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Research Article

A Compact Difference Scheme for a Class of Variable Coefficient Quasilinear Parabolic Equations with Delay

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A linearized compact difference scheme is provided for a class of variable coefficient parabolic systems with delay. The unique solvability, unconditional stability, and convergence of the difference scheme are proved, where the convergence order is four in space and two in time. A numerical test is presented to illustrate the theoretical results.

1. Introduction

From the twentieth century, more and more scholars have been attracted into the research on the theory of delay differential equations (DDEs) [1–4]. As we know, most DDEs have no analytical solutions; efficient numerical methods solving for DDEs and delay partial differential equations (DPDEs) need to be considered deeply. Recently, many scholars consider the numerical investigation on DPDEs. For instance, Marzban and Tabrizidooz [5] considered a hybrid approximation method for solving Hutchinson's equation; Jackiewicz and Zubik-Kowal [6] considered Chebyshev spectral collocation and waveform relaxation methods for nonlinear DPDEs and finite difference methods were considered to solve delay parabolic partial differential equations in [7–9]; Li et al. [10–12] constructed finite element methods to solve reaction-diffusion equations with delay. The numerical research of DPDEs focused on stability analysis can be referred to in [13].

The following variable coefficient parabolic systems with delay are considered in this paper:

$$r(x,t)\,u_t - du_{xx} = f(u(x,t), u(x,t-s), x, t), \quad (x,t) \in (0,1)\times(0,T], \tag{1}$$

$$u(x,t) = \phi(x,t), \quad x \in [0,1],\ t \in [-s,0], \tag{2}$$

$$u(0,t) = \alpha(t), \quad u(1,t) = \beta(t), \quad t \in (0,T], \tag{3}$$

where $d > 0$ is a constant and $s > 0$ is the delay term, $r(x,t) \in C((0,1)\times(0,T])$, $0 < c_0 \le r(x,t) \le c_1$. In the special case of $r(x,t) = 1$, numerical solutions of (1)–(3) have been considered in [14–17]. Ferreira and da Silva considered a backward Euler scheme and proved the stability and convergence by the energy method in [14]. A Crank-Nicolson scheme and a linearized compact difference scheme were proposed by Zhang and Sun in [15] and Sun and Zhang in [16], respectively. Q. Zhang and C. Zhang considered a new linearized compact multisplitting scheme in [17]. Gu and Wang constructed a Crank-Nicolson scheme in [18] to solve a special case of (1), where $f = f(u(x,t-s))$. In this paper, a linearized compact difference scheme solving for (1)–(3) will be constructed. The unique solvability, unconditional stability, and convergence of the difference scheme are proved, where the convergence order is four in space and two in time. A numerical test is presented to illustrate the theoretical results.

The paper is organized as follows. In Section 2, a linearized compact difference scheme is constructed to solve (1)–(3). Section 3 considers the solvability, stability, and convergence of the provided difference scheme.

In Section 4, a numerical test is presented to illustrate the theoretical results. Section 5 gives a brief discussion of this paper.

2. The Compact Difference Scheme and Local Truncation Error

Throughout this paper, the following assumptions are assumed to be true.

(H1) Let m be an integer satisfying $ms \le T < (m+1)s$, denote $I_l = (ls, ls+s)$, $l = -1, 0, \dots, m-1$, $I_m = (ms, T)$, and $I = \cup_{p=-1}^{m} I_p$, assume that (1)–(3) has a unique solution $u \in C^{6,4}(I \times (0,T])$ and that u and its partial derivatives are all bounded by a constant c_2;

(H2) $f(\mu, \nu, x, t)$ has bounded first-order continuous partial derivatives, and we denote

$$\begin{aligned} c_3 &= \max_{|\varepsilon_1| \le \varepsilon_0, |\varepsilon_2| \le \varepsilon_0} \left| f_\mu \left(u(x,t) + \varepsilon_1, u(x, t-s) + \varepsilon_2, x, t \right) \right|, \\ c_4 &= \max_{|\varepsilon_1| \le \varepsilon_0, |\varepsilon_2| \le \varepsilon_0} \left| f_\nu \left(u(x,t) + \varepsilon_1, u(x, t-s) + \varepsilon_2, x, t \right) \right|, \end{aligned} \tag{4}$$

where $\varepsilon_0 > 0$, c_3, and c_4 are constants, $(x,t) \in (0,1) \times (0,T]$.

First let M and j be two positive integers; then, we take $h = 1/M$, $\tau = s/j$, $x_i = ih$, $t_k = k\tau$, $t_{k+1/2} = (t_k + t_{k+1})/2$. Define $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i \mid 0 \le i \le M\}$, $\Omega_\tau = \{t_k \mid -j \le k \le N\}$, $N = [T/\tau]$. Denote $U_i^k = u(x_i, t_k)$, $0 \le i \le M$, $-j \le k \le N$, throughout this paper. Let

$$\mathscr{W} = \left\{ v_i^k \mid 0 \le i \le M, -j \le k \le N \right\} \tag{5}$$

be the grid function space defined on $\Omega_{h\tau}$. The following notations are made:

$$\begin{aligned} v_i^{k+(1/2)} &= \frac{v_i^k + v_i^{k+1}}{2}, \\ \delta_t v_i^{k+(1/2)} &= \frac{v_i^{k+1} - v_i^k}{\tau}, \\ \delta_x v_{i+(1/2)}^k &= \frac{v_{i+1}^k - v_i^k}{h}, \\ \delta_x^2 v_i^k &= \frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2}, \\ \mathscr{A} v_i^k &= \frac{1}{12} \left(v_{i-1}^k + 10 v_k^k + v_{i+1}^k \right). \end{aligned} \tag{6}$$

Considering (1) at the point $(x_i, t_{k+(1/2)})$, we have

$$\begin{aligned} r(x_i, t_{k+(1/2)}) &\frac{\partial u}{\partial t}(x_i, t_{k+(1/2)}) - d \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+(1/2)}) \\ &= f\left(u(x_i, t_{k+(1/2)}), u(x_i, t_{k+(1/2)-j}), x_i, t_{k+(1/2)} \right), \\ &\qquad 0 \le i \le M,\ 0 \le k \le N-1. \end{aligned} \tag{7}$$

From Taylor expansion, we have

$$\begin{aligned} \frac{\partial u}{\partial t}(x_i, t_{k+(1/2)}) &= \delta_t U_i^{k+(1/2)} - \frac{\tau^2}{24} \frac{\partial^3 u}{\partial t^3}(x_i, \eta_i^k), \\ &\qquad \eta_i^k \in (t_k, t_{k+1}), \\ \frac{\partial^2 u}{\partial x^2}&(x_i, t_{k+(1/2)}) \\ &= \frac{1}{2} \left[\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1}) \right] \\ &\quad - \frac{\tau^2}{8} \frac{\partial^4 u}{(\partial x^2 \partial t^2)}(x_i, \gamma_i^k) \\ &= \frac{1}{2} \left(\delta_x^2 U_i^k + \delta_x^2 U_i^{k+1} \right) \\ &\quad - \frac{h^2}{24} \left[\frac{\partial^4 u}{\partial x^4}(\xi_i^k, t_k) + \frac{\partial^4 u}{\partial x^4}(\xi_i^{k+1}, t_{k+1}) \right] \\ &\quad - \frac{\tau^2}{8} \frac{\partial^4 u}{\partial x^2 \partial t^2}(x_i, \gamma_i^k), \\ \xi_i^k, \xi_i^{k+1} &\in (x_{i-1}, x_{i+1}), \quad \gamma_i^k \in (t_k, t_{k+1}), \\ f\big(u(x_i, t_{k+(1/2)}), &u(x_i, t_{k+(1/2)-j}), x_i, t_{k+(1/2)}\big) \\ &= f\left(\frac{3}{2} U_i^k - \frac{1}{2} U_i^{k-1}, \frac{1}{2} U_i^{k+1-j} + \frac{1}{2} U_i^{k-j}, x_i, t_{k+(1/2)} \right) \\ &\quad + \frac{3\tau^2}{8} \frac{\partial^2 u(x_i, \rho^k)}{\partial t^2} f_\mu(\zeta_i^k, \varsigma_i^k, x_i, t_{k+(1/2)}) \\ &\quad - \frac{\tau^2}{8} \frac{\partial^2 u(x_i, \varrho^k)}{\partial t^2} f_\nu(\zeta_i^k, \varsigma_i^k, x_i, t_{k+(1/2)}), \end{aligned} \tag{8}$$

where $\rho^k \in (t_{k-1}, t_{k+1/2})$, $\varrho^k \in (t_{k-j}, t_{k+1-j})$, ζ_i^k is between $u(x_i, t_{k+(1/2)})$ and $(3/2)U_i^k - (1/2)U_i^{k-1}$, and ς_i^k is between $u(x_i, t_{k+(1/2)-j})$ and $(1/2)U_i^{k+1-j} + (1/2)U_i^{k-j}$. Substituting (8) into (7), denote $r_i^{k+(1/2)} = r(x_i, t_{k+(1/2)})$; we obtain

$$\begin{aligned} r_i^{k+(1/2)} \delta_t U_i^{k+(1/2)} &- \frac{d}{2} \left[\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1}) \right] \\ &= f\left(\frac{3}{2} U_i^k - \frac{1}{2} U_i^{k-1}, \frac{1}{2} U_i^{k+1-j} + \frac{1}{2} U_i^{k-j}, x_i, t_{k+(1/2)} \right) \\ &\quad + \tau^2 \overline{R}_i^k, \quad 1 \le i \le M,\ 0 \le k \le N-1, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \overline{R}_i^k &= \frac{r_i^{k+(1/2)}}{24} \frac{\partial^3 u}{\partial t^3}(x_i, \eta_i^k) - \frac{d}{8} \frac{\partial^4 u}{\partial x^2 \partial t^2}(x_i, \gamma_i^k) \\ &\quad + \frac{3}{8} \frac{\partial^2 u(x_i, \rho^k)}{\partial t^2} f_\mu(\zeta_i^k, \varsigma_i^k, x_i, t_{k+(1/2)}) \\ &\quad - \frac{1}{8} \frac{\partial^2 u(x_i, \varrho^k)}{\partial t^2} f_\nu(\zeta_i^k, \varsigma_i^k, x_i, t_{k+(1/2)}). \end{aligned} \tag{10}$$

Acting operator $\mathscr{A}$ on both sides of (9), we have

$$\begin{aligned}&\mathscr{A} r_i^{k+(1/2)} \delta_t U_i^{k+(1/2)} \\ &\quad - \frac{d}{2}\left[\mathscr{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \mathscr{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1})\right] \\ &= \mathscr{A} f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-j} + \frac{1}{2}U_i^{k-j}, x_i, t_{k+(1/2)}\right) \\ &\quad + \tau^2 \mathscr{A}\overline{R}_i^k, \quad 1 \le i \le M-1,\ 0 \le k \le N-1.\end{aligned} \tag{11}$$

Resorting to the following Lemma, we can obtain the estimation of the operator $\mathscr{A}$.

Lemma 1 (see [19, 20]). *Suppose that $q(x) \in C^6[x_{i-1}, x_{i+1}]$; then, we have*

$$\begin{aligned}&\frac{1}{12}\left[q''(x_{i-1}) + 10q''(x_i) + q''(x_{i+1})\right] \\ &\quad - \frac{1}{h^2}\left[q(x_{i-1}) - 2q(x_i) + q(x_{i+1})\right] \\ &= \frac{h^4}{240} q^{(6)}(\omega_i),\end{aligned} \tag{12}$$

where $\omega_i \in (x_{i-1}, x_{i+1})$.

From Lemma 1 and Taylor expansion, we obtain

$$\begin{aligned}&\mathscr{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_k) = \delta_x^2 U_i^k + \frac{h^4}{240}\frac{\partial^6 u}{\partial x^6}(\theta_i^k, t_k), \\ &\qquad \theta_i^k \in (x_{i-1}, x_{i+1}), \\ &\mathscr{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1}) = \delta_x^2 U_i^{k+1} + \frac{h^4}{240}\frac{\partial^6 u}{\partial x^6}(\theta_i^{k+1}, t_{k+1}), \\ &\qquad \theta_i^{k+1} \in (x_{i-1}, x_{i+1}).\end{aligned} \tag{13}$$

Inserting (13) into (11), we have

$$\begin{aligned}&\mathscr{A} r_i^{k+(1/2)} \delta_t U_i^{k+(1/2)} - d\delta_x^2 U_i^{k+1/2} \\ &= \mathscr{A} f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-j} + \frac{1}{2}U_i^{k-j}, x_i, t_{k+(1/2)}\right) \\ &\quad + R_i^k, \quad 1 \le i \le M-1,\ 0 \le k \le N-1,\end{aligned} \tag{14}$$

where

$$R_i^k = \tau^2 \mathscr{A}\overline{R}_i^k + \frac{dh^4}{480}\left[\frac{\partial^6 u}{\partial x^6}(\theta_i^k, t_k) + \frac{\partial^6 u}{\partial x^6}(\theta_i^{k+1}, t_{k+1})\right]. \tag{15}$$

From $0 < c_0 \le r_i^{k+1/2} \le c_1$ and assumptions (H1) and (H2), we have

$$\left|\overline{R}_i^k\right| \le c_5, \quad 1 \le i \le M-1,\ 0 \le k \le N-1, \tag{16}$$

such that

$$\left|R_i^k\right| \le c_6\left(\tau^2 + h^4\right), \quad 1 \le i \le M-1,\ 0 \le k \le N-1. \tag{17}$$

Discretizing the initial and boundary conditions of (2) and (3), we obtain

$$U_i^k = \phi(x_i, t_k), \quad 0 \le i \le M,\ -j \le k \le 0, \tag{18}$$

$$U_0^k = \alpha(t_k), \quad U_M^k = \beta(t_k), \quad 1 \le k \le N. \tag{19}$$

Replacing U_i^k by u_i^k in (14) and omitting R_i^k, we obtain the following compact difference scheme:

$$\begin{aligned}&\mathscr{A} r_i^{k+(1/2)} \delta_t u_i^{k+(1/2)} - d\delta_x^2 u_i^{k+1/2} \\ &= \mathscr{A} f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-j} + \frac{1}{2}u_i^{k-j}, x_i, t_{k+(1/2)}\right), \\ &\qquad 1 \le i \le M-1,\ 0 \le k \le N-1,\end{aligned} \tag{20}$$

$$u_i^k = \phi(x_i, t_k), \quad 0 \le i \le M,\ -j \le k \le 0, \tag{21}$$

$$u_0^k = \alpha(t_k), \quad u_M^k = \beta(t_k), \quad 1 \le k \le N. \tag{22}$$

3. The Solvability, Convergence, and Stability of the Compact Difference Scheme

Define the following grid function space on Ω_h:

$$V = \{v \mid v = (v_0, v_1, \ldots, v_M), v_0 = v_M = 0\}. \tag{23}$$

If $v \in V$, we introduce the following notations:

$$\begin{aligned}\|v\| &= \sqrt{h\sum_{i=1}^{M-1}(v_i)^2}, \\ |v|_1 &= \sqrt{h\sum_{i=1}^{M}\left(\frac{v_i - v_{i-1}}{h}\right)^2}, \\ \|v\|_\infty &= \max_{0\le i\le M}|v_i|.\end{aligned} \tag{24}$$

By [16, 17, 19], we have the following two inequalities:

$$\|v\|_\infty \le \frac{1}{2}|v|_1, \tag{25}$$

$$\|v\| \le \frac{1}{\sqrt{6}}|v|_1. \tag{26}$$

For the analysis of the difference scheme, the following Lemma is introduced.

Lemma 2 (see [16, 17, 19]). *Assume that $\{F^k \mid k \ge 0\}$ to be nonnegative sequence and satisfies*

$$F^{k+1} \le A + B\tau\sum_{i=1}^{k} F^l, \quad k = 0, 1, \ldots; \tag{27}$$

then

$$F^{k+1} \le A\exp(Bk\tau), \quad k = 0, 1, 2, \ldots, \tag{28}$$

where A and B are nonnegative constants.

Theorem 3. *Under the condition that $5c_0-c_1>0$, the compact difference scheme* (20)–(22) *has a unique solution.*

Proof. Denote that $\lambda=d\tau/h^2$; then, difference scheme (20)–(22) can be reformed as

$$\begin{aligned}&\left(\frac{1}{12}r_{i-1}^{k+1/2}-\frac{\lambda}{2}\right)u_{i-1}^{k+1}+\left(\frac{10}{12}r_i^{k+1/2}+\lambda\right)u_i^{k+1}\\&\quad+\left(\frac{1}{12}r_{i+1}^{k+1/2}-\frac{\lambda}{2}\right)u_{i+1}^{k+1}\\&=\left(\frac{1}{12}r_{i-1}^{k+1/2}+\frac{\lambda}{2}\right)u_{i-1}^{k}\\&\quad+\left(\frac{10}{12}r_i^{k+1/2}-\lambda\right)u_i^{k}+\left(\frac{1}{12}r_{i+1}^{k+1/2}+\frac{\lambda}{2}\right)u_{i+1}^{k}\\&\quad+\frac{\tau}{12}f\left(\frac{3}{2}u_{i-1}^{k}-\frac{1}{2}u_{i-1}^{k-1},\frac{1}{2}u_{i-1}^{k+1-j}\right.\\&\qquad\left.+\frac{1}{2}u_{i-1}^{k-j},x_{i-1},t_{k+(1/2)}\right)\\&\quad+\frac{10\tau}{12}f\left(\frac{3}{2}u_{i}^{k}-\frac{1}{2}u_{i}^{k-1},\frac{1}{2}u_{i}^{k+1-j}\right.\\&\qquad\left.+\frac{1}{2}u_{i}^{k-j},x_{i},t_{k+(1/2)}\right)\\&\quad+\frac{\tau}{12}f\left(\frac{3}{2}u_{i+1}^{k}-\frac{1}{2}u_{i+1}^{k-1},\frac{1}{2}u_{i+1}^{k+1-j}\right.\\&\qquad\left.+\frac{1}{2}u_{i+1}^{k-j},x_{i+1},t_{k+(1/2)}\right).\end{aligned}\tag{29}$$

The mathematical induction method will be used in the proof of this theorem. Denote

$$u^k=\left(u_0^k,u_1^k,\dots,u_M^k\right).\tag{30}$$

Notice that u^k $(-j\le k\le 0)$ is determined by the initial condition (21). Suppose that u^l has been determined.

Let $k=l$ in (20); the linear algebraic equations with respect to u^{l+1} can be obtained. Under the condition that $5c_0-c_1>0$, we have

$$\begin{aligned}&\left|\frac{10}{12}r_i^{l+1/2}+\lambda\right|-\left|\frac{1}{12}r_{i-1}^{l+1/2}-\frac{\lambda}{2}\right|-\left|\frac{1}{12}r_{i+1}^{l+1/2}-\frac{\lambda}{2}\right|\\&\quad\ge\frac{10}{12}r_i^{l+1/2}-\frac{1}{12}r_{i-1}^{l+1/2}-\frac{1}{12}r_{i+1}^{l+1/2}\\&\quad\ge\frac{10c_0-2c_1}{12}\\&\quad>0.\end{aligned}\tag{31}$$

Thus, the coefficient matrix of the linear algebraic system is strictly diagonally dominant and then there exists a unique solution u^{l+1}. By the inductive principle, the proof ends. □

Denote $e_i^k=U_i^k-u_i^k$, $0\le i\le M$, $-j\le k\le N$; subtracting (20)–(22) from (14), (18), and (19), respectively, the following error equations can be obtained:

$$\begin{aligned}&\mathscr{A}r_i^{k+(1/2)}\delta_te_i^{k+(1/2)}-d\delta_x^2e_i^{k+1/2}\\&\quad=\mathscr{A}\left[f\left(\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-j}+\frac{1}{2}U_i^{k-j},x_i,t_{k+(1/2)}\right)\right.\\&\qquad\left.-f\left(\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-j}+\frac{1}{2}u_i^{k-j},x_i,t_{k+(1/2)}\right)\right]\\&\qquad+R_i^k,\quad 1\le i\le M-1,\ 0\le k\le N-1,\end{aligned}\tag{32}$$

$$e_i^k=0,\quad 0\le i\le M,\ -j\le k\le 0,\tag{33}$$

$$e_0^k=0,\quad e_M^k=0,\quad 1\le k\le N.\tag{34}$$

Theorem 4. *Denote*

$$C=c_6\sqrt{\frac{3T}{d\left(9c_0-c_1\right)}}\exp\left(\frac{2\left(5c_3^2+c_4^2\right)}{d\left(9c_0-c_1\right)}T\right).\tag{35}$$

If the following conditions are satisfied:

$$\tau\le\left(\frac{\epsilon_0}{4C}\right)^{1/2},\qquad h\le\left(\frac{\epsilon_0}{4C}\right)^{1/4},\tag{36}$$

then we have

$$\left\|e^k\right\|_\infty\le C\left(\tau^2+h^4\right),\quad 0\le k\le N,\tag{37}$$

where $\epsilon_0>0$ is a constant.

Proof. Acting $h\delta_te_i^{k+(1/2)}$ on (32) and summing up for i from 1 to $M-1$, we obtain

$$\begin{aligned}&h\sum_{i=1}^{M-1}\left(\mathscr{A}r_i^{k+(1/2)}\delta_te_i^{k+(1/2)}\right)\delta_te_i^{k+(1/2)}\\&\quad-dh\sum_{i=1}^{M-1}\left(\delta_x^2e_i^{k+1/2}\right)\delta_te_i^{k+(1/2)}\\&\quad=h\sum_{i=1}^{M-1}\left\{\mathscr{A}\left[f\left(\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-j}\right.\right.\right.\\&\qquad\left.+\frac{1}{2}U_i^{k-j},x_i,t_{k+(1/2)}\right)\\&\qquad-f\left(\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-j}\right.\\&\qquad\left.\left.\left.+\frac{1}{2}u_i^{k-j},x_i,t_{k+(1/2)}\right)\right]\right\}\delta_te_i^{k+(1/2)}\\&\qquad+h\sum_{i=1}^{M-1}\left(R_i^k\right)\delta_te_i^{k+(1/2)},\quad 1\le i\le M-1,\ 0\le k\le N-1.\end{aligned}\tag{38}$$

Mathematical induction will be used to prove this theorem. Notice that $\|e^k\|_\infty=0$ $(-j\le k\le 0)$ and suppose that

(37) is true for $0 \le k \le l$; we will show that (37) is also true for $k = l+1$.

In the following, each term of (38) will be estimated:

$$\begin{aligned}
& h\sum_{i=1}^{M-1}\left(\mathscr{A} r_i^{k+(1/2)}\delta_t e_i^{k+(1/2)}\right)\delta_t e_i^{k+(1/2)} \\
&\quad \ge \frac{h}{12}\sum_{i=1}^{M-1}\left[9r_i^{k+(1/2)} - \frac{\left(r_{i-1}^{k+(1/2)} + r_{i+1}^{k+(1/2)}\right)}{2}\right]\left(\delta_t e_i^{k+(1/2)}\right)^2 \\
&\quad \ge \frac{9c_0 - c_1}{12}\left\|\delta_t e^{k+(1/2)}\right\|^2, \\
& -dh\sum_{i=1}^{M-1}\left(\delta_x^2 e_i^{k+(1/2)}\right)\delta_t e_i^{k+(1/2)} = \frac{d}{2\tau}\left(\left|e^{k+1}\right|_1^2 - \left|e^k\right|_1^2\right), \\
& h\sum_{i=1}^{M-1}\left(R_i^k\right)\delta_t e_i^{k+(1/2)} \\
&\quad \le \frac{6}{9c_0 - c_1}h\sum_{i=1}^{M-1}\left(R_i^k\right)^2 + \frac{9c_0 - c_1}{24}h\sum_{i=1}^{M-1}\left(\delta_t e_i^{k+(1/2)}\right)^2 \\
&\quad \le \frac{6}{9c_0 - c_1}c_6^2\left(\tau^2 + h^4\right)^2 + \frac{9c_0 - c_1}{24}\left\|\delta_t e^{k+(1/2)}\right\|^2.
\end{aligned} \tag{39}$$

From the inductive assumption and (36), we have

$$\left\|e^k\right\|_\infty \le C\left(\tau^2 + h^4\right) \le \frac{\epsilon_0}{2}, \quad 0 \le k \le l. \tag{40}$$

From (H2), we have

$$\begin{aligned}
& \left| f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-j} + \frac{1}{2}U_i^{k-j}, x_i, t_{k+(1/2)}\right)\right. \\
&\quad \left. - f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-j} + \frac{1}{2}u_i^{k-j}, x_i, t_{k+(1/2)}\right)\right| \\
&\quad \le c_3\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_4\left|\frac{1}{2}e_i^{k+1-j} + \frac{1}{2}e_i^{k-j}\right|, \\
&\qquad\qquad 1 \le i \le M,\ 0 \le k \le l.
\end{aligned} \tag{41}$$

It then follows that

$$\begin{aligned}
& \left|\mathscr{A}\left[f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-j} + \frac{1}{2}U_i^{k-j}, x_i, t_{k+(1/2)}\right)\right.\right. \\
&\quad \left.\left. - f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-j} + \frac{1}{2}u_i^{k-j}, x_i, t_{k+(1/2)}\right)\right]\right| \\
&\quad \le \mathscr{A}\left(c_3\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_4\left|\frac{1}{2}e_i^{k+1-j} + \frac{1}{2}e_i^{k-j}\right|\right), \\
&\qquad\qquad 1 \le i \le M-1,\ 0 \le k \le l.
\end{aligned} \tag{42}$$

From the inequality above, we obtain

$$\begin{aligned}
& h\sum_{i=1}^{M-1}\left\{\mathscr{A}\left[f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-j} + \frac{1}{2}U_i^{k-j}, x_i, t_{k+(1/2)}\right)\right.\right. \\
&\qquad \left.\left. - f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-j} + \frac{1}{2}u_i^{k-j}, x_i, t_{k+(1/2)}\right)\right]\right\} \\
&\quad \times \delta_t e_i^{k+(1/2)} \\
& \le h\sum_{i=1}^{M-1}\left\{\mathscr{A}\left(c_3\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_4\left|\frac{1}{2}e_i^{k+1-j} + \frac{1}{2}e_i^{k-j}\right|\right)\right\} \\
&\quad \times \delta_t e_i^{k+(1/2)} \\
& \le \frac{6}{9c_0 - c_1}h \\
&\quad \times \sum_{i=1}^{M-1}\left\{\mathscr{A}\left(c_3\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_4\left|\frac{1}{2}e_i^{k+1-j} + \frac{1}{2}e_i^{k-j}\right|\right)\right\}^2 \\
&\quad + \frac{9c_0 - c_1}{24}\left\|\delta_t e_i^{k+(1/2)}\right\|^2 \\
& \le \frac{6}{9c_0 - c_1}h \\
&\quad \times \sum_{i=1}^{M-1}\left(c_3\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_4\left|\frac{1}{2}e_i^{k+1-j} + \frac{1}{2}e_i^{k-j}\right|\right)^2 \\
&\quad + \frac{9c_0 - c_1}{24}\left\|\delta_t e^{k+(1/2)}\right\|^2 \\
& \le \frac{12}{9c_0 - c_1} \\
&\quad \times \left[c_3^2 h\sum_{i=1}^{M-1}\left(\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right)^2\right. \\
&\qquad \left. + c_4^2 h\sum_{i=1}^{M-1}\left(\frac{1}{2}e_i^{k+1-j} + \frac{1}{2}e_i^{k-j}\right)^2\right] \\
&\quad + \frac{9c_0 - c_1}{24}\left\|\delta_t e^{k+(1/2)}\right\|^2 \\
& \le \frac{12}{9c_0 - c_1} \\
&\quad \times \left[\frac{5}{2}c_3^2 h\sum_{i=1}^{M-1}\left(\left(e_i^k\right)^2 + \left(e_i^{k-1}\right)^2\right)\right. \\
&\qquad \left. + \frac{1}{2}c_4^2 h\sum_{i=1}^{M-1}\left(\left(e_i^{k+1-j}\right)^2 + \left(e_i^{k-j}\right)^2\right)\right] \\
&\quad + \frac{9c_0 - c_1}{24}\left\|\delta_t e^{k+(1/2)}\right\|^2 \\
& = \frac{30}{9c_0 - c_1}c_3^2\left(\left\|e^k\right\|^2 + \left\|e^{k-1}\right\|^2\right)
\end{aligned}$$

$$+\frac{6}{9c_0-c_1}c_4^2\left(\left\|e^{k+1-j}\right\|^2+\left\|e^{k-j}\right\|^2\right)$$
$$+\frac{9c_0-c_1}{24}\left\|\delta_t e^{k+(1/2)}\right\|^2,\quad 0\le k\le l. \tag{43}$$

Inserting (39)–(43) into (38), we obtain

$$\begin{aligned}\frac{d}{2\tau}\left(\left|e^{k+1}\right|_1^2-\left|e^k\right|_1^2\right) &\le \frac{30}{9c_0-c_1}c_3^2\left(\left\|e^k\right\|^2+\left\|e^{k-1}\right\|^2\right)\\ &\quad+\frac{6}{9c_0-c_1}c_4^2\left(\left\|e^{k+1-j}\right\|^2+\left\|e^{k-j}\right\|^2\right)\\ &\quad+\frac{6}{9c_0-c_1}c_6^2\left(\tau^2+h^4\right)^2,\quad 0\le k\le l.\end{aligned} \tag{44}$$

The above inequality has the following form:

$$\begin{aligned}\left|e^{k+1}\right|_1^2 &\le \left|e^k\right|_1^2+\frac{2\tau}{d}\frac{30}{9c_0-c_1}c_3^2\left(\left\|e^k\right\|^2+\left\|e^{k-1}\right\|^2\right)\\ &\quad+\frac{2\tau}{d}\frac{6}{9c_0-c_1}c_4^2\left(\left\|e^{k+1-j}\right\|^2+\left\|e^{k-j}\right\|^2\right)\\ &\quad+\frac{2\tau}{d}\frac{6}{9c_0-c_1}c_6^2\left(\tau^2+h^4\right)^2,\quad 0\le k\le l.\end{aligned} \tag{45}$$

Summing up (45) for k, noticing (33), and exploiting (26), we have

$$\begin{aligned}\left|e^{k+1}\right|_1^2 &\le \left|e^0\right|_1^2+\frac{2\tau}{d}\frac{30}{9c_0-c_1}c_3^2\\ &\quad\times\sum_{m=0}^{k}\left(\left\|e^m\right\|^2+\left\|e^{m-1}\right\|^2\right)+\frac{2\tau}{d}\frac{6}{9c_0-c_1}c_4^2\\ &\quad\times\sum_{m=0}^{k}\left(\left\|e^{m+1-j}\right\|^2+\left\|e^{m-j}\right\|^2\right)\\ &\quad+\frac{2\tau}{d}\frac{6}{9c_0-c_1}c_6^2\sum_{m=0}^{l}\left(\tau^2+h^4\right)^2\\ &\le \frac{24\left(5c_3^2+c_4^2\right)}{d\left(9c_0-c_1\right)}\tau\sum_{m=1}^{k}\left\|e^m\right\|^2+\frac{12c_6^2\left(l+1\right)\tau}{d\left(9c_0-c_1\right)}\left(\tau^2+h^4\right)^2\\ &\le \frac{4\left(5c_3^2+c_4^2\right)}{d\left(9c_0-c_1\right)}\tau\\ &\quad\times\sum_{m=1}^{k}\left|e^m\right|^2+\frac{12c_6^2T}{d\left(9c_0-c_1\right)}\left(\tau^2+h^4\right)^2,\quad 0\le k\le l.\end{aligned} \tag{46}$$

By Lemma 2, we have

$$\left|e^{l+1}\right|_1^2\le\frac{12c_6^2T}{d\left(9c_0-c_1\right)}\exp\left(\frac{4\left(5c_3^2+c_4^2\right)}{d\left(9c_0-c_1\right)}T\right)\left(\tau^2+h^4\right)^2. \tag{47}$$

Table 1: Numerical results of (54) when $h=1/10$, $\tau=1/100$.

(x,t)	Numerical solution	Exact solution	$\lvert u(x_i,t_k)-u_i^k\rvert$
(0.5, 0.1)	0.667184	0.667184	$8.652e-009$
(0.5, 0.2)	0.727837	0.727837	$1.623e-008$
(0.5, 0.3)	0.788490	0.788490	$2.258e-008$
(0.5, 0.4)	0.849143	0.849143	$2.805e-008$
(0.5, 0.5)	0.909796	0.909796	$3.288e-008$
(0.5, 0.6)	0.970449	0.970449	$3.725e-008$
(0.5, 0.7)	1.031102	1.031102	$4.129e-008$
(0.5, 0.8)	1.091755	1.091755	$4.509e-008$
(0.5, 0.9)	1.152408	1.152408	$4.870e-008$
(0.5, 1.0)	1.213061	1.213061	$5.218e-008$

From (25), we obtain

$$\left\|e^{l+1}\right\|_\infty\le c_6\sqrt{\frac{3T}{d\left(9c_0-c_1\right)}}\exp\left(\frac{2\left(5c_3^2+c_4^2\right)}{d\left(9c_0-c_1\right)}T\right)\left(\tau^2+h^4\right). \tag{48}$$

By the inductive principle, this completes the proof. □

To discuss the stability of the difference scheme (20)–(22), we consider the following problem:

$$\begin{aligned}&r(x,t)v_t-dv_{xx}=f\left(v(x,t),v(x,t-s),x,t\right),\\ &\qquad\qquad (x,t)\in(0,1)\times(0,T],\\ &v(x,t)=\phi(x,t)+\psi(x,t),\quad x\in[0,1],\ t\in[-s,0],\\ &v(0,t)=\alpha(t),\quad v(1,t)=\beta(t),\quad t\in(0,T],\end{aligned} \tag{49}$$

where $\psi(x,t)$ is the perturbation caused by $\phi(x,t)$. The following difference scheme solving for (49) can be obtained:

$$\begin{aligned}&\mathscr{A}r_i^{k+(1/2)}\delta_t v_i^{k+(1/2)}-d\delta_x^2 v_i^{k+1/2}\\ &\quad=\mathscr{A}f\left(\frac{3}{2}v_i^k-\frac{1}{2}v_i^{k-1},\frac{1}{2}v_i^{k+1-j}+\frac{1}{2}v_i^{k-j},x_i,t_{k+(1/2)}\right),\\ &\qquad\qquad 1\le i\le M-1,\ 0\le k\le N-1,\\ &v_i^k=\phi\left(x_i,t_k\right)+\psi_i^k,\quad 0\le i\le M,\ -j\le k\le 0,\\ &v_0^k=\alpha\left(t_k\right),\quad v_M^k=\beta\left(t_k\right),\quad 1\le k\le N.\end{aligned} \tag{50}$$

Similar to the proof of Theorem 4, the following stability result can be obtained

Theorem 5. *Denote*

$$\eta_i^k=v_i^k-u_i^k,\quad 0\le i\le M,\ -j\le k\le N. \tag{51}$$

Then, there exist constants c_7 and c_8 such that

$$\left\|\eta^k\right\|_\infty\le c_7\sqrt{\tau h\sum_{m=-j}^{0}\sum_{i=l}^{M-1}\left(\psi_i^k\right)^2} \tag{52}$$

under the condition that h and τ are small enough and $\max_{-j\le k\le 0,\,0\le i\le M}|\psi_i^k|\le c_8$.

TABLE 2: Maximum norm errors of (54) with different step-sizes.

h	τ	$E_\infty(h,\tau)$	$E_\infty(h,\tau)/E_\infty(h/2,\tau/2)$
1/10	1/100	$5.220e-008$	*
1/20	1/400	$3.295e-009$	15.842
1/40	1/1600	$2.059e-010$	15.999
1/80	1/6400	$1.283e-011$	16.046
1/160	1/25600	$6.566e-013$	19.546

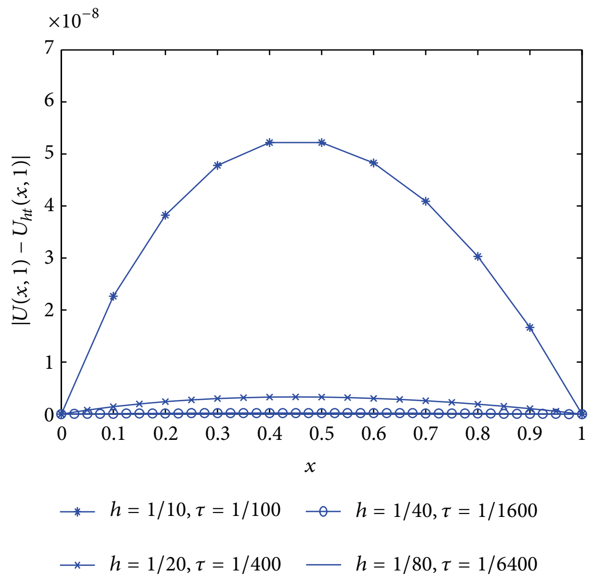

FIGURE 1: Error curves of difference scheme (20)–(22) solving for problem (54) with different step-sizes when $t = 1$.

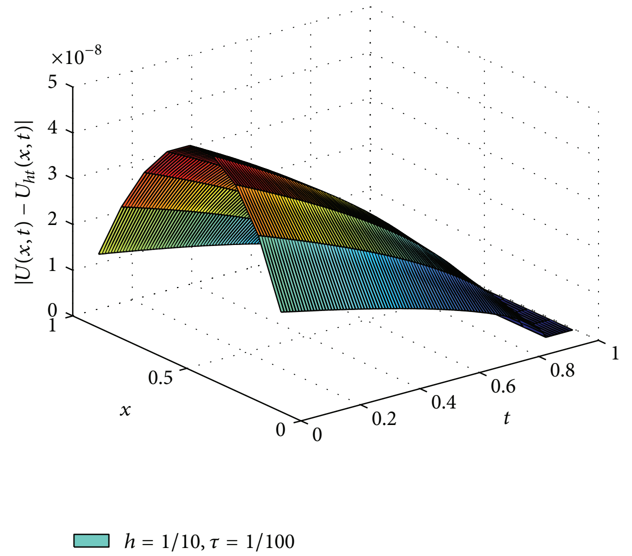

FIGURE 2: Error surface maps of difference scheme (20)–(22) solving for problem (54) with step-size $h = 1/10$; $\tau = 1/100$.

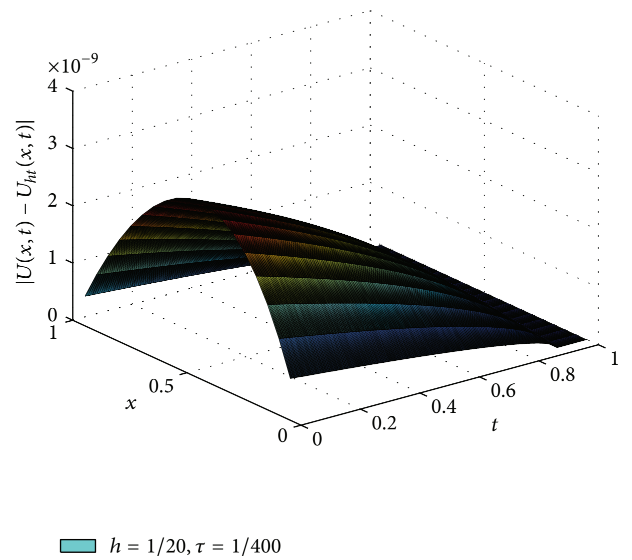

FIGURE 3: Error surface maps of difference scheme (20)–(22) solving for problem (54) with step-size $h = 1/20$; $\tau = 1/400$.

4. Numerical Test

In this section, a numerical test is considered to validate the algorithms provided in this paper, and the numerical solutions u_i^k of the example are obtained by exploiting scheme (20)–(22).

Define

$$E_\infty(h,\tau) = \max_{0\le i\le M,\,0\le k\le N} \left| u(x_i,t_k) - u_i^k \right|. \tag{53}$$

Example 1. Consider the following problem:

$$\begin{aligned} r(x,t)\,u_t - 2u_{xx} &= u(x,t) + u(x,t-0.1) + 2e^{-x}, \\ & \qquad x \in (0,1),\ t \in (0,1], \\ u(x,t) &= e^{-x}(1+t), \quad x \in (0,1),\ t \in [-0.1,0], \\ u(0,t) = 1+t, &\quad u(1,t) = e^{-1}(1+t), \quad t \in (0,1], \end{aligned} \tag{54}$$

where $r(x,t) = 4t + 5.9$. The exact solution of (54) is $u(x,t) = e^{-x}(1+t)$.

Table 1 provides some numerical results of difference scheme (20)–(22) solving for (54) with step-sizes $h = 1/10$ and $\tau = 1/100$. Table 2 gives the maximum absolute errors between numerical solutions and exact solutions with different step-sizes. From Table 2, we can see that when the space step-size and the time step-size are reduced by a factor of 1/2 and 1/4, respectively, then the maximum absolute errors are reduced by a factor of approximately 1/16.

Figure 1 provides us the error curves of numerical solutions for (54) at $t = 1$ by using scheme (20)–(22). Figures 2 and 3 give the error surface of the numerical solutions with step-sizes $h = 1/10$, $\tau = 1/100$, and $h = 1/20$, $\tau = 1/400$, respectively.

Generally speaking, from the results of the tables and the figures provided, we can see that the numerical results are coincident with the theoretical results.

5. Conclusion

In this paper, a compact difference scheme is constructed to solve a type of variable coefficient delay partial differential equations, and the difference scheme is proved to be unconditionally stable and convergent. Finally, a numerical test is presented to illustrate the theoretical results.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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