{2n}\right|, \qquad d_n = \frac{2^{2n}-1}{(2n)!}2^{2n}p^{2n}\left|B_{2n}\right|. \tag{30}$$

Similarly, we only need to show that d_n/c_n is decreasing if $p \in (0, 1/2]$ and increasing if $p \in [1/\sqrt{3}, 1]$. In fact, simple computation leads to

$$\begin{aligned}\frac{d_{n+1}}{c_{n+1}}-\frac{d_n}{c_n} &= p^{2n+2}\frac{4^{n+1}-1}{2n+3}-p^{2n}\frac{4^n-1}{2n+1}\\ &= \frac{4^{n+1}-1}{2n+3}p^{2n}\left(p^2-\frac{2n+3}{2n+1}\frac{4^n-1}{4^{n+1}-1}\right).\end{aligned} \tag{31}$$

It is easy to obtain that, for $n \in \mathbb{N}$,

$$\frac{d_{n+1}}{c_{n+1}}-\frac{d_n}{c_n}\begin{cases}\geq 0 & \text{if } p^2 \geq \max_{n\in\mathbb{N}}\left(\dfrac{2n+3}{2n+1}\dfrac{4^n-1}{4^{n+1}-1}\right)=\dfrac{1}{3},\\ \leq 0 & \text{if } p^2 \leq \min_{n\in\mathbb{N}}\left(\dfrac{2n+3}{2n+1}\dfrac{4^n-1}{4^{n+1}-1}\right)=\dfrac{1}{4},\end{cases} \tag{32}$$

which proves the monotonicity of c_n/d_n.

Making use of the monotonicity of F_p and the facts that

$$F_p(0^+) = \frac{1}{p^2}, \qquad F_p\left(\frac{\pi^-}{2}\right) = \frac{\ln 2 - \ln\pi - 1}{\ln(\cos(\pi p/2))} \tag{33}$$

we get inequality (27) and its reverse immediately. □

Lemma 7 (see [14, 15]). *For $t \in [0,\pi/2]$ and $p \in [0,1)$, let $U_p(t)$, $V_p(t)$, $W_p(t)$, and $R_p(t)$ be defined by*

$$U_p(t) = (\cos pt)^{1/p} \quad \text{if } p \neq 0, U_0(t) = 1, \tag{34}$$

$$V_p(t) = (\cos pt)^{1/p^2} \quad \text{if } p \neq 0, V_0(t) = e^{-t^2/2}, \tag{35}$$

$$W_p(t) = (\cos pt)^{1/\ln(\cos(\pi p/2))} \quad \text{if } p \neq 0, W_0(t) = e^{4t^2/\pi^2}, \tag{36}$$

$$R_p(t) = \left(\frac{\cos pt}{\cos(\pi p/2)}\right)^{1/p^2} \quad \text{if } p \neq 0, R_0(t) = e^{(\pi^2-4t^2)/8}. \tag{37}$$

Then, $U_p(t)$, $V_p(t)$, and $W_p(t)$ are decreasing with respect to $p \in [0,1)$, while $R_p(t)$ is increasing with respect to p on $[0,1)$.

Proof. It was proved in [14, 15] that the functions $U_p(t)$ and $V_p(t)$ are decreasing with respect to $p \in [0,1)$. Now, we prove that $W_p(t)$ has the same property. Logarithmic differentiation gives that, for $p \in (0,1)$,

$$\begin{aligned}
&\frac{\cos(p\pi/2)}{\sin(p\pi/2)}\ln^2\left(\cos\frac{p\pi}{2}\right) \times \frac{\partial \ln W_p(t)}{\partial p} \\
&\quad = \frac{\pi}{2}\ln(\cos pt) - t\frac{\sin pt}{\cos pt}\frac{\cos(p\pi/2)}{\sin(p\pi/2)}\ln\left(\cos\frac{p\pi}{2}\right) \\
&\quad := \phi_1(p), \\
&\frac{2\cos^2 pt\sin^2(p\pi/2)}{t\ln(\cos(p\pi/2))}\phi_1'(p) = \frac{\pi}{2}\sin 2pt - t\sin\pi p \\
&\qquad := \phi_2(p), \\
&\phi_2'(p) = \pi t(\cos 2pt - \cos p\pi) \geq 0 \quad \text{for } t \in \left[0, \frac{\pi}{2}\right].
\end{aligned} \tag{38}$$

Clearly, $\phi_2'(p) > 0$ for $t \in [0,\pi/2]$ and $p \in (0,1)$, which yields $\phi_2(p) > \phi_2(0) = 0$, and so $\phi_1'(p) \leq 0$. This gives $\phi_1(p) < \phi_1(0) = 0$ and $\partial \ln W_p(t)/\partial p < 0$.

Similarly, we get

$$\begin{aligned}
\frac{\partial \ln W_p(t)}{\partial p} &= \frac{4}{3p^3}\ln\left(\cos\frac{p\pi}{2}\right) - \frac{4}{3p^3}\ln(\cos pt) \\
&\quad - \frac{2t}{3p^2}\tan pt + \frac{\pi}{3p^2}\tan\frac{p\pi}{2}, \\
\frac{\partial^2 \ln W_p(t)}{\partial t\partial p} &= -\frac{2}{3}\frac{2pt - \sin 2pt}{p^2\cos^2 pt} < 0 \\
&\quad \text{for } t \in \left[0, \frac{\pi}{2}\right], \ p \in (0,1),
\end{aligned} \tag{39}$$

which implies that $\partial \ln W_p(t)/\partial p$ is decreasing with respect to t on $[0,\pi/2]$. Therefore,

$$\frac{\partial \ln W_p(t)}{\partial p} > \left.\frac{\partial \ln W_p(t)}{\partial p}\right|_{t=\pi/2} = 0, \tag{40}$$

which proves the desired result. □

3. Main Results

3.1. The First Sharp Bounds for $e^{t\cot t-1}$. In this subsection, we present the sharp bounds for $e^{t\cot t-1}$ in terms of $(\cos pt)^{1/p}$, which give the trigonometric versions of inequalities (6) and (7).

Theorem 8. *For $t \in (0,\pi/2)$, the two-side inequality*

$$\left(\cos\frac{2t}{3}\right)^{3/2} < e^{t\cot t-1} < (\cos p_1 t)^{1/p_1} \tag{41}$$

holds with the best possible constants 2/3 and $p_1 = 0.6505\ldots$, where p_1 is the unique root of the equation

$$1 + \frac{1}{p}\ln\left(\cos\frac{p\pi}{2}\right) = 0 \tag{42}$$

on $(0,1)$. Moreover, one has

$$\left(\cos\frac{2t}{3}\right)^{3/2} < e^{t\cot t-1} < \left(\cos\frac{2t}{3}\right)^{1/\ln 2} < \frac{2\sqrt{2}}{e}\left(\cos\frac{2t}{3}\right)^{3/2}, \tag{43}$$

$$(\cos p_1 t)^{2/(3p_1^2)} < e^{t\cot t-1} < (\cos p_1 t)^{1/p_1}, \tag{44}$$

where the exponents $3/2$, $1/\ln 2$ and coefficients 1, $2\sqrt{2}/e$ in (43) are the best possible constants and so is $p_1 \approx 0.6505536$ in (44).

Proof. (i) We first prove that the left inequality in (41) for $t \in (0,\pi/2)$ and 2/3 is the best possible constant. Letting $p = 2/3 \in [\sqrt{10}/5, 1]$ in (19), then we get the first inequality in (41) and the second inequality in (43). If there exists $p < 2/3$ such that $e^{t\cot t-1} > (\cos pt)^{1/p}$ for $t \in (0,\pi/2)$, then

$$\lim_{t\to 0^+}\frac{t\cot t - 1 - (1/p)\ln(\cos pt)}{t^2} \geq 0. \tag{45}$$

Using power series expansion gives

$$t\cot t-1-\frac{1}{p}\ln(\cos pt)=t^2\left(\frac{1}{2}p-\frac{1}{3}\right)+o\left(t^2\right). \quad (46)$$

Therefore,

$$\lim_{t\to 0^+}\frac{t\cot t-1-(1/p)\ln(\cos pt)}{t^2}=\frac{1}{2}\left(p-\frac{2}{3}\right)\geq 0, \quad (47)$$

which derives a contradiction. Hence, 2/3 is the best possible constant.

(ii) From Lemma 7, we clearly see that the function $p\mapsto 1+(1/p)\ln(\cos(p\pi/2))$ is decreasing on $(0,1)$. Note that

$$\lim_{p\to 0^+}\left(1+\frac{1}{p}\ln\left(\cos\frac{p\pi}{2}\right)\right)=1,$$
$$\lim_{p\to 1^-}\left(1+\frac{1}{p}\ln\left(\cos\frac{p\pi}{2}\right)\right)=-\infty. \quad (48)$$

Therefore, (42) has a unique root $p_1\in(0,1)$. Numerical calculation gives $p_1\approx 0.6505536$. Letting $p=p_1\in[\sqrt{10}/5,1]$ in Lemma 5 yields

$$-\frac{1}{\ln(\cos(\pi p_1/2))}<\frac{t\cot t-1}{\ln(\cos p_1 t)}<\frac{2}{3p_1^2}. \quad (49)$$

The above inequalities can be rewritten as

$$\begin{aligned}\frac{2}{3p_1^2}\ln(\cos p_1 t)&<t\cot t-1\\&<-\frac{1}{\ln(\cos(\pi p_1/2))}\ln(\cos p_1 t)\\&=\frac{1}{p_1}\ln(\cos p_1 t),\end{aligned} \quad (50)$$

where the equality is due to the fact that p_1 is the unique root of (42). Therefore, we get the right inequality in (41) and the first inequality in (44). We clearly see that p_1 is the best possible constant.

(iii) The third inequality in (43) easily follows from

$$\frac{1}{\ln 2}\ln\left(\cos\frac{2t}{3}\right)-\ln\frac{2\sqrt{2}}{e}-\frac{3}{2}\ln\left(\cos\frac{2t}{3}\right)<0, \quad (51)$$

which holds due to $\ln(\cos(2t/3))>\ln(\cos(\pi/3))=-\ln 2$ and $1/\ln 2<3/2$. From

$$\lim_{t\to 0^+}\frac{e^{t\cot t-1}}{(\cos(2t/3))^{3/2}}=1,\qquad \lim_{t\to\pi/2^-}\frac{e^{t\cot t-1}}{(\cos(2t/3))^{3/2}}=\frac{2\sqrt{2}}{e} \quad (52)$$

we clearly see that the coefficients 1 and $2\sqrt{2}/e$ are the best possible constants.

This completes the proof. □

Recently, Yang [16] proved that the inequalities

$$(\cos pt)^{1/p}<\frac{\sin t}{t}<(\cos qt)^{1/q} \quad (53)$$

hold for $t\in(0,\pi/2)$ if and only if $p\in[p_0,1)$ and $q\in(0,1/3]$, where $p_0\approx 0.3473$. Making use of Theorem 8 and Lemma 7, we have the following.

Corollary 9. *For $t\in(0,\pi/2)$, the chain of inequalities*

$$\begin{aligned}\cos t<\cdots<\left(\cos\frac{2t}{3}\right)^{3/2}<e^{t\cot t-1}<(\cos p_1 t)^{1/p_1}\\<\cdots<(\cos p_0 t)^{1/p_0}<\frac{\sin t}{t}<\left(\cos\frac{t}{3}\right)^3\cdots<1\end{aligned} \quad (54)$$

hold with the best possible constants 2/3, $p_1\approx 0.6505$, $p_0\approx 0.3473$, and 1/3.

3.2. The Second Sharp Bounds for $e^{t\cot t-1}$. In this subsection, we give the sharp bounds for $e^{t\cot t-1}$ in terms of $(\cos pt)^{2/(3p^2)}$, which give the trigonometric versions of inequalities (8).

Theorem 10. *For $t\in(0,\pi/2)$, the two-side inequality*

$$\left(\cos\frac{2t}{\sqrt{10}}\right)^{5/3}<e^{t\cot t-1}<(\cos p_2 t)^{2/(3p_2^2)} \quad (55)$$

holds with the best possible constants $2/\sqrt{10}$ and $p_2\approx 0.6210901$, where p_2 is the unique solution of the equation

$$H_p\left(\frac{\pi^-}{2}\right)=-1-\frac{2}{3p^2}\ln\left(\cos\frac{p\pi}{2}\right)=0 \quad (56)$$

on $(1/2,1)$. Moreover, the inequalities

$$(\cos pt)^{2/(3p^2)}<e^{t\cot t-1}<(\cos pt)^{\alpha_p}<\beta_p(\cos pt)^{2/(3p^2)} \quad (57)$$

hold for $p\in[\sqrt{10}/5,1]$, where the exponents

$$\frac{2}{(3p^2)},\quad \alpha_p=-\frac{1}{\ln(\cos(\pi p/2))} \quad (58)$$

and the coefficients

$$1,\qquad \beta_p=e^{-1}\left(\cos\frac{p\pi}{2}\right)^{-2/(3p^2)} \quad (59)$$

are the best possible constants. Also, the first member in (57) is decreasing with respect to p on $(0,1)$, while the third and fourth members are increasing with respect to p on $(0,1)$. The reverse inequality of (57) holds if $p\in(0,1/2]$.

Proof. For $t\in(0,\pi/2)$ and $p\in(0,1)$, we define

$$H_p(t):=t\frac{\cos t}{\sin t}-1-\frac{2}{3p^2}\ln(\cos pt). \quad (60)$$

To prove the desired results, we need two assertions. The first one is

$$\lim_{t\to 0^+}\frac{H_p(t)}{t^4}=\frac{1}{18}\left(p^2-\frac{2}{5}\right), \quad (61)$$

which follows by expanding in power series

$$H_p(t)=\frac{1}{90}\left(5p^2-2\right)t^4+o\left(t^4\right). \quad (62)$$

The second one states that the equation $H_p(\pi/2^-) = 0$, that is, (56), has a unique solution $p_2 \approx 0.6210901$ such that $H_p(\pi/2^-) < 0$ for $p \in (0, p_2)$ and $H_p(\pi/2^-) > 0$ for $p \in (p_2, 1)$. Indeed, Lemma 7 implies that $p \mapsto H_p(\pi/2^-)$ is increasing on $(0,1)$, which together with the facts that

$$H_{1/2}\left(\frac{\pi^-}{2}\right) = \frac{4}{3}\ln 2 - 1 < 0, \qquad H_1\left(\frac{\pi^-}{2}\right) = \infty \tag{63}$$

indicates the second assertion. By using mathematical software, we find $p_2 \approx 0.6210901$.

(i) Now, we prove that the first inequality in (55) holds with the best constant $2/\sqrt{10}$. Letting $p = 2/\sqrt{10}$ in Lemma 5 yields the first inequality in (55). Due to the decreasing property of $p \mapsto (\cos pt)^{2/(3p^2)}$ on $(0,1)$ given by Lemma 7, we assume that there is a $p' \in (0,1)$ with $p' < 2/\sqrt{10}$ such that the left inequality in (55) holds for $t \in (0, \pi/2)$; then we have $\lim_{t\to 0^+} t^{-4} H_{p'}(t) \ge 0$, which together with the relation (61) leads to $(p'^2 - 2/5) \ge 0$. It is clearly impossible. Hence, $2/\sqrt{10}$ is the best constant.

(ii) We next show that the second inequality in (55) holds with the best constant p_2. Let us introduce an auxiliary function h_{p_2} defined on $(0, \pi/2)$ by

$$h_{p_2}(t) = \frac{H'_{p_2}(t)}{t^3}. \tag{64}$$

Expanding in power series gives

$$\begin{aligned} H'_{p_2}(t) &= \frac{\cos t}{\sin t} - \frac{t}{\sin^2 t} + \frac{2}{3p_2}\frac{\sin p_2 t}{\cos p_2 t} \\ &= \left(\frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}\left|B_{2n}\right| t^{2n-1}\right) \\ &\quad - t\left(\frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{(2n-1)\,2^{2n}}{(2n)!}\left|B_{2n}\right| t^{2n-2}\right) \\ &\quad + \frac{2}{3p_2}\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}p_2^{2n-1}\left|B_{2n}\right| t^{2n-1} \\ &:= \sum_{n=2}^{\infty}\frac{r_n}{3\,(2n)!}2^{2n+1}\left|B_{2n}\right| t^{2n-1}, \end{aligned} \tag{65}$$

where

$$r_n = \left(\left(2^{2n}-1\right)p_2^{2n-2} - 3n\right). \tag{66}$$

Therefore, we have

$$h_{p_2}(t) = \frac{H'_{p_2}(t)}{t^3} = \sum_{n=2}^{\infty}\frac{2^{2n+1}\left|B_{2n}\right|}{3\,(2n)!} r_n t^{2n-4}. \tag{67}$$

Differentiation again yields

$$h'_{p_2}(t) = \sum_{n=3}^{\infty}\frac{(2n-4)\,2^{2n+1}\left|B_{2n}\right|}{3\,(2n)!} r_n t^{2n-5}. \tag{68}$$

We claim that $h'_{p_2}(t) > 0$ for $t \in (0, \pi/2)$. It suffices to show that $r_n > 0$ for $n \ge 3$. In fact, $r_3 = 63(p_2^4 - 1/7) > 0$, and r_n satisfies the recursive relation

$$\begin{aligned} \frac{r_{n+1}}{2^{2n+2}-1} - p_2^2\frac{r_n}{2^{2n}-1} &= \frac{3n}{2^{2n}-1}\left(p_2^2 - \frac{n+1}{n}\frac{2^{2n}-1}{2^{2n+2}-1}\right) \\ &:= \frac{3n}{2^{2n}-1}\left(p_2^2 - r'_n\right). \end{aligned} \tag{69}$$

A direct check leads to

$$\begin{aligned} r'_n - r'_{n+1} &= \frac{16\times 2^{4n} - \left(9n^2 + 18n + 17\right)\times 2^{2n} + 1}{n\left(16\times 2^{2n} - 1\right)\left(4\times 2^{2n}-1\right)(n+1)} \\ &:= \frac{r''_n}{n\left(16\times 2^{2n} - 1\right)\left(4\times 2^{2n}-1\right)(n+1)} > 0, \end{aligned} \tag{70}$$

due to $r''_3 = 55809$ and r''_n satisfies the recursive relation

$$r''_{n+1} - 16 r''_n = 12\left(12n + 9n^2 + 8\right)\times 2^{2n} - 15 > 0 \quad \text{for } n \ge 3. \tag{71}$$

Hence, r'_n is decreasing for $n \ge 3$, and so

$$\frac{1}{4} = \lim_{n\to\infty}\frac{n+1}{n}\frac{2^{2n}-1}{2^{2n+2}-1} < r'_n < \left[\frac{n+1}{n}\frac{2^{2n}-1}{2^{2n+2}-1}\right]_{n=3} = \frac{28}{85}, \tag{72}$$

which yields $p_2^2 - r'_n > p_2^2 - 28/85 > 0$. From the recursive relation (69), we get $r_n > 0$ for $n \ge 3$, which proves that $h'_{p_2}(t) > 0$ for $t \in (0, \pi/2)$. Note that

$$h_{p_2}\left(0^+\right) = \lim_{t\to 0^+} h_{p_2}(t) = \frac{2}{9}\left(p_2^2 - \frac{2}{5}\right) < 0. \tag{73}$$

We also assert that $h_{p_2}(\pi/2^-) > 0$. If not, that is, $h_{p_2}(\pi/2^-) \le 0$, then there must be $H'_{p_2}(t) < 0$ for $t \in (0, \pi/2)$, which yields $H_{p_2}(t) < H_{p_2}(0^+) = 0$ and $H_{p_2}(t) > H_{p_2}(\pi/2^-) = 0$ due to p_2 being the solution of the equation $H_p(\pi/2^-) = 0$. This is obviously a contradiction. It follows that there is a $t_1 \in (0, \pi/2)$ such that $h_{p_2}(t) < 0$ for $t \in (0, t_1)$ and $h_{p_2}(t) > 0$ for $t \in (t_1, \pi/2)$, which also implies that H_{p_2} is decreasing on $(0, t_1)$ and increasing on $(t_1, \pi/2)$. Therefore,

$$\begin{aligned} H_{p_2}(t) &< H_{p_2}\left(0^+\right) = 0 \quad \text{for } t \in \left(0, t_1\right), \\ H_{p_2}(t) &< H_{p_2}\left(\frac{\pi^-}{2}\right) = 0 \quad \text{for } t \in \left(t_1, \frac{\pi}{2}\right); \end{aligned} \tag{74}$$

that is, $H_{p_2}(t) < 0$ for $t \in (0, \pi/2)$.

It remains to prove that p_2 is the best possible constant. If there is a $p'_2 \in (0,1)$ with $p'_2 > p_2$ such that the right inequality in (55) holds for $t \in (0, \pi/2)$, then, by the second assertion proved previously, we have $H_{p'_2}(\pi/2^-) > 0$, which yields a contradiction.

(iii) The first and second inequalities in (57) and their reverse ones are clearly the direct consequences of Lemma 5.

It remains to prove the third one. We have to determine the sign of $D_p(t)$ defined by

$$D_p(t) := \alpha_p \ln(\cos pt) - \ln \beta_p - \frac{2}{3p^2} \ln(\cos pt) \tag{75}$$

for $t \in (0, \pi/2)$ and $p \in (0,1)$. Arranging leads to

$$\begin{aligned} D_p(t) &= -\frac{\ln(\cos pt)}{\ln(\cos(\pi p/2))} + 1 + \frac{2}{3p^2}\ln(\cos(\pi p/2)) \\ &\quad - \frac{2}{3p^2}\ln(\cos pt) \\ &= -\left(1 + \frac{2}{3p^2}\ln(\cos(\pi p/2))\right) \\ &\quad \times \frac{\ln(\cos pt) - \ln\ln(\cos(\pi p/2))}{\ln(\cos(\pi p/2))} \\ &= H_p\left(\frac{\pi^-}{2}\right)\frac{\ln(\cos pt) - \ln\ln(\cos(\pi p/2))}{\ln(\cos(\pi p/2))}. \end{aligned} \tag{76}$$

As shown previously, $H_p(\pi/2^-) < 0$ for $p \in (0, p_2)$ and $H_p(\pi/2^-) > 0$ for $p \in (p_2, 1)$, which together with $\ln(\cos pt) > \ln\ln(\cos(\pi p/2))$ and $\ln(\cos(\pi p/2)) < 0$ gives the desired result.

Lemma 7 reveals that the monotonicity of the first, second, and third members in (57) with respect to p on $(0,1)$ due to

$$\begin{gathered} (\cos pt)^{2/(3p^2)} = V_p(t)^{2/3}, \qquad (\cos pt)^{\alpha_p} = W_p(t)^{-1}, \\ \beta_p(\cos pt)^{2/(3p^2)} = e^{-1} R_p(t)^{2/3}. \end{gathered} \tag{77}$$

Finally, we show that β_p is the best possible constant. It easily follows that

$$\begin{gathered} \lim_{t \to 0^+} \frac{e^{t\cot t - 1}}{(\cos pt)^{2/(3p^2)}} = 1, \\ \lim_{t \to \pi/2^-} \frac{e^{t\cot t - 1}}{(\cos pt)^{2/(3p^2)}} = \frac{e^{-1}}{(\cos(\pi p/2))^{2/(3p^2)}} = \beta_p. \end{gathered} \tag{78}$$

Thus, we complete the proof. □

Remark 11. Letting $t = x/2$ and $p_2 = 2p_3$ in Theorem 10 and then taking squares, we deduce that the two-side inequality

$$\cos^{10/3}\frac{x}{\sqrt{10}} < e^{x\cot(x/2) - 2} < (\cos p_3 x)^{1/(3p_3^2)} \tag{79}$$

holds for $x \in (0, \pi)$, where $p_3 = p_2/2 \approx 0.31055$.

From the proof of Theorem 10, we clearly see that the constant $1/\sqrt{10}$ in (79) is the best possible constant, but $p_3 = p_2/2$ is not.

In [15, Theorems 1, 2, and 3], Yang proved that the chain of inequalities

$$\begin{aligned} (\cos p_0^* t)^{1/(3p_0^{*2})} &< \frac{\sin t}{t} < \left(\cos\frac{t}{\sqrt{5}}\right)^{5/3} \\ &< \cdots < e^{-t^2/6} < \frac{2 + \cos t}{3} \end{aligned} \tag{80}$$

holds for $t \in (0, \pi/2)$ with the best constants $1/\sqrt{5}$ and $p_0^* \approx 0.45346$. The monotonicity of the function $p \mapsto (\cos pt)^{1/(3p^2)}$ on $(0,1)$ given in Lemma 7 and Remark 11 lead to the following.

Corollary 12. *For $t \in (0, \pi/2)$, the chain of inequalities*

$$\begin{aligned} (\cos t)^{1/3} &< \cdots < \cos^{5/6}\frac{2t}{\sqrt{10}} < \sqrt{e^{t\cot t - 1}} < (\cos p_2 t)^{1/(3p_2^2)} \\ &< \cdots < (\cos p_0^* t)^{1/(3p_0^{*2})} < \frac{\sin t}{t} < \cos^{5/3}\frac{t}{\sqrt{5}} \\ &< \cdots < \cos^{10/3}\frac{t}{\sqrt{10}} < e^{t\cot(t/2) - 2} < (\cos p_3 t)^{1/(3p_3^2)} \\ &< \cdots < e^{-t^2/6} < \frac{2 + \cos t}{3} \end{aligned} \tag{81}$$

holds with the best possible constants $2/\sqrt{10} \approx 0.63246$, $p_2 \approx 0.6210901$, $p_0^* \approx 0.45346$, $1/\sqrt{5} \approx 0.44721$ and $1/\sqrt{10} \approx 0.31623$, and $p_3 \approx 0.31055$.

Using certain known inequalities and the corollary above, we can obtain the following novel inequalities chain for trigonometric functions.

Corollary 13. *For $t \in (0, \pi/2)$, one has*

$$\begin{aligned} (\cos t)^{1/3} &< \left(\frac{\sin t}{t}\cos t\right)^{1/4} < \left(\frac{1}{2}\frac{\sin t}{t} + \frac{1}{2}\cos t\right)^{1/2} \\ &< \sqrt{\frac{2}{3}\cos t + \frac{1}{3}} \\ &< \left(\cos\frac{2t}{3}\right)^{3/4} < \sqrt{e^{t\cot t - 1}} < \left(\cos\frac{t}{2}\right)^{4/3} < \frac{\sin t}{t} \\ &< \frac{2\cos(t/2) + \cos^2(t/2)}{3} < \left(\frac{2}{3}\cos\frac{t}{2} + \frac{1}{3}\right)^2 \\ &< \left(\cos\frac{t}{3}\right)^3 < e^{t\cot(t/2) - 2} < \left(\cos\frac{t}{4}\right)^{16/3} \\ &< \left(\cos\frac{t}{6}\right)^{12} < e^{-t^2/6} < \frac{2}{3} + \frac{1}{3}\cos t < \frac{e^{t\cot t - 1} + 1}{2}. \end{aligned} \tag{82}$$

Proof. The first, second, and third inequalities in (82) are due to Neuman [17, Theorem 1].

The fourth one in (82) is equivalent to

$$l(t) := \left(\frac{2}{3}\cos t + \frac{1}{3}\right)\left(\cos\frac{2t}{3}\right)^{-3/2} < 1, \tag{83}$$

which holds due to

$$l'(t) = -\frac{2}{3}\left(\cos\frac{2t}{3}\right)^{-5/2}\left(\sin\frac{t}{3}\right)\left(1-\cos\frac{t}{3}\right) < 0 \tag{84}$$

for $t \in (0, \pi/2)$.

The eighth one is derived from Neuman and Sándor [18, (2.5)].

The ninth one easily follows from

$$\begin{aligned}&\frac{2\cos(t/2)+\cos^2(t/2)}{3} - \left(\frac{2}{3}\cos\frac{t}{2}+\frac{1}{3}\right)^2\\ &\quad = -\frac{1}{9}\left(\cos\frac{t}{2}-1\right)^2 < 0.\end{aligned} \tag{85}$$

The tenth, eleventh, and twelfth ones can be obtained by [19, (3.9)].

Except the last one, other ones are obviously deduced from Corollary 12.

The last one is equivalent to

$$e^{t\cot t-1} > \frac{2}{3}\cos t + \frac{1}{3}, \tag{86}$$

which follows from the inequality connecting the fourth and sixth members in (82) proved previously.

Thus, the proof is complete. □

Remark 14. Sándor [20, page 81, Lemma 2.2] proved that the inequality

$$\ln\frac{t}{\sin t} < \frac{\sin t - t\cos t}{2\sin t} \tag{87}$$

holds for $t \in (0, \pi/2)$. Clearly, the sixth and seventh inequalities in (82), that is, for $t \in (0, \pi/2)$,

$$\sqrt{e^{t\cot t-1}} < \left(\cos\frac{t}{2}\right)^{4/3} < \frac{\sin t}{t}, \tag{88}$$

are a refinement of Sándor's inequality.

Remark 15. Using the decreasing property of the function l defined by (83) proved in Corollary 13, we also get $1 = l(0^+) > l(t) > l(\pi/2^-) = 2\sqrt{2}/3$ for $t \in (0, \pi/2)$, which can be rewritten as

$$\frac{2}{3}\cos t + \frac{1}{3} < \left(\cos\frac{2t}{3}\right)^{3/2} < \frac{2}{2\sqrt{2}}\cos t + \frac{1}{2\sqrt{2}}. \tag{89}$$

This in conjunction with (43) gives

$$\begin{aligned}\frac{2\cos t+1}{3} &< \left(\cos\frac{2t}{3}\right)^{3/2} < e^{t\cot t-1} < \left(\cos\frac{2t}{3}\right)^{1/\ln 2}\\ &< \frac{2\sqrt{2}}{e}\left(\cos\frac{2t}{3}\right)^{3/2} < \frac{2\cos t+1}{e}.\end{aligned} \tag{90}$$

From

$$\lim_{t\to 0^+}\frac{e^{t\cot t-1}}{2\cos t+1} = \frac{1}{3}, \qquad \lim_{t\to\pi/2^-}\frac{e^{t\cot t-1}}{2\cos t+1} = \frac{1}{e}, \tag{91}$$

we conclude that $1/3$ and $1/e$ are also the best possible constants.

Further, we conjecture that

$$\begin{aligned}\frac{2\cos t+1}{3} &< \left(\cos\frac{2t}{3}\right)^{3/2} < e^{t\cot t-1}\\ &< \left(\cos\frac{2t}{3}\right)^{1/\ln 2} < \left(\frac{2\cos t+1}{3}\right)^{1/\ln 3}\end{aligned} \tag{92}$$

hold for $t \in (0, \pi/2)$, where all exponents are optimal.

Taking $p = 1/2$, $1/3$, and 0^+ in (57), we get the following.

Corollary 16. *For $t \in (0, \pi/2)$, we have*

$$\begin{aligned}e^{-4t^2/\pi^2} &< \left(\cos\frac{t}{3}\right)^{2/(\ln 4-\ln 3)} < \left(\cos\frac{t}{2}\right)^{2/\ln 2}\\ &< e^{t\cot t-1} < \left(\cos\frac{t}{2}\right)^{8/3} < \left(\cos\frac{t}{3}\right)^{6} < e^{-t^2/3},\end{aligned} \tag{93}$$

$$\begin{aligned}e^{(\pi^2-12)/12}e^{-t^2/3} &< \frac{64}{27e}\left(\cos\frac{t}{3}\right)^{6} < \frac{2\sqrt[3]{2}}{e}\left(\cos\frac{t}{2}\right)^{8/3}\\ &< e^{t\cot t-1} < \left(\cos\frac{t}{2}\right)^{8/3} < \left(\cos\frac{t}{3}\right)^{6} < e^{-t^2/3},\end{aligned} \tag{94}$$

where $\alpha_{1/2} = 2/\ln 2 \approx 2.8854$, $\alpha_{1/3} = 2/(\ln 4 - \ln 3) \approx 6.9521$ and $\beta_{1/2} = 2\sqrt[3]{2}e^{-1} \approx 0.92700$, $\beta_{1/3} = 64e^{-1}/27 \approx 0.87201$ are the best possible constants.

Remark 17. The inequalities connecting the first, fourth, and seventh members in (93) state that, for $t \in (0, \pi/2)$,

$$e^{-4t^2/\pi^2} < e^{t\cot t-1} < e^{-t^2/3}, \tag{95}$$

which can be written as

$$1 - \frac{4t^2}{\pi^2} < \frac{t}{\tan t} < 1 - \frac{t^2}{3} \tag{96}$$

or

$$\frac{3}{3-t^2} < \frac{\tan t}{t} < \frac{\pi^2}{\pi^2 - 4t^2}. \tag{97}$$

It is easy to check that this double inequality is stronger than the new Redheffer-type one for $\tan t$ proved by Zhu and Sun [21, Theorem 3]; that is, for $t \in (0, \pi/2)$,

$$\left(\frac{\pi^2+4t^2}{\pi^2-4t^2}\right)^{\pi^2/24} < \frac{3}{3-t^2} < \frac{\tan t}{t} < \frac{\pi^2}{\pi^2-4t^2} < \frac{\pi^2+4t^2}{\pi^2-4t^2}. \tag{98}$$

Remark 18. Making use of the double inequalities

$$\left(\cos\frac{t}{2}\right)^{4/3} < \frac{\sin t}{t} < \left(\cos\frac{t}{2}\right)^{2(\ln\pi-\ln 2)/\ln 2},$$
$$\left(\cos\frac{t}{2}\right)^{4/3} < \frac{\sin t}{t} < \frac{2^{5/3}}{\pi}\left(\cos\frac{t}{2}\right)^{4/3}, \quad (99)$$

for $t \in (0, \pi/2)$ proved in [22] and [15, Corollary 3], respectively and taking into account (93) and (94), we easily obtain

$$\left(\frac{\sin t}{t}\right)^{1/\ln(\pi/2)} < \left(\cos\frac{t}{2}\right)^{2/\ln 2} < e^{t\cot t-1}$$
$$< \left(\cos\frac{t}{2}\right)^{8/3} < \left(\frac{\sin t}{t}\right)^{2}, \quad (100)$$

$$\frac{\pi^2}{4e}\left(\frac{\sin t}{t}\right)^{2} < \frac{2^{4/3}}{e}\left(\cos\frac{t}{2}\right)^{8/3} < e^{t\cot t-1}$$
$$< \left(\cos\frac{t}{2}\right)^{8/3} < \left(\frac{\sin t}{t}\right)^{2}. \quad (101)$$

3.3. The Sharp Bounds for $(\sin t/t)e^{t\cot t-1}$. In this subsection, we establish sharp inequalities between $(t^{-1}\sin t)e^{t\cot t-1}$ and $(\cos pt)^{1/p^2}$ and prove the trigonometric version of inequalities (9) and (10). Employing Lemmas 6 and 7, we have the following.

Theorem 19. *For* $t \in (0, \pi/2)$, *the two-side inequality*

$$\cos^3\frac{t}{\sqrt{3}} < \frac{\sin t}{t}e^{t\cot t-1} < \left(\cos p_4 t\right)^{1/p_4^2} \quad (102)$$

holds with the best possible constants $1/\sqrt{3}$ *and* $p_4 \approx 0.5763247$, *where* p_4 *is the unique root of the equation*

$$J_p\left(\frac{\pi^-}{2}\right) = \ln\frac{2}{\pi} - 1 - \frac{1}{p^2}\ln\left(\cos\frac{p\pi}{2}\right) = 0 \quad (103)$$

on $(1/2, 1)$. *Moreover, the inequalities*

$$\left(\cos pt\right)^{1/p^2} < \frac{\sin t}{t}\exp\left(t\cot t - 1\right) < \left(\cos pt\right)^{\gamma_p}$$
$$< \delta_p\left(\cos pt\right)^{1/p^2} \quad (104)$$

hold for $p \in [1/\sqrt{3}, 1]$, *where the exponents*

$$\frac{1}{p^2}, \qquad \gamma_p = \frac{\ln 2 - \ln\pi - 1}{\ln\left(\cos\left(\pi p/2\right)\right)} \quad (105)$$

and the coefficients

$$1, \qquad \delta_p = \frac{2}{\pi e}\left(\cos\frac{p\pi}{2}\right)^{-1/p^2} \quad (106)$$

are the best possible constants. Also, the first member in (104) *is decreasing with respect to p on* $(0,1)$, *while the third and fourth members are increasing with respect to p on* $(0,1)$. *The reverse of* (104) *holds if* $p \in (0, 1/2]$.

Proof. For $t \in (0, \pi/2)$ and $p \in (0,1)$, we define

$$J_p(t) := \ln\frac{\sin t}{t} + \left(t\frac{\cos t}{\sin t} - 1\right) - \frac{1}{p^2}\ln\left(\cos pt\right). \quad (107)$$

To prove the desired results, we need two assertions. The first is the limit relation

$$\lim_{t\to 0^+}\frac{J_p(t)}{t^4} = \frac{1}{12}\left(p^2 - \frac{1}{3}\right), \quad (108)$$

which follows by expanding in power series

$$J_p(t) = \frac{1}{36}\left(3p^2 - 1\right)t^4 + o\left(t^4\right). \quad (109)$$

The second one states that the equation $J_p(\pi/2^-) = 0$, that is, (103), has a unique solution $p_4 \approx 0.5763247$ such that $J_p(\pi/2) < 0$ for $p \in (0, p_4)$ and $J_p(\pi/2^-) > 0$ for $p \in (p_4, 1)$. In fact, Lemma 7 implies that $p \mapsto J_p(\pi/2^-)$ is increasing on $(0,1)$, which in conjunction with the facts that

$$J_{1/2}\left(\frac{\pi}{2}\right) = 3\ln 2 - \ln\pi - 1 < 0, \qquad J_1\left(\frac{\pi}{2}\right) = \infty \quad (110)$$

indicates the second one. By using mathematical software, we find $p_2 \approx 0.5763247$.

(i) Now we show that the first inequality in (102) holds for $t \in (0, \pi/2)$ with the best constants $1/\sqrt{3}$. In fact, the first inequality in (102) follows by Lemma 6. On the other hand, due to the decreasing property of $p^{-2}\ln(\cos pt)$ with respect to p on $(0,1)$, if there is a smaller $p^* \in (0,1)$ with $p^* < 1/\sqrt{3}$ such that the first inequality in (102) holds for $t \in (0, \pi/2)$, then there must be $\lim_{t\to 0^+}t^{-4}J_{p^*}(t) \geq 0$, which by the relation (108) gives $p^* \geq 1/\sqrt{3}$. This yields a contradiction. Consequently, the constants $1/\sqrt{3}$ is optimal.

(ii) We next prove that the second inequality in (102) holds for $t \in (0, \pi/2)$, where p_4 is the best possible constant. We introduce an auxiliary function j_{p_4} defined on $(0, \pi/2)$ by

$$j_{p_4}(t) = \frac{J'_{p_4}(t)}{t^3}. \quad (111)$$

Expanding in power series leads to

$$\begin{aligned} J'_{p_4}(t) &= 2\frac{\cos t}{\sin t} - \frac{t}{\sin^2 t} + \frac{1}{p_4}\frac{\sin p_4 t}{\cos p_4 t} - \frac{1}{t} \\ &= 2\left(\frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}\left|B_{2n}\right|t^{2n-1}\right) \\ &\quad - t\left(\frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{(2n-1)\,2^{2n}}{(2n)!}\left|B_{2n}\right|t^{2n-2}\right) \\ &\quad + \frac{1}{p_4}\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}p_4^{2n-1}\left|B_{2n}\right|t^{2n-1} - \frac{1}{t} \\ &= \sum_{n=2}^{\infty}\frac{\left(2^{2n}-1\right)p_4^{2n-2} - (2n+1)}{(2n)!}2^{2n}\left|B_{2n}\right|t^{2n-1} \\ &:= \sum_{n=2}^{\infty}\frac{2^{2n}\left|B_{2n}\right|}{(2n)!}s_n t^{2n-1}, \end{aligned} \quad (112)$$

where

$$s_n = \left(2^{2n} - 1\right) p_4^{2n-2} - (2n+1). \tag{113}$$

Therefore, we have

$$j_{p_4}(t) = \frac{J'_{p_4}(t)}{t^3} = \sum_{n=2}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} s_n t^{2n-4}. \tag{114}$$

Differentiation again yields

$$j'_{p_4}(t) = \sum_{n=3}^{\infty} \frac{(2n-4)\,2^{2n}|B_{2n}|}{(2n)!} s_n t^{2n-5},$$
$$\left(\frac{j'_{p_4}(t)}{t}\right)' = \sum_{n=4}^{\infty} \frac{(2n-4)(2n-6)\,2^{2n}|B_{2n}|}{(2n)!} s_n t^{2n-7}. \tag{115}$$

We assert that $(t^{-1}j'_{p_4}(t))' > 0$ for $t \in (0,\pi/2)$. It suffices to show that $s_n > 0$ for $n \geq 4$. In fact, $s_4 = 3(85p_4^6 - 3) > 0$, and s_n satisfies the recursive relation

$$\frac{s_{n+1}}{2^{2n+2}-1} - p_4^2 \frac{s_n}{2^{2n}-1} = \frac{2n+1}{2^{2n}-1}\left(p_4^2 - \frac{2n+3}{2n+1}\frac{2^{2n}-1}{2^{2n+2}-1}\right) := \frac{2n+1}{2^{2n}-1}\left(p_4^2 - s'_n\right). \tag{116}$$

A direct check gives $[64\times 2^{4n} - (36n^2+108n+113)2^{2n}+4]_{n=4} = 3907332$,

$$s'_n - s'_{n+1} = \frac{64\times 2^{4n} - \left(36n^2+108n+113\right)2^{2n}+4}{(2n+3)(2n+1)\left(16\times 2^{2n}-1\right)\left(4\times 2^{2n}-1\right)} := \frac{s''_n}{(2n+3)(2n+1)\left(16\times 2^{2n}-1\right)\left(4\times 2^{2n}-1\right)} > 0 \tag{117}$$

due to $s''_4 = 3907332$ and s''_n satisfies the recursive relation

$$s''_{n+1} - 16 s''_n = 12\left(36n^2+84n+65\right)2^{2n} - 60 > 0 \quad \text{for } n \geq 4. \tag{118}$$

Hence, s'_n is decreasing for $n \geq 4$, and so

$$\frac{1}{4} = \lim_{n\to\infty} \frac{n+1}{n}\frac{2^{2n}-1}{2^{2n+2}-1} < s'_n < \left[\frac{2n+3}{2n+1}\frac{2^{2n}-1}{2^{2n+2}-1}\right]_{n=4} = \frac{85}{279}, \tag{119}$$

which yields $p_4^2 - s'_n > p_4^2 - 85/279 > 0$. From the recursive relation (116), we get $s_n > 0$ for $n \geq 4$, which proves that $(t^{-1}j'_{p_4}(t))' > 0$ for $t \in (0,\pi/2)$. Therefore, we get

$$\operatorname{sgn}\lim_{t\to 0^+}\left(t^{-1}j'_{p_4}(t)\right) = \operatorname{sgn} s_3 = \operatorname{sgn}\left(7\left(3p_4^2-1\right)\left(3p_4^2+1\right)\right) < 0. \tag{120}$$

Next, we divide the proof into two cases.

Case 1 ($(t^{-1}j'_{p_4}(t))|_{t=\pi/2^-} < 0$). In this case, we clearly see that $t^{-1}j'_{p_4}(t) < 0$ for $t \in (0,\pi/2)$ and $j'_{p_4}(t) < 0$ for $t \in (0,\pi/2)$. Hence, $j_{p_4}(t) < j_{p_4}(0^+) = (3p_4^2-1)/9 < 0$, and so $J'_{p_4}(t) < 0$ for $t \in (0,\pi/2)$, which reveals that $J_{p_4}(t) < J_{p_4}(0^+) = 0$ and $J_{p_4}(t) > J_{p_4}(\pi/2^-) = 0$ for $t \in (0,\pi/2)$, where $J_{p_4}(\pi/2^-) = 0$ due to p_4 being the unique root of (103). This is impossible.

Case 2 ($(t^{-1}j'_{p_4}(t))|_{t=\pi/2^-} > 0$). In this case, we see that there is a $t_2 \in (0,\pi/2)$ such that $t^{-1}j'_{p_4}(t) < 0$ for $t \in (0,t_2)$ and $t^{-1}j'_{p_4}(t) > 0$ for $t \in (t_2,\pi/2)$. This indicates that j_{p_4} is decreasing on $(0,t_2)$ and increasing on $(t_2,\pi/2)$. Thus, we have $j_{p_4}(t) < j_{p_4}(0^+) = (3p_4^2-1)/9 < 0$ for $t \in (0,t_2)$.

If $j_{p_4}(\pi/2^-) < 0$, then $j_{p_4}(t) < 0$ for $t \in (0,\pi/2)$. Similar to Case 1, this also yields a contradiction.

If $j_{p_4}(\pi/2^-) > 0$, then there is a $t_3 \in (t_2,\pi/2)$ such that $j_{p_4}(\pi/2^-) < 0$ for $t \in (0,t_3)$ and $j_{p_4}(\pi/2^-) > 0$ for $t \in (t_3,\pi/2)$, which together with (111) shows that J_{p_4} is decreasing on $(0,t_3)$ and increasing on $(t_3,\pi/2)$. Therefore,

$$J_{p_4}(t) < J_{p_4}\left(0^+\right) = 0 \quad \text{for } t \in \left(0,t_3\right),$$
$$J_{p_4}(t) < J_{p_4}\left(\frac{\pi}{2^-}\right) = 0 \quad \text{for } t \in \left(t_3,\frac{\pi}{2}\right); \tag{121}$$

that is, $J_{p_4}(t) < 0$ for $t \in (0,\pi/2)$.

On the other hand, if there is a $p^* \in (0,1)$ with $p^* > p_4$ such that the second inequality in (55) holds for $t \in (0,\pi/2)$, then by the second assertion proved previously, we have $J_{p^*}(\pi/2^-) > 0$, which leads to a contradiction. This proves that the constant p_4 is the best possible constant.

(iii) The first and second inequalities in (57) and their reverse ones are clearly the direct consequences of Lemma 6. It remains to prove the third one. We have to determine the sign of $E_p(t)$ defined by

$$E_p(t) := \gamma_p \ln(\cos pt) - \ln\delta_p - \frac{1}{p^2}\ln(\cos pt) \tag{122}$$

for $t \in (0,\pi/2)$ and $p \in (0,1)$. Simplifying leads to

$$\begin{aligned} E_p(t) &= \frac{\ln 2 - \ln\pi - 1}{\ln(\cos(\pi p/2))}\ln(\cos pt) - \ln\frac{2}{\pi e} + \frac{1}{p^2}\ln\left(\cos\left(\frac{\pi p}{2}\right)\right) - \frac{1}{p^2}\ln(\cos pt) \\ &= \left(\ln\frac{2}{\pi e} - \frac{1}{p^2}\ln\left(\cos\left(\frac{\pi p}{2}\right)\right)\right)\frac{\ln(\cos pt)}{\ln(\cos(\pi p/2))} - \ln\frac{2}{\pi e} + \frac{1}{p^2}\ln\left(\cos\left(\frac{\pi p}{2}\right)\right) \\ &= J_p\left(\frac{\pi^-}{2}\right)\frac{\ln(\cos pt) - \ln\ln(\cos(\pi p/2))}{\ln(\cos(\pi p/2))}. \end{aligned} \tag{123}$$

As shown previously, $J_p(\pi/2^-) < 0$ for $p \in (0,p_4)$ and $J_p(\pi/2^-) > 0$ for $p \in (p_4,1)$, which in combination with

$\ln(\cos pt) > \ln(\cos(\pi p/2))$ and $\ln(\cos(\pi p/2)) < 0$ gives the desired result.

Lemma 7 reveals the monotonicity of the first, second, and third members in (104) with respect to p on $(0,1)$ due to

$$(\cos pt)^{1/p^2} = V_p(t), \qquad (\cos pt)^{\gamma_p} = W_p(t)^{\ln(2/\pi e)},$$
$$\delta_p(\cos pt)^{2/(3p^2)} = \frac{2}{\pi e}R_p(t). \tag{124}$$

Finally, we prove that β_p is the best possible constant. It can be deduced from

$$\lim_{t\to 0^+}\frac{(\sin t/t)\,e^{t\cot t-1}}{(\cos pt)^{1/p^2}} = 1,$$
$$\lim_{t\to\pi/2^-}\frac{(\sin t/t)\,e^{t\cot t-1}}{(\cos pt)^{1/p^2}} = \frac{2}{\pi e}\left(\cos\frac{p\pi}{2}\right)^{-1/p^2} = \delta_p. \tag{125}$$

Thus, the proof is complete. □

We note that (102) can be written as

$$\cos\frac{t}{\sqrt{3}} < \left(\frac{\sin t}{t}e^{t\cot t-1}\right)^{1/3} < (\cos p_4 t)^{1/(3p_4^2)}. \tag{126}$$

Making use of the monotonicity of the function $p \mapsto (\cos pt)^{1/(3p^2)}$ on $(0,1)$ given in Lemma 7 together with Corollary 12 and Theorem 19, we obtain the following.

Corollary 20. *For $t \in (0,\pi/2)$, the chain of inequalities*

$$\begin{aligned}(\cos t)^{1/3} &< \cdots < \cos^{5/6}\frac{2t}{\sqrt{10}} < \sqrt{e^{t\cot t-1}}\\ &< (\cos p_2 t)^{1/(3p_2^2)} < \cdots\\ < \cos\frac{t}{\sqrt{3}} &< \left(\frac{\sin t}{t}e^{t\cot t-1}\right)^{1/3} < (\cos p_4 t)^{1/(3p_4^2)} < \cdots\\ &< (\cos p_0^* t)^{1/(3p_0^{*2})} < \frac{\sin t}{t} < \cos^{5/3}\frac{t}{\sqrt{5}}\\ &< \cdots < \cos^{10/3}\frac{t}{\sqrt{10}} < e^{t\cot(t/2)-2} < (\cos p_3 t)^{1/(3p_3^2)}\\ &< \cdots < e^{-t^2/6} < \frac{2+\cos t}{3}\end{aligned} \tag{127}$$

holds, where $2/\sqrt{10} \approx 0.63246$, $p_2 \approx 0.6210901$, $1/\sqrt{3} \approx 0.57735$, $p_4 \approx 0.5763247$, $p_0^* \approx 0.45346$, $1/\sqrt{5} \approx 0.44721$, and $1/\sqrt{10} \approx 0.31623$ are the best possible constants, and $p_3 \approx 0.31055$.

Remark 21. From the above corollary, we clearly see that

$$\begin{aligned}(\cos t)^{1/3} &< \sqrt{e^{t\cot t-1}} < \left(\frac{\sin t}{t}e^{t\cot t-1}\right)^{1/3}\\ &< \left(\cos\frac{t}{2}\right)^{4/3} < \frac{\sin t}{t} < e^{-t^2/6} < \frac{2+\cos t}{3}\end{aligned} \tag{128}$$

for $t \in (0,\pi/2)$. The relation connecting the first, third, and fourth members in (128) can be written as

$$\sqrt{\cos t} < \sqrt{\frac{\sin t}{t}e^{t\cot t-1}} < \cos^2\frac{t}{2}. \tag{129}$$

Taking $p = 1/2, 0^+$ in Theorem 19, we have the following.

Corollary 22. *For $t \in (0,\pi/2)$, the inequalities*

$$\frac{8}{\pi e}\cos^4\frac{t}{2} < \left(\cos\frac{t}{2}\right)^{\gamma_{1/2}} < \frac{\sin t}{t}e^{t\cot t-1} < \cos^4\frac{t}{2}, \tag{130}$$

$$\frac{2}{\pi e}e^{(\pi^2-4t^2)/8} < \left(\frac{2}{\pi e}\right)^{4t^2/\pi^2} < \frac{\sin t}{t}e^{t\cot t-1} < e^{-t^2/2} \tag{131}$$

hold, where the exponents $\gamma_{1/2} = 2(\ln(\pi e/2))/\ln 2 \approx 4.1884$ and 4 and the coefficients 1 and $8/(\pi e) \approx 0.93680$ are the best possible constants.

Theorem 23. *For $t \in (0,\pi/2)$, we have*

$$\frac{(e(\pi-2)/\pi)\,e^{t\cot t-1} + \sin t/t}{2} < \frac{1+\cos t}{2} < \frac{\sin t/t + e^{t\cot t-1}}{2}, \tag{132}$$

$$1 < \frac{\sin t/t + e^{t\cot t-1}}{1+\cos t} < e^{-1} + \frac{2}{\pi}, \tag{133}$$

$$\frac{2}{\pi} - e^{-1} < \frac{\sin t/t - e^{t(\cos t/\sin t)-1}}{1-\cos t} < \frac{1}{3}, \tag{134}$$

where $e(\pi-2)/\pi$, 1, $(e^{-1}+2/\pi)$, $2/\pi - e^{-1}$, and $1/3$ are the best possible constants.

Proof. (i) We first prove (132). For this purpose, let us define

$$k(t) = (t\cot t - 1) - \ln\left(1 + \cos t - \frac{\sin t}{t}\right). \tag{135}$$

Differentiating $k(t)$ gives

$$k'(t) = \frac{t-\sin t}{t(t-\sin t + t\cos t)}k_1(t), \tag{136}$$

where

$$k_1(t) = -\frac{\cos t + 1}{\sin^2 t}t^2 + \frac{t}{\sin t} + 1. \tag{137}$$

Using double angle formula and Lemma 4, we have

$$\begin{aligned}k_1(t) &= -\frac{t^2}{2\sin^2(t/2)} + \frac{t}{\sin t} + 1\\ &= -\frac{t^2}{2}\left(\frac{1}{(t/2)^2} + \sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}}{(2n)!}|B_{2n}|\left(\frac{t}{2}\right)^{2n-2}\right)\\ &\quad + t\left(\frac{1}{t} + \sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|t^{2n-1}\right) + 1\\ &= \sum_{n=1}^{\infty}\frac{4^{n-1}-n}{(2n)!}|B_{2n}|t^{2n} > 0.\end{aligned} \tag{138}$$

Hence, $k'(t) > 0$ for $t \in (0, \pi/2)$, and so

$$0 = \lim_{t\to 0^+} k(t) < k(t) < \lim_{t\to\pi/2^-} k(t) = \ln\frac{\pi}{e(\pi-2)}, \quad (139)$$

which implies the desired inequalities.

(ii) Now, we prove (133). Differentiation leads to

$$\begin{aligned}\left(\frac{\sin t/t + e^{t\cot t-1}}{1+\cos t}\right)' &= -\frac{(\cos t+1)(t-\sin t)}{(\sin^2 t)(\cos t+1)^2}e^{t\cot t-1} \\ &\quad + \frac{(\cos t+1)(t-\sin t)}{t^2(\cos t+1)^2} \\ &= \frac{t-\sin t}{(\sin^2 t)(\cos t+1)^2}\left(\frac{\sin^2 t}{t^2} - e^{t\cot t-1}\right) > 0,\end{aligned} \quad (140)$$

where the inequality holds for $t \in (0, \pi/2)$ due to (88). Therefore,

$$\begin{aligned}1 = \lim_{t\to 0^+}\frac{\sin t/t + e^{t\cot t-1}}{1+\cos t} &< \frac{\sin t/t + e^{t\cot t-1}}{1+\cos t} \\ &< \lim_{t\to\pi/2^-}\frac{\sin t/t + e^{t\cot t-1}}{1+\cos t} = e^{-1} + \frac{2}{\pi},\end{aligned} \quad (141)$$

which deduces (133).

(iii) Similarly, we have

$$\begin{aligned}&\left(\frac{\sin t/t - e^{t(\cos t/\sin t)-1}}{1-\cos t}\right)' \\ &\quad = -\frac{t+\sin t}{(1-\cos t)\sin^2 t}\left(\frac{\sin^2 t}{t^2} - e^{t\cot t-1}\right) < 0,\end{aligned} \quad (142)$$

which gives

$$\begin{aligned}\frac{2}{\pi} - e^{-1} = \lim_{t\to 0^+}&\frac{\sin t/t - e^{t(\cos t/\sin t)-1}}{1-\cos t} \\ &< \frac{\sin t/t - e^{t(\cos t/\sin t)-1}}{1-\cos t} \\ &< \lim_{t\to\pi/2^-}\frac{\sin t/t - e^{t(\cos t/\sin t)-1}}{1-\cos t} = \frac{1}{3}.\end{aligned} \quad (143)$$

□

Using inequalities (129) and (133), we get immediately the trigonometric version of (9).

Corollary 24. *For $t \in (0, \pi/2)$, we have*

$$\begin{aligned}\sqrt{\cos t} &< \left(\frac{\sin t}{t}e^{t\cot t-1}\right)^{1/2} < \frac{1+\cos t}{2} \\ &< \frac{\sin t/t + e^{t\cot t-1}}{2} < \left(e^{-1} + \frac{2}{\pi}\right)\frac{1+\cos t}{2}.\end{aligned} \quad (144)$$

3.4. The Third Sharp Bounds for $e^{t\cot t-1}$. The trigonometric versions of (1.6) and (1.7) are contained in the following theorem.

Theorem 25. *Let $t \in (0, \pi/2)$. Then the following statements are true:*

(i) *if $p \geq 6/5$, then the two-side inequality*

$$\alpha\cos^p t + (1-\alpha) < e^{p(t\cot t-1)} < \beta\cos^p t + (1-\beta) \quad (145)$$

holds if and only if $\alpha \geq 1 - e^{-p}$ and $\beta \leq 2/3$;

(ii) *if $0 < p \leq 1$, then the double inequality* (145) *holds if and only if $\alpha \geq 2/3$ and $\beta \leq 1 - e^{-p}$;*

(iii) *if $p < 0$, then the double inequality* (145) *holds if and only if $\alpha \leq 0$ and $\beta \geq 2/3$;*

(iv) *the double inequality*

$$M_p\left(\cos t, 1; \frac{2}{3}\right) < e^{t\cot t-1} < M_q\left(\cos t, 1; \frac{2}{3}\right) \quad (146)$$

holds if and only if $p \leq \ln 3$ and $q \geq 6/5$, where $M_p(x, y; w)(w \in (0,1))$ is the weighted power mean of order r of x and y defined by

$$\begin{aligned}M_r(x, y; w) &= \left(wx^r + (1-w)y^r\right)^{1/r} \\ \text{if } r \neq 0,\ M_0(x, y; w) &= x^w y^{1-w}.\end{aligned} \quad (147)$$

Proof. For $t \in (0, \pi/2)$ and $p \neq 0$, we define

$$u(t) = \frac{1 - e^{p(t\cot t-1)}}{1-\cos^p t} := \frac{u_1(t)}{u_2(t)}. \quad (148)$$

Since $u_1(0^+) = u_2(0^+) = 0$, $u(t)$ can be written as

$$u(t) = \frac{u_1(t) - u_1(0^+)}{u_2(t) - u_2(0^+)}. \quad (149)$$

Differentiation gives

$$\frac{u_1'(t)}{u_2'(t)} = \frac{e^{p(t\cot t-1)}(t - \cos t\sin t)/\sin^2 t}{\cos^{p-1}t\sin t} := u_3(t), \quad (150)$$

$$\begin{aligned}u_3'(t) &= \frac{e^{p(t\cot t-1)}}{\sin^5 t\cos^p t}\left(p \times u_4(t) - u_5(t)\right) \\ &= \frac{e^{p(t\cot t-1)}}{\sin^5 t\cos^p t}u_4(t)\left(p - \frac{u_5(t)}{u_4(t)}\right),\end{aligned} \quad (151)$$

where

$$\begin{aligned}u_4(t) &= -t^2\cos t + 2t\cos^2 t\sin t + t\sin^3 t - \cos t\sin^2 t, \\ u_5(t) &= t\left(3\cos^2 t\sin t + \sin^3 t\right) - 3\cos t\sin^2 t.\end{aligned} \quad (152)$$

Clearly, if we prove that $u_3'(t) > 0$ for $p \geq 6/5$ and $u_3'(t) < 0$ for $p \leq 1$ with $p \neq 0$, then, by Lemma 2, we know that u is increasing if $p \geq 6/5$ and decreasing if $p \leq 1$ with $p \neq 0$, and

$$\begin{aligned} \frac{2}{3} &= \lim_{t\to 0^+} u(t) < u(t) = \frac{1-e^{p(t\cot t-1)}}{1-\cos^p t} \\ &< \lim_{t\to\pi/2^-} u(t) = 1-e^{-p} \quad \text{for } p \geq \frac{6}{5}, \\ 1-e^{-p} &= \lim_{t\to\pi/2^-} u(t) < u(t) = \frac{1-e^{p(t\cot t-1)}}{1-\cos^p t} \\ &< \lim_{t\to 0^+} u(t) = \frac{2}{3} \quad \text{for } 0 < p \leq 1, \\ 0 &= \lim_{t\to\pi/2^-} u(t) < u(t) = \frac{1-e^{p(t\cot t-1)}}{1-\cos^p t} \\ &< \lim_{t\to 0^+} u(t) = \frac{2}{3} \quad \text{for } p < 0, \end{aligned} \tag{153}$$

which yield the first, second, and third results in this theorem.

Now, we show that $u_3'(t) > 0$ if $p \geq 6/5$ and $u_3'(t) < 0$ if $p \leq 1$ with $p \neq 0$. Simple computations lead to

$$u_4(t) = (\sin t - t\cos t)(t - \cos t \sin t) > 0 \tag{154}$$

for $t \in (0, \pi/2)$. Using (15)–(17), we have

$$\begin{aligned} \frac{u_5(t)}{\cos t \sin^2 t} &= 3t\frac{\cos t}{\sin t} + t\frac{\sin t}{\cos t} - 3 \\ &= \sum_{n=1}^{\infty} \frac{4^n-4}{(2n)!} 2^{2n} \left|B_{2n}\right| t^{2n}, \\ \frac{u_4(t)}{\cos t \sin^2 t} &= 2t\frac{\cos t}{\sin t} - t^2\frac{1}{\sin^2 t} + t\frac{\sin t}{\cos t} - 1 \\ &= \sum_{n=1}^{\infty} (4^n - 2n - 2)\frac{2^{2n}}{(2n)!}\left|B_{2n}\right| t^{2n}, \\ \frac{u_5(t)}{u_4(t)} &= \frac{\sum_{n=1}^{\infty}\left((4^n-4)/(2n)!\right) 2^{2n}\left|B_{2n}\right| t^{2n}}{\sum_{n=1}^{\infty}(4^n-2n-2)\left(2^{2n}/(2n)!\right)\left|B_{2n}\right| t^{2n}} \\ &:= \frac{\sum_{n=2}^{\infty} a_n t^{2n}}{\sum_{n=2}^{\infty} b_n t^{2n}}. \end{aligned} \tag{155}$$

By Lemma 3, in order to prove the monotonicity of $u_5(t)/u_4(t)$, it suffices to get the monotonicity of a_n/b_n. Note that

$$\frac{a_n}{b_n} = \frac{4^n-4}{4^n-2n-2} := c(n). \tag{156}$$

Differentiating $c(x)$, we get

$$c'(x) = -2\frac{4^x((x-1)\ln 4 - 1) + 4}{(4^x-2x-2)^2} < 0 \tag{157}$$

for $x \geq 2$. The function $t \mapsto u_5(t)/u_4(t)$ is decreasing on $(0, \pi/2)$, and we conclude that

$$1 = \lim_{t\to\pi/2^-}\frac{u_5(t)}{u_4(t)} < \frac{u_5(t)}{u_4(t)} < \lim_{t\to 0^+}\frac{u_5(t)}{u_4(t)} = \frac{6}{5}. \tag{158}$$

Thus, $u_3'(t) > 0$ if $p \geq 6/5$ and $u_3'(t) < 0$ if $p \leq 1$ with $p \neq 0$.

Finally, we prove the fourth result. The first part implies that the right-hand side inequality in (146) holds if $q \geq 6/5$. While the necessity can be obtained from the following limit relation:

$$\lim_{t\to 0^+}\frac{t\cot t - 1 - \ln M_q(\cos t, 1; 2/3)}{t^4} \leq 0, \tag{159}$$

in fact, power series expansion leads to

$$t\cot t - 1 - \ln M_q\left(\cos t, 1; \frac{2}{3}\right) = -\frac{1}{36}\left(q - \frac{6}{5}\right)t^4 + o\left(t^4\right). \tag{160}$$

Now, we prove that the left-hand side inequality holds if and only if $p \leq \ln 3$. The necessity follows easily from

$$\begin{aligned} &\lim_{t\to\pi/2^-}\left(t\cot t - 1 - \ln M_p\left(\cos t, 1; \frac{2}{3}\right)\right) \\ &\quad = \begin{cases} -1 + \dfrac{1}{p}\ln 3 & \text{if } p > 0 \\ \infty & \text{if } p \leq 0 \end{cases} \geq 0. \end{aligned} \tag{161}$$

Next, we deal with the sufficiency. We divide the proof into two cases.

Case 1 ($p \leq 1$). The sufficiency follows immediately from the second and third results proved previously.

Case 2 ($1 < p \leq \ln 3$). It was proved previously that the function $t \mapsto u_5(t)/u_4(t)$ is decreasing on $(0, \pi/2)$, and so the function $t \mapsto (p - u_5(t)/u_4(t)) := u_6(t)$ is increasing on the same interval. The monotonicity $u_6(t)$ together with

$$u_6(0^+) = p - \frac{6}{5} < 0, \qquad u_6\left(\frac{\pi^-}{2}\right) = p - 1 > 0 \tag{162}$$

leads to the conclusion that there exists unique $t_0 \in (0, \pi/2)$ such that $u_6(t) < 0$ for $t \in (0, t_0)$ and $u_6(t) > 0$ for $t \in (t_0, \pi/2)$; then, from (151), we know that u_3 is decreasing on $(0, t_0)$ and increasing on $(t_0, \pi/2)$. It follows from Lemma 2 that u is decreasing on $(0, t_0)$, and so we have

$$u(t_0) \leq u(t) = \frac{1-e^{p(t\cot t-1)}}{1-\cos^p t} < u(0^+) = \frac{2}{3} \quad \text{for } t \in (0, t_0], \tag{163}$$

which can be rewritten as

$$e^{p(t\cot t-1)} > \frac{2}{3}\cos^p t + \frac{1}{3} \quad \text{for } t \in (0, t_0]. \tag{164}$$

On the other hand, Lemma 2 also implies that

$$t \longmapsto \frac{u_1(t) - u_1(\pi/2^-)}{u_2(t) - u_2(\pi/2^-)} = \frac{e^{p(t\cot t-1)} - e^{-p}}{\cos^p t} := v(t) \tag{165}$$

is increasing on $(t_0, \pi/2)$. Therefore,

$$\begin{aligned} v(t) = \frac{e^{p(t\cot t-1)} - e^{-p}}{\cos^p t} &> \frac{e^{p(t_0\cot t_0-1)} - e^{-p}}{\cos^p t_0} \\ &\text{for } t \in \left(t_0, \frac{\pi}{2}\right), \end{aligned} \tag{166}$$

which implies that

$$e^{p(t\cot t-1)} > \frac{e^{p(t_0\cot t_0-1)}-e^{-p}}{\cos^p t_0}\cos^p t+e^{-p} \quad \text{for } t\in\left(t_0,\frac{\pi}{2}\right). \tag{167}$$

Clearly, if we can prove that the right-hand side in (167) is also greater than the right-hand side in (164), then the proof is completed. Since t_0 satisfies (164), for $t\in(t_0,\pi/2)$, we have

$$\begin{aligned} e^{p(t\cot t-1)} &> \frac{e^{p(t_0\cot t_0-1)}-e^{-p}}{\cos^p t_0}\cos^p t+e^{-p} \\ &> \frac{((2/3)\cos^p t_0+1/3)-e^{-p}}{\cos^p t_0}\cos^p t+e^{-p} \\ &= \frac{2}{3}\cos^p t+\frac{1}{3}+\left(e^{-p}-\frac{1}{3}\right)\frac{\cos^p t_0-\cos^p t}{\cos^p t_0} \\ &\geq \frac{2}{3}\cos^p t+\frac{1}{3}, \end{aligned} \tag{168}$$

where the last inequality holds due to $p\in(1,\ln 3]$ and $t\in(t_0,\pi/2)$.

Thus, the proof is finished. □

4. Some Corresponding Inequalities for Means

The Schwab-Borchardt mean of two numbers $a\geq 0$ and $b>0$ is defined by

$$\mathrm{SB}=\mathrm{SB}(a,b)=\begin{cases}\dfrac{\sqrt{b^2-a^2}}{\arccos(a/b)}, & 0\leq a<b,\\[2mm] \dfrac{\sqrt{a^2-b^2}}{\operatorname{arccosh}(a/b)}, & b<a,\\ a, & a=b,\end{cases} \tag{169}$$

(see [23, Theorem 8.4], [24, (2.3)], and [25, (1.1)]). It is clear that $\mathrm{SB}(a,b)$ is not symmetric in its variables and is a homogeneous function of degree 1 in a and b. More properties of this mean can be found in [25–27]. Very recently, Yang [19, Definitions 3.2, 4.2, and 5.2] defined three families of two-parameter trigonometric means. For convenience, we recall the definition of two-parameter sine mean as follows.

Definition 26. Let $b\geq a>0$ and $p,q\in[-2,2]$ such that $0\leq p+q\leq 3$, and let $\widetilde{S}(p,q,t)$ be defined by

$$\widetilde{S}(p,q,t)=\begin{cases}\left(\dfrac{q}{p}\dfrac{\sin pt}{\sin qt}\right)^{1/(p-q)} & \text{if } pq(p-q)\neq 0,\\[2mm] \left(\dfrac{\sin pt}{pt}\right)^{1/p} & \text{if } q=0, p\neq 0,\\[2mm] \left(\dfrac{\sin qt}{qt}\right)^{1/q} & \text{if } p=0, q\neq 0,\\[2mm] e^{t\cot pt-1/p} & \text{if } p=q\neq 0,\\ 1 & \text{if } p=q=0.\end{cases} \tag{170}$$

Then $\mathscr{S}_{p,q}(a,b)$ defined by

$$\begin{aligned}\mathscr{S}_{p,q}(a,b)&=b\times\widetilde{S}\left(p,q,\arccos\left(\frac{a}{b}\right)\right)\\ &\text{if } a\neq b,\ \mathscr{S}_{p,q}(a,a)=a\end{aligned} \tag{171}$$

is called a two-parameter sine mean of a and b.

In particular, for $b\geq a>0$,

$$\begin{aligned}\mathscr{S}_{1,0}(a,b)&=\left.\frac{\sin t}{t}\right|_{t=\arccos(a/b)}=\frac{\sqrt{b^2-a^2}}{\arccos(a/b)}=\mathrm{SB}(a,b),\\ \mathscr{S}_{1,1}(a,b)&=\left.be^{t\cot t-1}\right|_{t=\arccos(a/b)}\\ &=b\exp\left(\frac{a}{\mathrm{SB}(a,b)}-1\right):=SY(a,b)\end{aligned} \tag{172}$$

are means of a and b. Similarly, according to the definition of two-parameter cosine mean (see [19, Definition 4.2]),

$$\mathscr{C}_{p,0}(a,b)=b\times U_p\left(\arccos\left(\frac{a}{b}\right)\right) \tag{173}$$

is also a mean of a and b, where $U_p(t)$ is defined by (34).

Further, we have the following.

Proposition 27. *For $b\geq a>0$ and $\alpha\in(0,1]$, the function*

$$\begin{aligned}\widehat{C}_{p,\alpha}(a,b)&=b\times V_p^{\alpha}\left(\arccos\left(\frac{a}{b}\right)\right)\\ &\text{if } a\neq b,\ \widehat{C}_{p,\alpha}(a,a)=a \text{ if } a=b\end{aligned} \tag{174}$$

is also a mean of a and b, where $V_p(t)$ is defined by (35).

Proof. It suffices to prove that the double inequality

$$a<\widehat{C}_{p,\alpha}(a,b)=b\times V_p^{\alpha}\left(\left(\arccos\left(\frac{a}{b}\right)\right)\right)<b \tag{175}$$

holds for $b>a>0$, which is equivalent to

$$\cos t<V_p^{\alpha}(t)<1, \tag{176}$$

where $t=(\arccos(a/b))\in(0,\pi/2)$.

Using the decreasing property proved in Lemma 7, we see that

$$\cos t<\cos^{\alpha}t<V_p^{\alpha}(t)<V_0^{\alpha}(t)=e^{-\alpha t^2/2}<1, \tag{177}$$

which proves the assertion. □

If we replace t by $\arccos(a/b)$ and then multiply b or b^{λ} for suitable λ in each sides of the inequalities in previous section, then we can get the corresponding inequalities for bivariate means. For example, Theorems 10, 19, and 25 can be rewritten as follows.

Theorem 10′. *For $b \ge a > 0$, the two-side inequality*

$$b\left(\cos\frac{2\arccos(a/b)}{\sqrt{10}}\right)^{5/3} < SY(a,b) < b\left(\cos\left(p_2\arccos\left(\frac{a}{b}\right)\right)\right)^{2/(3p_2^2)} \tag{178}$$

holds with the best possible constants $2/\sqrt{10}$ and $p_2 \approx 0.6210901$, where p_2 is the unique solution of (56) *on* $(1/2, 1)$.

Theorem 19′. *For $b \ge a > 0$, the two-side inequality*

$$b\left(\cos\frac{\arccos(a/b)}{\sqrt{3}}\right)^{3/2} < \sqrt{SB(a,b)\,SY(a,b)} < b\left(\cos\left(p_4\arccos\left(\frac{a}{b}\right)\right)\right)^{1/(2p_4^2)} \tag{179}$$

holds with the best constants $1/\sqrt{3}$ and $p_4 \approx 0.5763247$, where p_4 is the unique root of (103) *on* $(1/2, 1)$.

Theorem 25′. *Let $b \ge a > 0$. Then the following statements are true:*

(i) *if $p \ge 6/5$, then the two-side inequality*

$$\alpha a^p + (1-\alpha)b^p < SY(a,b)^p < \beta a^p + (1-\beta)b^p \tag{180}$$

holds if and only if $\alpha \ge 1 - e^{-p}$ and $\beta \le 2/3$;

(ii) *if $0 < p \le 1$, then the double inequality* (180) *holds if and only if $\alpha \ge 2/3$ and $\beta \le 1 - e^{-p}$;*

(iii) *if $p < 0$, then the double inequality* (180) *holds if and only if $\alpha \le 0$ and $\beta \ge 2/3$;*

(iv) *the double inequality*

$$\left(\frac{2}{3}a^p + \frac{1}{3}b^p\right)^{1/p} < SY(a,b) < \left(\frac{2}{3}a^q + \frac{1}{3}b^q\right)^{1/q} \tag{181}$$

holds if and only if $p \le \ln 3$ and $q \ge 6/5$, where the left hand side in (181) *is defined as $a^{2/3}b^{1/3}$ if $p = 0$.*

Similar to $SB(a,b)$, these bivariate means mentioned previously are not symmetric in their variables and are homogeneous of degree 1 in a and b. But they can generate more symmetric means by making certain substitutions; for example, Neuman and Sándor [25, (1.1)] proved that $\mathrm{SB}(G,A) = P$, $\mathrm{SB}(A,Q) = T$, where Q, A, G, P, and T denote the quadratic, arithmetic, geometric, first, and second Seiffert means [28, 29] of a and b given by

$$\begin{gathered} Q = Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \qquad A = A(a,b) = \frac{a+b}{2}, \\ G = G(a,b) = \sqrt{ab}, \\ P = P(a,b) = \frac{a-b}{2\arcsin((a-b)/(a+b))}, \\ T = T(a,b) = \frac{a-b}{2\arctan((a-b)/(a+b))}, \end{gathered} \tag{182}$$

respectively. In same way, we have

$$SY(G,A) = Ae^{G/P-1} = X(a,b) \equiv X, \tag{183}$$

which is a Sándor mean introduced in [20, page 82], [30]. Also, we get

$$SY(A,Q) = Qe^{A/T-1} := B(a,b) \equiv B, \tag{184}$$

which is also a new mean, and it satisfies the double inequality $A < B < Q$.

There are many inequalities involving means Q, A, G, P, and T; we quote [15, 20, 25, 27, 31–44]. Inequalities for Sándor's mean X can be found in [20, pages 86–93] and [19, Section 6].

We now deduce some inequalities involving these means from the inequalities for trigonometric functions established in Section 3.

Step 1. Put $t = \arccos(a/b)$, where $b \ge a > 0$.

Step 2. Put $(a,b) = (m(x,y), M(x,y))$, where $m(x,y), M(x,y)$ are means of positive numbers x and y, and $m(x,y) \le M(x,y)$ for all $x, y > 0$.

Let $(m,M) = (G,A)$ and $(m,M) = (A,Q)$. Then the following variable substitutions follows from Steps 1 and 2.

(i) Substitution 1: $t = \arccos(G/A)$. Then

$$\frac{\sin t}{t} = \frac{P}{A}, \qquad \cos t = \frac{G}{A}, \qquad e^{t\cot t - 1} = e^{G/P-1}. \tag{185}$$

(ii) Substitution 2: $t = \arccos(A/Q)$. Then

$$\frac{\sin t}{t} = \frac{T}{Q}, \qquad \cos t = \frac{A}{Q}, \qquad e^{t\cot t - 1} = e^{A/T-1}. \tag{186}$$

For simplicity in expressions, we only select the functions involving $(\sin t)/t$, $\cos t$, and $\cos(t/2)$ in a chain of inequalities given in Section 3.

The following follows from (88).

Proposition 28. *For $b \ge a > 0$, the inequalities*

$$\sqrt{bSY(a,b)} < b^{1/3}\left(\frac{a+b}{2}\right)^{2/3} < SB(a,b) \tag{187}$$

hold. Moreover, replacing (a,b) by (G,A), we have

$$\sqrt{AX} < A^{1/3}\left(\frac{G+A}{2}\right)^{2/3} < P; \tag{188}$$

replacing (a,b) by (A,Q), we get

$$\sqrt{QB} < Q^{1/3}\left(\frac{A+Q}{2}\right)^{2/3} < T. \tag{189}$$

Remark 29. The second inequalities in (188) and (189) are due to Sándor [31, 33], while the one connecting P and $\sqrt{AX}$ first appeared in [20, page 82, (2.6)].

From inequalities (90), we have the following.

Proposition 30. *For $b \ge a > 0$, the double inequality*

$$\frac{2a+b}{3} < SY(a,b) < \frac{2a+b}{e} \tag{190}$$

is valid, where 3 and e are the best possible constants. Moreover, replacing (a,b) by (G,A) and (A,Q), we get

$$\frac{2G+A}{3} < X < \frac{2G+A}{e}, \tag{191}$$

$$\frac{2A+Q}{3} < B < \frac{2A+Q}{e}. \tag{192}$$

Remark 31. The left-hand side inequalities in (190), (191), and (192) can be found in [19, Example 6.1]. But the left-hand side inequality in (191) is weaker than

$$\frac{AG}{P} < \frac{A(P+G)}{3P-G} < X \tag{193}$$

proved by Sándor [20, page 89, (2.14)].

Inequalities (100) can be written as the corresponding inequalities for certain bivariate means as follows.

Proposition 32. *For $b \ge a > 0$, the inequalities*

$$\begin{aligned}\frac{SB(a,b)^{1/\ln(\pi/2)}}{b^{1/\ln(\pi/2)-1}} &< \frac{((a+b)/2)^{2/\ln 2}}{b^{2/\ln 2-1}}\\ &< SY(a,b) < \frac{((a+b)/2)^{4/3}}{b^{1/3}} < \frac{SB(a,b)^2}{b}\end{aligned} \tag{194}$$

hold true with the best possible exponents and coefficients. Moreover, replacing (a,b) by (G,A) and (A,Q), we have

$$\begin{aligned}\frac{P^{1/\ln(\pi/2)}}{A^{1/\ln(\pi/2)-1}} &< \frac{((G+A)/2)^{2/\ln 2}}{A^{2/\ln 2-1}}\\ &< X < \frac{((G+A)/2)^{4/3}}{A^{1/3}} < \frac{P^2}{A},\end{aligned} \tag{195}$$

$$\begin{aligned}\frac{T^{1/\ln(\pi/2)}}{Q^{1/\ln(\pi/2)-1}} &< \frac{((A+Q)/2)^{2/\ln 2}}{Q^{2/\ln 2-1}}\\ &< X < \frac{((A+Q)/2)^{4/3}}{Q^{1/3}} < \frac{T^2}{Q}.\end{aligned} \tag{196}$$

Remark 33. The fourth inequality in (195) was first proved by Sándor in [31].

From (130) in Corollary 22, we clearly see the following.

Proposition 34. *For $b \ge a > 0$, the inequalities*

$$\begin{aligned}\sqrt{\frac{8}{\pi e}}\frac{a+b}{2} &< b^{1-\gamma_{1/2}/4}\left(\frac{a+b}{2}\right)^{\gamma_{1/2}/4}\\ &< \sqrt{SB(a,b)\,SY(a,b)} < \frac{a+b}{2}\end{aligned} \tag{197}$$

hold, where the exponents $\gamma_{1/2}/4 = (\ln(\pi e/2))/\ln 4 \approx 1.0471$ and 1 and the coefficients $\sqrt{8/(\pi e)} \approx 0.96788$ and 1 are the best possible constants. Moreover, replacing (a,b) by (G,A) and (A,Q), we have

$$\begin{aligned}\sqrt{\frac{8}{\pi e}}\frac{G+A}{2} &< A^{1-\gamma_{1/2}/2}\left(\frac{G+A}{2}\right)^{\gamma_{1/2}/2} < \sqrt{PX} < \frac{G+A}{2},\\ \sqrt{\frac{8}{\pi e}}\frac{A+Q}{2} &< Q^{1-\gamma_{1/2}/2}\left(\frac{A+Q}{2}\right)^{\gamma_{1/2}/2} < \sqrt{TB} < \frac{A+Q}{2}.\end{aligned} \tag{198}$$

Inequalities (134) lead to the following.

Proposition 35. *For $b \ge a > 0$, the sharp inequalities*

$$\frac{2}{\pi} - e^{-1} < \frac{SB(a,b) - SY(a,b)}{b-a} < \frac{1}{3} \tag{199}$$

hold true. Moreover, replacing (a,b) by (G,A), (A,Q), we have

$$\begin{aligned}\frac{2}{\pi} - e^{-1} &< \frac{P-X}{A-G} < \frac{1}{3},\\ \frac{2}{\pi} - e^{-1} &< \frac{T-B}{Q-A} < \frac{1}{3}.\end{aligned} \tag{200}$$

Inequalities (144) lead to the following conclusion.

Proposition 36. *For $b \ge a > 0$, the inequalities*

$$\begin{aligned}\sqrt{ab} &< \sqrt{SB(a,b)\,SY(a,b)} < \frac{a+b}{2}\\ &< \frac{SB(a,b)+SY(a,b)}{2} < \left(e^{-1}+\frac{2}{\pi}\right)\frac{a+b}{2}\end{aligned} \tag{201}$$

are valid, where $e^{-1}+2/\pi \approx 1.0045$ is the best possible constant. In particular, replacing (a,b) by (G,A) and (A,Q), we get

$$\sqrt{GA} < \sqrt{PX} < \frac{G+A}{2} < \frac{P+X}{2} < \left(e^{-1}+\frac{2}{\pi}\right)\frac{G+A}{2}, \tag{202}$$

$$\sqrt{AQ} < \sqrt{TB} < \frac{A+Q}{2} < \frac{T+B}{2} < \left(e^{-1}+\frac{2}{\pi}\right)\frac{A+Q}{2}. \tag{203}$$

Remark 37. The first inequality in (202) was established by Sándor in [20, page 87, (2.2)].

Conflict of Interests

The authors declare that they have no competing interests.

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