Research Article

Solution of the Differential-Difference Equations by Optimal Homotopy Asymptotic Method

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We applied a new analytic approximate technique, optimal homotopy asymptotic method (OHAM), for treatment of differential-difference equations (DDEs). To see the efficiency and reliability of the method, we consider Volterra equation in different form. It provides us with a convenient way to control the convergence of approximate solutions when it is compared with other methods of solution found in the literature. The obtained solutions show that OHAM is effective, simpler, easier, and explicit.

1. Introduction

In physical science the DDEs play a vital role in modeling of the complex physical phenomena. The DDEs models are used in vibration of particles in lattices, the flow of current in a network, and the pulses in biological chains. The models containing DDEs have been investigated by numerical techniques such as discretizations in solid state physics and quantum mechanics. In the last decades most of the research work has been done on DDEs. Levi and Yamilov [1, 2] attribute their work to classification of DDEs and connection of integrable partial differential equation (PDEs) and DDEs [3, 4]. In [5–8] the exact solutions of the DDEs have been studied. For the solution of the DDEs, Zou et al. extended the homotopy analysis method (HAM) [9, 10]. Wang et al. extended Adomian decomposition method (ADM) for solving nonlinear difference-differential equations (NDDEs) and got a good accuracy with the analytic solution [11]. Recently, Marinca et al. introduced optimal homotopy asymptotic method (OHAM) [12–16] for the solution of nonlinear problems. The validity of OHAM is independent of whether or not the nonlinear problems contain small parameters.

The motivation of this paper is to extend OHAM for the solution of NDDEs. In [17–20] OHAM has been proved to be useful for obtaining an approximate solution of nonlinear boundary value problems of differential equations. In this work, we have proved that OHAM is also useful and reliable for the solution of NDDEs, hence showing its validity and great potential for the solution of NDDEs phenomenon in science and engineering.

In the succeeding section, the basic idea of OHAM [12–16] is formulated for the solution of boundary value problems. In Section 3, the effectiveness of the modified formulation of OHAM for NDDEs has been studied.

2. Basic Mathematical Theory of OHAM for NDDE

Let us take OHAM to the following differential-difference equation:

$$\mathscr{A}\left(v_n\left(t\right)\right) + f\left(t\right) = 0, \quad n \in \Omega, \tag{1}$$

with boundary conditions:

$$\mathscr{B}\left(v_n\left(t\right), \frac{\partial v_n\left(t\right)}{\partial t}\right) = 0. \quad n \in \Gamma, \tag{2}$$

where $\mathscr{A}$ is a differential operator, $v_n(t)$ is an unknown function, n and t denote spatial and temporal independent variables, respectively, Γ is the boundary of Ω, and $f_n(t)$ is a known analytic function. $\mathscr{A}$ can be divided into two parts $\mathscr{L}$ and $\mathscr{N}$ such that

$$\mathscr{A} = \mathscr{L} + \mathscr{N}, \tag{3}$$

where $\mathscr{L}$ is simpler part of the partial differential equation which is easier to solve and $\mathscr{N}$ contains the remaining part of $\mathscr{A}$. So (1) can be written as

$$\mathscr{L}\left(v_n\left(t\right)\right) + f\left(t\right) + \mathscr{N}\left(v_n\left(t\right), v_{n-k}\left(t\right), v_{n+k}\left(t\right)\right) = 0. \quad (4)$$

According to OHAM, one can construct an optimal homotopy $\phi_n(t, q) : \Omega \times [0, 1] \to \mathfrak{R}$ satisfying

$$(1 - q)\left[\mathscr{L}\left(\phi_n\left(t, q\right)\right) + f\left(t\right)\right]$$

$$= H\left(q\right)\left[\mathscr{L}\left(\phi_n\left(t, q\right)\right) + f\left(t\right)\right. \quad (5)$$

$$\left. + \mathscr{N}\left(\phi_n\left(t, q\right), \phi_{n-k}\left(t, q\right), \phi_{n+k}\left(t, q\right)\right)\right],$$

$$\mathscr{B}\left(\phi_n\left(t, q\right), \frac{\partial \phi_n\left(t, q\right)}{\partial t}\right) = 0, \quad (6)$$

where $q \in [0, 1]$ is an embedding parameter, $\phi_n(t, q)$ is an unknown function, and $H(q)$ is a nonzero auxiliary function. The auxiliary function $H(q)$ is nonzero for $q \neq 0$ and $H(0) = 0$. Equation (5) is the structure of OHAM homotopy. Clearly we have

$$q = 0 \implies H\left(\phi_n\left(t; 0\right), 0\right)$$

$$= \mathscr{L}\left(\phi_n\left(t; 0\right)\right) + f\left(t\right) = 0,$$

$$q = 1 \implies H\left(\phi_n\left(t; 1\right), 1\right) \quad (7)$$

$$= H\left(1\right)\left\{\mathscr{A}\left(\phi_n\left(t; 1\right)\right) + f\left(t\right)\right\} = 0.$$

Obviously, when $q = 0$ and $q = 1$, we obtain

$$q = 0 \implies \phi_n\left(t, 0\right) = v_{n0}\left(t\right),$$

$$q = 1 \implies \phi_n\left(t, 1\right) = v_n\left(t\right), \quad (8)$$

respectively. Thus, as q varies from 0 to 1, the solution $\phi_n(t, q)$ varies from $v_{n0}(t)$ to $v_n(t)$, where $v_{n0}(t)$ is the zeroth order solution and can be obtained from (10).

Expanding $\phi_n(t, q, C_i)$, $\phi_{n-k}(t, q, C_i)$, $\phi_{n+k}(t, q, C_i)$ by Taylors series and choosing $H(q)$ as given

$$H\left(q\right) = qC_1 + q^2 C_2 + q^3 C_3 + \cdots + q^m C_m,$$

$$\phi_{n-k}\left(t, q, C_i\right) = v_{(n-k)0}\left(t\right)$$

$$+ \sum_{m=1}^{\infty} v_{(n-k)m}\left(t, C_i\right) q^m,$$

$$\phi_n\left(t, q, C_i\right) = v_{(n)0}\left(t\right) \quad (9)$$

$$+ \sum_{m=1}^{\infty} v_{(n)m}\left(t, C_i\right) q^m,$$

$$\phi_{n+k}\left(t, q, C_i\right) = v_{(n+k)0}\left(t\right)$$

$$+ \sum_{m=1}^{\infty} v_{(n+k)m}\left(t, C_i\right) q^m,$$

where C_i, $i = 1, 2, 3, \ldots$ are constants and to be determined and $m \in N$.

Now substituting (9) into (5) and equating the coefficient of like powers of q, we obtain

$$\mathscr{L}\left(v_{n0}\left(t\right)\right) + f\left(t\right) = 0, \qquad \mathscr{B}\left(v_{n0}\left(t\right), \frac{\partial v_{n0}\left(t\right)}{\partial t}\right) = 0, \quad (10)$$

$$\mathscr{L}\left(v_{n1}\left(t\right)\right) - \mathscr{L}\left(v_{n0}\left(t\right)\right)$$

$$= C_1\left[\mathscr{L}\left(v_{n0}\left(t\right)\right) + \mathscr{N}\left(v_{(n-k)0}\left(t\right), v_{n0}\left(t\right), v_{(n+k)0}\left(t\right)\right)\right],$$

$$\mathscr{B}\left(v_{n1}\left(t\right), \frac{\partial v_{n1}\left(t\right)}{\partial t}\right) = 0, \quad (11)$$

$$\mathscr{L}\left(v_{n2}\left(t\right)\right) - \mathscr{L}\left(v_{n1}\left(t\right)\right)$$

$$= C_1\left[\mathscr{L}\left(v_{n1}\left(t\right)\right) + \mathscr{N}\left(v_{(n-k)1}\left(t\right), v_{n1}\left(t\right), v_{(n+k)1}\left(t\right)\right)\right]$$

$$+ C_2\left[\mathscr{L}\left(v_{n0}\left(t\right)\right) + \mathscr{N}\left(v_{(n-k)0}\left(t\right), v_{n0}\left(t\right), v_{(n+k)0}\left(t\right)\right)\right],$$

$$\mathscr{B}\left(u_{n2}\left(n, t\right), \frac{\partial u_{n2}\left(n, t\right)}{\partial t}\right) = 0, \quad (12)$$

$$\mathscr{L}\left(v_{n3}\left(t\right)\right) - \mathscr{L}\left(v_{n2}\left(t\right)\right)$$

$$= C_1\left[\mathscr{L}\left(v_{n2}\left(t\right)\right) + \mathscr{N}\left(v_{(n-k)2}\left(t\right), v_{n2}\left(t\right), v_{(n+k)2}\left(t\right)\right)\right]$$

$$+ C_2\left[\mathscr{L}\left(v_{n1}\left(t\right)\right) + \mathscr{N}\left(v_{(n-k)1}\left(t\right), v_{n1}\left(t\right), v_{(n+k)1}\left(t\right)\right)\right]$$

$$+ C_3\left[\mathscr{L}\left(v_{n0}\left(t\right)\right) + \mathscr{N}\left(v_{(n-k)0}\left(t\right), v_{n0}\left(t\right), v_{(n+k)0}\left(t\right)\right)\right],$$

$$\mathscr{B}\left(v_{n3}\left(t\right), \frac{\partial v_{n3}\left(t\right)}{\partial t}\right) = 0,$$

$$\vdots \quad (13)$$

$$\mathscr{L}\left(v_{nj}\left(t\right)\right) - \mathscr{L}\left(v_{n(j-1)}\left(t\right)\right)$$

$$= \sum_{i=1}^{m} \sum_{j=m-1}^{0} C_i\left[\mathscr{L}\left(v_{nj}\left(t\right)\right)\right.$$

$$\left. + \mathscr{N}\left(v_{nj}\left(t\right), v_{(n-k)j}\left(t\right), v_{(n+k)j}\left(t\right)\right)\right],$$

$$\mathscr{B}\left(v_{nj}\left(t\right), \frac{\partial v_{nj}\left(t\right)}{\partial t}\right) = 0. \quad (14)$$

We obtained the zeroth order solution and first, second, third, and the general order solutions by solving (10)–(14). In the same way the remaining solutions can be determined. It has been observed that the convergence of the series (9) depends on the auxiliary constants C_i. If it is convergent at $q = 1$, one has

$$v_n\left(t; C_i\right) = v_{n0}\left(x, t\right) + \sum_{k \geq 1} v_{nk}\left(t; C_i\right). \quad (15)$$

Substituting (15) into (1), it results in the following expression for residual:

$$R\left(t, C_i\right) = \mathscr{L}\left(\tilde{v}_n\left(t, C_i\right)\right) + f\left(t\right) + \mathscr{N}\left(\tilde{v}_n\left(t, C_i\right)\right). \quad (16)$$

If $R(t, C_i) = 0$, then $\tilde{v}_n(t, C_i)$ is the exact solution of the problem. Generally it does not happen, especially in nonlinear problems.

For the determinations of auxiliary constants, C_i, $i = 1, 2, 3, \ldots$, there are different methods like Galerkin's method, Ritz method, least squares method, and collocation method. One can apply the method of least squares as follows:

$$J(C_i) = \int_a^b \int_\Omega R^2(t, C_i) \, dn \, dt, \tag{17}$$

where a and b are two values, depending on the nature of the given problem.

The auxiliary constants C_i can be optimally calculated as

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \frac{\partial J}{\partial C_3} \cdots \frac{\partial J}{\partial C_m} = 0. \tag{18}$$

The mth order approximate solution can be obtained by these obtained constants.

The convergence of OHAM is directly proportional to the number of optimal constants C_i.

3. Application of OHAM to Difference-Differential Equations

To show the validity and effectiveness of OHAM formulation to difference-differential equation, we apply it to Voltera equations in different form.

Model 1. We consider Voltera equation of the form

$$\frac{\partial v_n(t)}{\partial t} = v_n(t)(v_{n-1}(t) - v_{n+1}(t)), \quad n > 0, \ t \in [0, 1], \tag{19}$$

with initial conditions

$$v_n(0) = n. \tag{20}$$

The exact solution of (3.1) is given by

$$v_n(t) = \frac{n}{1 + nt}, \tag{21}$$

and we take

$$v_n(t) = v_{n0}(t) + p v_{n1}(t) + p^2 v_{n2}(t) + p^3 v_{n3}(t),$$

$$v_{n+1}(t) = v_{(n+1)0}(t) + p v_{(n+1)1}(t) + p^2 v_{(n+1)2}(t) + p^3 v_{(n+1)3}(t),$$

$$v_{n-1}(t) = v_{(n-1)0}(t) + p v_{(n-1)1}(t) + p^2 v_{(n-1)2}(t) + p^3 v_{(n-1)3}(t). \tag{22}$$

According to (1), we have

$$\mathcal{L} = \frac{\partial v_n(t)}{\partial t}, \qquad f(t) = 0,$$

$$\mathcal{N} = -v_n(t)(v_{n-1}(t) - v_{n+1}(t)). \tag{23}$$

Using (19), (20), and (22) into (5) and using the method discussed in Section 2 lead to the following.

Zeroth Order Problem. Consider

$$\frac{\partial v_{n0}(t)}{\partial t} = 0, \tag{24}$$

with

$$v_{n0}(0) = n, \tag{25}$$

from which we obtain

$$v_{n0}(t) = n. \tag{26}$$

First Order Problem. One has

$$\frac{\partial v_{n1}(t)}{\partial t} = (1 + C_1) \frac{\partial v_{n0}(t)}{\partial t} + C_1(v_{(n+1)0}(t) - v_{(n-1)0}(t)) v_{n0}(t), \tag{27}$$

with

$$v_{n1}(0) = 0. \tag{28}$$

Its solution is

$$v_{n1}(t) = 2C_1 n t. \tag{29}$$

Second Order Problem. Consider

$$\frac{\partial v_{n2}(t)}{\partial t} = (1 + C_1) \frac{\partial v_{n1}(t)}{\partial t} + C_2 \frac{\partial v_{n0}(t)}{\partial t}$$

$$+ C_1(v_{(n+1)0}(t) - v_{(n-1)0}(t)) v_{n1}(t)$$

$$+ C_1(v_{(n+1)1}(t) - v_{(n-1)1}(t)) v_{n0}(t)$$

$$+ C_2(v_{(n+1)0}(t) - v_{(n-1)0}(t)) v_{n0}(t), \tag{30}$$

with

$$v_{n2}(0) = 0, \tag{31}$$

$$v_{n2}(t) = 2(C_1 n t + C_1^2 n t + C_2 n t + 2C_1^2 n t^2). \tag{32}$$

Third Order Problem. One has

$$\frac{\partial v_{n3}(t)}{\partial t} = (1 + C_1) \frac{\partial v_{n2}(t)}{\partial t} + C_2 \frac{\partial v_{n1}(t)}{\partial t} + C_3 \frac{\partial v_{n0}(t)}{\partial t}$$

$$+ C_1(v_{(n+1)0}(t) - v_{(n-1)0}(t)) v_{n2}(t)$$

$$+ C_1(v_{(n+1)1}(t) - v_{(n-1)1}(t)) v_{n1}(t)$$

$$+ C_2(v_{(n+1)0}(t) - v_{(n-1)0}(t)) v_{n1}(t)$$

$$+ C_1(v_{(n+1)2}(t) - v_{(n-1)2}(t)) v_{n0}$$

$$+ C_2(v_{(n+1)1}(t) - v_{(n-1)1}(t)) v_{n0}(t)$$

$$+ C_3(v_{(n+1)0}(t) - v_{(n-1)0}(t)) v_{n0}(t), \tag{33}$$

Table 1: Comparisons of OHAM, exact, and ADM results at different values of n at $t = 1$.

n	OHAM	Exact	ADM	Abs error
1	0.333333	0.333333	0.333333	7.17103×10^{-7}
2	0.666665	0.666665	0.666665	1.43421×10^{-6}
3	0.999998	0.999998	0.999998	2.15131×10^{-6}
4	1.33333	1.33333	1.33333	2.86841×10^{-6}
5	1.66666	1.66666	1.66666	3.58551×10^{-6}
6	2	2	2	4.30262×10^{-6}
7	2.33333	2.33333	2.33333	5.01972×10^{-6}
8	2.66666	2.66666	2.66666	5.73682×10^{-6}
9	2.99999	2.99999	2.99999	6.45392×10^{-6}
10	3.33333	3.33333	3.33333	7.17103×10^{-6}

Table 2: Comparisons of first, second, and third order and exact solutions at different values of n at $t = 1$.

| n	First order	Second order	Third order	Exact sol.
1	0.527142	0.858619	0.333333	0.333333
2	1.05428	1.71724	0.666665	0.666665
3	1.58143	2.57586	0.999998	0.999998
4	2.10857	3.43447	1.33333	1.33333
5	2.63571	4.29309	1.66666	1.66666
6	3.16285	5.15171	2	2
7	3.68999	6.01033	2.33333	2.33333
8	4.21713	6.86895	2.66666	2.66666
9	4.74428	7.72757	2.99999	2.99999
10	5.27142	8.58619	3.33333	3.33333

with

$$v_{n3}(0) = 0, \tag{34}$$

$$v_{n3}(t) = \frac{2}{3}\left(3C_1 nt + 6C_1^2 nt + 3C_1^3 nt + 3C_2 nt + 3C_3 nt \right.$$
$$\left. + 12C_1^2 nt^2 + 12C_1^3 nt^2 + 6C_1 C_2 nt^2 + 4C_1^3 nt^3 \right). \tag{35}$$

Adding (26), (29), (32), and (35), we obtain

$$v_n(t) = v_{n0}(t) + v_{n1}(t, C_1) + v_{n2}(t, C_1, C_2)$$
$$+ v_{n3}(t, C_1, C_2, C_3), \tag{36}$$

$$v_n(t) = n + 2C_1 nt + 2\left(C_1 nt + C_1^2 nt + C_2 nt + 2C_1^2 nt^2\right)$$
$$+ \frac{2}{3}\left(3C_1 nt + 6C_1^2 nt + 3C_1^3 nt + 3C_2 nt + 3C_3 nt \right.$$
$$\left. + 12C_1^2 nt^2 + 12C_1^3 nt^2 + 6C_1 C_2 nt^2 + 4C_1^3 nt^3 \right). \tag{37}$$

For the computation of the constants C_1, C_2, and C_3 applying the method of least square mentioned in (16)–(18), we get

$$C_1 = -0.23642908394894221,$$
$$C_2 = 0.2344713680405294, \tag{38}$$
$$C_3 = -0.4015040325725273 \quad \text{for } n = 1.$$
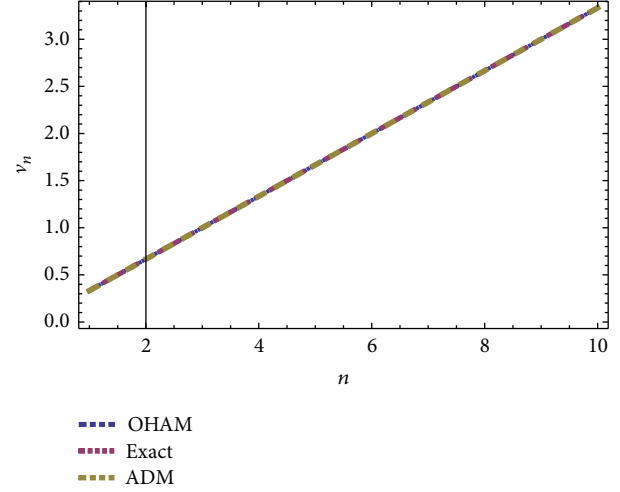
Figure 1: Comparison of OHAM, exact, and ADM solutions at $t = 1$.
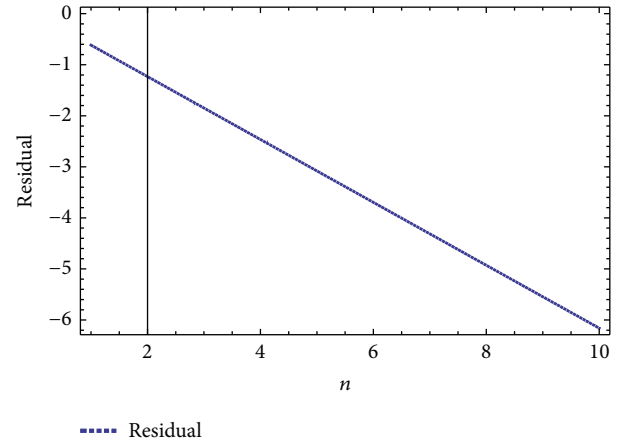
Figure 2: Residual of (19) at $t = 1$.

Putting these values in (37), we obtained the approximate solution of the form

$$v_n(t) = n - 0.472858nt + 2\left(0.05nt + 0.111797nt^2\right)$$
$$+ \frac{2}{3}\left(-0.914641nt + 0.179576nt^2 - 0.0528643nt^3\right). \tag{39}$$

The exact solution of the problem (3.1) is [11]

$$v_n(t) = \frac{n}{1 + nt}. \tag{40}$$

The Adomian solution with Padé approximation of (3.1) is given by [11]

$$v_n(t) = \frac{n}{1 + nt}. \tag{41}$$

Model 2. Now consider another form of Volterra equation

$$\frac{\partial v_n(t)}{\partial t} = v_n(t)\left(v_{n-1}(t) - v_{n+1}(t)\right)$$
$$+ v_n(t)\left(v_{n-2}(t) - v_{n+2}(t)\right), \quad n > 0, \; t \in [0,1], \tag{42}$$

with initial conditions

$$v_n(0) = n. \tag{43}$$

The exact solution of (42) is given by

$$v_n(t) = \frac{n}{1 + 6t}. \tag{44}$$

We take

$$v_n(t) = v_{n0}(t) + p v_{n1}(t) + p^2 v_{n2}(t)$$
$$+ p^3 v_{n3}(t),$$
$$v_{n+1}(t) = v_{(n+1)0}(t) + p v_{(n+1)1}(t)$$
$$+ p^2 v_{(n+1)2}(t) + p^3 v_{(n+1)3}(t),$$
$$v_{n-1}(t) = v_{(n-1)0}(t) + p v_{(n-1)1}(t)$$
$$+ p^2 v_{(n-1)2}(t) + p^3 v_{(n-1)3}(t), \tag{45}$$
$$v_{n+2}(t) = v_{(n+2)0}(t) + p v_{(n+2)1}(t)$$
$$+ p^2 v_{(n+2)2}(t) + p^3 v_{(n+2)3}(t),$$
$$v_{n-2}(t) = v_{(n-2)0}(t) + p v_{(n-2)1}(t)$$
$$+ p^2 v_{(n-2)2}(t) + p^3 v_{(n-2)3}(t).$$

According to (1), we have

$$\mathscr{L} = \frac{\partial v_n(t)}{\partial t}, \qquad f(t) = 0,$$
$$\mathscr{N} = -v_n(t)\left(v_{n-1}(t) - v_{n+1}(t)\right) \tag{46}$$
$$- v_n(t)\left(v_{n-2}(t) - v_{n+2}(t)\right).$$

Using (42), (43), and (45) into (5) and using the method discussed in Section 2 lead to the following.

Zeroth Order Problem. Consider

$$\frac{\partial v_{n0}(t)}{\partial t} = 0, \tag{47}$$

with

$$v_{n0}(0) = n, \tag{48}$$

from which we obtain

$$v_{n0}(t) = n. \tag{49}$$

First Order Problem. One has the following:

$$\frac{\partial v_{n1}(t)}{\partial t} = (1 + C_1)\frac{\partial v_{n0}(t)}{\partial t}$$
$$+ C_1\left(v_{(n+2)0}(t) - C_1 v_{(n-2)0}(t)\right) v_{n0}(t) \tag{50}$$
$$+ C_1\left(v_{(n+1)0}(t) - C_1 v_{(n-1)0}(t)\right) v_{n0}(t),$$

with

$$v_{n1}(0) = 0. \tag{51}$$

Its solution is

$$v_{n1}(t) = 6C_1 nt. \tag{52}$$

Second Order Problem. Consider

$$\frac{\partial v_{n2}(t)}{\partial t} = (1 + C_1)\frac{\partial v_{n1}(t)}{\partial t} + C_2\frac{\partial v_{n0}(t)}{\partial t}$$
$$+ C_1\left(v_{(n+2)1}(t) - v_{(n-2)1}(t)\right) v_{n0}(t)$$
$$+ C_2\left(v_{(n+2)0}(t) - v_{(n-2)0}(t)\right) v_{n0}$$
$$+ C_1\left(v_{(n+2)0}(t) - v_{(n-2)0}(t)\right) v_{n1}(t) \tag{53}$$
$$+ C_1\left(v_{(n+1)0}(t) - v_{(n-1)0}(t)\right) v_{n1}(t)$$
$$+ C_2\left(v_{(n+1)0}(t) - v_{(n-1)0}(t)\right) v_{n0}(t)$$
$$+ C_1\left(v_{(n+1)1}(t) - v_{(n-1)1}(t)\right) v_{n0}(t),$$

with

$$v_{n2}(0) = 0, \tag{54}$$
$$v_{n2}(t) = 6nt\left(C_1 + C_2 + C_1^2(1 + 6t)\right). \tag{55}$$

Third Order Problem. One has

$$\frac{\partial v_{n3}(t)}{\partial t} = (1 + C_1)\frac{\partial v_{n2}(t)}{\partial t} + C_2\frac{\partial v_{n1}(t)}{\partial t} + C_3\frac{\partial v_{n0}(t)}{\partial t}$$
$$+ C_1\left(v_{(n+2)2}(t) - v_{(n-2)2}(t)\right) v_{n0}(t)$$
$$+ C_2\left(v_{(n+2)1}(t) - v_{(n-2)1}(t)\right) v_{n0}(t)$$
$$+ C_3\left(v_{(n+2)0}(t) - v_{(n-2)0}(t)\right) v_{n0}(t)$$
$$+ C_1\left(v_{(n+2)1}(t) - v_{(n-2)1}(t)\right) v_{n1}(t)$$

$$+ C_2 \left(v_{(n+2)0}(t) - v_{(n-2)0}(t) \right) v_{n1}(t)$$

$$+ C_1 \left(v_{(n+2)0}(t) - v_{(n-2)0}(t) \right) v_{n2}(t)$$

$$+ C_1 \left(v_{(n+1)2}(t) - v_{(n-1)2}(t) \right) v_{n0}(t)$$

$$+ C_1 \left(v_{(n+1)0}(t) - v_{(n-1)0}(t) \right) v_{n2}(t)$$

$$+ C_1 \left(v_{(n+1)1}(t) - v_{(n-1)1}(t) \right) v_{n1}(t)$$

$$+ C_2 \left(v_{(n+1)0}(t) - v_{(n-1)0}(t) \right) v_{n1}(t)$$

$$+ C_1 \left(v_{(n+1)2}(t) - v_{(n-1)2}(t) \right) v_{n0}(t)$$

$$+ C_2 \left(v_{(n+1)1} - v_{(n-1)1} \right) v_{n0}(n,t)$$

$$+ C_3 \left(v_{(n+1)0}(n,t) - v_{(n-1)0}(n,t) \right) v_{n0}(t), \tag{56}$$

with

$$v_{n3}(0) = 0, \tag{57}$$

$$v_{n3}(t) = 6nt \left(C_2 + C_3 + 2C_1^2 (1 + 6t) \right.$$
$$\left. + C_1^3 (1 + 6t)^2 + C_1 \left(1 + 2C_2 (1 + 6t) \right) \right). \tag{58}$$

Adding (49), (52), (55), and (58), we obtain

$$v_n(t) = v_{n0}(t) + v_{n1}(t, C_1) + v_{n2}(t, C_1, C_2)$$
$$+ v_{n3}(t, C_1, C_2, C_3),$$

$$v_{n3}(t) = n + 6C_1 nt + 6nt \left(C_1 + C_2 + C_1^2 (1 + 6t) \right)$$
$$+ 6nt \left(C_2 + C_3 + 2C_1^2 (1 + 6t) + C_1^3 (1 + 6t)^2 \right.$$
$$\left. + C_1^3 (1 + 6t)^2 + C_1 \left(1 + 2C_2 (1 + 6t) \right) \right). \tag{59}$$

For the computation of the constants C_1, C_2, and C_3 applying the method of least square mentioned in (16)–(18), we get

$$C_1 = -0.23640805995815892,$$

$$C_2 = -0.039496242739004025, \tag{60}$$

$$C_3 = -0.012135699742215487 \quad \text{for } n = 1.$$

The exact solution of problem (42) is [9]

$$v_n(t) = \frac{n}{1 + 6t}. \tag{61}$$

The HAM solution of (42) is given by [9]

$$v_n(t) = (n + t) + h(n + 1)t + 3ht^2 + th(1 + h)(1 + 6n)$$
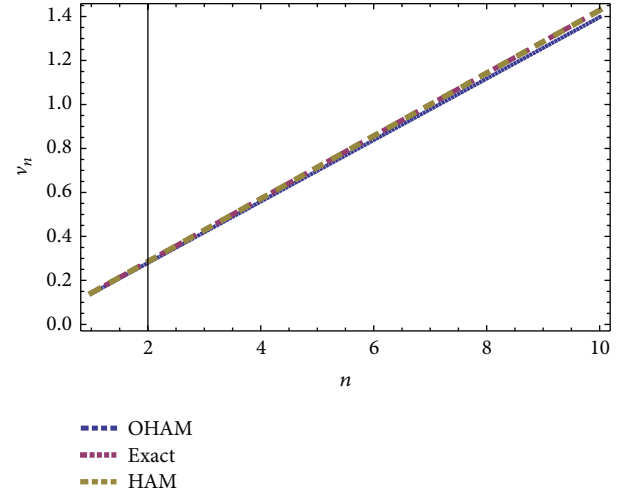$$+ \left(36h^2 n + 6h^2 + 3h \right) t^2 + 18h^2 t^3. \tag{62}$$

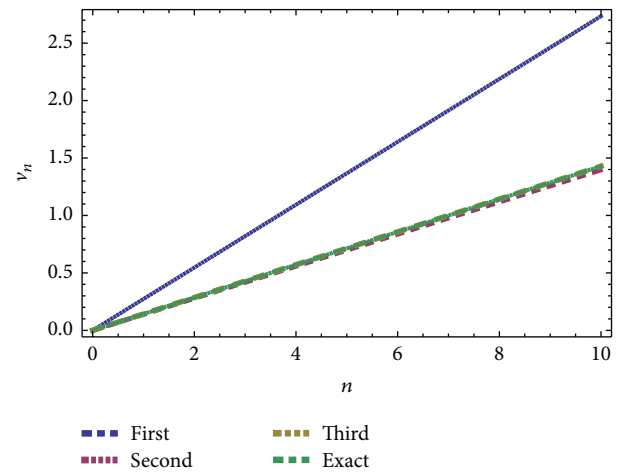Figure 3: Comparisons of OHAM, HAM and exact solutions at $t = 1$, $h = -1$.

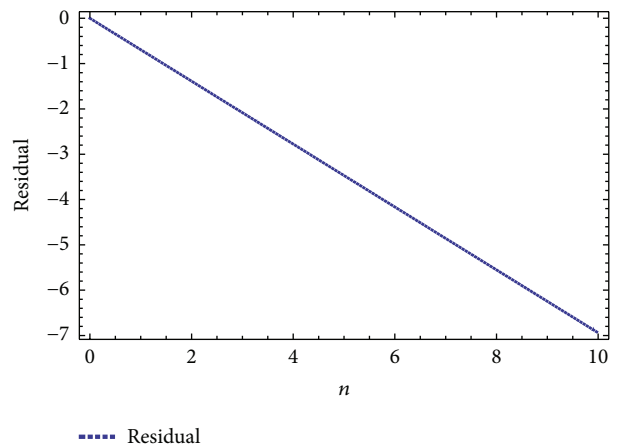Figure 4: Comparison of first, second, third of OHAM, HAM and exact solutions at $t = 1$, $h = -1$.

Figure 5: Residual of (42) at $t = 1$.

4. Results and Discussions

The formulation presented in Section 2 provides highly accurate solutions for the problems demonstrated in Section 3. We have used Mathematica 7 for most of our computational work. Table 1 and Figure 1 give the comparisons of OHAM with ADM and exact solutions for Model 1. Also the absolute errors at different values of n at $t = 1$ are given (Figure 4). Figure 3 shows the comparisons of OHAM results with HAM, exact at different values of n at $t = 1$ for Model 2. The convergence of approximate orders to exact solutions is given in Table 2 for Model 1 and Figure 5 for Model 2. The residuals have been plotted for Model 1 in Figure 2 and for Model 2 in Figure 5. We have concluded that the results obtained by OHAM are identical to the results obtained by ADM, HAM, and exact. OHAM converges rapidly with increasing of the order of approximation.

5. Conclusion

In this work, we have seen the effectiveness of OHAM [12–16] to differential-difference equations. By applying the basic idea of OHAM to differential-difference equations, we found it simpler in applicability, more convenient to control convergence, and involved less computational overhead. Therefore, OHAM shows its validity and great potential for the differential-difference equations arising in science and engineering.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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