Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 684594, 5 pages
http://dx.doi.org/10.1155/2014/684594

Research Article

Jørgensen's Inequality and Algebraic Convergence Theorem in Quaternionic Hyperbolic Isometry Groups

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Received 21 March 2014; Accepted 5 May 2014; Published 18 May 2014

Academic Editor: Luigi Rodino

We obtain an analogue of Jørgensen's inequality in quaternionic hyperbolic space. As an application, we prove that if the r-generator quaternionic Kleinian group satisfies I-condition, then its algebraic limit is also a quaternionic Kleinian group. Our results are generalizations of the counterparts in the n-dimensional real hyperbolic space.

1. Introduction

Jørgensen's inequality [1] gives a necessary condition for a nonelementary two-generator subgroup of $\mathrm{SL}(2,\mathbb{C})$ to be discrete, which involves the traces of one of the generators and the commutator of both generators, as follows.

Theorem A. *Let $f, g \in \mathrm{SL}(2,\mathbb{C})$. If each two-generator subgroup $\langle f, g\rangle$ is discrete and nonelementary, then*

$$\left|\mathrm{tr}^2(f) - 4\right| + \left|\mathrm{tr}([f,g]) - 2\right| < 1, \tag{1}$$

where $[f,g] = fgf^{-1}g^{-1}$ is the commutator of f and g and $\mathrm{tr}(\cdot)$ is the trace function.

Jørgensen's inequality has been generalized in many ways in real hyperbolic space [2, 3], complex hyperbolic space [4–6], and quaternionic hyperbolic space [7–9] and plays an important role in studying discreteness and algebraic convergence for real, complex, or quaternionic hyperbolic isometry group [10–14]. However, due to the noncommutative multiplication of the quaternions, Jørgensen's inequality in quaternionic hyperbolic isometry groups is relatively more complicated. To carry the results holding in real or complex hyperbolic geometry over to the quaternionic hyperbolic geometry, one sometimes has to reconsider these results involving the use of commutativity or the fact that purely imaginary complex numbers are isomorphic to $\mathbb{R}$.

In quaternionic hyperbolic space, the first step to generalize Jørgensen's inequality was taken by Kim and Parker [7] who gave a quaternionic hyperbolic version of Basmajian and Miner's stable basin theorem. Subsequently, Markham [9] and Kim [8] independently gave versions of Jørgensen's inequality for $\mathrm{Sp}(2,1)$. Recently, Cao, Tan, and Parker, [15, 16] obtained analogues of Jørgensen's inequality for nonelementary groups of isometries of quaternionic hyperbolic n-space generated by two elements, one of which is elliptic or loxodromic.

Shimizu's lemma deals with two-generator subgroup $\langle f, g\rangle$ with g being parabolic element and there are some generalizations to quaternionic hyperbolic space [7–9] for some special kinds of parabolic elements. But for g being screw parabolic, we only have analogues in the setting of 2-dimensional complex hyperbolic space [4, 6] and so forth. This gap is the main obstacle to investigate the discreteness and algebraic convergence theorem of groups in quaternionic hyperbolic space.

Our first aim is to erect generalizations of Jørgensen's inequality for two-generator nonelementary subgroup with some special kinds of elements in higher dimensional quaternionic hyperbolic isometry group $\mathrm{Sp}(n,1)$.

On the other hand, convergence of nonelementary subgroups of real or complex hyperbolic isometry groups is also another important problem. Let $\mathbb{G}$ be the n-dimensional sense-preserving Möbius group $M(\overline{\mathbb{R}}^n)$ or unitary group $U(1,n;\mathbb{C})$.

Definition 1. Let $\{G_{r,i}\}$ $(i = 1,2,\ldots)$ be a sequence of subgroups in group $\mathbb{G}$, where $G_{r,i}$ is generated by $g_{1,i}, g_{2,i}, \ldots, g_{r,i}$ and $r = 1,2,\ldots$. If, for each $t(1 \le t \le r)$,

$$g_{t,i} \longrightarrow g_t \in \mathbb{G} \quad \text{as } i \longrightarrow \infty, \tag{2}$$

then one says that $\{G_{r,i}\}$ algebraically converges to $G_r = \langle g_1, g_2, \ldots, g_r\rangle$.

The problem that under which condition the limit group G_r is also a Kleinian group if, for each i, $G_{r,i}$ is a Kleinian group was intensely studied. Using the well-known Jørgensen's inequality, Jørgensen and Klein [17] proved the following.

Theorem B. *If each $G_{r,i}$ is r-generator Kleinian subgroup of $\mathbb{G}$, where $\mathbb{G}$ is $M(\overline{\mathbb{R}}^2)$, then the limit group G_r is also a Kleinian group.*

However, the examples in [18] show that Theorem B could not be extended to n-dimensional cases $(n \ge 3)$ without any modifications. The reason for this phenomenon is that there is a distinction between the fixed point sets of elliptic elements in $M(\overline{\mathbb{R}}^2)$ and $M(\overline{\mathbb{R}}^n)$ $(n \ge 3)$. The reasoning mechanism in $M(\overline{\mathbb{R}}^2)$ mainly relies on the fact that each elliptic element has only two fixed points in $\mathbb{R}^2$. Because the fixed point set of an elliptic element of $M(\overline{\mathbb{R}}^n)$ $(n \ge 3)$ may be empty set or subset of $\overline{\mathbb{R}}^n$, we cannot use the same reasoning mechanism as in $n = 2$. By adding some condition(s) to control the fixed point set of elliptic element and using generalized Jørgensen's inequality, several authors have obtained their analogues in $M(\overline{\mathbb{R}}^n)$ when $n \ge 3$.

Martin [2] proved the following theorem.

Theorem C. *Let G_r be the algebraic limit group of a sequence of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^n)$ of uniformly bounded torsion. Then G_r is a Kleinian group.*

Martin also asked how one might weaken the hypothesis of uniformly bounded torsion. Fang and Nai [19] first gave condition A to consider such a question. Recently, Wang [20] and Yang [21] used EP-condition and condition A, respectively, to weaken Martin's uniformly bounded torsion and proved the following.

Theorem D. *Let $r < \infty$ and G_r be the algebraic limit group of a sequence of r-generator Kleinian groups $\{G_{r,i}\}$ of $M(\overline{\mathbb{R}}^n)$. If $\{G_{r,i}\}$ satisfies EP-condition (or condition A), then G_r is a Kleinian group.*

See details for the definitions of uniformly bounded torsion, EP-condition, and condition A in [18, 19, 21].

In [10], Cao gave a convergence theorem about algebraic limit group of complex Kleinian groups under IP-condition, as follows.

Theorem E. *Let G_r be the algebraic limit group of complex Kleinian groups $\{G_{r,i}\}$ of $U(1,n;\mathbb{C})$. If $\{G_{r,i}\}$ satisfies IP-condition, then G_r is a complex Kleinian group.*

Here, $\{G_{r,i}\}$ satisfying IP-condition means that, for any sequence $\{f_{i_k}\}(f_{i_k} \in G_{r,i_k})$, if, for each k, $\mathrm{card}(\mathrm{fix}(f_{i_k})) = \infty$ and $f_{i_k} \to f$ as $k \to \infty$ with f being parabolic or the identity, then $\{f_{i_k}\}$ has uniformly bounded torsion.

Our second aim is to investigate analogous condition mentioned above that an algebraic convergence theorem holds in the quaternionic hyperbolic space. We define the concept of uniformly bounded torsion as follows: a subset H of $\mathrm{Sp}(n,1)$ is said to have *uniformly bounded torsion* if there exists an integer M such that $\forall g \in H$,

$$\mathrm{ord}(g) \le M \quad \text{or} \quad \mathrm{ord}(g) = \infty. \tag{3}$$

And we call a nonelementary and discrete subgroup G of $\mathrm{Sp}(n,1)$ *a quaternionic Kleinian group*.

For a sequence of subgroups $\{G_{r,i}\}$ of $\mathrm{Sp}(n,1)$, we introduce the following condition.

Definition 2. One says that $\{G_{r,i}\}$ satisfies I-condition if any sequence $\{f_{i_k}\}$ $(f_{i_k} \in G_{r,i_k})$, satisfying the condition that, for each k, $\mathrm{card}[\mathrm{fix}(f_{i_k})] = \infty$ and $f_{i_k} \to I$ as $k \to \infty$, has uniformly bounded torsion. Here $\mathrm{card}(M)$ denotes the cardinality of a set M.

Our main results are the following theorems.

Theorem 3. *Suppose that f and $g \in \mathrm{Sp}(n,1)$ generate a discrete and nonelementary group. Then*

(i) *if f is parabolic or loxodromic, one has*

$$\max\left\{\|f - I_{n+1}\|, \|[f,g] - I_{n+1}\|\right\} \ge \frac{\left(2-\sqrt{3}\right)}{\sqrt{2}}, \tag{4}$$

$$\max\left\{\|f - I_{n+1}\|, \left\|g^{-1}fg - I_{n+1}\right\|\right\} \ge \frac{\left(2-\sqrt{3}\right)}{\sqrt{2}}; \tag{5}$$

(ii) *if f is elliptic, one has*

$$\begin{aligned}\max&\left\{\|f - I_{n+1}\|, \left\|\left[f,g^i\right] - I_{n+1}\right\| : i = 1,2,\ldots,n\right\}\\ &\ge \frac{\left(2-\sqrt{3}\right)}{\sqrt{2}},\end{aligned} \tag{6}$$

$$\max\left\{\left\|g^{-i}fg^i - I_{n+1}\right\| : i = 0,1,\ldots,n\right\} \ge \frac{\left(2-\sqrt{3}\right)}{\sqrt{2}}, \tag{7}$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm of an element.

Theorem 4. *Let G_r be the algebraic limit group of quaternionic Kleinian groups $\{G_{r,i}\}$ of $\mathrm{Sp}(n,1)$. If $\{G_{r,i}\}$ satisfies I-condition, then G_r is a quaternionic Kleinian group.*

2. Several Lemmas

Let $\mathbb{F}$ denote the field $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. We adopt the same notations and definitions as in [7, 16, 22, 23] such as $H^n_{\mathbb{H}}$, $U(1, n+1; \mathbb{F})$, discrete groups, limit sets, and elementary and nonelementary.

We first discuss some properties of elliptic elements. As in [22], for an elliptic element g, let Λ_0 and Λ_i, $i = 1, 2, \ldots, n$ be its negative and positive class of eigenvalues, respectively. Let $\mathrm{fix}(f)$ denote the set of fixed point(s) of f in $H^n_{\mathbb{F}}$. Then the fixed point set of g in $H^n_{\mathbb{F}}$ contains only one fixed point if $\Lambda_0 \neq \Lambda_i$, $i = 1, 2, \ldots, n$, and is a totally geodesic submanifold which is equivalent to $H^m_{\mathbb{F}}$ (resp., $H^m_{\mathbb{C}}$) if $\Lambda_0 \subset \mathbb{R}$ (resp., $\Lambda_0 \not\subseteq \mathbb{R}$) coincides with exact m of the class Λ_i, $i = 1, 2, \ldots, n$. In the latter case, the fixed point set of f in $\overline{H^n_{\mathbb{F}}}$ is $\overline{H^m_{\mathbb{F}}}$ or $\overline{H^m_{\mathbb{C}}}$, and we define $\dim(\mathrm{fix}(f)) = m$. The elliptic elements with only one fixed point in $H^m_{\mathbb{F}}$ are called regular elliptic elements, while the other elements are called boundary elliptic elements. We call an elliptic element g an irrational rotation if $e^{i\theta} \in \Lambda_t$ with irrational θ for some t.

Since $U(1, n+1; \mathbb{F})$ does not act effectively in $H^n_{\mathbb{F}}$, one always consider its projective group $PU(1, n; \mathbb{F}) = U(1, n; \mathbb{F})/Z(1, n; \mathbb{F})$. It is well known that the n-dimensional Möbius group $M(n)$ is isomorphic to *the identity component* of $PU(1, n+1; \mathbb{R})$, the projective orthogonal group. Each elliptic element is conjugate to an element with the form

$$\mathrm{diag}\left(\lambda_0, A\right) \in U(1; \mathbb{F}) \times U(n; \mathbb{F}), \quad \text{where } \lambda_0 \in \Lambda_0. \tag{8}$$

When $\mathbb{F} = \mathbb{R}$ and n is even, there are elliptic elements with $\lambda_0 = 1$ and the eigenvalues of positive class form $n/2$ conjugated pairs of complex numbers of norm 1. Those elements correspond to the so-called *fixed-point-free* elements in $M(\overline{\mathbb{R}}^{n-1})$. However, when n is odd, by our above isomorphism, $(n-1)$-dimensional Möbius group $M(n-1)$ cannot contain any *fixed-point-free* elements. In contrast to real hyperbolic space, we have regular elliptic elements in any dimensional complex and quaternionic hyperbolic space.

Using the quaternionic version in [24] of Schur's unitary triangularization theorem, we can prove the following lemma.

Lemma 5. *Let $f \in \mathrm{Sp}(n, 1)$ be an elliptic element of order m. If $2 \leq m < M$, then there is a constant $\delta(M)$ such that*

$$\left\| f - I_{n+1} \right\| > \delta(M). \tag{9}$$

Proof. Let the right complex eigenvalues of f be $\lambda_j = e^{i\theta_j}$ $(j = 1, \ldots, n+1)$. By Schur's unitary triangularization theorem of quaternionic version in [24], there is a matrix $U \in U(n+1; \mathbb{H})$ such that

$$Uf\overline{U}^T = \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_{n+1} \end{pmatrix}. \tag{10}$$

Hence $\|f - I_{n+1}\|^2 \geq \sum_{j=1}^{n+1} |\lambda_j - 1|^2 = 2(n+1) - 2\sum_{j=1}^{n+1} \cos\theta_j$. It follows from $f^m = I_{n+1}$ that there is j $(j = 1, 2, \ldots, n+1)$ such that $|\cos\theta_j| \neq 1$ and $\theta_j = 2p\pi/m$ (here p and m are prime). Hence

$$1 - \cos\theta_j \geq 1 - \left|\cos\theta_j\right| > 1 - \left|\cos\frac{\pi}{m}\right|. \tag{11}$$

Set $\delta(M) = \sqrt{1 - |\cos(\pi/m)|}$. Then $\delta(M)$ is the desired number. □

By the above lemma, we know that if the sequence $\{g_i\}$ of nontrivial unitary quaternionic transformations converges to the identity, then the orders of g_i converge to infinity. So a family of groups $\{G_{r,i}\}$ satisfies I-condition if there is no sequence $\{f_{i_k}\}(f_{i_k} \in G_{r,i_k})$ converging to the identity, such that $\mathrm{card}(\mathrm{fix}(f_{ik})) = \infty$ for each k.

When working in the matrix algebra, one has two choices, whether to use the spectral norm or the Hilbert-Schmidt norm. Following the ideas of Martin [2], we choose the Hilbert-Schmidt norm to construct our version of Jørgensen's inequality (Theorem 3) in $\mathrm{Sp}(n, 1)$.

The following lemma is a classification of elementary subgroups of $\mathrm{Sp}(n, 1)$.

Lemma 6 (cf. [25]). (1) *If G contains a parabolic element but no loxodromic element, then G is elementary if and only if it fixes a point in $\partial H^n_{\mathbb{H}}$;*

(2) *if G contains a loxodromic element, then G is elementary if and only if it fixes a point in $\partial H^n_{\mathbb{H}}$ or a point-pair $\{x, y\} \subset \partial H^n_{\mathbb{H}}$;*

(3) *G is purely elliptic; that is, each nontrivial element of G is elliptic; then G is elementary and fixes a point in $\overline{H^n_{\mathbb{H}}}$.*

By Lemma 6, we have the following lemmas.

Lemma 7. *If $G \subset \mathrm{Sp}(n, 1)$ is discrete nilpotent group without elliptic element, then G is elementary.*

Lemma 8 (cf. [2, Lemma 2.8]). *Let x and y be two distinct points in $\overline{H^n_{\mathbb{H}}}$. If $f \in \mathrm{Sp}(n, 1)$ interchanges x and y, then $\|f - I_{n+1}\| \geq \sqrt{2}$.*

The proofs of the following two lemmas follow from similar discussions in [2].

Lemma 9 (cf. [2, Lemma 4.1]). *Let $\langle f, g\rangle$ be discrete with f being parabolic or loxodromic element. If $\langle f, g^{-1}fg\rangle$ is elementary, then $\langle f, g\rangle$ is also elementary.*

Lemma 10 (cf.[2, Lemma 4.2]). *Let $\langle f, g\rangle$ be discrete with f being elliptic element. Let $m = \dim(\mathrm{fix}(f))$. If $G = \langle g^{-i}fg^i : i = 0, 1, \ldots, m+1\rangle$ is elementary, then $\langle f, g\rangle$ is elementary or $\|f - I_{n+1}\| \geq \sqrt{2}$.*

Lemma 11. *Suppose that f and $g \in \mathrm{Sp}(n, 1)$ generate a discrete and nonelementary group. Then*

$$\max\left\{\left\|f - I_{n+1}\right\|, \left\|g - I_{n+1}\right\|\right\} \geq \frac{\left(2 - \sqrt{3}\right)}{\sqrt{2}}, \tag{12}$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm of an element.

Proof of Lemma 11. We can choose $\mathbb{C}$ to be the subspace of $\mathbb{H}$ spanned by $\{1, \mathbf{i}\}$. With respect to this choice of $\mathbb{C}$ we can write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbf{j}$; that is, every element $a \in \mathbb{H}$ can be uniquely expressed as $a = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k} = (a_1 + a_2\mathbf{i}) + (a_3 + a_4\mathbf{i})\mathbf{j}$, where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$.

Similarly, $f \in \mathrm{Sp}(n,1)$ can be expressed as $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in M_{n+1}(\mathbb{C})$, the set of $(n+1)\times(n+1)$ complex matrices. This gives an embedding

$$\psi : \mathrm{Sp}(n,1) \longrightarrow \mathrm{GL}(2n+2,\mathbb{C}), \quad f \longmapsto \psi(f) = \begin{pmatrix} f_1 & f_2 \\ -\bar{f}_2 & \bar{f}_1 \end{pmatrix}. \tag{13}$$

We call $\psi(f)$ the complex representation of f. Obviously, ψ is an isomorphism between $\mathrm{Sp}(n,1)$ and $\psi(\mathrm{Sp}(n,1))$. Let $E = \mathrm{diag}(I_{n+1}, I_{n+1})$ and let $\Omega = \{f \in \mathrm{GL}(2n+2,\mathbb{C}) : \|f - E\| < 2 - \sqrt{3}\}$. Then Ω is the zassenhaus neighborhood [2] of $\mathrm{GL}(2n+2,\mathbb{C})$ and we have

$$\sqrt{2}\,\|f - I_{n+1}\| = \|\psi(f) - E\|. \tag{14}$$

Suppose that (12) does not hold. Then

$$\max\{\|\psi(f) - E\|, \|\psi(g) - E\|\} < 2 - \sqrt{3}. \tag{15}$$

By the property of Zassenhaus neighborhood, $\langle \psi(f), \psi(g)\rangle$ is nilpotent. Hence $\langle f, g\rangle$ is also nilpotent. By Selberg lemma, $\langle f, g\rangle$ contains a torsion free subgroup G with finite index. Hence G is nilpotent. By Lemma 7, G is elementary. By [22, Lemma 4.3.2], $L(G) = L(\langle f, g\rangle)$, which implies that $\langle f, g\rangle$ is elementary. This is a contradiction. The proof is complete. □

3. Proofs of Main Results

Proof of Theorem 3. (i) Suppose that (4) does not hold. Then

$$\max\{\|f - I_{n+1}\|, \|[f,g] - I_{n+1}\|\} < \frac{(2-\sqrt{3})}{\sqrt{2}}. \tag{16}$$

By Lemma 11, $\langle f, [f,g]\rangle$ is elementary. Since $\langle f, [f,g]\rangle = \langle f, gfg^{-1}\rangle$, by Lemma 9, $\langle f, g\rangle$ is elementary. This is a contradiction. Similarly, (5) holds.

(ii) Suppose that (6) does not hold. Then

$$\max\{\|\psi(f) - E\|, \left\|[\psi(f), \psi(g^i)] - E\right\| : i = 1, 2, \ldots, m+1\} < 2 - \sqrt{3}, \tag{17}$$

where m is the dimension of $\mathrm{fix}(f)$. Let $G = \langle \psi(f), [\psi(f), \psi(g^i)] : i = 1, 2, \ldots, m+1\rangle$. Then G is nilpotent. By the isomorphism of ψ, $\langle f, [f, g^i] : i = 1, 2, \ldots, m+1\rangle$ is also nilpotent. As in the reasoning in Lemma 11, $\langle f, [f, g^i] : i = 1, 2, \ldots, m+1\rangle$ is elementary. By Lemma 10, $\langle f, g\rangle$ is elementary. This is a contradiction. Similarly, (7) holds. The proof is complete. □

Proof of Theorem 4. We divide our proof into two parts.

(1) We first prove that G_r is discrete.

Suppose that G_r is not discrete. Then there is a sequence $\{g_j\}$ of G_r such that

$$g_j \longrightarrow I_{n+1} \quad \text{as } j \longrightarrow \infty, \tag{18}$$

and we can find a corresponding sequence $\{g_{j_k,i_k}\}$ such that

$$g_{j_k,i_k} \in G_{r,i_k}, \qquad g_{j_k,i_k} \longrightarrow I_{n+1} \quad \text{as } k \longrightarrow \infty. \tag{19}$$

Since $\{G_{r,i}\}$ satisfies I-condition and $G_{r,i}$ is discrete for each i, we may assume that, for each k, g_{j_k,i_k} is parabolic, loxodromic, or regular elliptic element.

If g_{j_k,i_k} is parabolic, loxodromic, for each k, there is at least one generator of G_{r,i_k}, say g_{1,i_k}, such that $\langle g_{1,i_k}, g_{j_k,i_k}\rangle$ is nonelementary, which is a contradiction to Theorem 3.

If g_{j_k,i_k} is a regular elliptic element, by Theorem 3, $\langle g_{t,i_k}, g_{j_k,i_k}\rangle$ is discrete and elementary for $t = 1, \ldots r$. By Lemma 6, each $\langle g_{t,i_k}, g_{j_k,i_k}\rangle$ is purely elliptic or contains a loxodromic element. If the latter case occurs, then $\{g^2_{j_k,i_k}\}$ are sequence of boundary elliptic elements which converges to the identity. This is a contradiction to our assumption of I-condition, while, for the first case, each g_{t,i_k} shares a fixed point in $H^n_{\mathbb{H}}$. This is also a contradiction.

The above proves the discreteness of G_r.

(2) We prove that G_r is nonelementary.

We assume that $G_{r,i} = \langle g_{1,i}, g_{2,i}, \ldots, g_{r,i}\rangle$ and $g_{t,i} \to g_t$ as $i \to \infty$; that is, $G_r = \langle g_1, g_2, \ldots, g_r\rangle$. The proof of part (1) implies that each g_t, $t = 1, \ldots, r$ is not the identity.

Since $G_{r,i}$ is discrete and nonelementary, there exist two loxodromic elements f_i and h_i having no common fixed points. Since f_i and h_i are words of the generators $\{g_{t,i}\}$, we can get the limit f and g by the word convergence of f_i and h_i, respectively. It remains to prove that $\langle f, h\rangle \subset G_r$ is nonelementary.

We first show that f is parabolic or loxodromic. Since $\langle f, g\rangle$ is discrete, f cannot be an irrational rotation. Suppose that there is a positive number M such that $f^M = I_{n+1}$. Then $f_i^M \neq I_{n+1}$ and

$$f_i^M \longrightarrow I_{n+1} \quad \text{as } i \longrightarrow \infty. \tag{20}$$

Hence for sufficiently large i,

$$\max\left\{\left\|f_i^M - I_{n+1}\right\|, \left\|[f_i^M, h_i^t] - I_{n+1}\right\| \mid t = 1, 2, \ldots, n+1\right\} < \frac{(2-\sqrt{3})}{2}. \tag{21}$$

By Theorem 3, $\langle f_i^M, h_i\rangle$, which are subgroups of discrete group $\langle f_i, h_i\rangle$, are elementary for sufficiently large i. This implies that $\langle f_i, h_i\rangle$ is elementary, which is a contradiction.

We then show that $\langle f, h\rangle$ is nonelementary.

Suppose on the contrary that $\langle f, h\rangle$ is elementary. As in [2, Proposition 2.7], we can show that $\langle f, h\rangle$ is virtually Abelian. Thus there exist two integers t and s such that

$$\left[f^{t}, hf^{s}h^{-1}\right] = I_{n+1}. \tag{22}$$

Let $q_i = [f_i^t, h_i f_i^s h_i^{-1}]$. Then

$$q_i \in \langle f_i, h_i\rangle, \quad q_i \neq I_{n+1},\ q_i \longrightarrow I_{n+1} \quad \text{as } i \longrightarrow \infty. \tag{23}$$

As in the proof of part (1), we can get a contradiction. Thus $\langle f, h\rangle$ is nonelementary.

The proof is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by NSFs of China (11071063 and 10801107) and NSF of Guangdong Province (no. S2011010000735).

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