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Research Article

Commutator Theorems for Fractional Integral Operators on Weighted Morrey Spaces

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Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbb{R}^n)$ with Gaussian kernel bounds, and let $L^{-\alpha/2}$ be the fractional integrals of L for $0 < \alpha < n$. For any locally integrable function b, the commutators associated with $L^{-\alpha/2}$ are defined by $[b, L^{-\alpha/2}](f)(x) = b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x)$. When $b \in \mathrm{BMO}(\omega)$ (weighted BMO space) or $b \in \mathrm{BMO}$, the authors obtain the necessary and sufficient conditions for the boundedness of $[b, L^{-\alpha/2}]$ on weighted Morrey spaces, respectively.

1. Introduction and Main Results

Morrey [1] introduced the classical Morrey spaces to investigate the local behavior of solutions to second order elliptic partial differential equations. Chiarenza and Frasca [2] established the boundedness of the Hardy-Littlewood maximal operator, the fractional operator, and a singular integral operator on the Morrey spaces. On the other hand, Coifman and Fefferman [3] and Muckenhoupt [4] studied the boundedness of these operators on weighted L^p spaces. Motivated by these works, Komori and Shirai [5] introduced the following weighted Morrey space and investigated the boundedness of classical operators in harmonic analysis, that is, the Hardy-Littlewood maximal operator, a Calderón-Zygmund operator, the fractional integral operator, and so forth.

Let $1 \le p < \infty$ and $0 \le k < 1$. Then for two weights μ and ν, the weighted Morrey space is defined by

$$L^{p,k}(\mu,\nu) = \left\{ f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{L^{p,k}(\mu,\nu)} < \infty \right\}, \tag{1}$$

where

$$\|f\|_{L^{p,k}(\mu,\nu)} = \sup_Q \left(\frac{1}{\nu(Q)^k} \int_Q |f(x)|^p \mu(x)\,dx \right)^{1/p}, \tag{2}$$

and the supremum is taken over all balls Q in $\mathbb{R}^n$.

If $\mu = \nu$, then we have the classical Morrey space $L^{p,k}(\mu)$ with measure μ. When $k = 0$, then $L^{p,k}(\mu,\nu) = L^p(\mu)$ is the Lebesgue space with measure μ.

Suppose that L is a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup e^{-tL} with a kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \le \frac{C}{t^{n/2}} e^{-c(|x-y|^2/t)} \tag{3}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

For $0 < \alpha < n$, the fractional integral $L^{-\alpha/2}$ of the operator L is defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f) \frac{dt}{t^{-\alpha/2+1}}(x). \tag{4}$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α which plays important roles in many fields. It is well known that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $p > 1$, $1/q = 1/p - \alpha/n > 0$ and is also of weak type $(1, n/(n-\alpha))$.

Let $1 \leq p < \infty$ and ω be a weight function. A locally integrable function b is said to be in $\mathrm{BMO}_p(\omega)$ if

$$\|b\|_{\mathrm{BMO}_p(\omega)} = \sup_Q \left(\frac{1}{\omega(Q)} \int_Q \left| b(x) - b_Q \right|^p \omega(x)^{1-p} dx \right)^{1/p} \leq C < \infty, \tag{5}$$

where $b_Q = (1/|Q|) \int_Q b(y)dy$ and the supremum is taken over all balls $Q \in \mathbb{R}^n$.

Let $\omega \in A_1$; García-Cuerva [6] proved that the spaces $\mathrm{BMO}_p(\omega)$ coincide, and the norms of $\|\cdot\|_{\mathrm{BMO}_p(\omega)}$ are equivalent with respect to different values provided that $1 \leq p < \infty$.

Let b be a locally integrable function on $\mathbb{R}^n$; we consider the commutator $[b, L^{-\alpha/2}]$ defined by

$$\left[b, L^{-\alpha/2}\right](f)(x) = b(x) L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x). \tag{6}$$

Chanillo [7] proved that the commutator $[b, I_\alpha]$ of the multiplication operator by $b \in$ BMO is bounded on L^p for $1 < p < \infty$.

Duong and Yan [8] proved that $[b, L^{-\alpha/2}]$ is bounded from L^p to L^q, where $b \in$ BMO, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \alpha < n$.

Mo and Lu [9] proved that the multilinear commutator generated by $\vec{b}$ and $L^{-\alpha/2}$ is bounded from L^p to L^q, where $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \alpha < 1$, $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in$ BMO, for $i = 1, \ldots, m$.

Lu et al. [10] proved that $[b, I_\alpha]$ is bounded from L^p to L^q if and only if $b \in$ BMO.

Wang [11] proved that $[b, I_\alpha]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$, where $b \in \mathrm{BMO}(\omega)$, $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < k < p/q$, and $\omega^{q/p} \in A_1$.

Inspired by the above results, we study the boundedness properties of the commutator $[b, L^{-\alpha/2}]$ on weighted Morrey spaces in this work. The main theorems are stated as follows.

Theorem 1. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 \leq k < p/q$, $\omega^{q/p} \in A_1$, and $r_\omega > ((1-k)/(p/q-k))$, where r_ω denotes the critical index of ω for the reverse Hölder condition. Then the following conditions are equivalent.*

(a) *$b \in BMO(\omega)$.*

(b) *$[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$.*

In particular, when $k = 0$ in Theorem 1, we get the following.

Corollary 2. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $\omega^{q/p} \in A_1$, and $r_\omega > q/p$, where r_ω denotes the critical index of ω for the reverse Hölder condition. Then the following conditions are equivalent.*

(a) *$b \in BMO(\omega)$.*

(b) *$[b, L^{-\alpha/2}]$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-(1-\alpha/n)q})$.*

Furthermore, if $L = -\Delta$ is the Laplacian, then the following conditions are equivalent.

(a′) *$b \in BMO(\omega)$.*

(b′) *$[b, I_\alpha]$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-(1-\alpha/n)q})$.*

Theorem 3. *Let $0 < \alpha < n$, $0 \leq k < p/q$, $1/q = 1/p - \alpha/n$, and $1 < r, s < \infty$ such that $1 < rs < p < n/\alpha$, $\omega^{rs} \in A_{p/rs, q/rs}$. Then the following conditions are equivalent.*

(a) *$b \in BMO$.*

(b) *$[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$.*

In particular, when $k = 0$ in Theorem 3, we obtain the following.

Corollary 4. *Let $0 < \alpha < n$, $1/q = 1/p - \alpha/n$, and $1 < r, s < \infty$ such that $1 < rs < p < n/\alpha$, $\omega^{rs} \in A_{p/rs, q/rs}$. Then the following conditions are equivalent.*

(a) *$b \in BMO$.*

(b) *$[b, L^{-\alpha/2}]$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$.*

Furthermore, if $L = -\Delta$ is the Laplacian, then the following conditions are equivalent.

(a′) *$b \in BMO$.*

(b′) *$[b, I_\alpha]$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$.*

Remark 5. It is easy to see that our results extend the results in [7, 8, 10, 11] significantly.

2. Prerequisite Material

Let us first recall some definitions.

Definition 6. The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy. \tag{7}$$

Let ω be a weight. The weighted maximal operator M_ω is defined by

$$M_\omega(f)(x) = \sup_{x \in Q} \frac{1}{\omega(Q)} \int_Q |f(y)| \omega(y) dy. \tag{8}$$

A variant of this maximal operator will become the main tool in our scheme; for $1 < r < \infty$,

$$M_{r,\omega}(f)(x) = M_\omega\left(|f|^r\right)^{1/r}(x). \tag{9}$$

For $0 < \alpha < n$, $r \geq 1$, the fractional maximal operator $M_{\alpha,r}$ is defined by

$$M_{\alpha,r}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\alpha r/n}} \int_Q |f(y)|^r dy \right)^{1/r}, \tag{10}$$

and the fractional weighted maximal operator $M_{\alpha,r,\omega}$ is defined by

$$M_{\alpha,r,\omega}(f)(x)=\sup_{x\in Q}\left(\frac{1}{\omega(Q)^{1-\alpha r/n}}\int_Q|f(y)|^r\omega(y)\,dy\right)^{1/r}. \quad (11)$$

For any $f\in L^p(\mathbb{R}^n)$, $p\geq 1$, the sharp maximal function $M_L^\sharp f$ associated with the generalized approximations to the identity $\{e^{-tL}\}_{t>0}$ is given by

$$M_L^\sharp f(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q\left|f(y)-e^{-t_QL}f(y)\right|dy, \quad (12)$$

where $t_Q=r_Q^2$ and r_Q is the radius of the ball Q.

In the above definitions, the supremum is taken over all balls Q containing x.

Definition 7. A weight function ω is said to be in the Muckenhoupt class A_p with $1<p<\infty$ if, for every ball Q in $\mathbb{R}^n$, there exists a positive constant C which is independent of Q such that

$$\left(\frac{1}{|Q|}\int_Q\omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q\omega(x)^{-1/(p-1)}dx\right)^{p-1}\leq C. \quad (13)$$

When $p=1$, $\omega\in A_1$, if

$$\left(\frac{1}{|Q|}\int_Q\omega(x)\,dx\right)\leq C\operatorname{ess}\inf_{x\in Q}\omega(x). \quad (14)$$

When $p=\infty$, $\omega\in A_\infty$, if there exist positive constants δ and C such that, given a ball Q and a measurable subset E of Q,

$$\frac{\omega(E)}{\omega(Q)}\leq C\left(\frac{|E|}{|Q|}\right)^{\delta}. \quad (15)$$

Definition 8. A weight function ω belongs to $A_{p,q}$ for $1<p<q<\infty$ if, for every ball Q in $\mathbb{R}^n$, there exists a positive constant C which is independent of Q such that

$$\left(\frac{1}{|Q|}\int_Q\omega(x)^q dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q\omega(x)^{-p'}dx\right)^{1/p'}\leq C, \quad (16)$$

where p' denotes the conjugate exponent of $p>1$, that is, $1/p+1/p'=1$.

Definition 9. A weight function ω belongs to the reverse Hölder class RH_r if there exist two constants $r>1$ and $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|Q|}\int_Q\omega(x)^r dx\right)^{1/r}\leq C\left(\frac{1}{|Q|}\int_Q\omega(x)\,dx\right) \quad (17)$$

holds for every ball Q in $\mathbb{R}^n$.

It is well known that if $\omega\in A_p$ with $1\leq p<\infty$, then there exists $r>1$ such that $\omega\in\mathrm{RH}_r$. It follows from Hölder's inequality that $\omega\in\mathrm{RH}_r$ implies $\omega\in\mathrm{RH}_s$ for all $1<s<r$. Moreover, if $\omega\in\mathrm{RH}_r$, $r>1$, then we have $\omega\in\mathrm{RH}_{r+\epsilon}$ for some $\epsilon>0$. We thus write $r_w=\sup\{r>1:\omega\in\mathrm{RH}_r\}$ to denote the critical index of ω for the reverse Hölder condition.

We will make use of the following lemmas. We first provide a weighted version of the local good λ inequality for $M_L^\sharp$ which allows us to obtain an analog of the classical Fefferman-Stein (see [3, 12]) estimate on weighted Morrey spaces.

Lemma 10 (see [13]). *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (3). Take $\lambda>0$, $f\in L_0^1(\mathbb{R}^n)$, and a ball Q_0 such that there exists $x_0\in Q_0$ with $Mf(x_0)\leq\lambda$. Then, for every $\omega\in A_\infty$, $0<\eta<1$, one can find $\gamma>0$ (independent of λ, Q_0, f, x_0) and constant C_ω, $r>0$ (which only depend on ω), such that*

$$\omega\left\{x\in Q_0:Mf(x)>A\lambda,\ M_L^\sharp f(x)\leq\gamma\lambda\right\}\leq C_\omega\eta^r\omega(Q_0), \quad (18)$$

where $A>1$ is a fixed constant which depends only on n.

As a consequence, by using the standard arguments, we have the following estimates.

For every $f\in L^{p,k}(\mu,\nu)$, with $1<p<\infty$. if $\mu,\nu\in A_\infty$, $1<p<\infty$, $0\leq k<1$, then

$$\|f\|_{L^{p,k}(\mu,\nu)}\leq\|Mf\|_{L^{p,k}(\mu,\nu)}\leq C\left\|M_L^\sharp f\right\|_{L^{p,k}(\mu,\nu)}. \quad (19)$$

In particular, when $\mu=\nu=\omega$ and $\omega\in A_\infty$, we have

$$\|f\|_{L^{p,k}(\omega)}\leq\|Mf\|_{L^{p,k}(\omega)}\leq C\left\|M_L^\sharp f\right\|_{L^{p,k}(\omega)}. \quad (20)$$

Lemma 11 (see [11]). *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, and $\omega^{q/p}\in A_1$. Then if $0<k<p/q$ and $r_\omega>(1-k)/(p/q-k)$, one has*

$$\|M_{\alpha,1}f\|_{L^{q,kq/p}(\omega^{q/p},\omega)}\leq C\|f\|_{L^{p,k}(\omega)}. \quad (21)$$

The same conclusion still holds for I_α.

Lemma 12 (see [11]). *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, and $\omega^{q/p}\in A_1$. Then if $0<k<p/q$, $1<r<p$, and $r_\omega>(1-k)/(p/q-k)$, one has*

$$\|M_{r,\omega}f\|_{L^{q,kq/p}(\omega^{q/p},\omega)}\leq C\|f\|_{L^{q,kq/p}(\omega^{q/p},\omega)}. \quad (22)$$

Lemma 13 (see [11]). *Consider $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $0<k<p/q$, and $\omega\in A_\infty$. For any $1<r<p$, one has*

$$\|M_{\alpha,r,\omega}f\|_{L^{q,kq/p}(\omega)}\leq C\|f\|_{L^{p,k}(\omega)}. \quad (23)$$

Remark 14. By checking the proof of Lemmas 11–13, we know that the three lemmas above still hold when $k=0$.

Lemma 15. *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, and $\omega^{q/p}\in A_1$. Then if $0\leq k<p/q$ and $r_\omega>(1-k)/(p/q-k)$, one has*

$$\left\|L^{-\alpha/2}f\right\|_{L^{q,kq/p}(\omega^{q/p},\omega)}\leq C\|f\|_{L^{p,k}(\omega)}. \quad (24)$$

Proof. Since the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (3), it is easy to check that $L^{-\alpha/2}(f)(x) \le CI_\alpha(|f|)(x)$ for all $x \in \mathbb{R}^n$. Using the boundedness property of I_α on weighted Morrey space (see Lemma 11), we have

$$\left\|L^{-\alpha/2}f\right\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \le \left\|I_\alpha f\right\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \le C\|f\|_{L^{p,k}(\omega)}, \quad (25)$$

where $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. □

Remark 16. Since I_α is of weak type $(1, n/(n-\alpha))$. from the proof of Lemma 15, we can get that $L^{-\alpha/2}$ is also of weak type $(1, n/(n-\alpha))$.

Lemma 17 (see [8, 14]). *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (3). Then for $0 < \alpha < n$, the difference operator $L^{-\alpha/2} - e^{-tL}L^{-\alpha/2}$ has an associated kernel $K_{\alpha,t}(x,y)$ which satisfies*

$$\left|K_{\alpha,t}(x,y)\right| \le \frac{C}{|x-y|^{n-\alpha}}\frac{t}{|x-y|^2}, \quad (26)$$

for some positive constant C.

Lemma 18. *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (3), and let $b \in BMO(\omega)$, $\omega \in A_1$. Then, for every function $f \in L^p(\mathbb{R}^n)$, $p > 1$, and for all $x \in \mathbb{R}^n$, one has*

$$\begin{aligned}\sup_{x\in Q}\frac{1}{|Q|}\int_Q\left|e^{-t_QL}(b(y)-b_Q)f(y)\right|dy \\ \le C\|b\|_{BMO(\omega)}\omega(x)M_{r,\omega}(f)(x),\end{aligned} \quad (27)$$

where $t_Q = r_Q^2$, r_Q being the radius of Q.

Proof. For any $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ and $x \in Q$. We have

$$\begin{aligned}&\frac{1}{|Q|}\int_Q\left|e^{-t_QL}((b(\cdot)-b_Q)f)(y)\right|dy \\ &\quad\le \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n}\left|p_{t_Q}(y,z)\right|\left|(b(z)-b_Q)f(z)\right|dz\,dy \\ &\quad\le \frac{1}{|Q|}\int_Q\int_{2Q}\left|p_{t_Q}(y,z)\right|\left|(b(z)-b_Q)f(z)\right|dz\,dy \\ &\qquad+\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^{k+1}Q\backslash 2^kQ}\left|p_{t_Q}(y,z)\right| \\ &\qquad\qquad\times\left|(b(z)-b_Q)f(z)\right|dz\,dy \\ &\quad\doteq \mathbf{M}+\mathbf{N}.\end{aligned} \quad (28)$$

For any $y \in Q$ and $z \in 2Q$. We have

$$\left|p_{t_Q}(y,z)\right| \le Ct_Q^{-n/2} \le C\frac{1}{|2Q|}. \quad (29)$$

Thus,

$$\begin{aligned}\mathbf{M} &\le C\frac{1}{|2Q|}\int_{2Q}\left|(b(z)-b_Q)f(z)\right|dz \\ &\le C\frac{1}{|2Q|}\left(\int_{2Q}\left|b(z)-b_Q\right|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'} \\ &\quad\times\left(\int_{2Q}\left|f(z)\right|^r\omega(z)\,dz\right)^{1/r} \\ &\le C\|b\|_{\mathrm{BMO}(\omega)}\frac{\omega(2Q)}{|2Q|}\left(\frac{1}{\omega(2Q)}\int_{2Q}\left|f(z)\right|^r\omega(z)\,dz\right)^{1/r} \\ &\le C\|b\|_{\mathrm{BMO}(\omega)}\omega(x)M_{r,\omega}f(x).\end{aligned} \quad (30)$$

Moreover, for any $y \in Q$ and $z \in 2^{k+1}Q\backslash 2^kQ$, we have $|y-z| \ge 2^{k-1}r_Q$ and $|p_{t_Q}| \le C(e^{-c2^{2(k-1)}}2^{(k+1)n}/|2^{k+1}Q|)$:

$$\begin{aligned}&\mathbf{N} \\ &= \frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^{k+1}Q\backslash 2^kQ}\left|p_{t_Q}(y,z)\right|\left|(b(z)-b_Q)f(z)\right|dz\,dy \\ &\le C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|(b(z)-b_Q)f(z)\right|dz \\ &\le C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|(b(z)-b_{2^{k+1}Q})f(z)\right|dz \\ &\quad+C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|(b_{2^{k+1}Q}-b_Q)f(z)\right|dz \\ &\doteq \mathbf{N}_1+\mathbf{N}_2.\end{aligned} \quad (31)$$

We estimate each term in turn. For $\mathbf{N}_1$, we apply Hölder's inequalities with exponent r. Then we have

$$\begin{aligned}\mathbf{N}_1 &\le C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}Q|} \\ &\quad\times\left(\int_{2^{k+1}Q}\left|b(z)-b_{2^{k+1}Q}\right|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'} \\ &\quad\times\left(\int_{2^{k+1}Q}\left|f(z)\right|^r\omega(z)\,dz\right)^{1/r} \\ &\le C\sum_{k=1}^{\infty}2^{(k+1)n}e^{-c2^{2(k-1)}}\|b\|_{\mathrm{BMO}(\omega)}\frac{\omega(2^{k+1}Q)}{|2^{k+1}Q|} \\ &\quad\times\left(\frac{1}{\omega(2^{k+1}Q)}\int_{2^{k+1}Q}\left|f(z)\right|^r\omega(z)\,dz\right)^{1/r} \\ &\le C\|b\|_{\mathrm{BMO}(\omega)}\omega(x)M_{r,\omega}f(x).\end{aligned} \quad (32)$$

Since $\omega \in A_1$, then $|b_{2^{k+1}Q} - b_Q| \le Ck\omega(x)\|b\|_{\mathrm{BMO}(\omega)}$. This fact together with Hölder's inequality implies

$$\begin{aligned}
\mathbf{N}_2 &\le C\sum_{k=1}^{\infty} 2^{(k+1)n} e^{-c2^{2(k-1)}} \frac{k}{|2^{k+1}Q|}\omega(x) \\
&\quad \times \|b\|_{\mathrm{BMO}(\omega)} \int_Q |f(z)|\,dz \\
&\le C\sum_{k=1}^{\infty} k2^{(k+1)n} e^{-c2^{2(k-1)}} \omega(x) \\
&\quad \times \|b\|_{\mathrm{BMO}(\omega)} \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} |f(z)|^r dz\right)^{1/r} \\
&= C\sum_{k=1}^{\infty} k2^{(k+1)n} e^{-c2^{2(k-1)}} \omega(x) \\
&\quad \times \|b\|_{\mathrm{BMO}(\omega)} \left(\frac{\omega(2^{k+1}Q)}{|2^{k+1}Q|}\frac{1}{\omega(2^{k+1}Q)}\int_{2^{k+1}Q} |f(z)|^r dz\right)^{1/r} \\
&\le C\sum_{k=1}^{\infty} k2^{(k+1)n} e^{-c2^{2(k-1)}} \omega(x) \\
&\quad \times \|b\|_{\mathrm{BMO}(\omega)} \left(\frac{1}{\omega(2^{k+1}Q)}\int_{2^{k+1}Q} |f(z)|^r \omega(x)\,dz\right)^{1/r} \\
&\le C\|b\|_{\mathrm{BMO}(\omega)}\omega(x) M_{r,\omega} f(x).
\end{aligned} \tag{33}$$

Then Lemma 18 is proved. □

Lemma 19. *Let $0 < \alpha < n$, $\omega \in A_1$, and $b \in BMO(\omega)$. Then for all $r > 1$ and for all $x \in \mathbb{R}^n$, one has*

$$\begin{aligned}
&M_L^\sharp\left(\left[b, L^{-\alpha/2}\right] f\right)(x) \\
&\quad \le C\|b\|_{BMO(\omega)}\left(\omega(x) M_{r,\omega}\left(L^{-\alpha/2} f\right)(x) + \omega(x)^{1-\alpha/n}\right. \\
&\qquad \left. \times M_{\alpha,r,\omega}(f)(x) + \omega(x) M_{\alpha,1}(f)(x)\right).
\end{aligned} \tag{34}$$

Proof. For any given $x \in \mathbb{R}^n$, fix a ball $Q = Q(x_0, r_Q)$ which contains x. We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$. Observe that

$$\begin{aligned}
\left[b, L^{-\alpha/2}\right] f(x) &= (b - b_Q) L^{-\alpha/2} f \\
&\quad - L^{-\alpha/2}(b - b_Q) f_1 \\
&\quad - L^{-\alpha/2}(b - b_Q) f_2, \\
e^{-t_Q L}\left(\left[b, L^{-\alpha/2}\right] f\right) &= e^{-t_Q L}\left[(b - b_Q) L^{-\alpha/2} f\right. \\
&\quad - L^{-\alpha/2}(b - b_Q) f_1 \\
&\quad \left. - L^{-\alpha/2}(b - b_Q) f_2\right].
\end{aligned} \tag{35}$$

Then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|\left[b, L^{-\alpha/2}\right] f(y) - e^{-t_Q L}\left[b, L^{-\alpha/2}\right] f(y)\right| dy \\
&\quad \le \frac{1}{|Q|}\int_Q \left|(b(y) - b_Q) L^{-\alpha/2} f(y)\right| dy \\
&\qquad + \frac{1}{|Q|}\int_Q \left|L^{-\alpha/2}(b - b_Q) f_1(y)\right| dy \\
&\qquad + \frac{1}{|Q|}\int_Q \left|e^{-t_Q L}\left((b - b_Q) L^{-\alpha/2} f\right)(y)\right| dy \\
&\qquad + \frac{1}{|Q|}\int_Q \left|e^{-t_Q L} L^{-\alpha/2}((b - b_Q) f_1(y))\right| dy \\
&\qquad + \frac{1}{|Q|}\int_Q \left|\left(L^{-\alpha/2} - e^{-t_Q L} L^{-\alpha/2}\right)((b - b_Q) f_2)(y)\right| dy \\
&\quad \doteq \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV} + \mathbf{V}.
\end{aligned} \tag{36}$$

We estimate each term separately.

Since $\omega \in A_1$, then it follows from Hölder's inequality that

$$\begin{aligned}
\mathbf{I} &\le \frac{1}{|Q|}\int_Q \left|(b(y) - b_Q) L^{-\alpha/2} f(y)\right| dy \\
&\le \frac{1}{|Q|}\left(\int_Q |b(y) - b_Q|^{r'} \omega(y)^{1-r'} dy\right)^{1/r'} \\
&\quad \times \left(\int_Q \left|L^{-\alpha/2} f(y)\right|^r \omega(y)\,dy\right)^{1/r} \\
&\le C\|b\|_{\mathrm{BMO}(\omega)} \frac{\omega(Q)}{|Q|}\left(\frac{1}{\omega(Q)}\int_Q \left|L^{-\alpha/2} f(y)\right|^r \omega(y)\,dy\right)^{1/r} \\
&\le C\|b\|_{\mathrm{BMO}(\omega)}\omega(x) M_{r,\omega}\left(L^{-\alpha/2} f\right)(x).
\end{aligned} \tag{37}$$

Applying Kolmogorov's inequality (see [15, page 485]), Hölder's inequality, and the continuity of $L^{-\alpha/2}$, we thus have

$$\begin{aligned}
\mathbf{II} &= \frac{1}{|Q|}\int_Q \left|L^{-\alpha/2}(b - b_Q) f_1(y)\right| dy \\
&\le C\frac{1}{|Q|^{1-\alpha/n}}\left\|L^{-\alpha/2}(b - b_Q) f_1\right\|_{L^{n/(n-\alpha),\infty}} \\
&\le C\frac{1}{|Q|^{1-\alpha/n}}\int_Q (b - b_Q) f_1(y)\,dy \\
&\le C\frac{1}{|Q|^{1-\alpha/n}}\left(\int_Q |b(y) - b_Q|^{r'} \omega(y)^{1-r'} dy\right)^{1/r'}
\end{aligned}$$

$$\begin{aligned}
&\times\left(\int_Q |f(y)|^r \omega(y)dy\right)^{1/r}\\
&\leq C\|b\|_{\mathrm{BMO}(\omega)}\frac{w(Q)^{1-\alpha/n}}{|Q|^{1-\alpha/n}}\\
&\times\left(\frac{1}{w(Q)^{1-r\alpha/n}}\int_Q |f(y)|^r \omega(y)\,dy\right)^{1/r}\\
&\leq C\|b\|_{\mathrm{BMO}(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha,r,\omega}(f)(x).
\end{aligned} \tag{38}$$

By Lemma 18, we have

$$\mathbf{III} \leq C\|b\|_{\mathrm{BMO}(\omega)}\omega(x)\,M_{r,\omega}\left(L^{-\alpha/2}f\right)(x). \tag{39}$$

For **IV**, using the estimate obtained in **II**, we get

$$\begin{aligned}
\mathbf{IV} &\leq \frac{1}{|Q|}\int_Q\int_{2Q}\left|p_{t_Q}(y,z)\right|\left|b(z)-b_Q\right|\left|f(z)\right|dz\,dy\\
&\leq \frac{1}{|2Q|}\int_{2Q}\left|L^{-\alpha/2}\left(\left(b(z)-b_Q\right)\right)f(z)\right|dz\\
&\leq C\|b\|_{\mathrm{BMO}(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha,r,\omega}(f)(x).
\end{aligned} \tag{40}$$

By virtue of Lemma 17, we have

$$\begin{aligned}
\mathbf{V} &\leq \frac{1}{|Q|}\int_Q\int_{(2Q)^c}\left|K_{\alpha,t_Q}(y,z)\right|\left|\left(b(z)-b_Q\right)f(z)\right|dz\,dy\\
&\leq C\sum_{k=1}^{\infty}\int_{2^k r_Q\leq|x_0-z|<2^{k+1}r_Q}\frac{1}{|x_0-z|^{n-\alpha}}\frac{t_Q}{|x_0-z|^2}\\
&\qquad\times\left|\left(b(z)-b_Q\right)f(z)\right|dz\\
&\leq C\sum_{k=1}^{\infty}2^{-2k}\frac{1}{|2^{k+1}Q|^{1-\alpha/n}}\int_{2^{k+1}Q}\left|\left(b(z)-b_Q\right)f(z)\right|dz\\
&\leq C\sum_{k=1}^{\infty}2^{-2k}\frac{1}{|2^{k+1}Q|^{1-\alpha/n}}\int_{2^{k+1}Q}\left|\left(b(z)-b_{2^{k+1}Q}\right)f(z)\right|dz\\
&\quad+C\sum_{k=1}^{\infty}2^{-2k}\left(b_{2^{k+1}Q}-b_Q\right)\frac{1}{|2^{k+1}Q|^{1-\alpha/n}}\int_{2^{k+1}Q}|f(z)|\,dz\\
&\doteq \mathbf{VI}+\mathbf{VII}.
\end{aligned} \tag{41}$$

For **VI**, applying the same arguments as in **II**, we get

$$\begin{aligned}
\mathbf{VI} &\leq C\|b\|_{\mathrm{BMO}(\omega)}\sum_{k=1}^{\infty}2^{-2k}\omega(x)^{1-\alpha/n}M_{\alpha,r,\omega}(f)(x)\\
&\leq C\|b\|_{\mathrm{BMO}(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha,r,\omega}(f)(x).
\end{aligned} \tag{42}$$

Since $\omega\in A_1$, then $|b_{2^{k+1}Q}-b_Q|\leq Ck\omega(x)\|b\|_{\mathrm{BMO}(\omega)}$. Thus,

$$\begin{aligned}
\mathbf{VII} &\leq C\|b\|_{\mathrm{BMO}(\omega)}\sum_{k=1}^{\infty}2^{-2k}k\omega(x)\,M_{\alpha,1}(f)(x)\\
&\leq C\|b\|_{\mathrm{BMO}(\omega)}\omega(x)\,M_{\alpha,1}(f)(x).
\end{aligned} \tag{43}$$

Then

$$\begin{aligned}
\mathbf{V} &\leq C\|b\|_{\mathrm{BMO}(\omega)}\\
&\quad\times\left(\omega(x)^{1-\alpha/n}M_{\alpha,r,\omega}(f)(x)+\omega(x)\,M_{\alpha,1}(f)(x)\right).
\end{aligned} \tag{44}$$

Combining the above estimates **I–V**, we get (34). The proof of Lemma 19 is complete. □

3. Proofs of the Main Results

In this section we prove our main results. We start with the proof of Theorem 1.

Proof. (a)⇒(b): Applying Lemmas 10 and 19, we get

$$\begin{aligned}
&\left\|\left[b,L^{-\alpha/2}\right]f\right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leq\left\|M_L^{\sharp}\left(\left[b,L^{-\alpha/2}\right]f\right)\right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leq C\|b\|_{\mathrm{BMO}(\omega)}\\
&\quad\times\left(\left\|\omega M_{r,\omega}\left(L^{-\alpha/2}f\right)\right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\right.\\
&\qquad+\left\|\omega^{1-\alpha/n}M_{\alpha,r,\omega}f\right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\qquad\left.+\left\|\omega M_{\alpha,1}f\right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\right)\\
&\leq C\|b\|_{\mathrm{BMO}(\omega)}\\
&\quad\times\left(\left\|M_{r,\omega}\left(L^{-\alpha/2}f\right)\right\|_{L^{q,kq/p}(\omega^{q/p},\omega)}\right.\\
&\qquad+\left\|M_{\alpha,r,\omega}f\right\|_{L^{q,kq/p}(\omega)}\\
&\qquad\left.+\left\|M_{\alpha,1}f\right\|_{L^{q,kq/p}(\omega^{q/p},\omega)}\right).
\end{aligned} \tag{45}$$

Since $0\leq k<p/q$, $\omega^{q/p}\in A_1$, and $r_\omega>(1-k)/(p/q-k)$, by making use of Lemmas 11–13, then we obtain

$$\begin{aligned}
&\left\|\left[b,L^{-\alpha/2}\right]f\right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\quad\leq C\|b\|_{\mathrm{BMO}(\omega)}\left(\left\|L^{-\alpha/2}f\right\|_{L^{q,kq/p}(\omega^{q/p},\omega)}+\|f\|_{L^{p,k}(\omega)}\right)\\
&\quad\leq C\|b\|_{\mathrm{BMO}(\omega)}\|f\|_{L^{p,k}(\omega)}.
\end{aligned} \tag{46}$$

The last inequality follows from Lemma 15. This completes the proof of (a)⇒(b).

(b)⇒(a): Let $L=-\Delta$ be the Laplacian on $\mathbb{R}^n$; then $L^{-\alpha/2}$ is the classical fractional integral I_α. Choose $Z_0\in\mathbb{R}^n$ so that $|Z_0|=3$. For $x\in Q(Z_0,2)$, $|x|^{-\alpha+n}$ can be written as the absolutely convergent Fourier series $|x|^{-\alpha+n}=\sum_{m\in Z_n}a_m e^{i\langle\nu_m,x\rangle}$

with $\sum_m |a_m| < \infty$ since $|x|^{-\alpha+n} \in C^\infty(Q(Z_0,2))$. For any $x_0 \in \mathbb{R}^n$ and $\rho > 0$, let $Q = Q(x_0,\rho)$ and $Q_{Z_0} = Q(x_0+Z_0\rho,\rho)$,

$$\begin{aligned}
&\int_Q \left|b(x) - b_{Q_{Z_0}}\right| dx \\
&= \frac{1}{|Q_{Z_0}|}\int_Q \left|\int_{Q_{Z_0}} (b(x)-b(y))\,dy\right| dx \\
&= \frac{1}{\rho^n}\int_Q s(x)\left(\int_{Q_{Z_0}} (b(x)-b(y))\,|x-y|^{\alpha-n}|x-y|^{n-\alpha}dy\right)dx,
\end{aligned} \tag{47}$$

where $s(x) = \operatorname{sgn}(\int_{Q_{Z_0}}(b(x)-b(y))dy)$. Fix $x \in Q$ and $y \in Q_{Z_0}$ and we have $(y-x)/\rho \in Q(Z_0,2)$; hence, we have

$$\begin{aligned}
&\frac{\rho^{-\alpha+n}}{\rho^n}\int_Q s(x)\left(\int_{Q_{Z_0}}(b(x)-b(y))\,|x-y|^{\alpha-n}\left(\frac{|x-y|}{\rho}\right)^{n-\alpha}dy\right)dx \\
&= \rho^{-\alpha}\sum_{m\in Z^n} a_m \\
&\quad\times \int_Q s(x)\left(\int_{Q_{Z_0}}(b(x)-b(y))\right. \\
&\qquad\left.\times |x-y|^{\alpha-n} e^{i\langle \nu_m, y/\rho\rangle}dy\right) e^{-i\langle \nu_m, x/\rho\rangle}dx \\
&\le \rho^{-\alpha}\left|\sum_{m\in Z^n}|a_m|\int_Q s(x)\left[b, L^{-\alpha/2}\right]\right. \\
&\qquad\left.\times\left(\chi_{Q_{Z_0}} e^{i\langle \nu_m,\cdot/\rho\rangle}\right)\chi_Q(x)\, e^{-i\langle \nu_m, x/\rho\rangle}dx\right| \\
&\le \rho^{-\alpha}\sum_{m\in Z^n}|a_m|\left\|\left[b,L^{-\alpha/2}\right]\left(\chi_{Q_{Z_0}}e^{i\langle \nu_m,\cdot/\rho\rangle}\right)\right\|_{L^{q,0}(\omega^{1-(1-(\alpha/n))q},\omega)} \\
&\quad\times\left(\int_Q \omega(x)^{q'[(1-(\alpha/n))-1/q]}dx\right)^{1/q'} \\
&\le C\rho^{-\alpha}\sum_{m\in Z^n}|a_m|\left\|\chi_{Q_{Z_0}}\right\|_{L^{p,0}(\omega)}\left(\int_Q \omega(x)^{q'(1/q'-\alpha/n)}dx\right)^{1/q'} \\
&\le C\omega(Q)^{1/p+1/q'-\alpha/n} \\
&= C\omega(Q).
\end{aligned} \tag{48}$$

This implies $b \in \mathrm{BMO}(\omega)$. Thus Theorem 1 is proved. □

Similarly, to prove Theorem 3, we need the following lemmas.

Lemma 20. *Let $0 < \alpha < n$, $1 < r$, $s < \infty$ such that $rs < p < n/\alpha$ and $b \in BMO$. Then for all $r > 1$ and for all $x \in \mathbb{R}^n$, one has*

$$\begin{aligned}
&M_L^\sharp\left(\left[b, L^{-\alpha/2}\right]f\right)(x) \\
&\quad\le C\|b\|_{BMO}\left(M_r\left(L^{-\alpha/2}f\right)(x) + M_{\alpha,rs}(f)(x)\right),
\end{aligned} \tag{49}$$

where $M_r(f)(x) = M(|f|^r)^{1/r}(x)$.

Proof. The case $0 < \alpha < 1$ was proved by Duong and Yan (see [8] for details). The general case $0 < \alpha < n$ follows by repeating the same steps as in Lemma 19. Since the main steps and the ideas are almost the same, here we omit the proof. □

Lemma 21 (see [5]). *If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < k < p/q$, and $\omega \in A_{p,q}$, then the fractional maximal operator $M_{\alpha,1}$ is bounded from $L^{p,k}(\omega^p,\omega^q)$ to $L^{q,kq/p}(\omega^q)$.*

Lemma 22 (see [5]). *If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < k < p/q$, and $\omega \in A_{p,q}$, then the fractional maximal operator I_α is bounded from $L^{p,k}(\omega^p,\omega^q)$ to $L^{q,kq/p}(\omega^q)$.*

Lemma 23 (see [5]). *If $1 < p < \infty$, $0 < k < 1$, and $\omega \in A_p$, then M is bounded on $L^{p,k}(\omega)$.*

Remark 24. By applying the same argument as in Lemma 15, we know that the conclusion in Lemma 22 still holds for $L^{-\alpha/2}$. We omit the proof here.

Remark 25. By checking the proof of Lemmas 21–23, we know that the three lemmas above still hold when $k = 0$.

Now we prove Theorem 3.

Proof. (a)⇒(b): Since $\omega^{rs} \in A_{p/rs,q/rs}$, then we get $\omega^q \in A_{q/rs}$ and $\omega^p \in A_{p/rs}$. Applying Lemmas 10 and 20–23, we get

$$\begin{aligned}
&\left\|\left[b,L^{-\alpha/2}\right]f\right\|_{L^{q,kq/p}(\omega^q)} \\
&\le \left\|M_L^\sharp\left(\left[b,L^{-\alpha/2}\right]f\right)\right\|_{L^{q,kq/p}(\omega^q)} \\
&\le C\|b\|_{\mathrm{BMO}}\left(\left\|M_r\left(L^{-\alpha/2}f\right)\right\|_{L^{q,kq/p}(\omega^q)} + \left\|M_{\alpha,rs}(f)\right\|_{L^{q,kq/p}(\omega^q)}\right) \\
&\le C\|b\|_{\mathrm{BMO}}\left(\left\|L^{-\alpha/2}f\right\|_{L^{q,kq/p}(\omega^q)} + \|f\|_{L^{p,k}(\omega^p,\omega^q)}\right) \\
&\le C\|b\|_{\mathrm{BMO}}\|f\|_{L^{p,k}(\omega^p,\omega^q)}.
\end{aligned} \tag{50}$$

In the last inequality, we used the fact that $L^{-\alpha/2}$ is bounded from $L^{p,k}(\omega^p,\omega^q)$ to $L^{q,kq/p}(\omega^q)$ (see Remark 24).

(b)⇒(a): Let $L = -\Delta$ be the Laplacian on $\mathbb{R}^n$; then $L^{-\alpha/2}$ is the classical fractional integral I_α. Let $k = 0$ and weight $\omega \equiv 1$, and then $L^{p,k}(\omega^p,\omega^q) = L^p$ and $L^{q,kq/p}(\omega^q) = L^q$. From [10] we deduce that the (L^p, L^q) boundedness of $[b, I_\alpha]$ implies $b \in \mathrm{BMO}$. Thus Theorem 3 is proved. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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